Analysis of Cascaded Canonical Dissipative Systems and LTI Filter Sections

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***Abstract*—A series of feedforward coupled hopf-type amplifiers and LTI filter sections are suitable in the cochlea modeling. From a more general point of view, we compare the usage of different canonical dissipative systems with Hopf-type bifurcations and analyze their nonlinear amplification characteristics.**

***Index Terms*—cochlea modeling, canonical dissipative system, bifurcation, nonlinear amplification**

I. Introduction

Various experiments revealed that the nonlinear amplification process in the cochlea is characteristic for a system close to a Hopf instability [1]. Thus, a Hopf-type amplifier was proposed as basic element in the cochlea modeling [2]. Among other complicate hydrodynamic models, a chain of alternating Hopf amplifiers and filters shows the desired accuracy to model the entire cochlea [3]. Thereby, the Hopf cells are described by a μ-family of complex differential equations

$$\dot{\xi} = (\mu + j)\omega_0\xi - \omega_0|\xi|^2\xi - \omega_0 F, \quad \xi,\, F \in \mathbb{C}, \tag{1}$$

or its real representation

$$\begin{aligned} \dot{x} &= \mu\omega_0 x - \omega_0 y - \omega_0 x(x^2+y^2) - \omega_0 p \\ \dot{y} &= \mu\omega_0 y + \omega_0 x - \omega_0 y(x^2+y^2) - \omega_0 q, \end{aligned} \tag{2}$$

where $\xi = x + jy$ and the external forcing $F = p + jq$. A main property of this Hopf cell is a μ-dependent nonlinear amplification of the input signal for small negative μ-values. This phenomenon arise also in a cochlear such that a Hopf cell is well-suited in cochlear modeling.

It is known (see [4]) that (2) can be reformulated, neglecting the forcing terms, as a canonical dissipative system (CDS)

$$\begin{aligned} \dot{x} &= -\frac{\partial H}{\partial y} - g_\mu^x(H)\frac{\partial H}{\partial x} \\ \dot{y} &= \quad \frac{\partial H}{\partial x} - g_\mu^y(H)\frac{\partial H}{\partial y}, \end{aligned} \tag{3}$$

where $H(x,y) := (\omega_0/2)(x^2+y^2)$ and $g_\mu^{x,y}(H) := (2/\omega_0)H - \mu$. Omitting the second terms of the r.h.s. of (3), that can be interpreted as damping terms, we obtain a energy preserving Hamilton system. In this case the system represents a simple linear oscillator. For positive μ-values the function $g_\mu(H)$ has a non-trivial zero set and a stable limit cycle arises (see [4]). For negative μ-values the damping terms are positive and zero is the only and furthermore asymptotically stable solution. Due to its amplification characteristic, that appears close to the bifurcation point $\mu = 0$, only the case of small negative μ-values is of interest.

Obviously, there are other CDS where H and $g_\mu^{x,y}$ have to be chosen such that we obtain a system with limit cycles. In this paper we consider the symmetric CDS (3) and a asymmetric variant of (3) where one of the terms g_μ^x or g_μ^y is omitted and a forcing term is added. Then we compare the transfer behavior of forced symmetric and asymmetric CDS. Especially, we assume the asymmetric CDS

$$\begin{aligned} \dot{x} &= \quad \omega_0 y \\ \dot{y} &= -\omega_0 x - \omega_0 y(x^2+y^2-\mu) + \omega_0 f, \end{aligned} \tag{4}$$

where f is the external forcing. At first we analyze the symmetric system (2). After setting the forcing term $F(t)$ to zero and linearizing the r.h.s. of (2) we calculate the eigenvalues of the corresponding Jacobian as $\lambda_{1,2} = (\mu \pm j)\,\omega_0$. We find that the imaginary parts of the eigenvalues are constant and only the real parts change linear in varying μ. If the asymmetric CDS (4) is linearized we obtain its eigenvalues as $\lambda_{1,2} = (\mu/2 \pm (1/2)\sqrt{\mu^2-4})\omega_0$. In this case the eigenvalues are complex only for $|\mu| < 2$. We emphasize that the transient solutions differ in dependence of μ. Assuming an external forcing term $F(t) = F_0 e^{j\omega t}$ in (1) results in a steady-state solution of the type $\xi(t) = \xi_0 e^{j(\omega t + \theta)}$; a corresponding real representation for (2) can be obtained. For the asymmetric CDS (4) we assume $f(t) = f_0\cos(\omega t)$ and the solution is of the form $x(t) = x_0\cos(\omega t + \varphi)$. Calculating the amplitudes of (1) and (4) for these input signals close to resonance, $\omega = \omega_0$, we obtain $F_0 = |\mu\xi_0 - \xi_0^3|$ and $f_0 = |\mu x_0 - x_0^3|$, respectively. Therefore, we have the same amplification characteristics for both systems.

Now, we analyze the behavior of the cascaded systems that consist of CDS and LTI filter sections, where each CDS_i has a different resonance frequency $\omega_{0,i}$ and each filter section has its own cutoff frequency $f_{\text{ch},i}$. The filters are realized by 6^{th}-order IIR Butterworth low-pass filters (see [5]). The numerical solutions of the systems are calculated by an explicit 4^{th}-order Runge-Kutta method. We implemented the different systems on a DSP development board. More

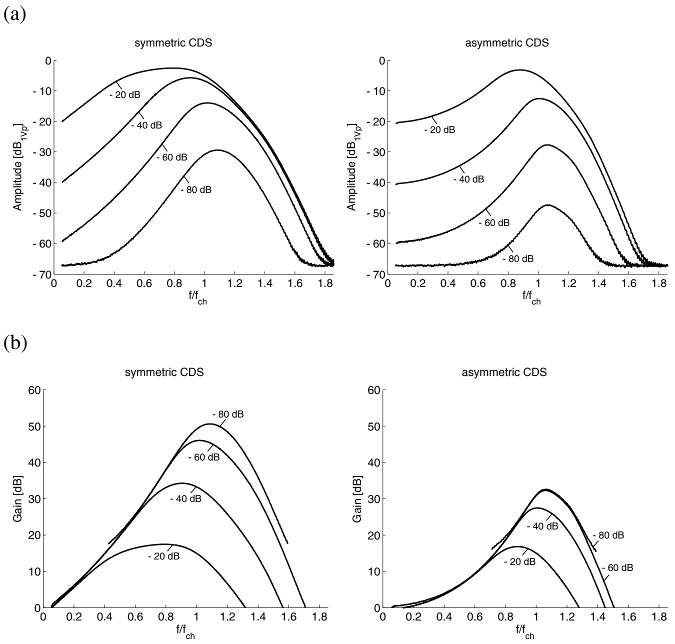

Fig. 1. Single tone responses, section 8, $\mathrm{f_{ch}} = 2960\,\mathrm{Hz}$ and $\mu_i = -0.2\ \forall i$.

details about the realization can be found in the final paper.

Measurement results for the cascaded systems with the symmetric and the asymmetric CDS are shown in Fig. 1. At the 8th section the response upon a single-tone stimulation is measured as a function of the stimulation frequency. Thereby, the input strength is scaled from $-20\,\mathrm{dB}$ down to $-80\,\mathrm{dB}$. Comparing the transfer behavior for small μ the phenomenon of nonlinear amplification arises in both systems and even the qualitative behavior in dependence of the frequency is similar. We expect that this behavior exists also in other CDS-filter chains. Some more detailed results can be found in the final paper.

REFERENCES

[1] C.D. Geisler, "From sound to synapse: Physiology of the mammalian ear", *Oxford Univ. Press*, 1998.

[2] V. M. Eguíluz, M. Ospeck, Y. Choe, A. J. Hudspeth, and M. O. Magnasco, "Essential nonlinearities in hearing", *Phys. Rev. Lett.*, vol. 84, pp. 5232–5235, 2000.

[3] R. Stoop, T. Jasa, Y. Uwate, and S. Martignoli, "From hearing to listening: Design and properties of an actively tunable electronic hearing sensor", *Sensors*, vol. 7, pp. 3287–3298, 2007.

[4] W. Ebeling, I.M. Sokolov, "Statistical thermodynamics and stochastic theory of nonequilibrium systems", *World Scientific*, Singapore 2004.

[5] M. Reit, R. Stoop, and W. Mathis, "Time-Discrete Nonlinear Cochlea Model Implemented on DSP for Auditory Studies", *Proceedings of Nonlinear Dynamics of Electronic Systems (NDES)*, 2012.