ROLLING RIGID OBJECTS

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ABSTRACT

Simulating the dynamics of rigid bodies plays an important role in virtual reality applications such as virtual assembly planning and ergonomy studies but also in the field of computer animation. In order to decrease the complexity of the object representations and to increase the accuracy of the simulation algorithms one goal is to deal with objects with curved surfaces directly instead of approximating them by polyhedra. One important aspect of the dynamic behaviour of objects with curved surfaces is the rolling process. In this paper we develop the dynamics equations that describe the rolling motion of arbitrarily shaped rigid objects that are in a one- or two-point contact with an arbitrary surface. As a method to keep track of the pairs of closest points we use techniques from differential geometry.

Keywords: rolling motion, constraint based simulation

1 INTRODUCTION

As virtual reality techniques are getting more and more popular to engineers who design complex mechanical systems as well as to people who work in the field of computer animation, the physically correct simulation of the dynamics of rigid bodies is getting more and more important. Thereby objects with curved surfaces are usually approximated by polyhedra. Obviously there is a trade off between the efficiency of the simulation algorithms and the accuracy of the approximation. The collision detection algorithms for example depend intensely on the complexity of the representation of the objects. On the other hand the correctness of the simulation also depends on the accuracy of the approximation. So it is very natural to ask whether it is possible to handle objects with curved surfaces directly instead of approximating them. In this paper we want to study one aspect of the dynamic behaviour of objects with curved surfaces, namely the rolling motion.

There are two fundamentally different approaches to simulate the dynamics of objects, namely the impulse based and the constraint based approach. The impulse based approach proposed by Mirtich ([Mirti96]) can simulate impacts between rigid bodies in consideration of friction. But his approach is limited to situations in which only one contact point occurs and he has to use a trick to cope with contacts that exist for a longer period of time. These non-transient contacts are modeled as a sequence of so called micro collisions and ballistic trajectories in between. In the constraint based approach geometric or dynamic constraints are added to the Newton-Euler dynamics equations. Geometric constraints for example prevent the objects from penetrating each other. Examples for dynamic constraints are equations that describe the friction between objects or the *rolling condition* which is introduced in section 3.

In order to simulate the rolling of rigid bodies we decided to use the constraint based approach. The reason for this decision is that the motion generated by the impulse based method is only an approximation of the rolling motion whereas the constraint based method uses a system of differential equations which describes the rolling mo-

tion exactly.
Our paper is organized as follows: In section 2 we present a revision of an idea from [Anite95] to keep track of the pairs of closest points between two objects. In section 3 we develop the dynamics equations for the rolling of an arbitrary object that touches an arbitrary surface in one or two points. In section 4 we give an evaluation example. We conclude with some remarks on further research in section 5.

2 TRACKING CLOSEST POINTS

In order to simulate the dynamics of rigid objects that get in contact with each other it is necessary to know the points of contact. These closest points are fairly hard to compute if the boundaries of the objects consist of curved surfaces. Computing the distance between two circles in 3D for example leads to the problem of finding the roots of a polynomial of degree eight, which is shown in [Neff90]. As the contact points must be known in each simulation step, always recomputing them is a very costly task. We will revise an approach that was taken from [Anite95] which avoids this repeated computation. The idea is to formulate the motion of the contact points as a function of the velocities of the objects.

We start by introducing some notation. Let A be a coordinate frame. Then $\mathbf{R}_A$ is defined to be the matrix with the axes of A as columns and the vector $\boldsymbol{p}_A$ is the origin of A in world coordinates. For two frames A and B let $\mathbf{R}_{AB}$ be the matrix with the axes of B in coordinates of A as columns. With this definition we have $\mathbf{R}_B = \mathbf{R}_A\mathbf{R}_{AB}$. Let $\boldsymbol{p}_{AB}$ be the origin of B in coordinates of A. We define the linear and angular velocity of B relative to A as $\boldsymbol{v}_{AB} = \mathbf{R}_{AB}^T\dot{\boldsymbol{p}}_{AB}$ and $\boldsymbol{\omega}^\times = \mathbf{R}_{AB}^T\dot{\mathbf{R}}_{AB}$. Now let A, B and C be coordinate frames and let $\mathbf{R}_{AB}, \mathbf{R}_{BC}, \boldsymbol{p}_{AB}$ and $\boldsymbol{p}_{BC}$ be defined as above. Then in [Monta88] is shown that the following holds:

$$\begin{aligned} \boldsymbol{v}_{AC} &= \mathbf{R}_{BC}^T\boldsymbol{v}_{AB} + \mathbf{R}_{BC}^T(\boldsymbol{\omega}_{AB}\times\boldsymbol{p}_{BC}) \\ &\quad +\boldsymbol{v}_{BC} \end{aligned} \quad (1)$$

$$\boldsymbol{\omega}_{AC} = \mathbf{R}_{BC}^T\boldsymbol{\omega}_{AB} + \boldsymbol{\omega}_{BC}. \quad (2)$$

We define the generalized velocity vector $\boldsymbol{V}_{AB} = [\boldsymbol{v}_{AB}^T, \boldsymbol{\omega}_{AB}^T]^T$ and the matrix

$$\mathbf{W}_{BC} = \begin{bmatrix} \mathbf{R}_{BC} & \boldsymbol{p}_{BC}^\times\mathbf{R}_{BC} \\ \mathbf{0} & \mathbf{R}_{BC} \end{bmatrix}.$$

Then the eqs. (1) and (2) can be rewritten as

$$\boldsymbol{V}_{AC} = \mathbf{W}_{BC}^{-1}\boldsymbol{V}_{AB} + \boldsymbol{V}_{BC}. \quad (3)$$

Now we are able to develop the kinematic equations for the closest points. We distinguish three cases: closest points between two curves, closest points between two surfaces and closest points between a curve and a surface.

2.1 Curve-Curve

Let B_1 and B_2 be the coordinate frames of two objects. Let $\boldsymbol{c}_i$ be a curve on object i and let $|\dot{\boldsymbol{c}}_i(t)| = 1$ for all values of t. We assume that the points $\boldsymbol{c}_i(t)$ are given in B_i-coordinates. Let $\boldsymbol{c}_i(\lambda_i)$ be the closest point on $\boldsymbol{c}_i$ and let d be the distance between these two points. We define the contact frames C_i as follows: The origin of C_i is $\boldsymbol{p}_{C_i} = \boldsymbol{c}_i(\lambda_i)$. The x-axis of C_i is the tangent vector to $\boldsymbol{c}_i$ in $\boldsymbol{p}_{C_i}$. The z-axis of C_i points along the minimum distance line between the curves and the y-axis is chosen such that C_i is a right-handed coordinate system. We further define the frame T_i to be the frenet-trihedron on $\boldsymbol{c}_i$ in $\boldsymbol{p}_{C_i}$. This means that the origin of T_i is the point $\boldsymbol{p}_{C_i}$ and that the axes of T_i are the tangent $\boldsymbol{t}_i$, the normal $\boldsymbol{n}_i$ and the binormal $\boldsymbol{b}_i$ in the point $\boldsymbol{p}_{C_i}$. Furthermore we define the angles φ_i and ψ. The angle φ_i is the angle between T_i and C_i measured around the common x-axis. ψ is chosen such that a rotation of one of the frames C_i with the angle ψ makes the x-axes of C_1 and C_2 coincide. Let finally κ_i and τ_i be the curvature and torsion of $\boldsymbol{c}_i$ in $\boldsymbol{p}_{C_i}$, respectively. With these definitions we observe that the following relationships hold:

$$\mathbf{R}_{C_1C_2} = \begin{bmatrix} \mathbf{R}_\psi & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix} \quad \text{with} \quad (4)$$

$$\mathbf{R}_\psi = \begin{bmatrix} \cos\psi & -\sin\psi \\ -\sin\psi & -\cos\psi \end{bmatrix}, \quad (5)$$

$$\boldsymbol{v}_{C_1C_2} = \begin{bmatrix} 0 \\ 0 \\ -\dot{d} \end{bmatrix}, \quad \boldsymbol{\omega}_{C_1C_2} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}, \quad (6)$$

$$\mathbf{R}_{B_iT_i} = [\boldsymbol{t}_i, \boldsymbol{n}_i, \boldsymbol{b}_i] \quad \text{and} \quad (7)$$

$$\mathbf{R}_{T_iC_i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_i & -\sin\varphi_i \\ 0 & \sin\varphi_i & \cos\varphi_i \end{bmatrix}. \quad (8)$$

Now the following lemma can be proven.

Lemma 1. *In the curve-curve case the linear and angular velocities of the contact frames relative to the object frames are given by*

$$\boldsymbol{v}_{B_iC_i} = [\dot{\lambda}_i, 0, 0]^T \quad \textit{and} \quad (9)$$

$$\boldsymbol{\omega}_{B_iC_i} = \begin{bmatrix} \tau_i\dot{\lambda}_i + \dot{\varphi}_i \\ \kappa_i\dot{\lambda}_i\sin\varphi_i \\ \kappa_i\dot{\lambda}_i\cos\varphi_i \end{bmatrix}, \quad (10)$$

respectively.

Proof. See [Anite95] □

Now we want to compute the velocities of the contact frame C_2 relative to C_1. Using (3) twice we obtain

$$\begin{aligned} \boldsymbol{V}_{C_1C_2} &= \mathbf{W}_{C_2B_2}\boldsymbol{V}_{C_1B_2} + \boldsymbol{V}_{B_2C_2} \\ &= \mathbf{W}_{C_2B_2}\mathbf{W}_{B_2B_1}\boldsymbol{V}_{C_1B_1} + \tilde{\boldsymbol{V}} + \boldsymbol{V}_{B_2C_2}. \end{aligned}$$

Herein we set $\tilde{\boldsymbol{V}} = \mathbf{W}_{C_2B_2}\boldsymbol{V}_{B_1B_2}$. After applying several transformations this yields

$$\begin{aligned} \boldsymbol{V}_{C_1C_2} = &\begin{bmatrix} \mathbf{R}_{C_1C_2}\left[\boldsymbol{\omega}_{B_1C_1} \times \begin{bmatrix} 0 \\ 0 \\ -d \end{bmatrix} - \boldsymbol{v}_{B_1C_1}\right] \\ -\mathbf{R}_{C_1C_2}\boldsymbol{\omega}_{B_1C_1} \end{bmatrix} \\ &+\tilde{\boldsymbol{V}} + \boldsymbol{V}_{B_2C_2}. \end{aligned} \tag{11}$$

From here it is easy to prove the following

Theorem 1. *With the above definitions the kinematic equations for the closest points between two curves are*

$$\begin{aligned} \mathbf{R}_\psi \begin{bmatrix} \dot{\lambda}_1 \\ 0 \end{bmatrix} - \mathbf{R}_\psi \begin{bmatrix} -\kappa_1 d\dot{\lambda}_1 \sin\varphi_1 \\ \tau_1 d\dot{\lambda}_1 + d\dot{\varphi}_1 \end{bmatrix} - \begin{bmatrix} \dot{\lambda}_2 \\ 0 \end{bmatrix} &= \begin{bmatrix} \tilde{v}_x \\ \tilde{v}_y \end{bmatrix}, \\ \mathbf{R}_\psi \begin{bmatrix} \tau_1\dot{\lambda}_1 + \dot{\varphi}_1 \\ \kappa_1\dot{\lambda}_1 \sin\varphi_1 \end{bmatrix} - \begin{bmatrix} \tau_2\dot{\lambda}_2 + \dot{\varphi}_2 \\ \kappa_2\dot{\lambda}_2 \sin\varphi_2 \end{bmatrix} &= \begin{bmatrix} \tilde{\omega}_x \\ \tilde{\omega}_y \end{bmatrix}, \\ \dot{d} &= -\tilde{v}_z \\ \dot{\psi} &= \tilde{\omega}_z + \kappa_2\dot{\lambda}_2\cos\varphi_2 + \kappa_1\dot{\lambda}_1\cos\varphi_1, \end{aligned}$$

with $[\tilde{v}_x, \tilde{v}_y, \tilde{v}_z, \tilde{\omega}_x, \tilde{\omega}_y, \tilde{\omega}_z]^T = \mathbf{W}_{C_2B_2}\boldsymbol{V}_{B_1B_2}$.

Proof. We insert (4), (5), (6), (9) and (10) into eq. (11). Then we get the result by separating the rows containing $\dot{d}$ and $\dot{\psi}$ from the others. □

Theorem 1 gives a system of first order ordinary differential equations with the unknowns $\lambda_1, \lambda_2, \varphi_1, \varphi_2, \psi$ and d. This system depends on the current velocities of the objects and can e.g. be solved with the adaptive Runge-Kutta method (see [Press94]).

2.2 Surface-Surface

As in section 2.1, let B_1 and B_2 be the coordinate frames of two objects. Let $\boldsymbol{f}_i$ be a surface on object i. We assume that the points $\boldsymbol{f}_i(\boldsymbol{u})$ are given in B_i-coordinates. Let $\boldsymbol{f}_i(\boldsymbol{\alpha}_i)$ be the closest point on $\boldsymbol{f}_i$ and let d be the distance between these two points. As before we define contact frames C_i as follows: The origin of C_i is $\boldsymbol{p}_{C_i} = \boldsymbol{f}_i(\boldsymbol{\alpha}_i)$. The x-axis of C_i is $\boldsymbol{f}_{iu_1}/|\boldsymbol{f}_{iu_1}|$. The z-axis of C_i is $(\boldsymbol{f}_{iu_1} \times \boldsymbol{f}_{iu_2})/|\boldsymbol{f}_{iu_1} \times \boldsymbol{f}_{iu_2}|$. So this axis points along the minimum distance line between the two surfaces. The y-axis is chosen such that C_i is a right-handed coordinate system. We denote the axes of C_i as $\boldsymbol{x}_i, \boldsymbol{y}_i$ and $\boldsymbol{z}_i$. Let the angle ψ be defined as in section 2.1. With these definitions the relationships (4), (5) and (6) hold. We want to state a similar result as in Lemma 1. Therefore we define the following matrices:

$$\begin{aligned} \mathbf{M}_i &= [\boldsymbol{x}_i, \boldsymbol{y}_i]^T \mathbf{J}_{\boldsymbol{f}_i}, \\ \mathbf{K}_i &= [\boldsymbol{x}_i, \boldsymbol{y}_i]^T \mathbf{J}_{\boldsymbol{z}_i}, \\ \mathbf{T}_i &= \boldsymbol{y}_i^T \mathbf{J}_{\boldsymbol{x}_i}, \end{aligned} \tag{12}$$

where for any function $\boldsymbol{g} : \mathbb{R}^m \to \mathbb{R}^n$ the matrix $\mathbf{J}_{\boldsymbol{g}}$ is the Jacobian matrix of $\boldsymbol{g}$. Now the proof of the following lemma is straightforward.

Lemma 2. *In the surface-surface case the linear and angular velocities of the contact frames relative to the object frames are given by*

$$\boldsymbol{v}_{B_iC_i} = \begin{bmatrix} \mathbf{M}_i\dot{\boldsymbol{\alpha}}_i \\ 0 \end{bmatrix}, \tag{13}$$

$$\boldsymbol{\omega}^{\times}_{B_iC_i} = \begin{bmatrix} \begin{matrix} 0 & -\mathbf{T}_i\dot{\boldsymbol{\alpha}}_i \\ \mathbf{T}_i\dot{\boldsymbol{\alpha}}_i & 0 \end{matrix} & \mathbf{K}_i\dot{\boldsymbol{\alpha}}_i \\ -(\mathbf{K}_i\dot{\boldsymbol{\alpha}}_i)^T & 0 \end{bmatrix},$$

$$\boldsymbol{\omega}_{B_iC_i} = \begin{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{K}_i\dot{\boldsymbol{\alpha}}_i \\ \mathbf{T}_i\dot{\boldsymbol{\alpha}}_i \end{bmatrix}. \tag{14}$$

To obtain a parallel result to theorem 1 we insert the relationships (4), (5), (6), (13) and (14) into eq. (11).

Theorem 2. *The kinematic equations for the closest points between two surfaces are*

$$\begin{aligned} \mathbf{R}_\psi(\mathbf{K}_1 d + \mathbf{M}_1)\dot{\boldsymbol{\alpha}}_1 - \mathbf{M}_2\dot{\boldsymbol{\alpha}}_2 &= \begin{bmatrix} \tilde{v}_x \\ \tilde{v}_y \end{bmatrix}, \\ \mathbf{R}_\psi\mathbf{K}_1\dot{\boldsymbol{\alpha}}_1 + \mathbf{K}_2\dot{\boldsymbol{\alpha}}_2 &= \begin{bmatrix} -\tilde{\omega}_y \\ \tilde{\omega}_x \end{bmatrix}, \\ \dot{d} &= -\tilde{v}_z, \\ \dot{\psi} &= \tilde{\omega}_z + \mathbf{T}_1\dot{\boldsymbol{\alpha}}_1 + \mathbf{T}_2\dot{\boldsymbol{\alpha}}_2, \end{aligned}$$

with $[\tilde{v}_x, \tilde{v}_y, \tilde{v}_z, \tilde{\omega}_x, \tilde{\omega}_y, \tilde{\omega}_z]^T = \mathbf{W}_{C_2B_2}\boldsymbol{V}_{B_1B_2}$.

As in section 2.1 this is a system of first order ordinary differential equations in the variables $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \psi$ and d. This system depends on the velocities of the objects.

2.3 Curve-Surface

Now we consider the last case where we have a surface $\boldsymbol{f}$ on object 1 and a curve $\boldsymbol{c}$ on object 2. As in the sections before the points $\boldsymbol{f}(\boldsymbol{u})$ and $\boldsymbol{c}(t)$ are given in the respective coordinate system. Let the coordinate frame C_1 be defined as in section 2.2 and the frame C_2 as in section 2.1. As in the curve-curve case T is defined to be the frenet-trihedron on $\boldsymbol{c}$ in the closest point and φ is the angle of rotation between the frames T and C_2 measured around the common x-axis. The angle ψ is defined as in the two sections before. We denote the curve parameter as λ and the surface parameters as $\boldsymbol{\alpha}$. Let κ and τ be the curvature and the torsion of the curve.

As before the relationships (4), (5) and (6) hold. we define the matrices $\mathbf{M}, \mathbf{K}$ and $\mathbf{T}$ as in (12). For the relative velocities $\boldsymbol{v}_{B_1C_1}$ and $\boldsymbol{\omega}_{B_1C_1}$ lemma 2 holds, whereas $\boldsymbol{v}_{B_2C_2}$ and $\boldsymbol{\omega}_{B_2C_2}$ are given by lemma 1. We obtain a similar result as in the two sections before by inserting into eq. (11).

Theorem 3. *The kinematic equations for the closest points between a curve and a surface are*

$$\begin{aligned}
\mathbf{R}_\psi(\mathbf{K}d + \mathbf{M})\dot{\boldsymbol{\alpha}} - \begin{bmatrix} \dot\lambda \\ 0 \end{bmatrix} &= \begin{bmatrix} \tilde v_x \\ \tilde v_y \end{bmatrix}, \\
\mathbf{R}_\psi \mathbf{K}\dot{\boldsymbol{\alpha}} + \begin{bmatrix} \kappa\dot\lambda \sin\varphi \\ -\tau\dot\lambda - \dot\varphi \end{bmatrix} &= \begin{bmatrix} -\tilde\omega_y \\ \tilde\omega_x \end{bmatrix}, \\
\dot d &= -\tilde v_z, \\
\dot\psi &= \tilde\omega_z + \mathbf{T}\dot{\boldsymbol{\alpha}} + \kappa\dot\lambda\cos\varphi.
\end{aligned}$$

Once again this is a system of first order ordinary differential equations in the variables $\boldsymbol{\alpha}, \lambda, \varphi, \psi$ and d.

3 ROLLING MOTION

We will now develop the differential equations that describe the rolling of a rigid object on a surface. First we concentrate on the case of a one-point contact, i.e. the object is in permanent contact with the surface in exactly one point. Next we consider the case of two contact points. We call the motion of an object *rolling motion*, if in each contact point $\boldsymbol{p}$ the relative contact velocity is zero, i.e.

$$\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r} = \mathbf{0}, \tag{15}$$

where $\boldsymbol{v}$ and $\boldsymbol{\omega}$ are the linear and angular velocity of the object and $\boldsymbol{r}$ is the vector that points from the center of mass to $\boldsymbol{p}$. We refer to constraint (15) as the *rolling condition*. With this condition it is not hard to see that in the more general case of arbitrarily many contacts all contact points must be collinear. Hence this general case can be reduced to the case of two contact points.

Note that the rolling condition also implies that all contacts are bilateral. But it is easily possible that a rolling object loses the contact to the surface and follows a ballistic trajectory. We will see how to detect these cases.

3.1 One contact point

The ideas used here are similar to those in [MacMi60], where the rolling motion of a sphere on a given surface is studied. As above, $\boldsymbol{r}$ denotes the vector pointing from the center of mass to the contact point. Let m be the mass and $\mathbf{I}$ be the inertia matrix of the object. Let further $\boldsymbol{f}$ be the sum of all external forces acting on the object. We assume that $\boldsymbol{f}$ acts on the center of mass, hence it does not cause a torque. Let $\boldsymbol{f}_r$ denote the reaction force acting in the contact point. The Newton-Euler dynamics equations are

$$m\dot{\boldsymbol{v}} = \boldsymbol{f} + \boldsymbol{f}_r \quad \text{and} \tag{16}$$
$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \boldsymbol{r} \times \boldsymbol{f}_r. \tag{17}$$

Derivating the rolling condition with respect to time yields

$$\dot{\boldsymbol{v}} = \dot{\boldsymbol{r}} \times \boldsymbol{\omega} + \boldsymbol{r} \times \dot{\boldsymbol{\omega}}. \tag{18}$$

We can easily eliminate the reaction force $\boldsymbol{f}_r$ by multiplying (16) by $\boldsymbol{r}^\times$ and subtracting the result from (17). We obtain

$$m\boldsymbol{r} \times \dot{\boldsymbol{v}} - \mathbf{I}\dot{\boldsymbol{\omega}} - \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \boldsymbol{r} \times \boldsymbol{f}. \tag{19}$$

Together with eq. (18) we get the system of differential equations

$$\begin{bmatrix} m\boldsymbol{r}^\times & -\mathbf{I} \\ \mathbf{E} & -\boldsymbol{r}^\times \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{v}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r} \times \boldsymbol{f} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} \\ \dot{\boldsymbol{r}} \times \boldsymbol{\omega} \end{bmatrix}, \tag{20}$$

where $\mathbf{E}$ denotes the 3×3-identity matrix. By the results of section 2 the vector $\dot{\boldsymbol{r}}$ can be viewed as a function of $\boldsymbol{v}$ and $\boldsymbol{\omega}$. So together with these results (20) describes the rolling motion of the object. Solving eq. (20) for $\boldsymbol{\omega}$ yields

$$\dot{\boldsymbol{\omega}} = \left(\mathbf{I} - m\boldsymbol{r}^\times \boldsymbol{r}^\times\right)^{-1} \cdot \boldsymbol{b} \tag{21}$$

for some vector $\boldsymbol{b}$. We see that the matrix $\mathbf{I} - m\boldsymbol{r}^\times\boldsymbol{r}^\times$ is positive definite and thus non-singular. Consequently the differential equation (20) is always non-singular.

As mentioned before the rolling condition constrains the contact to be bilateral. In order to detect whether the object loses the contact we simply compute the reaction force which is given by $\boldsymbol{f}_r = m\dot{\boldsymbol{v}} - \boldsymbol{f}$ according to eq. (16). Let $\boldsymbol{n}$ be the surface normal in the contact point. If the scalar product $\boldsymbol{f}_r^T \boldsymbol{n}$ is negative then $\boldsymbol{f}_r$ is an attractive force. So in this case the contact has to be released.

3.2 Two contact points

Now suppose that the object always touches the surface in two contact points. Let the two vectors pointing from the center of mass to the contact points be denoted as $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. For simplicity we write $\boldsymbol{r}_{12}$ for $\boldsymbol{r}_1 - \boldsymbol{r}_2$. As before $\boldsymbol{f}$ is the sum of all external forces and we assume that $\boldsymbol{f}$ does not cause a torque. We denote the reaction forces of the surface in the two contact points as $\boldsymbol{f}_{r_1}$ and $\boldsymbol{f}_{r_2}$. The Newton-Euler dynamics equations are

$$
\begin{aligned}
m\dot{\boldsymbol{v}} &= \boldsymbol{f} + \boldsymbol{f}_{r_1} + \boldsymbol{f}_{r_2} \quad \text{and} \quad (22)\\
\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} &= \boldsymbol{r}_1 \times \boldsymbol{f}_{r_1} + \boldsymbol{r}_2 \times \boldsymbol{f}_{r_2}. \quad (23)
\end{aligned}
$$

From the rolling conditions for the two contact points follows that the angular velocity is always parallel to the line between the contact points, i.e.

$$\boldsymbol{\omega} \times \boldsymbol{r}_{12} = \mathbf{0}. \tag{24}$$

Also from the rolling conditions we obtain by differentiation the two equations

$$
\begin{aligned}
\dot{\boldsymbol{v}} &= \dot{\boldsymbol{r}}_1 \times \boldsymbol{\omega} + \boldsymbol{r}_1 \times \dot{\boldsymbol{\omega}} \quad \text{and}\\
\dot{\boldsymbol{v}} &= \dot{\boldsymbol{r}}_2 \times \boldsymbol{\omega} + \boldsymbol{r}_2 \times \dot{\boldsymbol{\omega}}.
\end{aligned} \tag{25}
$$

If we multiply eq. (23) by $\boldsymbol{r}_{12}^T$ we obtain after a simple transformation and using (24)

$$(\boldsymbol{f}_{r_1} + \boldsymbol{f}_{r_2})^T (\boldsymbol{r}_1 \times \boldsymbol{r}_2) = \boldsymbol{r}_{12}^T \mathbf{I}\dot{\boldsymbol{\omega}}. \tag{26}$$

We use eq. (22) to eliminate the reaction forces from (26) and then we use the upper equation of (25) to eliminate $\dot{\boldsymbol{v}}$. After some transformations we have

$$
\begin{aligned}
&\big(\mathbf{I}\boldsymbol{r}_{12} - m(\boldsymbol{r}_1 \times \boldsymbol{r}_2) \times \boldsymbol{r}_1\big)^T \dot{\boldsymbol{\omega}}\\
&= (m\dot{\boldsymbol{r}}_1 \times \boldsymbol{\omega} - \boldsymbol{f})^T (\boldsymbol{r}_1 \times \boldsymbol{r}_2).
\end{aligned} \tag{27}
$$

By using the lower equation of (25) instead of the upper one we obtain a similar result which we add to (27) to obtain an equation which is symmetric in the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$:

$$
\begin{aligned}
&\big(2\mathbf{I}\boldsymbol{r}_{12} - m(\boldsymbol{r}_1 \times \boldsymbol{r}_2) \times (\boldsymbol{r}_1 + \boldsymbol{r}_2)\big)^T \dot{\boldsymbol{\omega}}\\
&= \big(m(\dot{\boldsymbol{r}}_1 + \dot{\boldsymbol{r}}_2) \times \boldsymbol{\omega} - 2\boldsymbol{f}\big)^T (\boldsymbol{r}_1 \times \boldsymbol{r}_2).
\end{aligned}
$$

We write this equation as

$$\boldsymbol{w}^T \dot{\boldsymbol{\omega}} = c. \tag{28}$$

Subtracting the two equations (25) from one another and yields

$$\boldsymbol{r}_{12} \times \dot{\boldsymbol{\omega}} = \boldsymbol{\omega} \times \dot{\boldsymbol{r}}_{12}. \tag{29}$$

We combine (29) and (28) to obtain a system of equations for the vector $\dot{\boldsymbol{\omega}}$:

$$
\begin{aligned}
\boldsymbol{r}_{12} \times \dot{\boldsymbol{\omega}} &= \boldsymbol{b}\\
\boldsymbol{w}^T \dot{\boldsymbol{\omega}} &= c,
\end{aligned} \tag{30}
$$

where we denoted the right hand side of (29) as $\boldsymbol{b}$ for the sake of convenience. In order to solve this system we first observe that

$$\boldsymbol{w}^T \boldsymbol{r}_{12} \neq 0. \tag{31}$$

This can be seen by applying some transformations to $\boldsymbol{w}^T \boldsymbol{r}_{12}$ which lead to

$$\boldsymbol{w}^T \boldsymbol{r}_{12} = 2\big(\alpha + m(\boldsymbol{r}_1 \times \boldsymbol{r}_2)^2\big)$$

with $\alpha = \boldsymbol{r}_{12}^T \mathbf{I} \boldsymbol{r}_{12}$. Since $\mathbf{I}$ is positive definite and $\boldsymbol{r}_1 \neq \boldsymbol{r}_2$ we have $\alpha \geq 0$ and hence $\boldsymbol{w}^T \boldsymbol{r}_{12} > 0$. Now we are able to solve the system (30). If $\boldsymbol{r}_{12}^T \boldsymbol{b} \neq 0$ then there is obviously no solution. This means that the object is in a situation in which rolling is impossible. Otherwise a solution is given by

$$\dot{\boldsymbol{\omega}} = \frac{\boldsymbol{b} \times \boldsymbol{r}_{12}}{\boldsymbol{r}_{12}^2} + \left(c + \frac{\boldsymbol{r}_{12}^T(\boldsymbol{b} \times \boldsymbol{w})}{\boldsymbol{r}_{12}^2}\right) \frac{\boldsymbol{r}_{12}}{\boldsymbol{r}_{12}^T \boldsymbol{w}},$$

which can easily be verified. The relationship (31) ensures that the system has rank 3 which implies that the solution is unique.

Together with (25) and with the results of section 2 we have a system of differential equations that describes the rolling of the object.

As in section 3.1 we want to detect the cases when a contact has to be released. We do so by looking at the components of the reaction forces in the direction of the contact normals. By multiplying eq. (22) with $\boldsymbol{r}_2^\times$ and subtracting the result from eq. (23) we obtain an equation of the form

$$\boldsymbol{r}_{12} \times \boldsymbol{f}_{r_1} = \boldsymbol{u}. \tag{32}$$

If $\boldsymbol{u} \neq \mathbf{0}$ we compute the component of the projection of $\boldsymbol{f}_{r_1}$ into the plane with normal $\boldsymbol{r}_{12}$ in the direction of $\boldsymbol{n}_1$. This can be achieved by multiplying (32) with $(\boldsymbol{r}_{12} \times \boldsymbol{n}_1)^T$, such that the left hand side becomes $\boldsymbol{n}_1^T\big(\boldsymbol{r}_{12} \times (\boldsymbol{f}_{r_1} \times \boldsymbol{r}_{12})\big)$. So $\boldsymbol{f}_{r_1}$ is an attractive force if $(\boldsymbol{r}_{12} \times \boldsymbol{n}_1)^T \boldsymbol{u} < 0$. In this case the respective contact has to be released. If $\boldsymbol{u} = \mathbf{0}$ we compute $\boldsymbol{f}_{r_1} + \boldsymbol{f}_{r_2}$ from eq. (22) and multiply this vector with $\boldsymbol{n}_1^T$. Again we release the contact if the result is negative. The check for the force $\boldsymbol{f}_{r_2}$ is completely analogous.

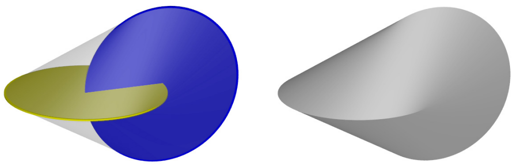

Figure 1: The left image shows a transparent Oloid with the two circles defining it inscribed. The right image shows the solid shaded Oloid.

4 EVALUATION

As an evaluation example we simulated the rolling of an Oloid[1] on an inclined plane. The Oloid is the convex hull of two circles that lie in perpendicular planes such that each of them contains the center of the other. Figure 1 shows the two circles defining the Oloid and the Oloid itself. In figure 2 you see a sequence of snapshots of the rolling motion. The dark curves are the curves of the endpoints of the line segment that touches the plane. In [Dirnb97] the development of the bounding torse of the Oloid has been computed, so we could verify that our result coincides with the curves given there.
This simulation was performed in real time on a Sun workstation with a 440 MHz processor.

5 FURTHER RESEARCH

The rolling condition implies that the object rolls perfectly on the surface. This means that there is no sliding. This corresponds to the assumption that the sticking friction between the object and the surface is infinite. So one aspect of our future work will be to extend the dynamics equations such that the sticking friction is not assumed to be infinite and both sliding and rolling can be simulated.
Another goal for the future is to model the effect of rolling friction.

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[1]The Oloid was invented by Paul Schatz (1898-1979) who took a patent on it in 1933 (Deutsches Reichspatent Nr. 589 452).

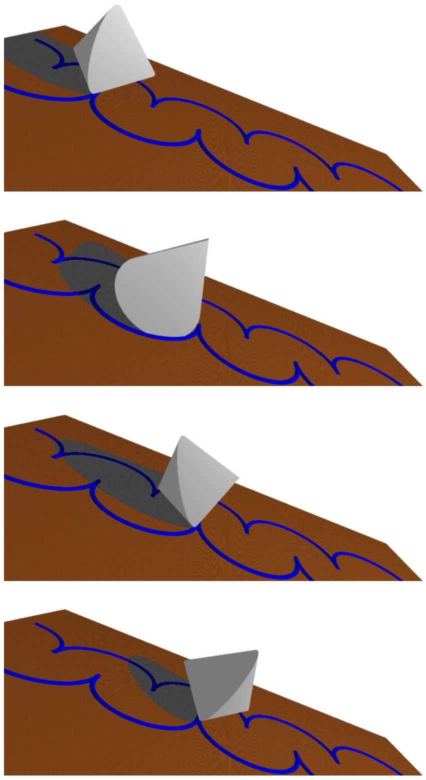

Figure 2: An Oloid rolling down an inclined plane and the contact curves.

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