A bi-quadratic smooth spline surface generation over irregular meshes

Mahdi Imanparast

School of Mathematics, Statistics
and Computer Science, Faculty of science,
University of Tehran,
Tehran, Iran

Majzi28@khayam.ut.ac.ir

Manouchehr Parsaei

School of Mathematics, Statistics
and Computer Science, Faculty of science,
University of Tehran,
Tehran, Iran

Parsaei@khayam.ut.ac.ir

ABSTRACT

A method for generating a smooth spline surface over an irregular mesh is presented. This method generates a smooth spline surface similar to the methods proposed by [Loo94-Pet00, ZZZ+05]. The rules applied to construct the control points in order to achieve the continuity conditions are simple and comprehensible.

Keywords

Spline surfaces, Irregular meshes, Bi-quadratic Bezier patch, Tangent plane continuity, Quad-net.

1. INTRODUCTION

The construction of smooth spline surfaces over control meshes has been a popular topic in computer graphics, geometrical modeling and CAGD. A control mesh consists of a set of vertices, edges and faces.

Irregular meshes differ from regular meshes in the following two ways. Either a vertex has other than four edges emanating from it or a face is defined by other than four edges. To overcome this limitation, different methods have been proposed for the construction of smooth surfaces of irregular topology.

These methods can be roughly categorized into two groups: subdivision surfaces and spline surfaces.

The earliest attempts to overcome the topological limitations of B-spline surfaces were based on the subdivision principle.

Some non-polynomial surface patches used to define B-spline-like surfaces over irregular meshes include the 3 and 5-sided patches defined in [Sab83] and *n*-sided S-patches in [Loo90]. Hahmann et al. [HB08] presents a piecewise bi-cubic parametric G^1 spline surface interpolating a quad irregular mesh.

A scheme proposed by peters [Pet93] adjusts irregularities by applying one or more refinement steps. Another scheme by Loop [Loo94] only uses a one refinement step and creates a spline surface. In general this is a composition of patches at most of degree 4. Peters [Pet00] generated a bi-cubic scheme using a Catmull-Clark. Recently, a method has been presented by Zheng et al. [ZZZ+05] in which the Zheng-Ball patches are used to generate a bi-quadratic B-spline-like surface.

In this paper we present a bi-quadratic spline surface, which is a generalization of [Loo94]. The method presented here can be applied in irregular meshes of arbitrary topology. Only one step of subdivision is used. The rules used to generate the control points are simple and comprehensible. It dose not go through the complicated computation process needed in [Loo94-Pet00]. That ensures the locality property and has a piecewise polynomial form. Straightforward conditions have been used to ensure smoothness.

The construction process of this method consists of three steps: the first step carries out a single refinement procedure over the initial mesh, resulting in a new mesh in which the valence of each vertex is four. In the second step a quad-net is constructed corresponding to each vertex of the new mesh. Then each quad-net is split into four sub-quad, and a bi-quadratic Bezier patch is constructed over each sub-quad area.

2. SPLINE SURFACE GENERATION

Constructing the spline surface begins with a user-defined control mesh M^0. The details of each phase of this method have been presented in the next three sections.

Initial mesh refinement

The first step is to carry out a refinement procedure over initial mesh M^0. Let F be a face of M^0 consisting of vertices $\{C_0, C_1, ..., C_{n-1}\}$ and the average of this points is:

$$O = \frac{1}{n}\sum_{i=0}^{n-1} C_i .$$

The point c_i of M^1 corresponding to $\{C_i, F\}$ found by:

$$c_i = \frac{1}{n}O + \frac{n-2}{n}C_i + \frac{1}{2n}C_{i-1} + \frac{1}{2n}C_{i+1} , \quad (1)$$

where all subscripts are taken modulo n.

The faces of M^1 are constructed from vertex, face or edge of M^0 [GRE01] (Figure 1).

Note, that all the vertices of M^1 are 4-valent and every non-4-sided face in new mesh M^1 is surrounded by 4-sided faces. Clearly, if in the initial closed mesh M^0 all vertex valences are already of degree four, this step is not needed and one can use the mesh directly.

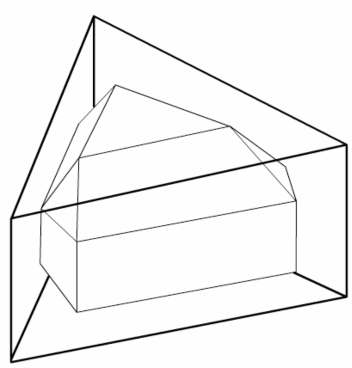

Figure 1. Mesh subdivision. Initial mesh (bold lines). After the subdivision (thin lines).

Generating the Quad-Nets

In the second step, a quad-net is constructed corresponding to each vertex of M^1.

Consider the vertex V in figure 3, to generate a quad-net on this vertex we will do as follows: The centriod points of four faces surrounding a vertex V are regarded as the corner points of a quad-net ($Q_{00}, Q_{03}, Q_{30}, Q_{33}$).

The points on the boundary of each quad-net are computed such that all quad-net points surrounding a quad-net corner point are coplanar.

The following theorem is the key to constructing quad-net points that satisfy this requirement.

Theorem 2.1: Let $p_0, p_1, \ldots, p_{n-1} \in \Re^3$ be a set of points in general position. The set of points $q_0, q_1, \ldots, q_{n-1}$ found by:

$$q_i = \frac{1}{n}\sum_{j=0}^{n-1} p_j (1 + \beta(\cos\frac{2\pi(j-i)}{n} + \tan\frac{\pi}{n}\sin\frac{2\pi(j-i)}{n})) \quad (2)$$

satisfy:

$$(1-\cos\frac{2\pi}{n})O + \cos\frac{2\pi}{n} q_i = \frac{1}{2}q_{i-1} + \frac{1}{2}q_{i+1} \quad (3)$$

where:

$$O = \frac{1}{n}\sum_{i=0}^{n-1} p_i$$

and are therefore coplanar.
Proof: take:

$$M_k = \beta(\cos\frac{2\pi k}{n} + \tan\frac{\pi}{n}\sin\frac{2\pi k}{n}).$$

Expand the right hand side of equation (3) as follows:

$$\begin{aligned}
&\frac{1}{2}q_{i-1} + \frac{1}{2}q_{i+1} \\
&= \frac{1}{2n}\sum_{j=0}^{n-1} p_j (1 + M_{j-(i-1)}) + \frac{1}{2n}\sum_{j=0}^{n-1} p_j (1 + M_{j-(i+1)}), \\
&= \frac{1}{2n}\sum_{j=0}^{n-1} p_j (2 + M_{j-(i-1)} + M_{j-(i+1)}), \\
&= \frac{1}{n}\sum_{j=0}^{n-1} p_j (1 - \cos\frac{2\pi}{n}) + \frac{1}{n}\sum_{j=0}^{n-1} p_j \cos\frac{2\pi}{n}(1 + M_{j-i}), \\
&= (1 - \cos\frac{2\pi}{n})O + \cos\frac{2\pi}{n} q_i .
\end{aligned}$$

Note that the well known trigonometric equations of sum of sinuses and cosines have been utilized in combining $M_{j-(i-1)} + M_{j-(i+1)}$ to get $2\cos\frac{2\pi}{n} M_{j-i}$.

From relation (3) we can see that q_{i+1} is a linear combination of points O, q_i, q_{i-1}, and by replacement we can find that each q_i $(i \geq 2)$ is a linear combination of three points O, q_0, q_1. Therefore each q_i lies in the plane made by these three points. Hence the collection of $q_i's$ obtained from the relation (2) are coplanar□.

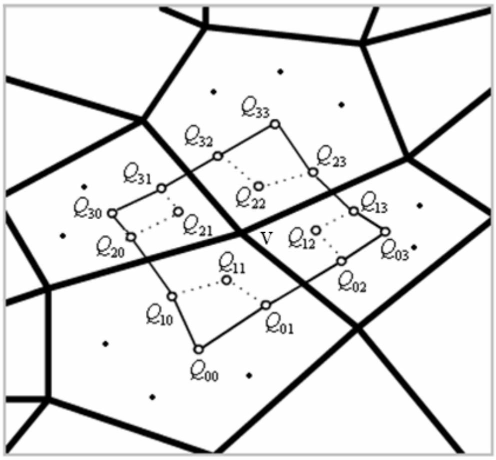

Figure 2. Quad-net.

In theorem 2.1 it can be observed that factor β is a free parameter which can be set arbitrarily.

Interpreting the points $p_0, p_1, \ldots, p_{n-1}$ as the vertices of a face blending to mesh M^1, the point O as Q_{00} and the points $q_0, q_1, \ldots, q_{n-1}$ as the quad-net points surrounding Q_{00}. It is immediately clear that all the quad-net points surrounding Q_{00} are coplanar. Also constraint (3) is satisfied by set:

$$(1-\cos\frac{2\pi}{n})Q_{00} + \cos\frac{2\pi}{n}Q_{01} = \frac{1}{2}Q_{10} + \frac{1}{2}\hat{Q}_{10}. \qquad (4)$$

Where $\hat{Q}_{10}$ is a point of an adjacent quad-net and n is the number of vertices of that face. Similar interpretation are used for the three points Q_{30}, Q_{03} and Q_{33}.

All the boundary quad-net points can be produced easily by applying theorem 2.1 to each one of the four faces surrounding each vertex of M^1.

The interior point Q_{11} is computed as follows:

$$Q_{11} = Q_{10} + Q_{01} - Q_{00} \qquad (5)$$

the other three interior points Q_{12}, Q_{21} *and* Q_{22} are found by symmetry.

Patch generation

Parametric surface patches are constructed in this step. They interpolate the information generated by quad-nets in the previous step. Each quad-net is accomplished by four bi-quadratic Bezier patches which are constructed as follows: Suppose a quad-net is divided into four bi-quadratic Bezier patches which labeled as shown in figure 3. First we set the corner points:

$$a_{00} = Q_{00},\ a_{04} = Q_{03},$$
$$a_{40} = Q_{30},\ a_{44} = Q_{33}.$$

and:

$$b_{11} = Q_{11}, b_{13} = Q_{12},$$
$$b_{31} = Q_{21}, b_{33} = Q_{22}.$$

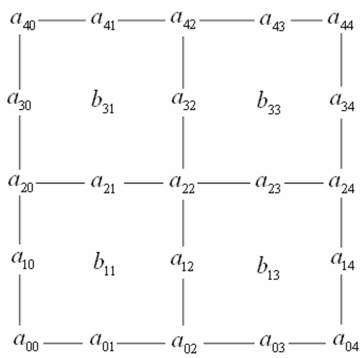

Figure 3. Split of a quad-net into four bi-quadratic Bezier patches.

Formulas for the control points of one of the patches are given here, similar formulas for the other three patches can be found by symmetry.

Internal control points of this patch are:

$$a_{12} = \frac{1}{2}(b_{11} + b_{13}),\ a_{21} = \frac{1}{2}(b_{11} + b_{31}),$$
$$a_{22} = \frac{1}{2}(a_{12} + a_{32}). \qquad (6)$$

the other control points are:

$$a_{01} = \frac{1}{2}(b_{11} + b_{31}^{(1)}),\ a_{10} = \frac{1}{2}(b_{11} + b_{13}^{(2)}). \qquad (7)$$

and:

$$a_{02} = \frac{1}{2}(a_{12} + a_{32}^{(1)}),\ a_{20} = \frac{1}{2}(a_{21} + a_{23}^{(2)}). \qquad (8)$$

in which the $b_{31}^{(1)}, b_{13}^{(2)}, a_{32}^{(1)}, a_{23}^{(2)}$ are the points of the two adjacent patches.

It can be observed that these constructions ensure that the triples $\{b_{11}, a_{21}, b_{31}\}, \{b_{11}, a_{12}, b_{13}\}, \{a_{12}, a_{22}, a_{32}\}$,

$\{a_{21}, a_{22}, a_{23}\}$ are collinear hence the rectangular patches generated inside each quad-net are C^1 continuous along their internal boundaries.

Two examples of initial meshes and their result surfaces are shown in figure 4.

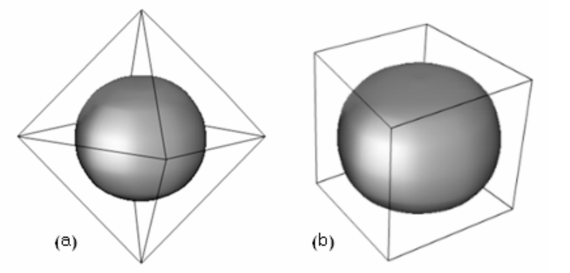

Figure 4. Two models generated by this method.

The same procedure is applied to a boundary quad-net, the only difference is in the control points along the boundary edge.

As it was pointed out in subsection 2.1 if all vertex valences in the initial mesh are of degree four we don't need a refinement step and can immediately create quad-nets, this leads to fewer patches. An example of this special case is presented in figure 4(a).

3. SMOOTHNEES

In this section we establish the smoothness conditions for the resulting surface. As can be seen from the relations of (6), (7) and (8) except in a control point on the corner of the corresponding quad-net each bi-quadratic patches satisfied the C^1 continuity condition over all its control points (shown in figure 5 by the orange points). We now demonstrate that the corner points have tangent plane continuity.

As it was pointed out in subsection 2.2 all the points around the corner of each quad-net found by the theorem 2.1 are coplanar and the internal quad-net points that are computed by the equation (5) lie on the same plane. Finally, the control points of the final patch around this corner point (computed by (7)) are also on the same plane, which includes corner point. This means that, in this corner point we have tangent plane continuity because all the points surrounding it have identical normal vectors.

Considering the symmetric procedure we used to generate the control points in each adjacent patches. It is easily to see that all the patches constructed in this method encode identical tangent planes along the common boundaries.

4. CONCLUSIONS

In this paper, a method is presented to construct a smooth surface over an irregular mesh by means of bi-quadratic Bezier patches. This method can be

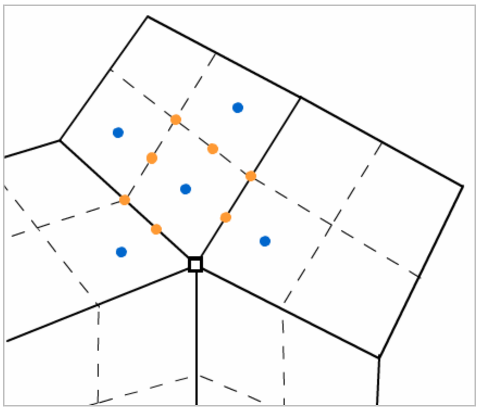

Figure 5. Five quad-nets, common in a corner point. Each bi-quadratic Bezier patch has C^1 continuity in its points (orange points) except in corner point of the quad-net (□) which has the tangent plane continuity.

implemented over both types of open and closed meshes and the result will be a smooth surface.

Following the steps of this method it can be seen that each step has a geometric construction involving weight average (affine combinations) of the points. Therefore, the spline surface is affine invariant.

If all internal vertex valences in the initial mesh are of degree four it should be possible to avoid the subdivision step as an optimization.

5. REFERENCES

[GRE01] Gohen, E., Riesenfeld, R.F., Elber, G.: Geometric Modeling with Splines: an Introduction. Natick, Mass:Ak peters, pp. 582-587, 2001.

[HB08] Hahmann, S., Bonneau, G.P., Caramiaux, B.: Bicubic G^1 interpolation of irregular quad meshes using a 4-split. Lecture Notes in Computer Science 4675, pp. 17-32, 2008.

[Loo90] Loop, C., DeRose, T.: Generalized B-spline surfaces of arbitrary topology. In: Proceedings of SIGGRAPH, pp.347-356, 1990.

[Loo94] Loop, C.: Smooth spline surfaces over irregular meshes. In: Proceedings of SIGGRAPH, pp.303–310, 1994.

[Pet93] Peters, J.: Smooth free-form surfaces over irregular meshes generalizing quadratic splines. Computer Aided Geometric Design. 10:347–361, 1993.

[Pet00] Peters, J.: Patching Catmull–Clark meshes. In: Proceedings of SIGGRAPH, pp. 255–258, 2000.

[Sab83] Sabin, M.A.: Non-rectangular surfaces suitable for inclusion in a B-spline surface. In: Hagen, T. (ed.) Eurographics, 56-69, 1983.

[ZZZ+05] Zheng, J.J., Zhang, J.J., Zhou, H.J., Shen, L.G.: Smooth spline surface generation over meshes of irregular topology .The visual computer, 21:858-864, 2005.