A new Image Interpolation Technique using Exponential B-Spline

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ABSTRACT

In this paper, we propose a new interpolation technique using exponential B-spline, which is super-set of the B-spline. An interpolation kernel of exponential B-spline was proposed using IIR based technique by Unser. As another approach, this paper presents an exponential B-spline interpolation kernel using simple mathematics based on Fourier approximation. A high signal to noise ratio can be achieved because exponential B-spline parameters can be set depending on the signal characteristics. The analysis of these interpolated kernels shows they have better performance in high and low frequency components as compared to other conventional nearest neighbor, linear, spline based methods.

Keywords

Exponential B-spline, Interpolation, Image zooming.

1. INTRODUCTION

Nowadays, interpolation technique has a large usage in the field of computer vision, digital photography, multimedia and electronic publishing for generating preview images. In image compression, digital zooming, computed tomography (CT), magnetic resonance imaging (MRI), image reconstruction requires interpolation to approximate the discrete data to get the enhanced image, hence there is a need of good interpolatory scheme that can efficiently restore the signal and can help to reduce the cost of systems.

Interpolation is a method of constructing new data points from a discrete set which are being fitted in the continuous curves and then sampling at a higher rate interpolates the given data. In the early years, simple algorithms, such as nearest neighbor or linear interpolation, were used for sampling. After the introduction of sinc function a revolutionary idea was born in the field of interpolation because of its acceptance as a ideal interpolation function. However, this ideal interpolator has an infinite impulse response (IIR) and is not suitable for local interpolation with finite impulse response (FIR).

The B-spline functions because of its close resemblance with the sinc function were being started to use prominently as an interpolation function. The term spline is used to refer to a wide class of piecewise polynomial function jointed at certain continuity points called as knots. Until now, in the spline family, extensive research is being done for polynomial spline [Uns99a]. However, the exponential splines are more general representation of these polynomial splines [Dah87a]. In the present work, the continuous exponential function is derived at equally spaced knots using truncated power functions and for the formulation of the exponential interpolated kernel this approximation function is convolved with Fourier approximation of the sampled exponential E-spline function [Leh99a]. The calculation of polynomial B-splines is a particular case, when the parameters of the exponents are set to be zero. The exponential B-spline interpolation function is derived for symmetric case taking different exponential parameter in consideration. The spatial and the frequency domain characteristics of the exponential spline interpolation function is discussed. Analyzing these characteristics it can be said exponential B-spline are less band-limited functions passing some of the high frequency components and reducing some of the low frequency components. This characteristic is useful for edge enhancement in various types of images. These spline-based algorithms have been found to be quite advantageous for image processing and medical imaging, because of restoring the high frequency components, especially in the context of high-quality interpolation

The paper is organized as follows. Section II starts with a brief review of B-spline and shows how this B-spline's can be linked to the extension of exponential splines. The exponential B-spline framework is ideally suited for performing some basic continuous-time signal processing operations in the discrete domain. Further characteristic of the spline is shown by analyzing it in the spatial and Fourier domain. Section III presents the results after applying the kernel of the exponential B-spline on one and two dimensional data followed by conclusion.

2. EXPONENTIAL B-SPLINES

2.1 Exponential B-splines Function

A continuous basic spline can be obtained by the convolution of $w^i(x), i = 1,..., n$ [Hsi78a],[Asa01a].

$$\beta^n(x) = (w^1 * w^2 * \cdots * w^n)(x) \qquad (1)$$

Where $*$ denotes the convolution operator. These weights are defined in the domain $x \in [-1/2, 1/2)$ and 0 otherwise for centered splines [Uns05b]. For shifted spline the domain is $x \in [0,1)$. In the case of polynomial splines, weight $w^i(x)$ is a rectangular function of height 1 and in the case of exponential B-splines the weight being exponential in nature are given as $w^i(x) = e^{\alpha_i x}$ with $\alpha_i \in C$ is the parameter of the exponential function. The exponential parameter vector is given as $\vec{\alpha}_n = (\alpha_i, ..., \alpha_n)^T$. Continuing from the B-spline the exponential B-spline can be obtained by convolving the weights. Also, the exponential B-spline can also be expressed as [Dah87a]:

$$\beta^{\vec{\alpha}_n}(x) = \sum_{k \in Z} d^{\vec{\alpha}_n}[k] \rho^{\vec{\alpha}_n}(x-k) \qquad (2)$$

Where the single and multiple discrete differences function are given by

$$d^{\alpha_1}[k] = \delta[k] - e^{\alpha_1} \delta[k-1]$$

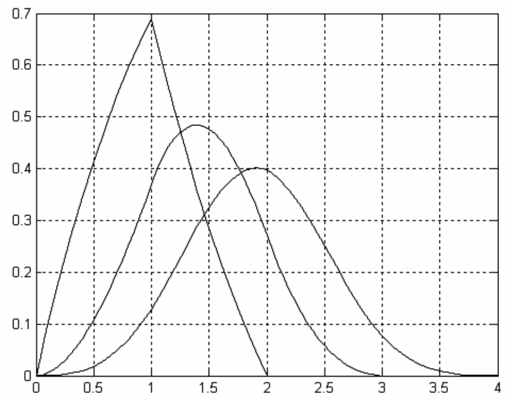

Figure 1. n-*th* order Exponential B-spline with n varying from 2.

$$d^{\vec{\alpha}_n}[k] = d^{\alpha_1} * d^{\alpha_2} * \cdots * d^{\alpha_n}[k] \qquad (3)$$

respectively, with $\delta[k]$ being the kronecker's delta function and $\rho^{\vec{\alpha}_n}(\bullet)$ is the continuous exponential truncated power function which can be expressed as:

$$\rho^{\alpha_1}(x) = 1_+(x) e^{\alpha_1 x}$$

And further can be recursively calculated using,

$$\rho^{\vec{\alpha}_n}(x) = (\rho^{\alpha_1} * \rho^{\alpha_2} * \cdots \rho^{\alpha_n})(x). \qquad (4)$$

Considering linear (n=2) case, we have vector $\vec{\alpha}_2 = (\alpha_1, \alpha_2)^T$ and from equation (3),

$$d^{\vec{\alpha}_2}[k] = d^{\alpha_1} * d^{\alpha_2}[k],$$

$$d^{\vec{\alpha}_2}[k] = \delta[k] - (e^{\alpha_1} + e^{\alpha_2}) \delta[k-1] + e^{\alpha_1 + \alpha_2} \delta[k-2].$$

And the truncated exponential from equation (4) can be simplified as :

$$\rho^{\vec{\alpha}_2}(x) = 1_+(x) \left\{ \frac{e^{\alpha_1 x}}{\alpha_2 - \alpha_1} + \frac{e^{\alpha_2 x}}{\alpha_1 - \alpha_2} \right\}$$

where $1_+(x) = \begin{cases} 1, x \geq 0 \\ 0, otherwise \end{cases}$

Therefore from equation (2),

$$\beta^{\vec{\alpha}_2}(x) = d^{\vec{\alpha}_2}[k] * \rho^{\vec{\alpha}_2}(x-k)$$

Hence,

$$\beta^{\vec{\alpha}_2}(x) = \begin{cases} \dfrac{e^{\alpha_2 x} - e^{\alpha_1 x}}{\alpha_2 - \alpha_1} & 0 \leq x < 1, \\ \dfrac{e^{\alpha_2 x - \alpha_2 + \alpha_1} - e^{\alpha_1 x + \alpha_2 - \alpha_1}}{\alpha_2 - \alpha_1} & 1 \leq x < 2. \end{cases} \qquad (5)$$

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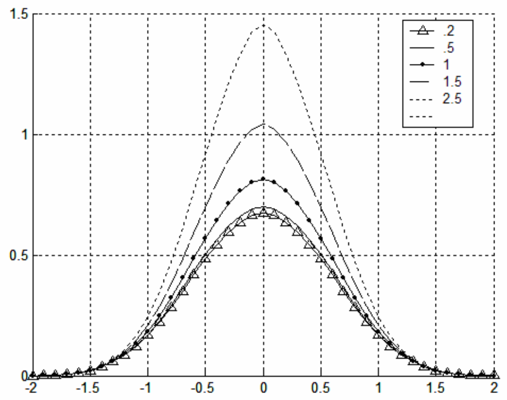

Figure 2. 4-*th* order Exponential B-spline for (α, α,-α, -α) where α = .2, .5, 1,1.5,2.5,3.

By similarly expanding equation (2), the piecewise function for the exponential B-splines of order n is given by the expression:

$$\sum_{k=1}^{n} d^{\vec{\alpha}_n}[k]x^k = \prod_{k=1}^{n}(1-e^{\alpha_k}x),$$

$$\rho^{\vec{\alpha}_n}(x) = 1_+(x)\sum_{i=1}^{n}\left\{\frac{e^{\alpha_i x}}{\prod_{j\neq i}(\alpha_j-\alpha_i)}\right\}.$$

The 4-th order exponential B-spline function considering exponential parameters (α, α, -α, -α) are symmetric for real values of α as shown in Fig. 2.

2.2 Interpolation Kernel Function

For interpolation, the steps involved must reconstruct a one dimensional continuous signal $s(x)$, from its discrete samples $s(k)$ with $s, x \in R$ and $k \in \mathrm{N}$. Thus the value at the position x must be estimated from its discrete neighbors. This can be formally described as the convolution of the discrete samples with the continuous impulse response given by

$$s(x) = \sum_{k} s(k)\cdot h(x-k) \qquad (6)$$

The 1-D ideal interpolation equals to the multiplication with a rectangular function in the Fourier domain and it can be realized by convolution with the sinc function in the spatial domain.

$$h_{ideal}(x) = \frac{\sin(\pi x)}{\pi x} = \sin c(x) \qquad (7)$$

Some fundamental properties of any interpolator can be derived from this ideal interpolation function. Therefore

$$h(0) \equiv 1,\ h(x) \equiv 0, |x| = 1,2. \qquad (8)$$

These zero crossings guarantee that the signal is not modified, avoids smoothing and preserves high frequencies.

As interpolating the discrete signal is equivalent to sampling the continuous signal. Due to aliasing of the high frequency in the lower ones, It is important to examine not only the continuous interpolation function $h(x)$ but also interpolated function $h(k)$ i.e., the sum of all samples should be one for any displacement $0 \le d < 1$

$$\sum_{k=-\infty}^{\infty} h(d+k) \equiv 1 \qquad (9)$$

This means that for any displacement d the summation of the function h should be unity. That's why the mean amplitude of the signal remains unaffected if the signal is resampled or interpolated.

2.3 Exponential B-spline Interpolation

To create an interpolating exponential B-spline kernel, the exponential B-spline approximator is applied to different sets of samples $t(k)$. Since the exponential B-spline kernel is symmetrical and separable, the reconstruction yields

$$s(x) = \sum_{k} t(k)\cdot h(x-k) \qquad (10)$$

with h being the basis function. The general case (10) reduces to equation (6) if the samples are taken from the given data: $t(k) \equiv s(k)$. Here, the $t(k)$ must be derived from the given data $s(k)$ in such a way that the resulting curve interpolates the given data.
So,

$$s(k) = \sum_{m=k-2}^{m=k+2} t(m)\cdot h(x-m) \qquad (11)$$

The function $h(x)$ is exponential B-spline function for n=4 case, so

$$h(x) = \beta(x)$$

Therefore,

$$s(k) = \beta(1)t(k-1) + \beta(0)t(k) + \beta(-1)t(k+1)$$

ignoring edge effects, results in a set of equations to solve

$$S = BT$$

The coefficient T can be evaluated by multiplying the known data points S with the inverse of the tridiagonal matrix B

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$$T = B^{-1}S \qquad (12)$$

To simplify its analytical derivation, the interpolated image $s(x)$ and the data samples $s(k)$, calling them u and v, respectively. From equation (11) we obtain,

$$v = t * B$$

$$v(x) = t(x) * (\beta(1)\delta(x-1) + \beta(0)\delta(x) + \beta(-1)\delta(x+1)) \qquad (13)$$

In frequency domain for symmetric case the equation reduces to

$$V(f) = T(f) \cdot B(f) \qquad (14)$$

Inversion of equation (14) yields,

$$T(f) = V(f) \cdot \frac{1}{\beta(0) + 2\beta(1)\cos(2\pi f)}$$

As $G(f) = \dfrac{1}{\beta(0) + 2\beta(1)\cos(2\pi f)}$ is even so

Taking Fourier series approximation we get

$$T(f) = V(f) \cdot (a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi f))$$

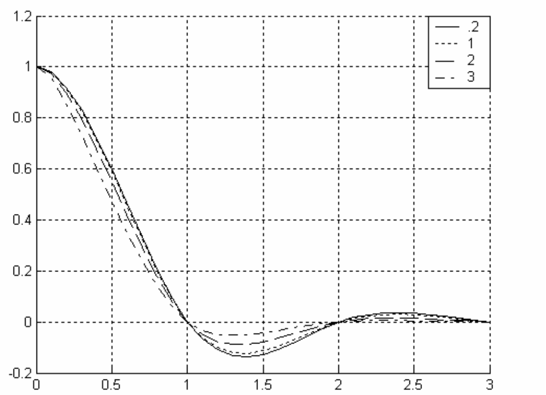


Figure 3. 4-*th* order Exponential B-spline interpolation kernel with (α, α,-α, -α).

where,

$$a_0 = \frac{1}{T}\int_0^T G(f)df$$

$$a_n = \frac{1}{T}\int_0^T G(f)\cos(2\pi f)df$$

Taking back in the spatial domain

$$t(x) = v(x) * (\frac{a_0}{2}\delta(x) + \sum_{n=1}^{\infty} a_n(\delta(x-n) + \delta(x+n)))$$

Hence equation (10) can be written as

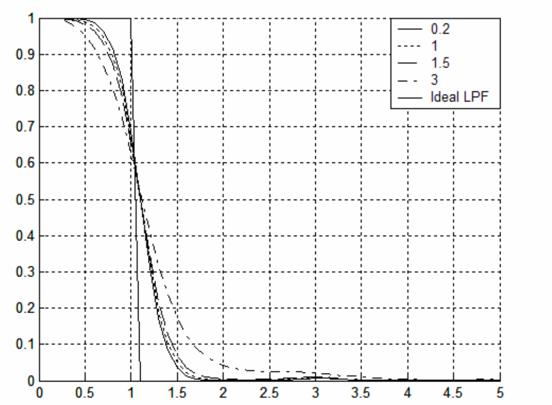
$$u = t * h = v * h_{Espline} \qquad (15)$$

Finally with (15) we obtain,

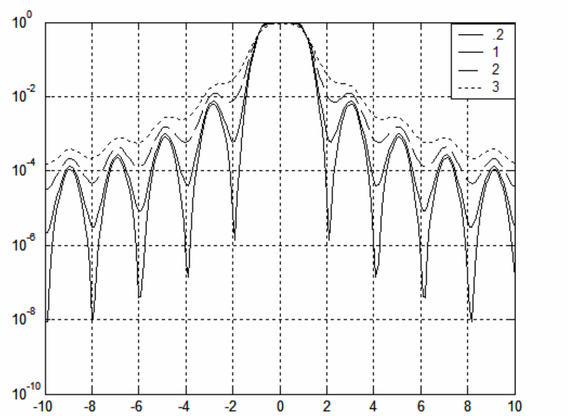
$$h_{Espline}(x) = \beta(x) * (\frac{a_0}{2}\delta(x) + \sum_{n=1}^{\infty} a_n(\delta(x-n) + \delta(x+n)))$$

The interpolation kernel is symmetric, passes through the integer points. Fig. 3 shows the interpolation kernel for only [0,3], moreover the kernel is symmetric around x=0.

The Fourier domain response of the interpolation kernel is shown in Fig. 4 for different values of exponential parameters. As the value of alpha is changed the filter response deviates from the ideal low pass filter. However, the interpolation kernel is band limited passing the high frequency components near the cut-off frequency, which can be used to preserve the edge information in the images. With the increase in value of alpha, the transition part of the filter decreases the magnitude of the low frequency and increases the number and magnitude of the high frequency components.



(a)



(b)

Figure 4. (a) Fourier domain magnitude plot and, (b) log plot of Exponential B-spline for (α, α,-α, -α) with different values of α.

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ringing effect is diminished by using the Exponential B-spline.

Exponential parameter *α*	Energy distributed between -3 to 3	Energy distributed between -2 to 2
0.2	0.999800	0.998423
1	0.999941	0.998964
2	0.999991	0.999711
3	1.000000	0.999961
4	1.000000	0.999997

Table 1. The percentage energy distribution in exponential B-spline interpolation kernel for different α after truncation.

The energy distribution in exponential B-spline interpolation kernel is minimum for lower value of exponential parameter. As the value of these parameter increases, energy decreases in both between -2 to 2 and -3 to 3 which is shown in Table 1.

Fig. 5 plots the sum of sampled interpolation kernel from equation (9) as a function of displacement *d*. The summation is done after truncating the kernel from -3 to 3. It is clear that for alpha closer to 1.2, the sum of sampled interpolated kernels is closer to 1, hence the value close to 1.2 give better interpolation. This can be verified with PSNR given in table 2.

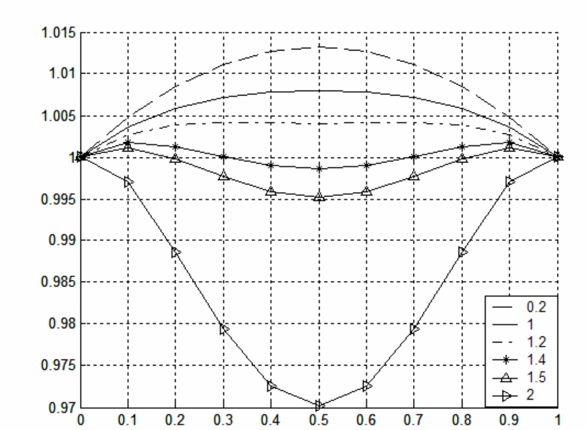

Figure 5. Sum of sampled interpolation kernels as a function of the displacement for different α.

3. SIMULATION AND RESULT

Here we have examined the exponential spline interpolation kernel with different 1-D and 2-D data's. The exponential B-spline interpolation function derived is experimented with unit step function. It is shown in Fig. 6 that Exponential B-spline results in less oscillation for higher values of exponential parameter. Similarly Fig. 7 and Fig. 8 show the interpolated Lena image by factor 2. In the case of the expansion using B-spline, some ringing effect near the edges can be observed, however these

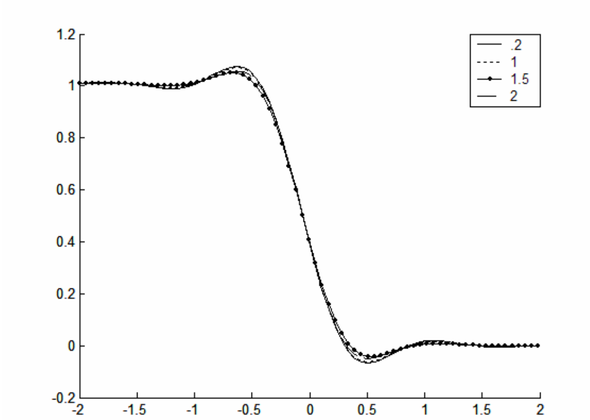

Figure 6. Interpolation of the discrete Step using exponential and polynomial B-splines functions for different α.

This Exponential B-spline interpolation is compared with the other conventional methods (nearest neighbor, bilinear, and B-spline). In the set of 2-D experiments with original Lena, Barbara and other gray scaled image exponential B-spline has performed better results. For calculating peak signal to noise ratio (PSNR), initially we have filtered and downsampled the image using wavelet decomposition with Daubechies's 9-7 tap filter and then the image is interpolated using exponential B-spline interpolation where PSNR=20log10(255/RMSE) and RMSE is root mean square error between the original signal and interpolated signal. The PSNR calculation is shown in Table 2, the shaded part shows the maximum PSNR values.

4. CONCLUSION

This paper is focused on interpolation methods using Exponential B-spline, the method is easy for calculation of Exponential B-spline (where the calculation of B-spline is a particular case). Continuing from the Exponential B-spline approximation to the derivation of Exponential B-spline interpolation this paper provided a unified framework for the theoretical analysis and performance of both approximation and interpolation kernels. We have applied this analysis to specific case that involves piecewise exponential B-spline approximation having exponential parameter (α, α, -α, -α). This interpolation kernel can be used for images containing high as well as low frequency components for different values of α. The Exponential B-spline interpolation is performed on various 2-D and 1-D examples. Experiments shows that as the value of *α*

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is increased the ringing and oscillatory behavior can be reduced at the edges.

Further the best performance of the kernel is near α=1.2 which is shown by PSNR in Table 2. The further improvement is to use some adaptive algorithm for image interpolation based on changing values of exponential parameter.

5. ACKNOWLEDGEMENT

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Image	Linear	B-spline	Exponential B-spline							
			α = 0.2	α= 0.5	α= 1	α= 1.2	α=1.3	α=1.4	α=1.5	α =2
Lena	30.49	31.34	31.37	31.41	31.54	31.55	31.53	31.46	31.33	29.31
Gold-hill	29.60	29.84	29.86	29.89	29.98	30.00	29.95	29.87	29.54	29.60
Pepper	29.85	30.88	30.90	30.92	30.98	30.97	30.94	30.88	30.77	29.20
Barbara	24.29	24.63	24.64	24.65	24.67	24.66	24.65	24.63	24.59	23.95
Boat	27.70	28.25	28.27	28.29	28.38	28.40	28.39	28.35	28.28	26.98

Table 2. PSNR calculation of different interpolation technique on gray images.

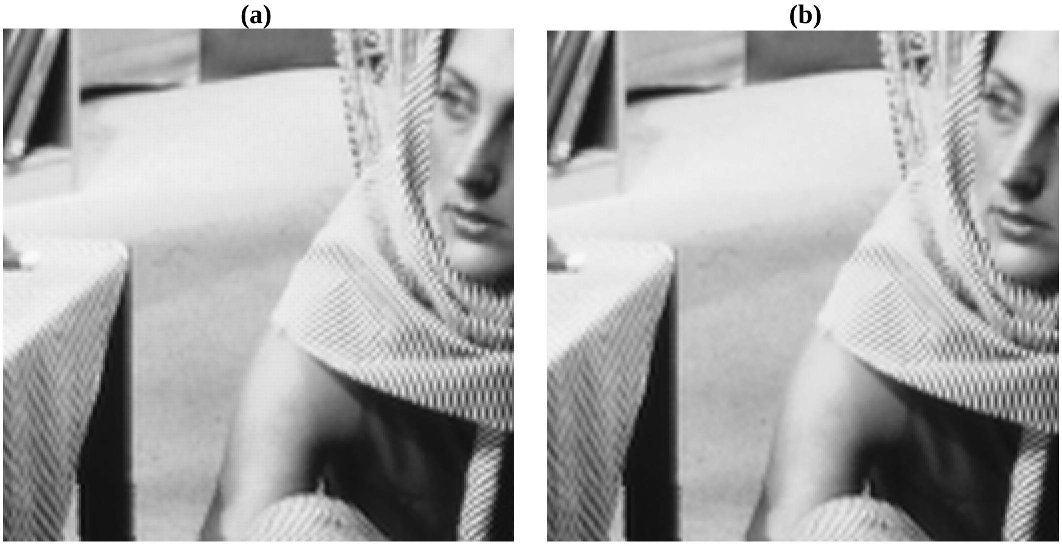
(a)

(b)

(c) **(d)**

Figure 7. (a)Original Lena Image (256*256), (b) Linear interpolated image (256*256), (c) Exponential B-spline interpolated image (256*256) with α=0.2, and (d) with α =1.2.

(a) **(b)**



(c) **(d)**

Figure 8. (a) Original Barbara Image (256*256), (b) Linear interpolated image (256*256), (c) Exponential B-spline interpolated image (256*256) with α=0.2, and (d) with α =1.2.

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