Determination of Positive Realization of Two Dmensional Systems Using Digraph Theory and GPU Computing Method

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***Abstract*— In the recent years many researchers were interested in positive two-dimensional (2D) linear systems. Analysis of positive 2D systems is more difficult than of positive one-dimensional (1D) systems. A lot of numerical problems that arised in positive 2D systems are unsolved completely, for examples: minimal positive realization problem, determination of lower and upper index reachability, determination of reachability index set, determination of state matrices from characteristic polynomial, etc. In many case this problems cannot be solved analytically by hand. To solve this problems we can use new computational method based on digraph theory and CPU or GPU computing method. A new method of determination positive realization of two-dimensional systems using digraph theory will be proposed. A procedure for computation of the state matrices will be given. The procedure will be illustrated by a numerical example.**

Keywords—positive systems, realizatiojns, digraph, GPU computing

I. INTRODUCTION

Let $\mathbb{R}_+^{n\times m}$ be the set of $n\times m$ real matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$. The set of nonnegative integers will be denoted by $\mathbb{Z}_+$ and the $n\times n$ identity matrix by $\mathbf{I}_n$.

Consider the two-dimension (2D) general model described by the equation [4]

$$\begin{aligned} &x_{i+1,j+1} = \\ &= \mathbf{A}_0 x_{ij} + \mathbf{A}_1 x_{i+1,j} + \mathbf{A}_2 x_{i,j+1} + \mathbf{B}_0 u_{ij} + \mathbf{B}_1 u_{i+1,j} + \mathbf{B}_2 u_{i,j+1} \\ &y_{ij} = \mathbf{C} x_{ij} + \mathbf{D} u_{ij}, \quad i,j \in \mathrm{Z}_+ = \{0,1,\ldots\} \end{aligned} \tag{1}$$

where $xij\in\mathbb{R}_n$, $u_{ij}\in\mathbb{R}_m$ and $y_{ij}\in\mathbb{R}^p$ are state, input and output vectors, respectively at the point (i, j), and $\mathbf{A}_k\in\mathbb{R}^{n\times n}$, $\mathbf{B}_k\in\mathbb{R}^{n\times m}$, k = 0,1,2, $\mathbf{C}\in\mathbb{R}^{p\times n}$, $\mathbf{D}\in\mathbb{R}^{p\times m}$.

In this paper special case of general model for $\mathbf{A}_0 = 0$ and $\mathbf{B}_0 = 0$ – the second Fornasini-Marchesini model (2) with many inputs

Definition 1. *The model is called (internally) positive second Fornasini-Marchesini model if for all boundary conditions*

$$x_{i0} \in \mathrm{R}_+^n, \quad i \in \mathrm{Z}_+ \quad \text{and} \quad x_{0j} \in \mathrm{R}_+^n, \quad j \in \mathrm{Z}_+ \tag{2}$$

and every sequence of inputs $u_{ij}\in\mathbb{R}_+^m$, $i,j\in\mathbb{Z}_+$, we have $x_{ij}\in\mathbb{R}_+^n$ and $y_{ij}\in\mathbb{R}_+^p$ for $i,j\in\mathbb{Z}_+$.

Theorem 1. *The second Fornasini-Marchesini model is internally positive if and only if*

$$\mathbf{A}_k \in \mathrm{R}_+^{n\times n},\ \mathbf{B}_k \in \mathrm{R}_+^{n\times m},\ k = 1,2,\ \mathbf{C} \in \mathrm{R}_+^{p\times n},\ \mathbf{D} \in \mathrm{R}_+^{p\times m} \tag{3}$$

In this paper we assume that the second Fornasini-Marchesini model is single-input single-output (m=p=1) model.

The transfer function $T(z_1,z_2)\in\mathbb{R}_+^{p\times m}$ of the second Fornasini-Marchesini model is given by

$$T(z_1,z_2) = \mathbf{C}\left[\mathbf{I}z_1z_2 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2\right]^{-1}\left(\mathbf{B}_1 z_1 + \mathbf{B}_2 z_2\right) + D \tag{4}$$

The transfer function (4) can be rewritten in the form

$$\begin{aligned} T(z_1,z_2) &= \frac{\mathbf{C}\ \mathrm{Adj}\ \mathbf{H}(z_1,z_2)\left(\mathbf{B}_1 z_1 + \mathbf{B}_2 z_2\right)}{\det \mathbf{H}(z_1,z_2)} + D = \\ &= \frac{n(z_1,z_2)}{d(z_1,z_2)} + D \end{aligned} \tag{5}$$

Our task is the following: for given transfer function (5) determine entries of the state matrices $\mathbf{A}_k$ and $\mathbf{B}_k$ for k = 1,2 and $\mathbf{C}$ using two dimensional $D^{(2)}$ digraphs theory and GPU computing method. The dimension of the state matrices must be the minimal among possible ones and cannot appear additional condition on the coefficients of the characteristic polynomial.

II. SOLUTION OF THE PROBLEM

The realization problem we can divide on the following tasks:

- Task 1 – Using characteristic polynomial determine entries of the state matrices $\mathbf{A}_k$, for k=1,2 using two dimensional $D^{(2)}$ digraphs theory.

- Task 2 – Determine entries of the matrices $\mathbf{B}_k$ for k=1,2 and matrix $\mathbf{C}$ using two dimensional $D^{(2)}$ digraphs theory and GPU computing method (solution of the linear equation set).

A. *Task 1 – determination matrix A_1 and A_2*

Theorem 2. *The state matrices* $\mathbf{A}_k$, k = 1,2 *of the positive second Fornasini-Marchesini model have the structure*

$$\mathbf{A}_1^{11} = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ d_{n,0} & 0 & \dots & 0 & d_{n,1} \\ 0 & 1 & \dots & 0 & d_{n,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & d_{n,n-1} \end{bmatrix} \in \mathrm{R}_+^{n\times n},$$

$$\mathbf{A}_1^{12} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \in \mathrm{R}_+^{n\times n-1}, \mathbf{A}_1^{21} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \in \mathrm{R}_+^{n-1\times n},$$

$$\mathbf{A}_1^{22} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ d_{n-2,n-1} & d_{n-3,n-1} & \dots & d_{1,n-1} & 0 \end{bmatrix} \in \mathrm{R}_+^{n-1\times n-1},$$

$$\mathbf{A}_2^{11} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ d_{n-1,1} & 0 & 0 & \dots & 0 \\ d_{n-1,2} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{n-1,n-2} & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathrm{R}_+^{n-1\times n-1},$$

B. *Task 2 – determination matrix B_1, B_2 and C*

In two dimensional digraph determinate in Task 1 we draw up source (which represent matrix $\mathbf{B}_k$, k=1,2). Then we can draw arcs from source s_p, p=1,2,…m to vertex $v_1,v_2,\dots,v_n$ and for every arcs we assign weight $w(s_p, v_i)_{B1}z_2^{-1}$, i=1,2,…,n when arcs came from matrix $\mathbf{B}_1$ and weight $w(s_p, v_i)_{B2}z_1^{-1}$, i=1,2,…,n when arcs came from matrix $\mathbf{B}_2$. In next step we decompose received two dimensional digraph $D^{(2)}$ on simple digraphs including finite path from source s_p p=1,2,…,m to finite vertex v_k (vertex v_k is show by the matrix $\mathbf{C}$). Compering coefficients polynomial we obtain set of linear equations. Solving linear equations set we obtain matrix $\mathbf{B}_k$, k=1,2 and $\mathbf{C}$ in the form

$$\mathbf{B}_1 = \begin{bmatrix} w\left(s_p, v_1\right)_{B_1} \\ w\left(s_p, v_2\right)_{B_1} \\ \vdots \\ w\left(s_p, v_n\right)_{B_1} \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} w\left(s_p, v_1\right)_{B_2} \\ w\left(s_p, v_2\right)_{B_2} \\ \vdots \\ w\left(s_p, v_n\right)_{B_2} \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \dots & 0 & v_k & 0 & \dots \end{bmatrix}$$

$$\mathbf{A}_k = \begin{bmatrix} \mathbf{A}_k^{11} & \mathbf{A}_k^{12} \\ \mathbf{A}_k^{21} & \mathbf{A}_k^{22} \end{bmatrix} \in \mathrm{R}_+^{(2n-1)\times(2n-1)}, \quad k = 1,2$$

where

$$\mathbf{A}_2^{12} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & d_{n-2,2} & 0 & \dots & 0 & 0 & 0 \\ 0 & d_{n-2,3} & d_{n-3,3} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & d_{n-2,n-2} & d_{n-3,n-2} & \dots & d_{2,n-2} & 0 & 0 \\ 0 & 0 & & \dots & 0 & 0 & 0 \end{bmatrix} \in \mathrm{R}_+^{n-1\times n},$$

$$\mathbf{A}_2^{21} = \begin{bmatrix} d_{n-1,n-1} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathrm{R}_+^{n\times n-1},$$

$$\mathbf{A}_1^{22} = \begin{bmatrix} d_{n-1,n} & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ d_{n-2,n} & d_{n-3,n} & \dots & d_{1,n} & d_{0,n} & 0 \end{bmatrix} \in \mathrm{R}_+^{n\times n}$$

Let be given m equation with n unknown in the following form

$$\begin{gathered} w_{11}t_1 + w_{12}t_2 + \ldots + w_{1n}t_n = p_1 \\ w_{21}t_1 + w_{22}t_2 + \ldots + w_{2n}t_n = p_2 \\ \vdots \\ w_{m1}t_1 + w_{m2}t_2 + \ldots + w_{mn}t_n = p_m \end{gathered}$$

Linear equation set we can write in matrix form

$$t = \mathbf{W}^{-1}p$$

Equation we can solve using m-function created in Matlab. Additionally, when we have large size matrix $\mathbf{W}$ and p to solve the linear equation set we can use Parallel Computing Toolbox (using GPU computing method and multi-core method). Additionally using graphic card to solve equation we obtain much more faster solution than using classic methods.