Upper and Lower Bounds on the Quality of the PCA Bounding Boxes

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Klaus Kriegel, Günter Rote

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$\mathbb{R}^2$

- Minimum-area bounding rectangle [Tousaint ’83]

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- Heuristics

  AABB (Axis Aligned Bounding Boxes)

  R-tree

    Packed R-tree [Rousopoulos, Leifker '85]

    R^+-tree [Sellis, Rousopoulos, Faloutsos '87]]

    R^*-tree [Beckmann, Kriegel, Schneider, Seeger '90]

  ...

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  PCA-bounding box $O(n)$, $O(n \log n)$, $O(n^{\lfloor \frac{d}{2} \rfloor + 1})$

  OBB-tree [Gottchalk, Lin, Manocha, '96]

  BOXTREE [Barequet, Chazelle, Guibas, Mitchell, Tal '96]

  $\ldots$

Principal Component Analysis

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$var(X, v) = \langle Cv, v\rangle$, $\quad C_{ij} = \frac{1}{m}\sum_{k=1}^{m}(x_{ik} - c_i)(x_{jk} - c_j)$.

PCA

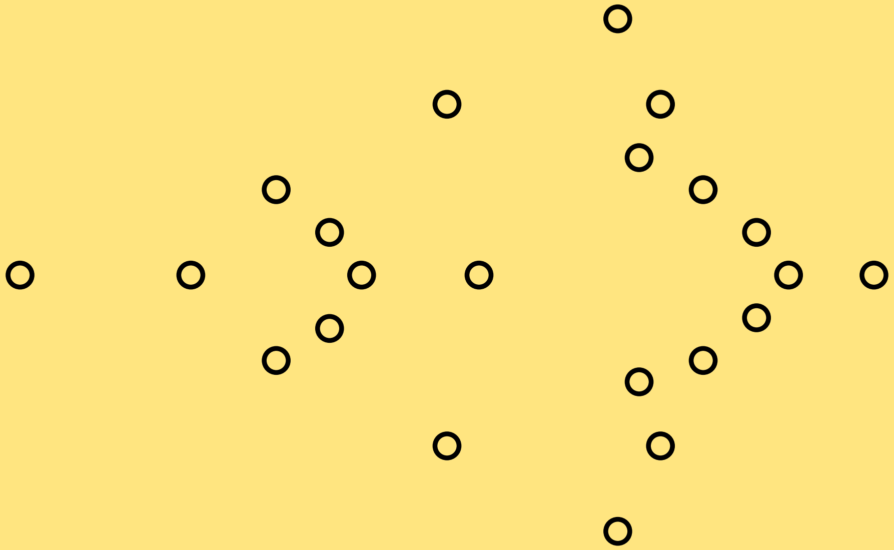
Lemma 1. *For 1 $\leq$ j $\leq$ d, let λ_j be the j-th largest eigenvalue of C and let v_j denote the unit eigenvector for λ_j. Let $B_j = \{v_1, v_2, \ldots, v_j\}$, $sp(B_j)$ be the linear subspace spanned by B_j, and $sp(B_j)^\perp$ be the orthogonal complement of $sp(B_j)$. Then $\lambda_1 = \max\{var(X, v)$: unit vector v in $\mathbb{R}^d\}$ and for any $2 \leq j \leq d$,*

i) $\lambda_j = \max\{var(X, v)$: *unit vector v in* $sp(B_{j-1})^\perp\}$.

ii) $\lambda_j = \min\{var(X, v)$: *unit vector v in* $sp(B_j)\}$.

iii) $var(X, B_j) \geq var(X, S)$ *for any set S of j orthogonal unit vectors.*

P



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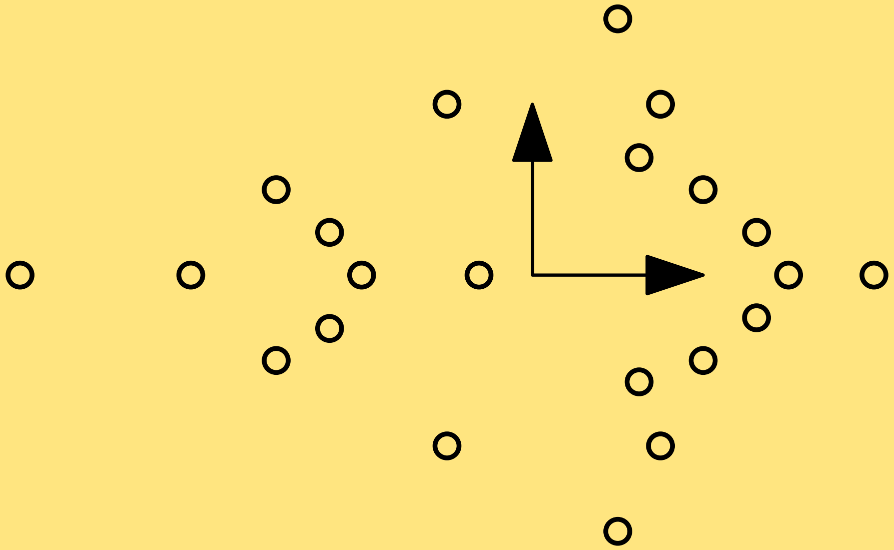

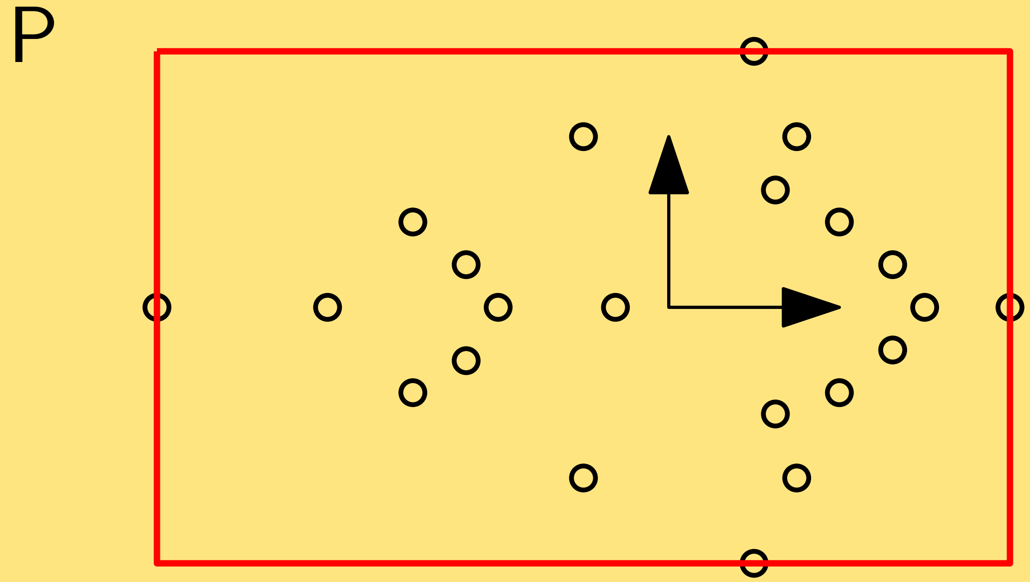

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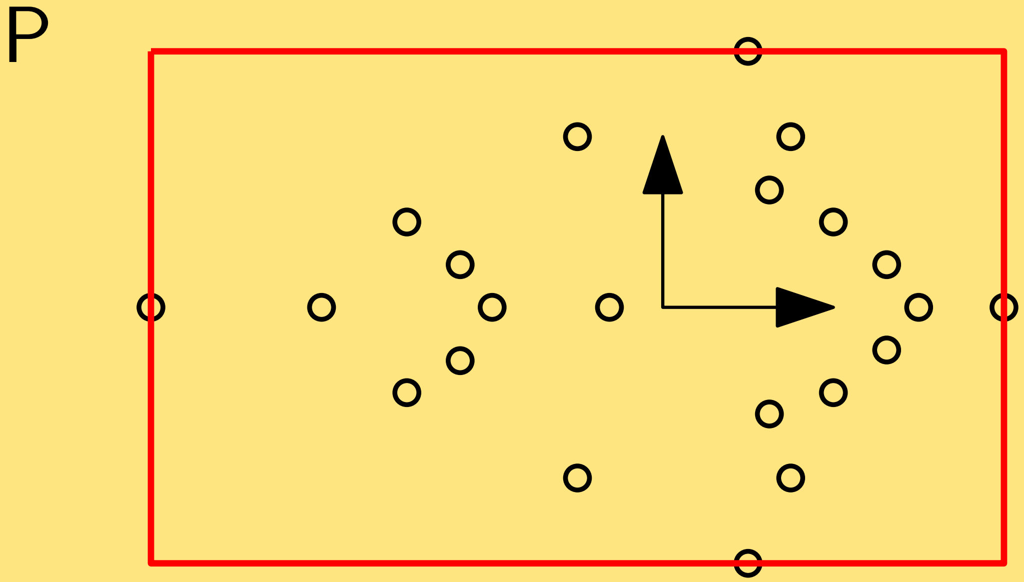

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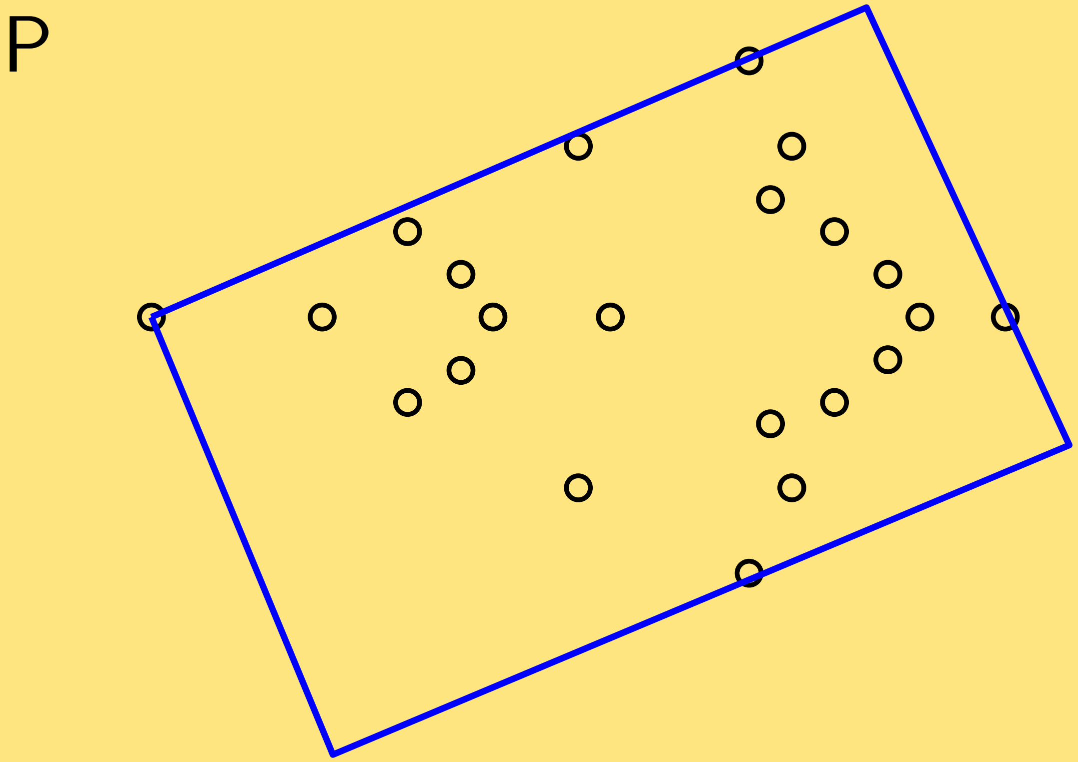

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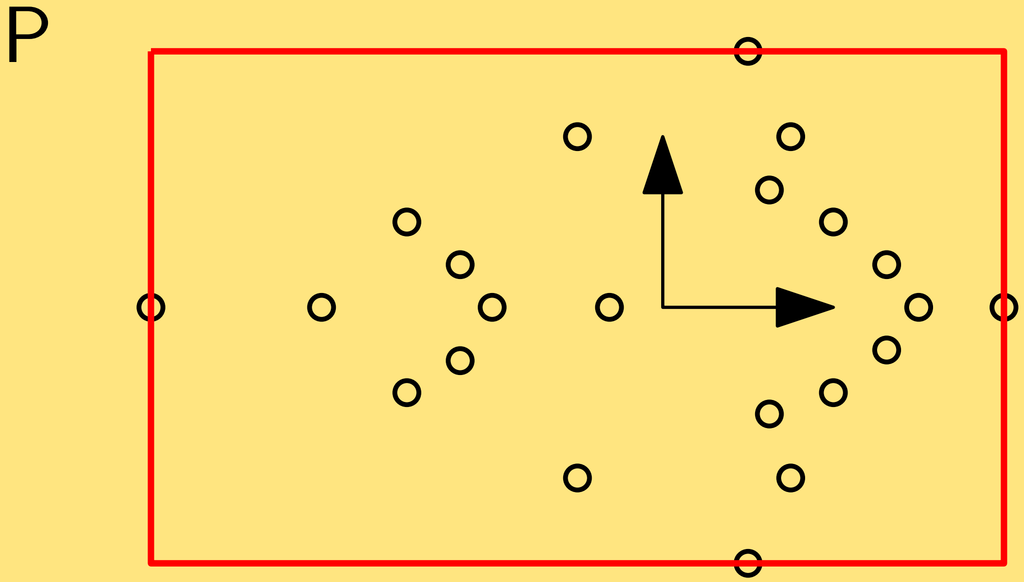

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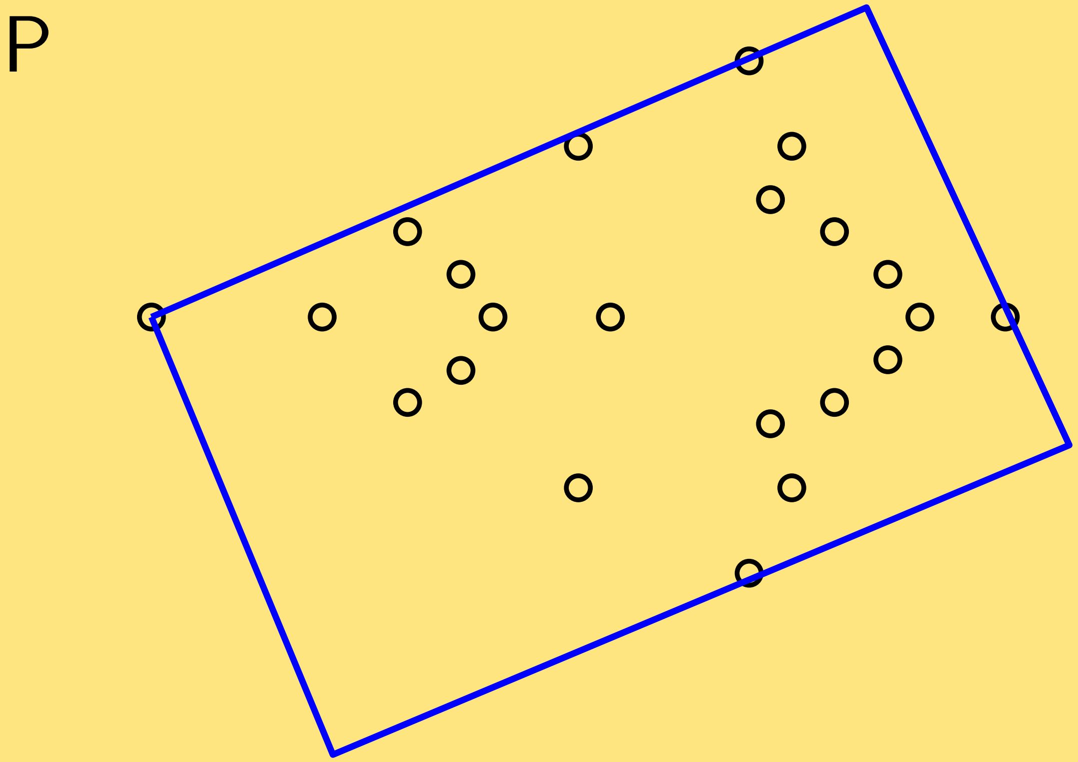

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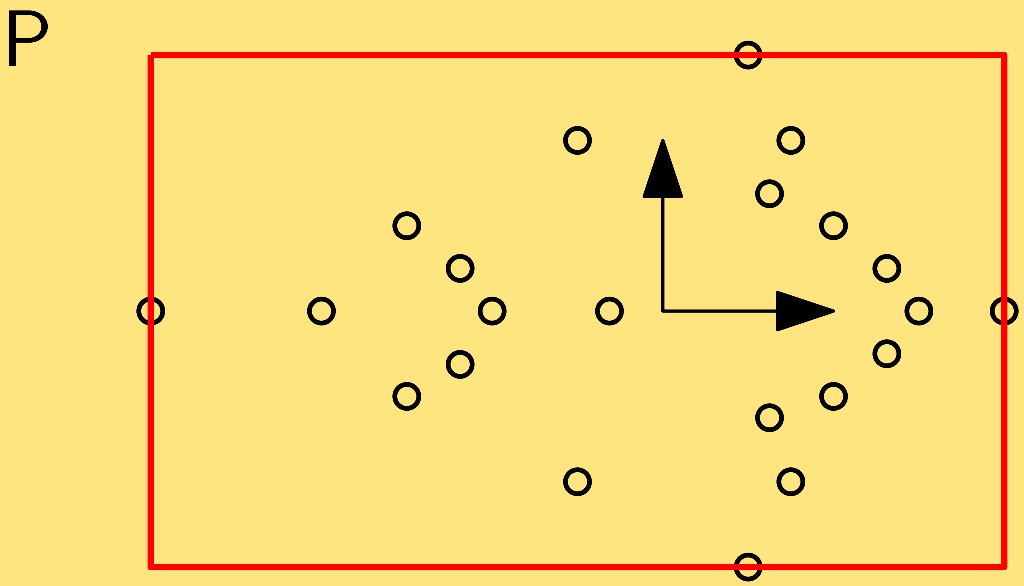
$$\lambda_d(P) = \frac{Vol(BB_{pca}(P))}{Vol(BB_{opt}(P))}$$



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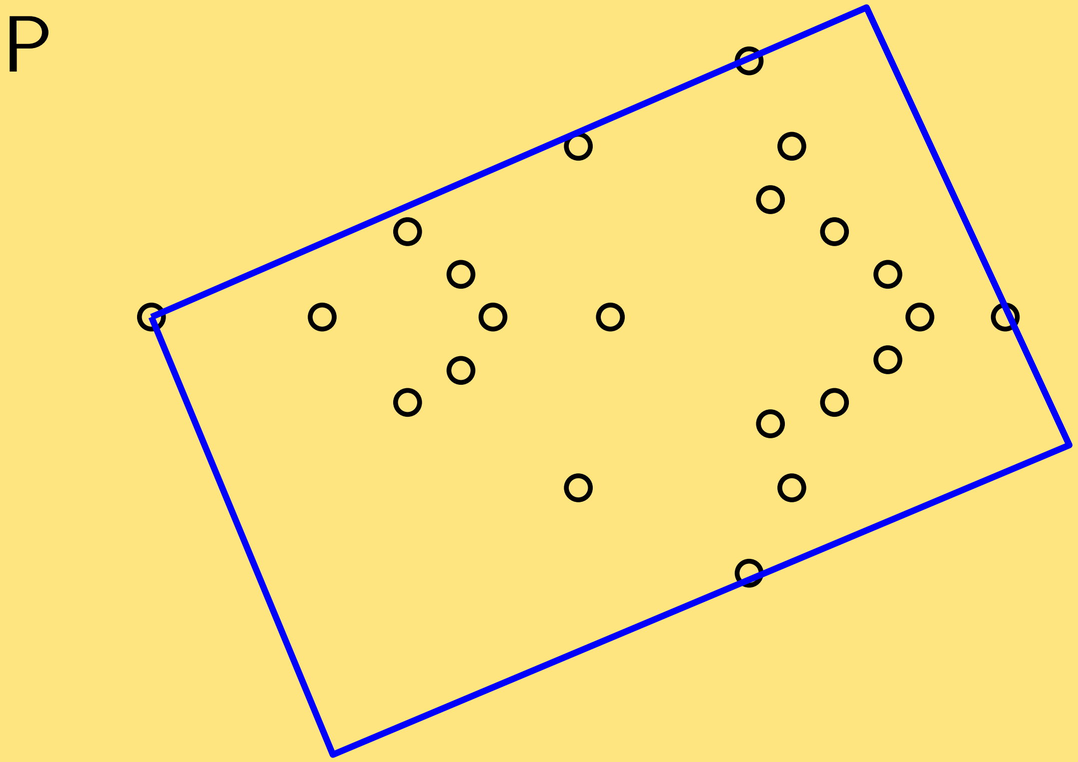

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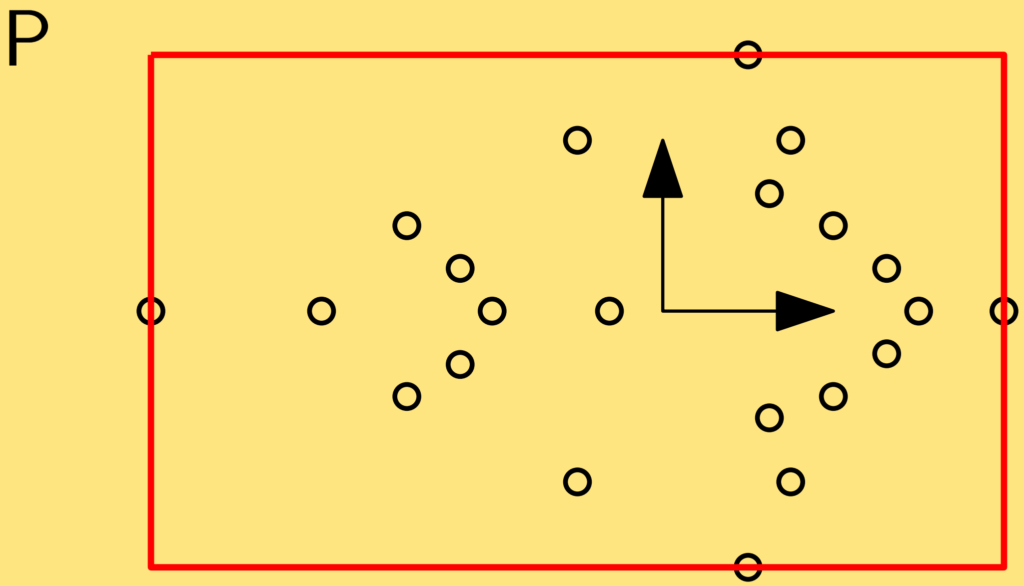

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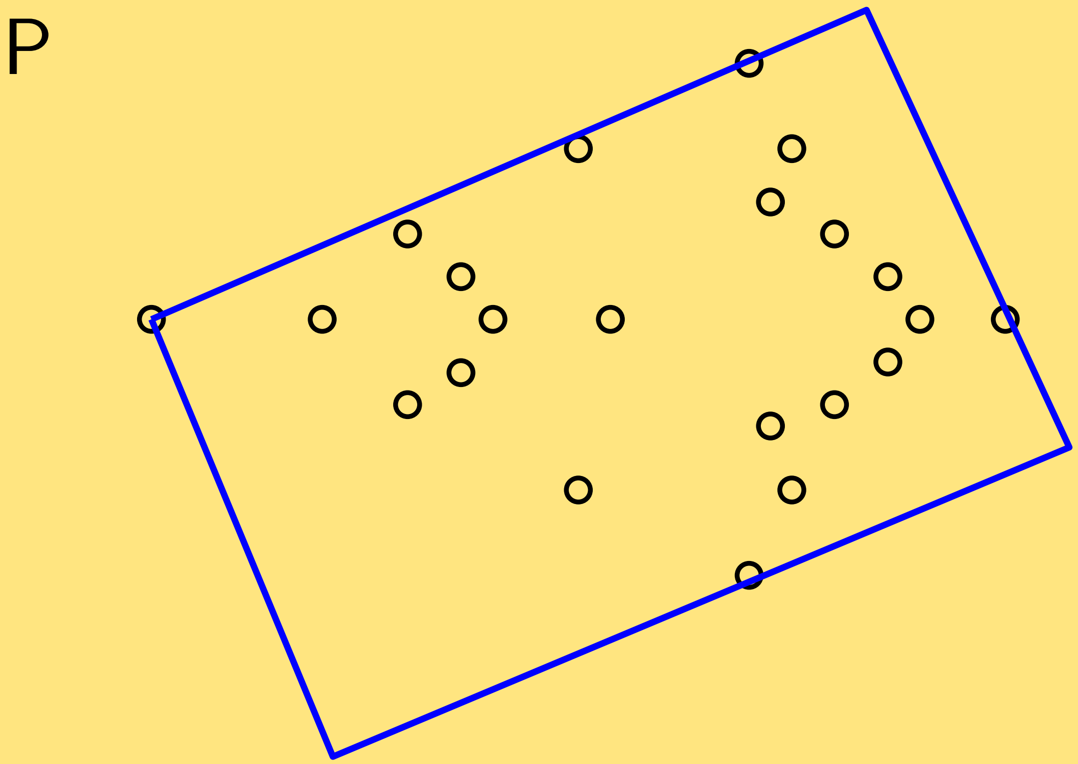

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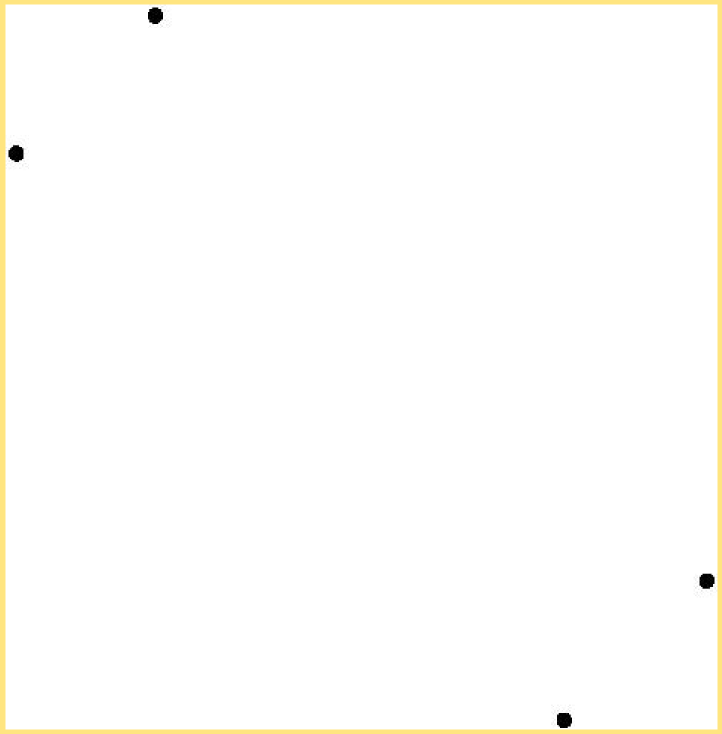

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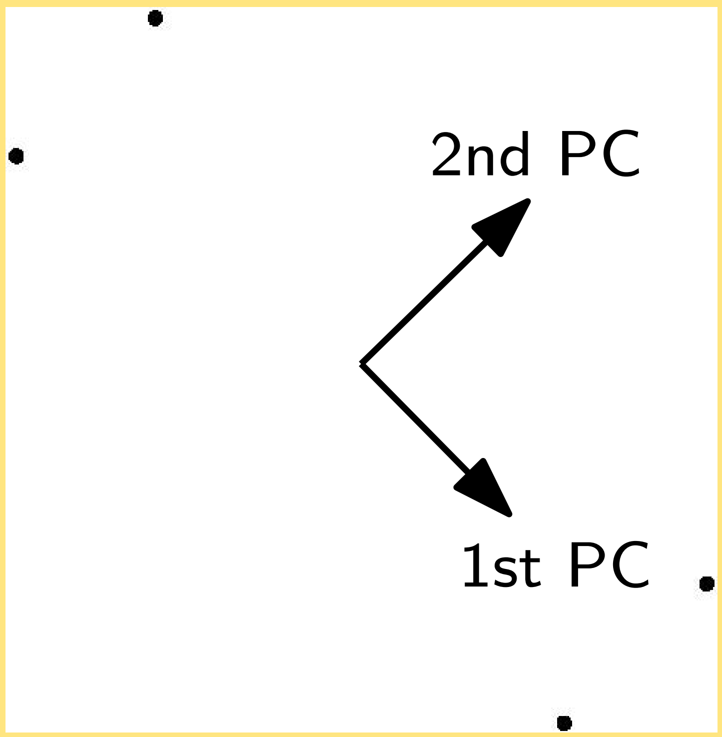

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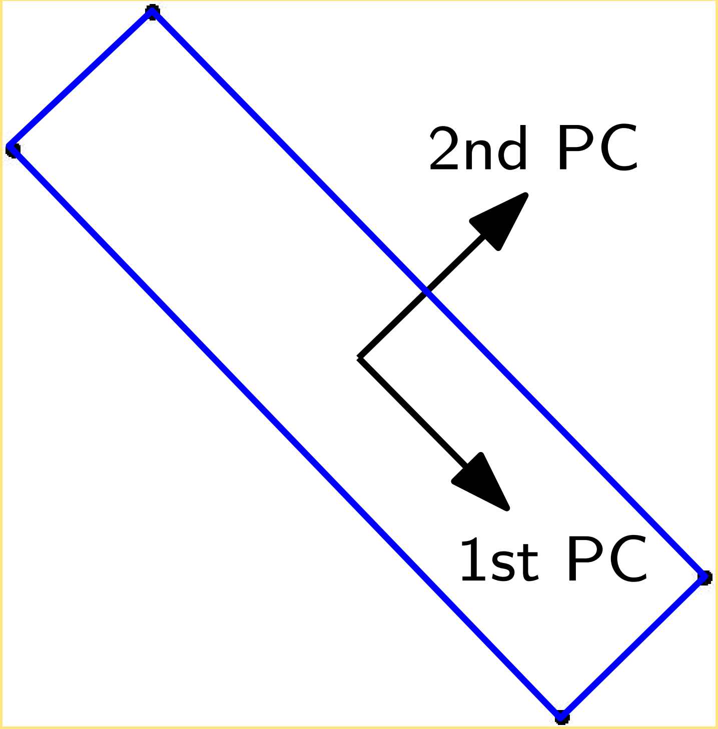

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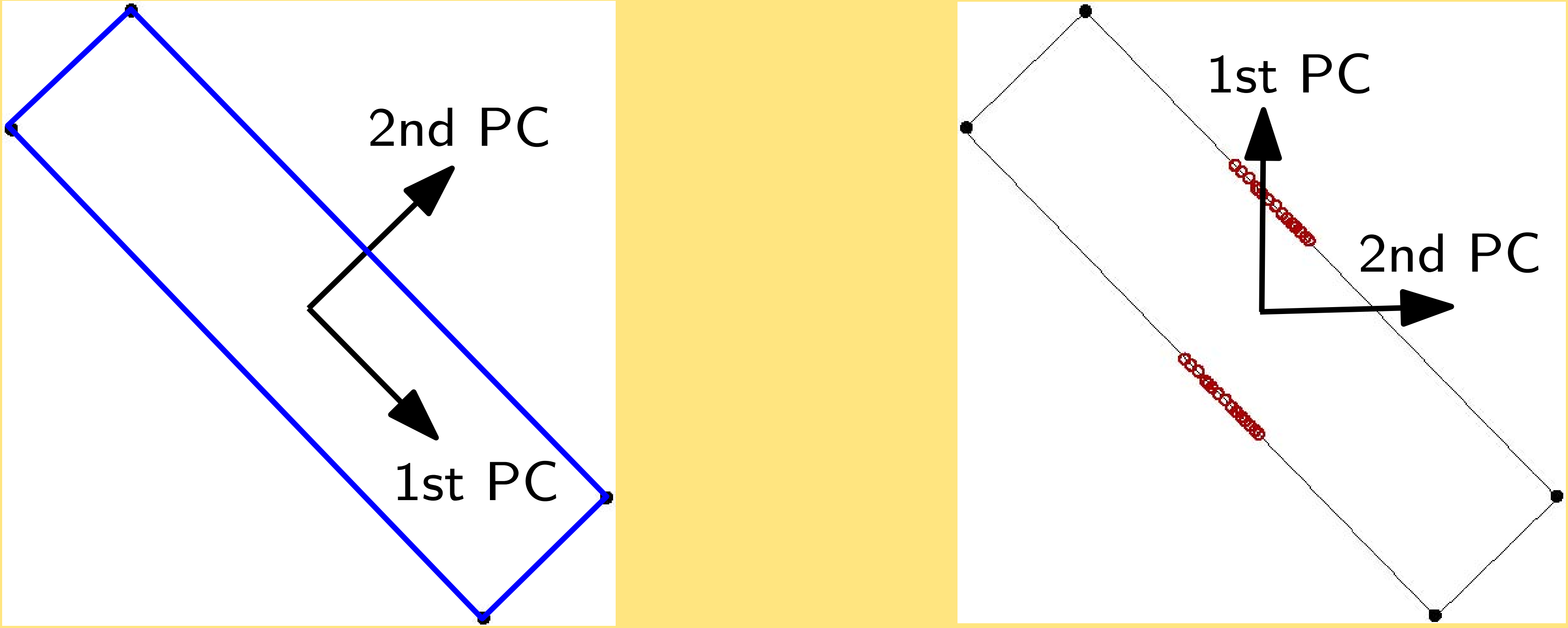

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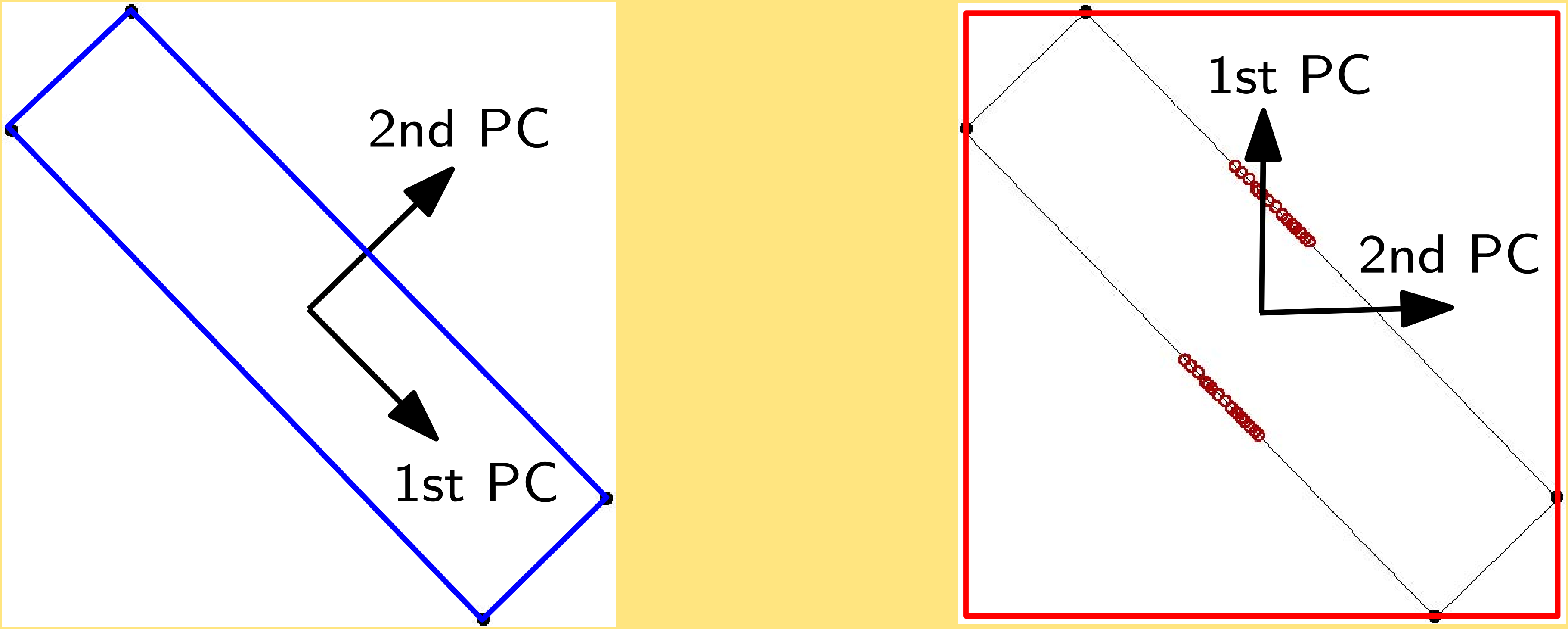

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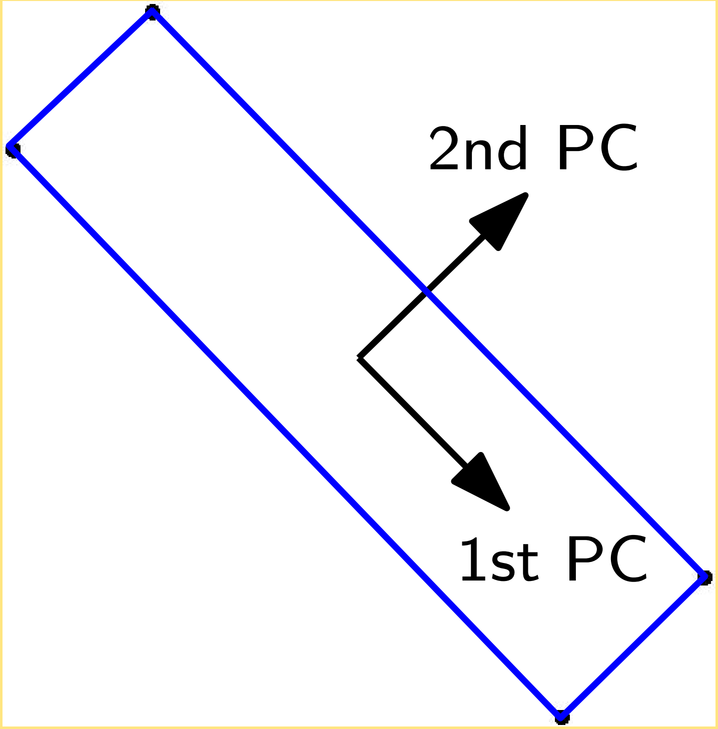

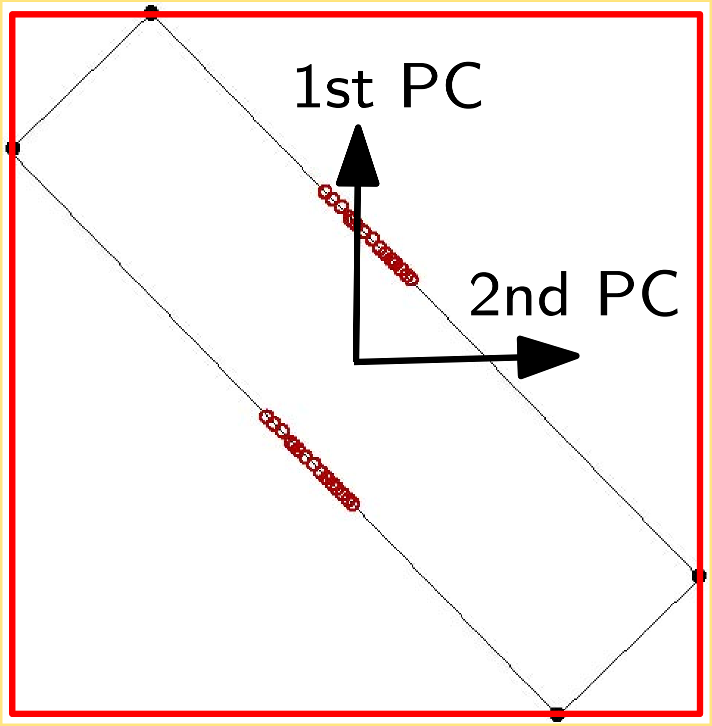

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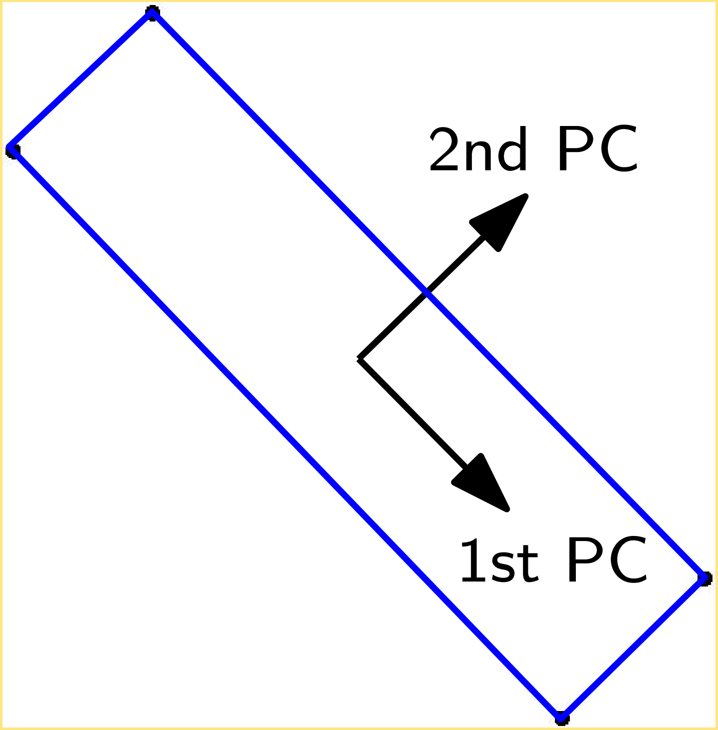

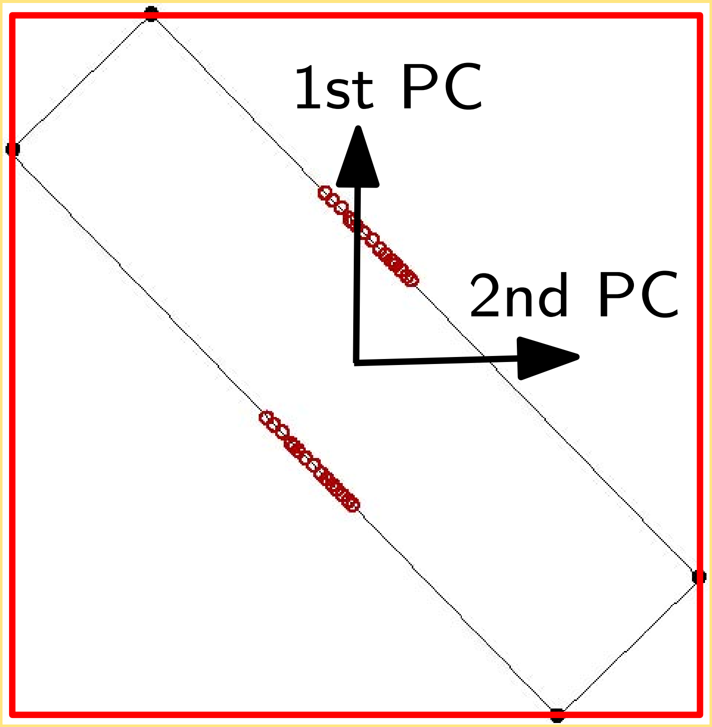

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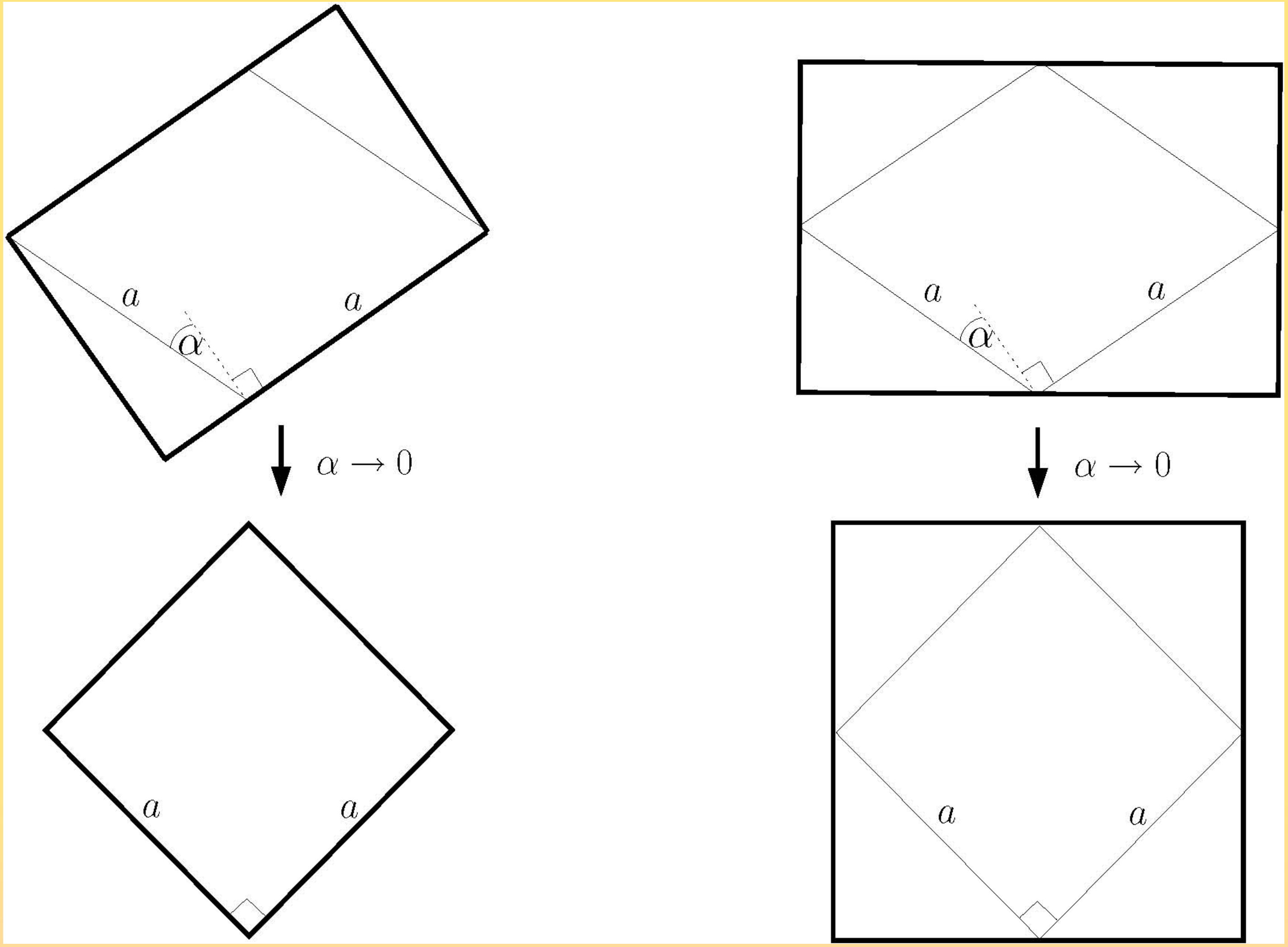

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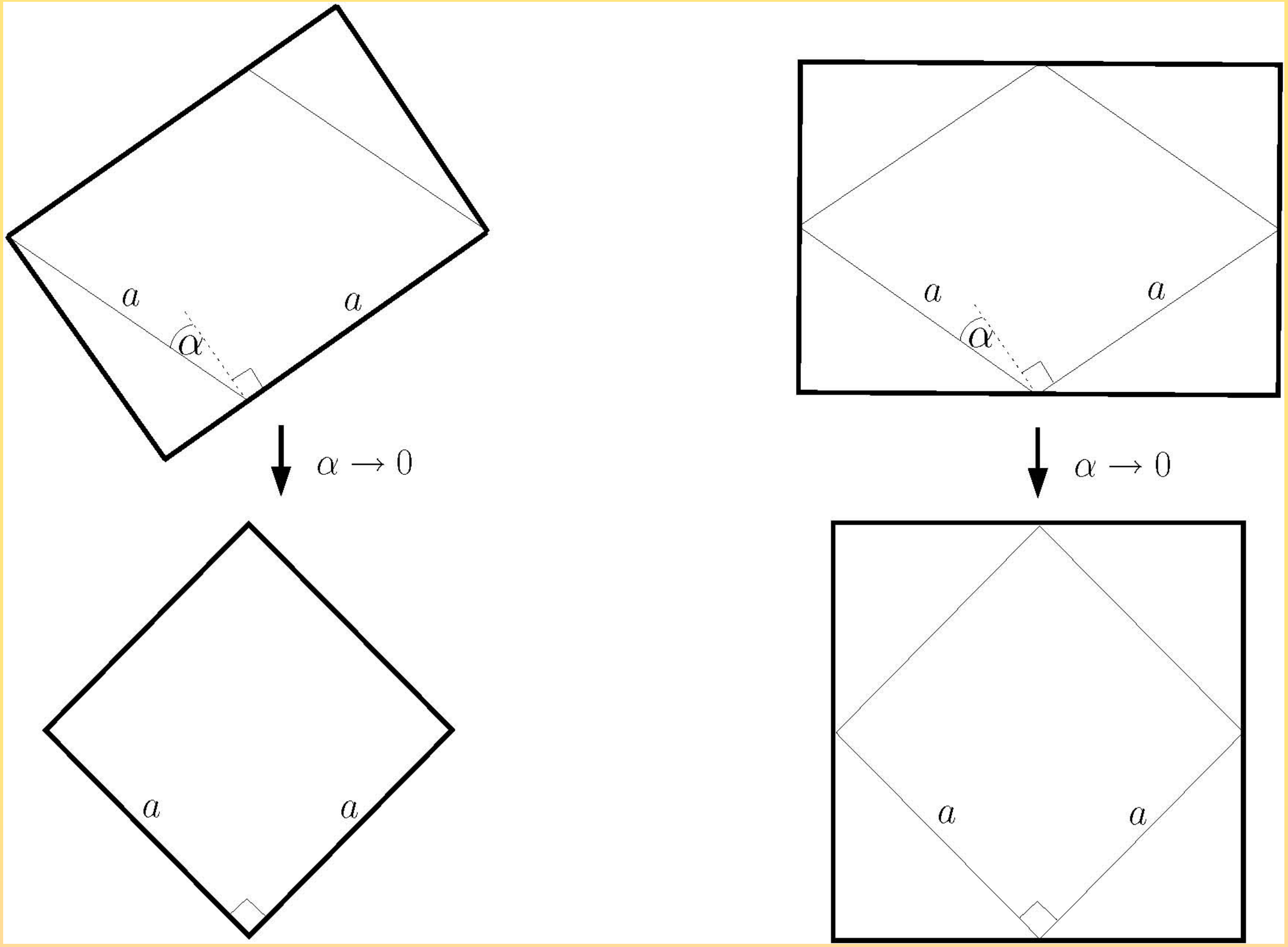

$$R_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

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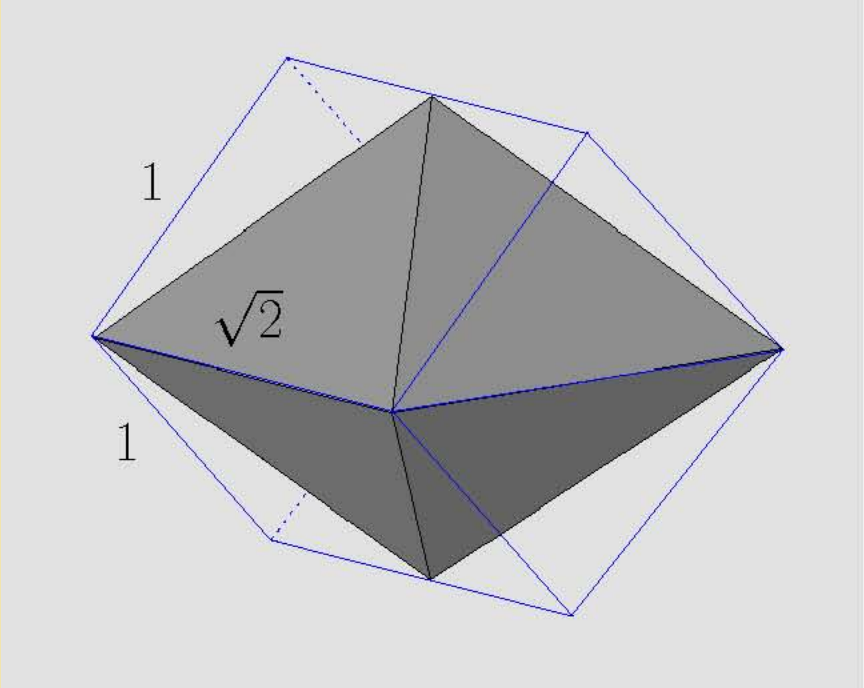

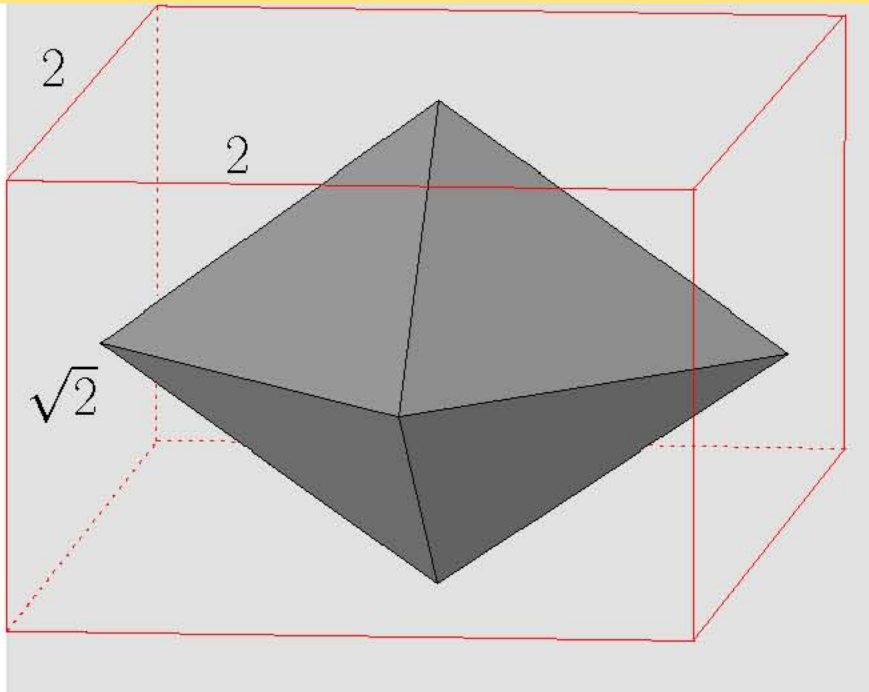

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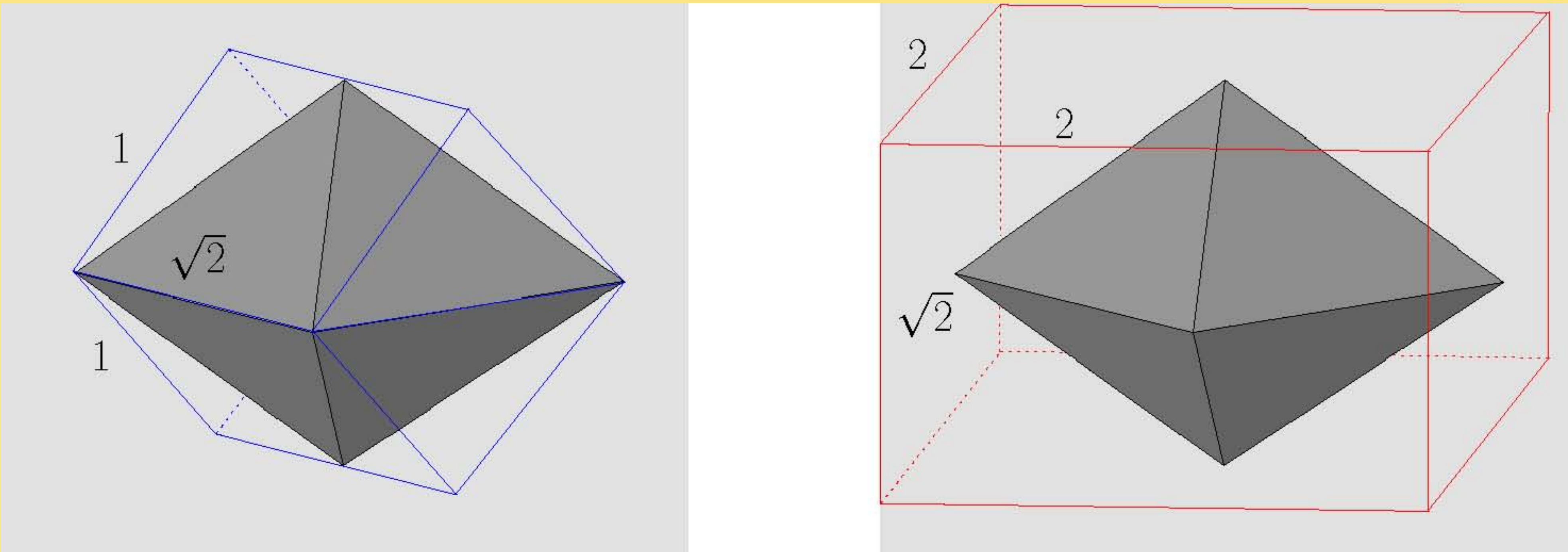

$$R_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

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$R_d(P_d)$ fits into unit cube $[-0.5, 0.5]^d$

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$\Longrightarrow$ $\lambda_{d_1} \cdot \lambda_{d_2}$ is a lower bound in $\mathbb{R}^{d_1+d_2}$

λ_{d_2} is a lower bound in $\mathbb{R}^{d_2}$

Lower bounds $\mathbb{R}^d$

$$\left.\begin{array}{l}\lambda_{d_1} \text{ is a lower bound in } \mathbb{R}^{d_1} \\ \\ \lambda_{d_2} \text{ is a lower bound in } \mathbb{R}^{d_2}\end{array}\right. \Longrightarrow \quad \lambda_{d_1} \cdot \lambda_{d_2} \text{ is a lower bound in } \mathbb{R}^{d_1+d_2}$$

dimension	$\mathbb{R}$	$\mathbb{R}^2$	$\mathbb{R}^3$	$\mathbb{R}^4$	$\mathbb{R}^5$	$\mathbb{R}^6$	$\mathbb{R}^7$	$\mathbb{R}^8$	$\mathbb{R}^9$	$\mathbb{R}^{10}$
lower bound	1	2	4	16	16	32	64	4096	4096	8192

Upper bound $\mathbb{R}^2$

Theorem 5. $\lambda_{2,1} \leq 2.737.$

Upper bound $\mathbb{R}^2$

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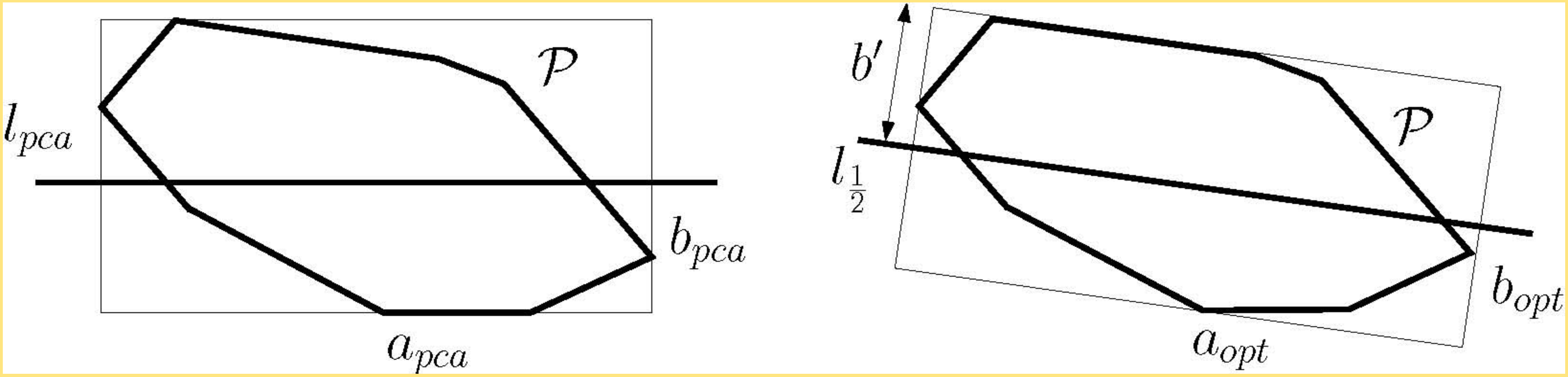

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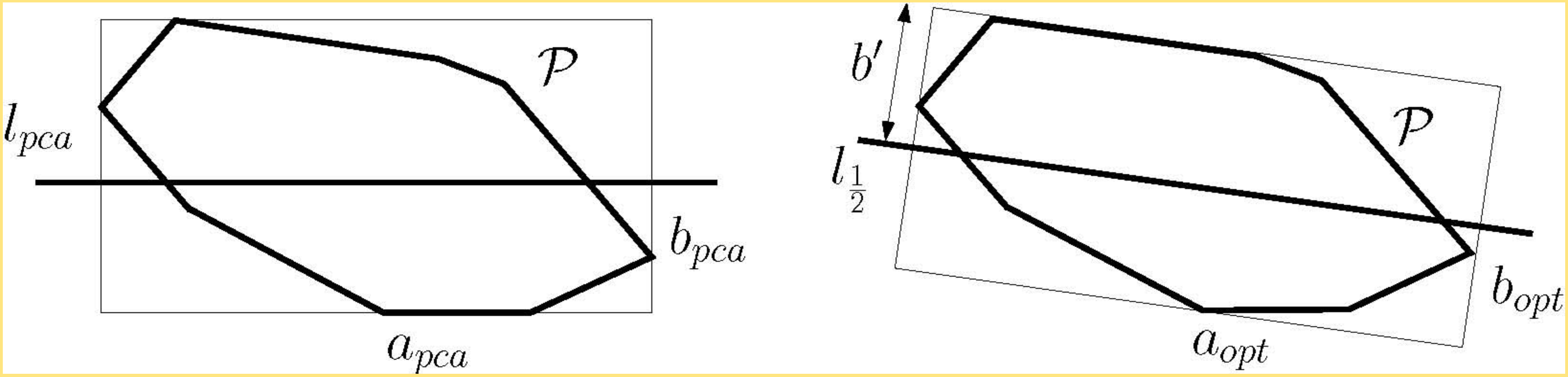

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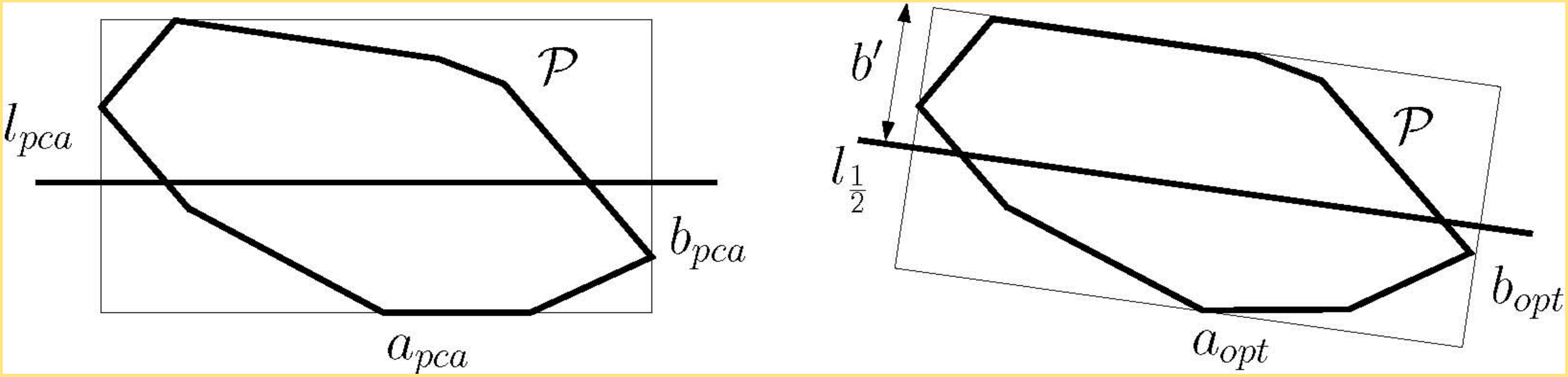

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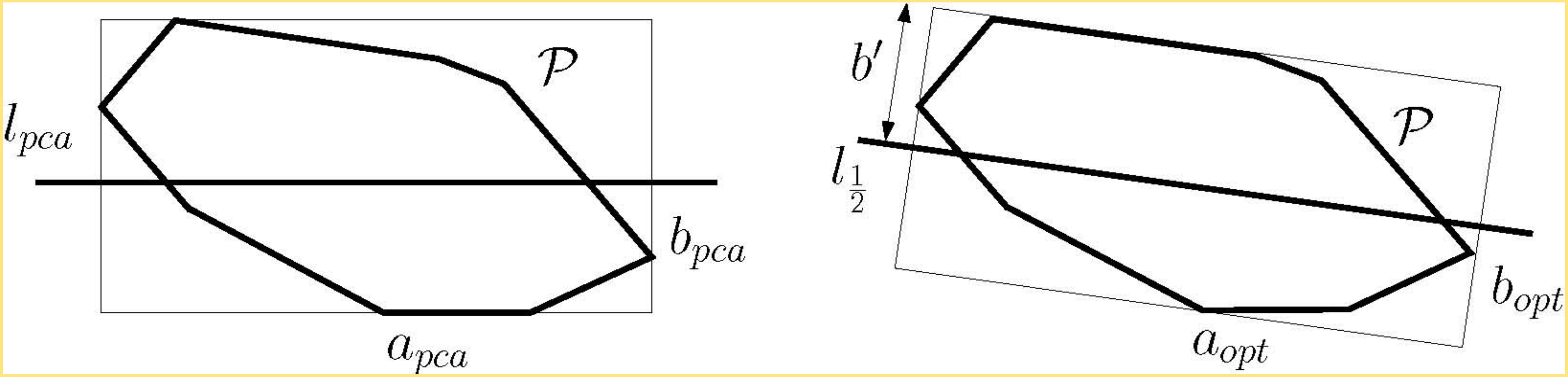

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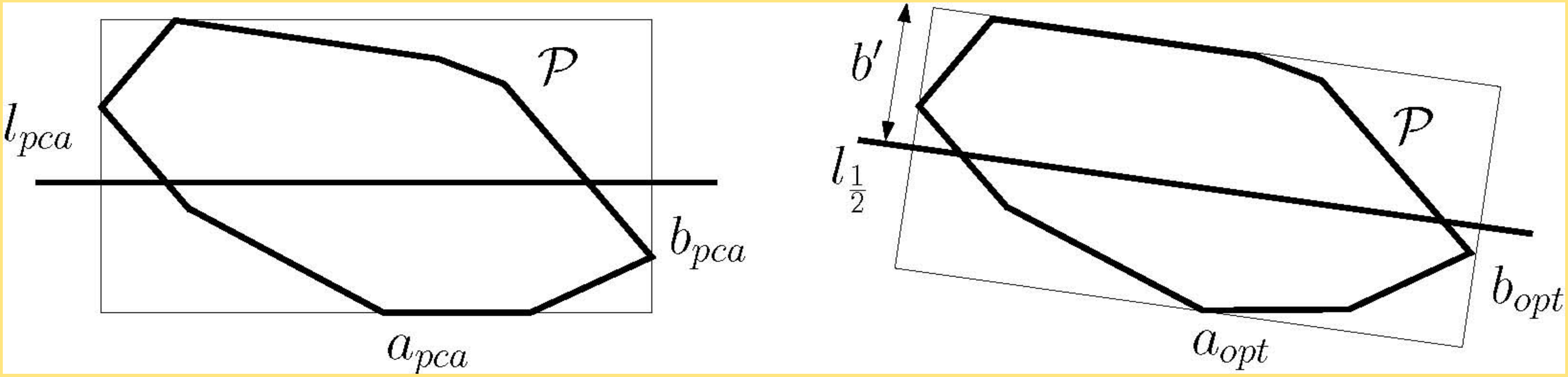

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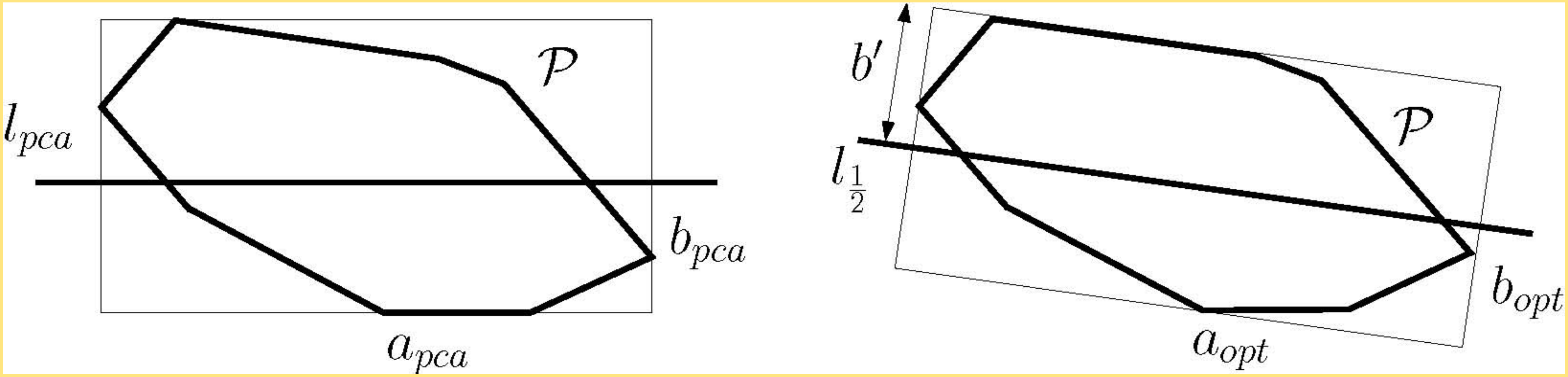

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$$\eta(P) = a_{opt}(P)/b_{opt}(P)$$

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Lemma 2. *$\lambda_{2,1}(P) \leq \eta + \frac{1}{\eta}$ and $\lambda_{2,2}(P) \leq \eta + \frac{1}{\eta}$ for any point set P with fixed aspect ratio $\eta(P) = \eta$.*

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Lemma 3. $\lambda_{2,1}(P) \leq \sqrt{\frac{6\eta+2}{\eta}\sqrt{1 + \frac{1}{\eta^2}}}$ *for any point set P with fixed aspect ratio $\eta(P) = \eta$.*

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$$\lambda_{2,1} \leq \sup_{\eta \geq 1} \left\{ \min \left(\eta + \frac{1}{\eta}, \ \sqrt{\frac{6\eta+2}{\eta}\sqrt{1+\frac{1}{\eta^2}}} \right) \right\}$$

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$$\alpha\beta \leq \eta \left(\sqrt{1 + \frac{1}{\eta^2}}\right)^2 = \eta + \frac{1}{\eta}$$

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$$d^2(\mathcal{P}, l_{\frac{1}{2}}) \geq d^2(\mathcal{P}, l_{pca})$$

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$$d^2(\mathcal{P}, l) = \int_{x\in\mathcal{P}} d^2(x, l)ds$$

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Lemma 3. $\lambda_{2,1}(P) \leq \sqrt{\frac{6\eta+2}{\eta}\sqrt{1+\frac{1}{\eta^2}}}$ for any point set P with fixed aspect ratio $\eta(P) = \eta$.

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Upper bound $\mathbb{R}^2$

Lemma 4. $d^2(\mathcal{P}, l_{\frac{1}{2}}) \leq d^2(\mathcal{BB}_{\mathcal{OPT}}, l_{\frac{1}{2}}) \quad (= \frac{{b_{opt}}^2 a_{opt}}{2} + \frac{{b_{opt}}^3}{6})$

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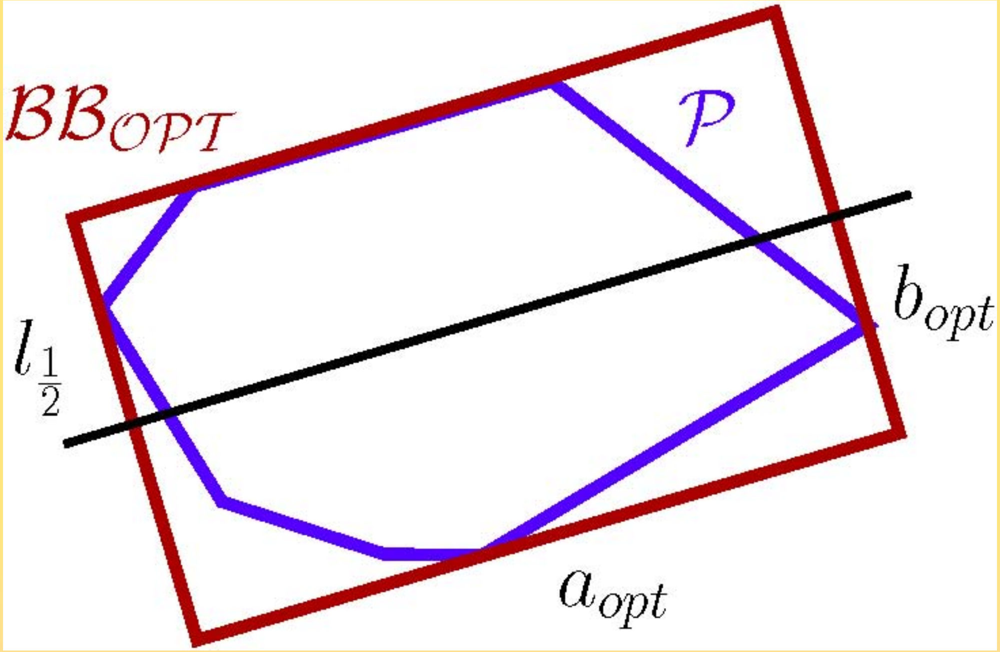
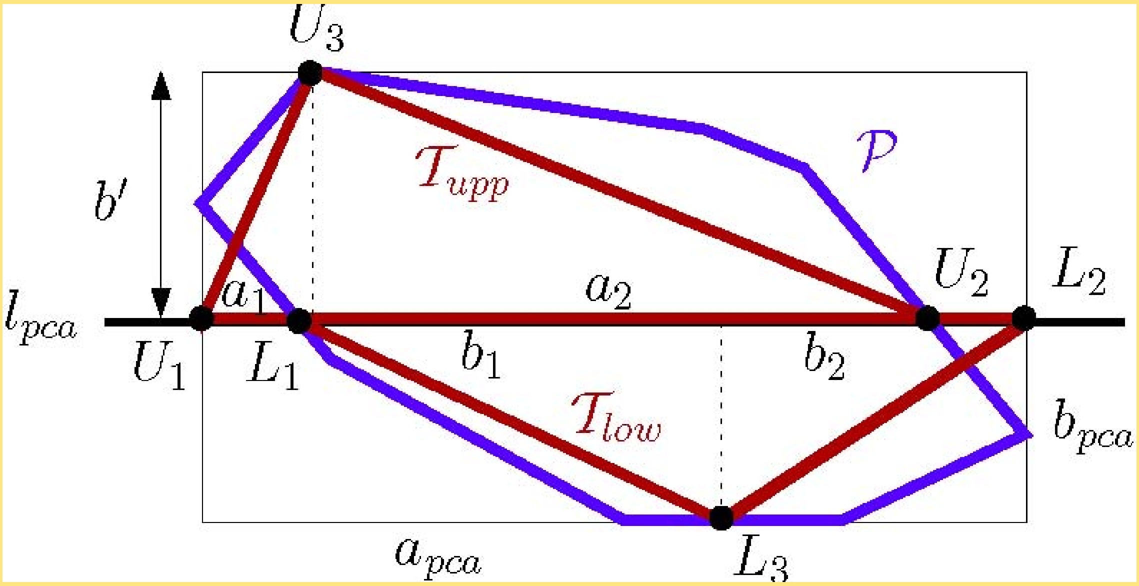
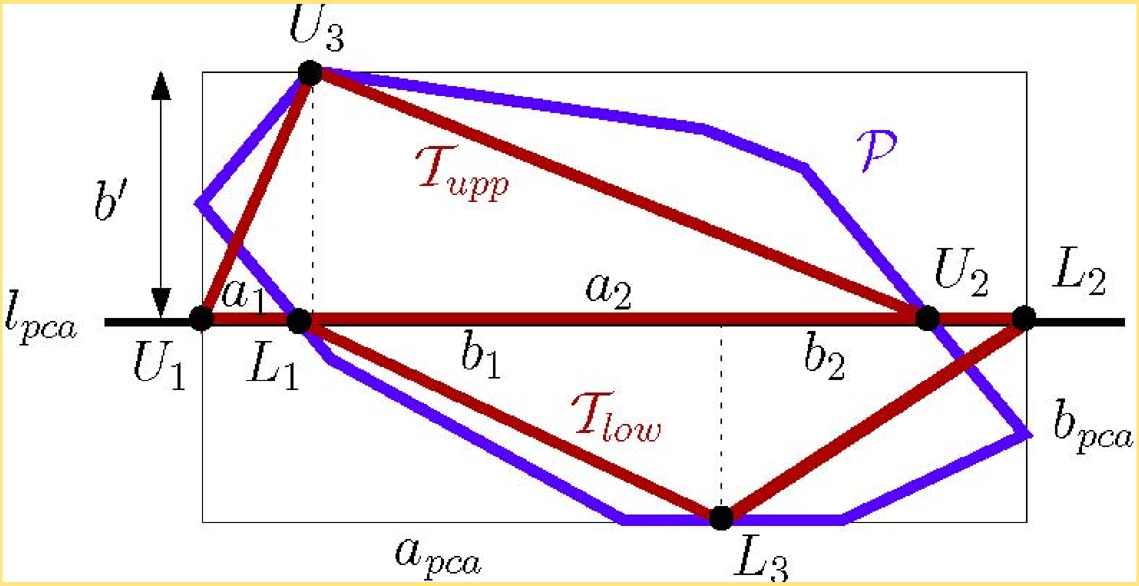


Upper bound $\mathbb{R}^2$



Upper bound $\mathbb{R}^2$



Lemma 5.

$$\begin{aligned} d^2(\mathcal{P}, l_{pca}) &\geq d^2(\mathcal{T}_{upp}, l_{pca}) + d^2(\mathcal{T}_{low}, l_{pca}) \\ &\geq \tfrac{b_{pca}^2}{12}\sqrt{a_{pca}^2 + 4b_{pca}^2}). \end{aligned}$$

Future work and open problems

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- Upper bound in $\mathbb{R}^3$
- Upper bounds for an approximation factor in arbitrary dimension

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