Biquadratic S-Patch in Bézier form

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ABSTRACT
Mutual conversions between triangular and quadrilateral meshes need the same degree of both diagonal and boundary curves of quadrilateral meshes. New approach to quadrilateral patches, S-Patches, offers such possibility. The Bézier approach of Smart patches (S-Patch) in the biquadratic case is analyzed. Dependencies among the control points are derived. BS-Patches are presented. Close relation between Bézier triangles and BS-Patches is found. Condition for smooth concatenation of biquadratic BS-Patches is formulated.

Keywords
Parametric modeling, S-Patch, Bézier patch, Bézier triangle.

1. INTRODUCTION
Two types of meshes, triangular and quadrilateral are used very often in various fields of computer graphic modeling [Pup_11]. Mutual conversions between them are the aim of interest for a long time, e.g. [Bru_80], [Far_86], [Gol_87], [Far_88]. Due to different geometric properties and incompatibility in these two types of meshes, it is difficult to use both kinds of patches in the same CAD system. Approximation techniques of the meshes mutual substitution are analyzed e.g. in [Lai_99]. Conversion of triangular patch to three quadrilateral ones is analyzed in [Hu_96]. Idea of degenerated rectangular meshes is used in [Hu_01]. Functional composition of the meshes is studied in [Fen_99] and [Las_02].
In [Hol_99] some properties of diagonal curve of the quadrilateral patch are analyzed. Importance of the main diagonal curves is recognized in [Ska_10], where the concept of Smart-Patches (S-Patch) is introduced. Here the main idea is to find suitable conditions, when both diagonal and boundary curves are the parametric curves of the same degree. It gives us a possibility to find simple and direct correlation between triangular and quadrilateral patches. In [Ska_10] bicubic patches in Hermit polynomial basis are analyzed.

In our approach we inspired with the idea mentioned above. We prefer Bernstein-Bézier form of polynomial basis functions. It is more convenient, due to the fact that we obtain the same formal description of both triangular and quadrilateral patches.
In this paper only the biquadratic case of S-Patches is analyzed in detail. (The importance and usefulness of biquadratic quadrilateral patches and quadratic triangular patches can be found e.g. in [Raz_05], [Boc_09].) Proves of main properties are presented in a very detailed way due to the fact, that in similar way the analysis of the patches of higher degree can be realized.
The rest of the paper is organized as follows. In section 2 biquadratic S-Patch is introduced in general form of simple polynomial basis functions (1, u, u^2). It gives us a basic form of S-Patch. In section 3 Bernstein-Bézier polynomial basis is used. Mutual dependencies of control points are analyzed. In section 4 it is shown when diagonal curves of S-Patch can be expressed as Bézier curves of proper 'diagonal' control points. Such patches are introduced as BS-Patches. In section 5 it is shown, that BS-Patches we can split to Bézier triangle patches. In section 6 conditions of smooth concatenation of BS-Patches are formulated.

2. PROBLEM FORMULATION
Let us consider biquadratic parametric patch

$$X(u,v) = \mathbf{uRv}^T = \left(1\,u\,u^2\right)\begin{pmatrix} R_{00} & R_{01} & R_{02} \\ R_{10} & R_{11} & R_{12} \\ R_{20} & R_{21} & R_{22} \end{pmatrix}\begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix}^T \qquad (1)$$

Our goal is to find out the conditions for all boundary lines and both main diagonals $D_1(u)$, $D_2(u)$ to be the lines of the same degree.

$$D_1(u) = X(u,u) = \mathbf{u}\,\mathbf{R}\,\mathbf{u}^T = A_0 + A_1 u + A_2 u^2 \tag{2}$$

$$\begin{aligned} D_2(u) &= X(u, 1-u) = \mathbf{uR}(\mathbf{1}-\mathbf{u})^T = \\ &= \mathbf{uR}\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix}\mathbf{u}^T = B_0 + B_1 u + B_2 u^2 \end{aligned} \tag{3}$$

Such patches are named S-Patches [Ska_10].

Theorem 1. Biquadratic patch (1) is a S-Patch iff

$$R_{12} = R_{21} = R_{22} = 0,$$

i.e.

$$X(u,v) = \mathbf{u}\begin{pmatrix} R_{00} & R_{01} & R_{02} \\ R_{10} & R_{11} & 0 \\ R_{20} & 0 & 0 \end{pmatrix}\mathbf{v}^T \tag{4}$$

Proof: Resulting matrix for (3) is

$$\begin{pmatrix} R_{00}+R_{01}+R_{02} & -R_{01}-2R_{02} & R_{02} \\ R_{10}+R_{11}+R_{12} & -R_{11}-2R_{12} & R_{12} \\ R_{20}+R_{21}+R_{22} & -R_{21}-2R_{22} & R_{22} \end{pmatrix}$$

So, the conditions (2) and (3) lead to the equations

$$\begin{array}{lllcl} R_{12} & +R_{21} & & = & 0 \\ & & R_{22} & = & 0 \\ R_{12} & -R_{21} & -2R_{22} & = & 0 \\ & & R_{22} & = & 0 \end{array}. \tag{5}$$

It is obvious, that linear system (5) has trivial solution only

$$R_{12} = R_{21} = R_{22} = 0.$$

QED.

Corollary. All parametric lines of a biquadratic S-Patch are curves of degree $d \le 2$.

Proof: Let us consider general parametric line of S-patch. Using standard transformations of (4) for $L(u) = X(u, a+bu)$ we obtain the resulting formula

$$L(u) = \mathbf{u}\begin{pmatrix} Q_{00} & Q_{01} & Q_{02} \\ Q_{10} & Q_{11} & 0 \\ Q_{20} & 0 & 0 \end{pmatrix}\mathbf{u}^T$$

where

$$\begin{aligned} Q_{00} &= R_{00} + aR_{01} + a^2 R_{02}, \\ Q_{01} &= bR_{01} + 2abR_{02}, \\ Q_{10} &= R_{10} + aR_{11}, \\ Q_{02} &= b^2 R_{02}, \\ Q_{11} &= bR_{11}, \\ Q_{20} &= R_{20}. \end{aligned}$$

Q.E.D.

3. BÉZIER FORM OF S-PATCH

Let us express the biquadratic S-Patch in Bézier form

$$X(u,v) = \mathbf{uRv}^T = \mathbf{u}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}\mathbf{P}\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{v}^T \tag{6}$$

From (6) we can find control points P_{ij}.

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}^{-1}\mathbf{R}\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \tag{7}$$

In more detailed way

$$\begin{pmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} R_{00} & R_{01} & R_{02} \\ R_{10} & R_{11} & 0 \\ R_{20} & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Explicit vector form of (7) gives

$$\begin{pmatrix} P_{00} \\ P_{01} \\ P_{02} \\ P_{10} \\ P_{11} \\ P_{12} \\ P_{20} \\ P_{21} \\ P_{22} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 4 & 0 & 0 \\ 4 & 0 & 2 & 0 & 0 & 0 \\ 4 & 2 & 2 & 0 & 0 & 1 \\ 4 & 4 & 2 & 4 & 0 & 2 \\ 4 & 0 & 4 & 0 & 4 & 0 \\ 4 & 2 & 4 & 0 & 4 & 2 \\ 4 & 4 & 4 & 4 & 4 & 4 \end{pmatrix}\begin{pmatrix} R_{00} \\ R_{01} \\ R_{10} \\ R_{02} \\ R_{20} \\ R_{11} \end{pmatrix} \tag{8}$$

Rank of the matrix **M** in (8), *rank*(**M**) = 6.

In the text below (Figures 2 – 4) we use vector indexing and Cartesian indexing of control points of a patch – Fig. 1.

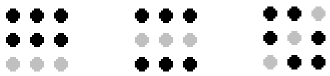
1	2	3		00	01	02
4	5	6		10	11	12
7	8	9		20	21	22
	a)				b)	

Figure 1. a) vector indexing, b) Cartesian indexing of control points.

The first six rows of matrix **M** in (8) (Fig. 2a)) are linearly dependent, as we can write

$$P_{12} = P_{00} - 2P_{01} + P_{02} - P_{10} + 2P_{11}.$$

Similarly, the sets of rows in (8) (i.e. the rows of matrix **M**) (1,2,3,7,8,9), (4,5,6,7,8,9), (1,4,7,2,5,8), (1,4,7,3,6,9), (2,5,8,3,6,9) are linearly dependent too.



a) b) c)

Figure 2. Configurations of dependent 6-element sets of control points P_{ij} (black).

For the configuration of points Fig. 2c), i.e. for the configuration of rows (1,2,4,6,8,9) of the matrix **M** in (8) we can find relation

$$P_{22} = P_{12} + P_{21} - P_{01} - P_{10} + P_{00}.$$

The symmetric configuration of rows (2,3,4,6,7,8) is linearly dependent too.

We can formulate the condition for the independency of sets of the control points P_{ij}.

Theorem 2. Only the eight six-element sets of control points mentioned above are linearly dependent.

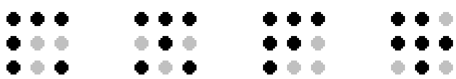
Proof can be done by computing the determinant of all $\binom{9}{6} = 84$ 6x6 submatrices of **M** in (8).

(Due to symmetries it is enough to parse not more than 21 cases.)

QED.

Such configurations of control points cannot be used for the patch determination.

Fig. 3 gives examples of independent sets of control points. Symmetrical cases are independent too.



a) b) c) d)

Figure 3. Some configurations of independent 6-element sets of control points (black).

Useful properties of some independent configurations of control points are:

1. configurations from Fig. 3a), 3b) involve all corner control points,
2. configurations from Fig. 3a), 3c) involve full information of the pair of neighbour boundary lines.

4. BS-PATCH

Let us analyze the relations between main diagonal $D_1(u)$ of S-Patch (4)

$$D_1(u) = \mathbf{u} \begin{pmatrix} R_{00} & R_{01} & R_{02} \\ R_{10} & R_{11} & 0 \\ R_{20} & 0 & 0 \end{pmatrix} \mathbf{u}^T \tag{9}$$

and proper Bézier diagonal – i.e. the curve defined on the set of 'diagonal' control points P_{00}, P_{11}, P_{22} – Bézier diagonal curve

$$D_{1B}(u) = \mathbf{u} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} P_{00} \\ P_{11} \\ P_{22} \end{pmatrix}. \tag{10}$$

Using the abbreviation

$$\Re = \begin{pmatrix} R_{00} & R_{01} & R_{10} & R_{02} & R_{20} & R_{11} \end{pmatrix}^T,$$

for (9) we obtain

$$D_1(u) = \mathbf{u} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \Re \tag{11}$$

On the other hand, for Bézier diagonal curve the resulting expression of (10) is

$$D_{1B}(u) = \mathbf{u} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 1 & 0.5 \end{pmatrix} \Re. \tag{12}$$

As the matrices in (11) and (12) differ in the last column only, the condition $R_{11} = 0$ must be fulfilled for both Bézier and S-patch diagonals to be identical.

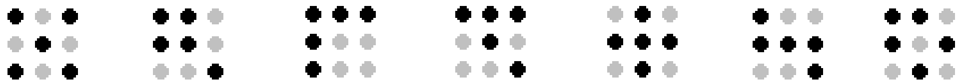
The same result we obtain for the diagonal curves $D_2(u)$ and $D_{2B}(u)$.

$$D_2(u) = \mathbf{u} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix} \Re, \tag{13}$$

$$D_{2B}(u) = \mathbf{u} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 1 & 0.5 \end{pmatrix} \Re. \tag{14}$$

Here the matrices in (13) and (14) also differ in the last column only.

Just proved relations among the diagonal lines can be formulated as the theorem below.



a) b) c) d) e) f) g)

Figure 4. 5-element sets of control-points. a),b) – non independent, c)–g) independent sets.

Theorem 3. $D_1(u)= D_{1B}(u)$ if and only if $R_{11}=0$. Moreover, equality of these diagonals automatically implies the equality of $D_2(u)= D_{2B}(u)$.

On the base of the Theorem 3 we can introduce biquadratic BS-Patch, i.e. patch in the form as follows

$$X(u,v)=\mathbf{u}\begin{pmatrix} R_{00} & R_{01} & R_{02} \\ R_{10} & 0 & 0 \\ R_{20} & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix}\mathbf{v}^T . \qquad (15)$$

In this case mutual relations among Bézier control points P_{ij} (8) are reduced to

$$\begin{pmatrix} P_{00} \\ P_{01} \\ P_{02} \\ P_{10} \\ P_{11} \\ P_{12} \\ P_{20} \\ P_{21} \\ P_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 2 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 2 & 0 \\ 2 & 0 & 2 & 0 & 2 \\ 2 & 1 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} R_{00} \\ R_{01} \\ R_{10} \\ R_{02} \\ R_{20} \end{pmatrix} . \qquad (16)$$

We can see that now the corner control points (Fig. 4a)) are dependent,

$$P_{22} = P_{20} + P_{02} - P_{00} .$$

It means that the corner control points create rhomboids.

Similarly the quaternion of neighbour control points (Fig. 4b)) is dependent too,

$$P_{11} = P_{10} + P_{01} - P_{00} .$$

Examples of non independent and independent 5-element sets of control points of BS-Patches are presented in Fig. 4 c) – g). E.g. for independent pentad from Fig. 4e) the rest of control points can be represented as

$$\begin{aligned} P_{00} &= P_{01} + P_{10} - P_{11} \\ P_{02} &= P_{01} + P_{12} - P_{11} \\ P_{20} &= P_{21} + P_{10} - P_{11} \\ P_{22} &= P_{21} + P_{12} - P_{11} \end{aligned} . \qquad (17)$$

5. BS-PATCH AND BÉZIER TRIANGLES

As both diagonal and boundary curves of BS-Patches are Bézier curves, it is meaningful to analyze the triangle patches. We shall demonstrate that there is a very close connection between the Cartesian BS-Patch and a pair of triangular Bézier patches. This relation is formulated for the case $n=2$.

Let us consider triangular mesh of nodes

$$P_{ijk} \quad 0 \le i, j, k \le n, \quad i + j + k = n$$

where nodes $P_{i_1 j_1 k_1}, P_{i_2 j_2 k_2}$ are neighbour, if

$$|i_1 - i_2| + |j_1 - j_2| + |k_1 - k_2| = 2 .$$

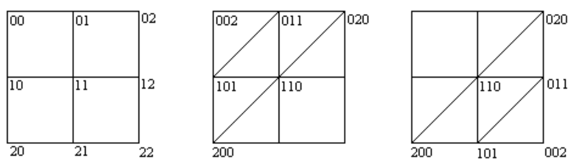
Bézier triangular patch is defined as

$$P_\Delta(u, v, w, \{P_{ijk}\}) = \sum_{(i,j,k)} \frac{n!}{i!\,j!\,k!} u^i v^j w^k P_{ijk} \qquad (18)$$

where

$0 \le u, v, w \le 1,\ u + v + w = 1,\ 0 \le i, j, k \le n,\ i + j + k = n$.

Let us consider quadratic BS-Patch defined on the set of control points $P_{ij}, 0 \le i, j \le 2$. Let us consider Cartesian and triangular indexing of these control points according to Fig. 5.



a) b) c)

Figure 5. Cartesian a) and triangular b), c) indexing of control nodes for *n* = 2.

Theorem 4. BS-Patch defined on control points $P_{ij}, 0 \le i, j \le 2$ is the same surface as the pair of triangular Bézier patches, defined on the sets of proper control points.

Proof. Let us use the independent set of control points according to Fig. 4e). Solving proper subsystem of (16)

$$\begin{pmatrix} P_{01} \\ P_{10} \\ P_{11} \\ P_{12} \\ P_{21} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 2 & 0 \\ 2 & 1 & 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} R_{00} \\ R_{01} \\ R_{10} \\ R_{02} \\ R_{20} \end{pmatrix}$$

and inserting the solution into (15) we obtain

$$\mathbf{R} = \begin{pmatrix} P_{01} + P_{10} - P_{11} & 2(P_{11} - P_{10}) & P_{10} + P_{12} - 2P_{11} \\ 2(P_{11} - P_{01}) & 0 & 0 \\ P_{01} + P_{21} - 2P_{11} & 0 & 0 \end{pmatrix}. \quad (19)$$

Let us express triangular patch on the nodes

$P_{00}, P_{01}, P_{02}, P_{10}, P_{11}, P_{20}$ – Fig. 5a).

Using the triangular indexing – Fig. 5b), we have

$$\begin{aligned} &P_\Delta(u,v,w) = \\ &= u^2 P_{200} + v^2 P_{020} + w^2 P_{002} + \\ &\quad + 2uvP_{110} + 2uwP_{101} + 2vwP_{011} \end{aligned} \quad (20)$$

Rewriting it to Cartesian indexes – Fig. 5a) we obtain

$$\begin{aligned} &P_\Delta(u,v,w) = \\ &= u^2 P_{20} + v^2 P_{02} + w^2 P_{00} + \\ &\quad + 2uvP_{11} + 2uwP_{10} + 2vwP_{01} \end{aligned}$$

Using (17) and excluding w, as $w = 1 - u - v$ leads to the final form of triangle patch

$$\begin{aligned} &P_\Delta(u,v,w) = P_{01} + P_{10} - P_{11} + \\ &+ 2u(P_{11} - P_{01}) + 2v(P_{11} - P_{10}) + \\ &+ u^2(P_{01} + P_{21} - 2P_{11}) + v^2(P_{10} + P_{12} - 2P_{11}). \end{aligned}$$

We have obtained the same relation as (19).

For the triangle defined on control points form Fig. 5c) the process of proving is similar. Difference is in used parametrization only: in (20) we use $(1-u)$ instead of u and $(1-v)$ instead of v.

For triangles with the diagonal defined on control points P_{00}, P_{11}, P_{22} in (20) we have to use parametrizations $(1-u), v$, $u, (1-v)$ respectively.

QED.

Corollary. Just proved theorem gives us an important generalizaton of the trivial fact that a bilinear patch can be decomposed to two triangles iff the quaternion of control points is planar.

6. SMOOTH CONCATENATION OF BS-PATCHES

Let us consider 5-element set of independent control points form Fig. 4e). Condition (17) says that the set of control points creates four rhomboids – Fig. 6. Here we can distinguish three types of control points: 'central', 'crosswise' and 'dependent'.

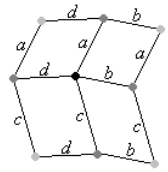
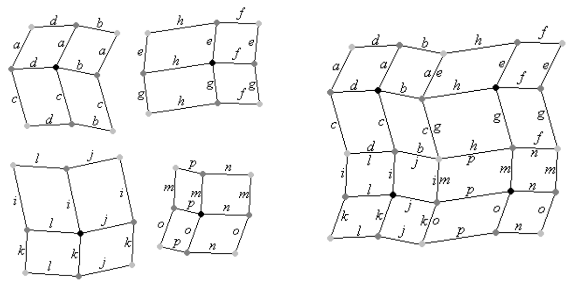


Figure 6. Resulting geometry of control points for BS-patch. Different types of control points are distinguished: black – central one, dark – crosswise ones, light – dependent ones.

Let us consider four general BS-patches – Fig. 7a). The conditions for concatenation of the patches are obvious – Fig 7b):

a=e, c=g, i=m, k=o, d=l, b=j, h=p, f=n.

This condition can be formulated more generally in the following way.



a) b)

Figure 7. Concatenation of BS-patches.

a) Four independent BS-patches. b) Concatenated BS-patches.

Let there are two open polylines

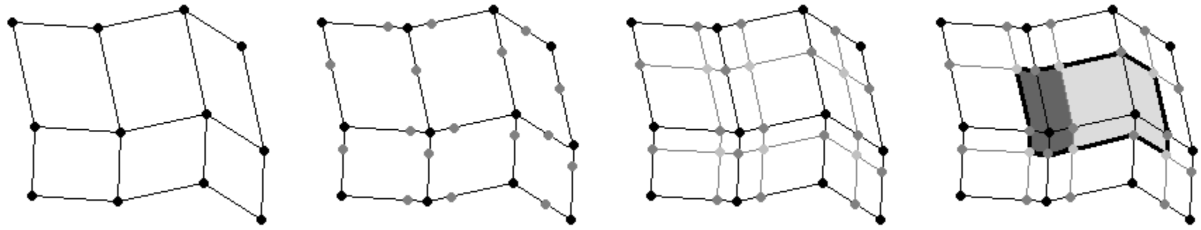
$\Lambda_1 = (P_0 P_1 P_2 \cdots P_n)$ and $\Lambda_2 = (R_0 R_1 R_2 \cdots R_m)$.

Let us consider the lattice of nodes

$$\Lambda = (Q_{i,j} : 0 \le i \le n, 0 \le j \le m)$$

where

$$x_{i,j} = x_{P_i} + x_{R_j} - x_{R_0}, \quad y_{i,j} = y_{R_j} + y_{P_i} - x_{P_0}.$$



a) b) c) d)

Figure 8. Smooth concatenation of BS-patches according to the steps a) – d) below.

Theorem 5. Surface is set of BS-Patches iff set of control points is a lattice of polylines.

Moreover, if we demand smooth concatenation of BS-Patches, edges *b,h* must be parallel. Similarly, edges *c,i* must be parallel too. It means that the quaternion of central control points from Fig. 7b) creates the vertices of rhomboid.

Construction

Given two polylines

$$\Lambda_1 = \left(P_0 P_1 \cdots P_n\right),\ \Lambda_2 = \left(R_0 R_1 \cdots R_m\right),$$

given two sets

$$\pi = \left(p_0, p_1, \cdots, p_{n-1}\right), \rho = \left(r_0, r_1, \cdots, r_{m-1}\right), 0 < p_i, r_j < 1,$$

we can construct smooth concatenation of BS-patches according to the steps below.

a) We suppose that the central control points of BS-patches create a lattice.
b) Crosswise control points can be found as a ratio of neighbour central control points.
c) Dependent control points (corners of BS-patches) are found according to the (17).
d) Concatenation consists of full-defined BC-patches.

Fig. 8 illustrates the above described construction.

7. CONCLUSIONS

In the presented study we have described the Bézier form of S-Patches in the biquadratic case.

- Dependencies among the control points are derived.
- BS-Patches are introduced.
- Close relation between Bézier triangles and BS-Patches is found.
- Condition for smooth concatenation of biquadratic BS-Patches is formulated.

We can see that biquadratic BS-patches are very convenient for mutual conversion between triangular and quadrilateral patches. On the other hand, smooth concatenation of such patches is 'too rigid' and perhaps it is hardly used for the shape expression in general case. Future work will be focused to

- more detailed analysis of relationship between S-Patches and BS-Patches,
- patches of higher degree.

8. ACKNOWLEDGMENTS

This work is supported by the grant GACR 105/09/1830 of the Grant Agency CR and the research plan AVOZ 30860518 of the Academy of Sciences of the Czech Republic.

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