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Comparison of various refined beam theories for the bending and free vibration analysis of thick beams

A. S. Sayyad[a,*]

[a] *Department of Civil Engineering, SRES's College of Engineering, Kopargaon-423601, M.S., India.*

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Abstract

In this paper, unified shear deformation theory is used to analyze simply supported thick isotropic beams for the transverse displacement, axial bending stress, transverse shear stress and natural frequencies. This theory enables the selection of different in-plane displacement components to represent shear deformation effect. The numbers of unknowns are same as that of first order shear deformation theory. The governing differential equations and boundary conditions are obtained by using the principle of virtual work. The results of displacement, stresses, natural bending and thickness shear mode frequencies for simply supported thick isotropic beams are presented and discussed critically with those of exact solution and other higher order theories. The study shows that, while the transverse displacement and the axial stress are best predicted by the models 1 through 5 whereas models 1 and 2 are overpredicts the transverse shear stress. The model 4 predicts the exact dynamic shear correction factor ($\pi^2/12 = 0.822$) whereas model 1 overpredicts the same.

Keywords: Thick beam, shear deformation, principle of virtual work, bending analysis, transverse shear stress, free flexural vibration, natural frequencies, dynamic shear correction factor

1. Introduction

Beams are common structural elements in most structures and they are analyzed using classical or refined shear deformation theories to evaluate static and dynamic characteristics. Elementary theory of beam bending underestimates deflections and overestimates the natural frequencies since it disregards the transverse shear deformation effect. Timoshenko [24] was the first to include refined effects such as rotatory inertia and shear deformation in the beam theory. This theory is now widely referred to as Timoshenko beam theory or first order shear deformation theory. In this theory transverse shear strain distribution is assumed to be constant through the beam thickness and thus requires problem dependent shear correction factor. The accuracy of Timoshenko beam theory for transverse vibrations of simply supported beam in respect of the fundamental frequency is verified by Cowper [6, 7] with a plane stress exact elasticity solution.

The limitations of elementary theory of beam and first order shear deformation theory led to the development of higher order shear deformation theories. Many higher order shear deformation theories are available in the literature for static and dynamic analysis of beams [2–5, 11, 15]. Levinson [17] has developed a new rectangular beam theory for the static and dynamic analysis of beam. Reddy [18] has developed well known third order shear deformation theory for the non-linear analysis of plates with moderate thickness. The trigonometric shear deformation theories are presented by Touratier [25], Vlasov and Leont'ev [26] and Stein [20] for thick

*Corresponding author. Tel.: +91 97 63 567 881, e-mail: attu_sayyad@yahoo.co.in.

beams. However, with these theories shear stress free boundary conditions are not satisfied at top and bottom surfaces of the beam. Ghugal and Shipmi [9] and Ghugal [10] has developed a trigonometric shear deformation theory which satisfies the shear stress free condition at top and bottom surfaces of the beam. Soldatos [22] has dveloped hyperbolic shear deformation theory for homogeneous monoclinic plates. Recently Ghugal and Sharma [8] employed hyperbolic shear deformation theory for the static and dynamic analysis of thick isotropic beams. A study of literature [1,12–14,19] indicates that the research work dealing with flexural analysis of thick beams using refined shear deformation theories is very scant and is still in infancy. Sayyad [21] has carried out comparison of various shear deformation theories for the free vibration analysis of thick isotropic beams.

In the present study, various shear deformation theories are used for the bending and free vibration analysis of simply supported thick isotropic beams.

2. Beam under consideration

Consider a beam made up of isotropic material as shown in Fig. 1. The beam can have any boundary and loading conditions. The beam under consideration occupies the region given by

$$0 \le x \le L, \qquad -b/2 \le y \le b/2, \qquad -h/2 \le z \le h/2, \tag{1}$$

where x, y, z are Cartesian co-ordinates, L is length, b is width and h is the total depth of the beam. The beam is subjected to transverse load of intensity $q(x)$ per unit length of the beam.

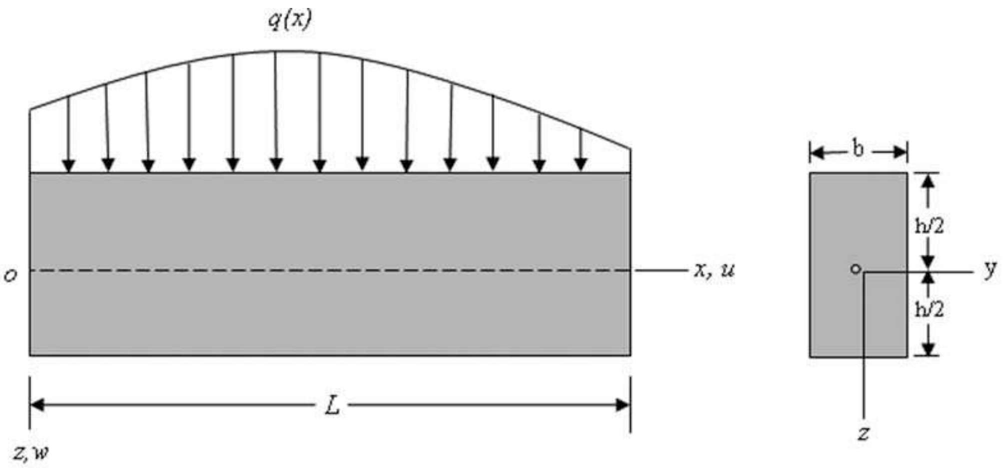

Fig. 1. Beam under bending in $x - z$ plane

2.1. Assumptions made in theoretical formulation

1. The in-plane displacement u in x direction consists of two parts:

   (a) A displacement component analogous to displacement in elementary beam theory of bending;

   (b) Displacement component due to shear deformation which is assumed to be parabolic, sinusoidal, hyperbolic and exponential in nature with respect to thickness coordinate.

2. The transverse displacement w in z direction is assumed to be a function of x coordinate.
3. One dimensional constitutive law is used.
4. The beam is subjected to lateral load only.

2.2. The displacement field

Based on the before mentioned assumptions, the displacement field of the present unified shear deformation theory is given as below

$$u(x,z,t) = -z\frac{\partial w}{\partial x} + f(z)\phi(x,t), \tag{2}$$
$$w(x,z,t) = w(x,t). \tag{3}$$

Here u and w are the axial and transverse displacements of the beam center line in x and z-directions respectively and t is the time. The ϕ represents the rotation of the cross-section of the beam at neutral axis which is an unknown function to be determined. The functions $f(z)$ assigned according to the shearing stress distribution through the thickness of the beam are given in Table 1.

Table 1. Functions $f(z)$ for different shear stress distribution

Model	Author	Function $f(z)$
Model 1	(Ambartsumian [2])	$f(z) = \left[\frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\right]$
Model 2	(Kruszewski [15])	$f(z) = \left[\frac{5z}{4}\left(1 - \frac{4z^2}{3h^2}\right)\right]$
Model 3	(Reddy [18])	$f(z) = z\left[1 - \frac{4}{3}\left(\frac{z}{h}\right)^2\right]$
Model 4	(Touratier [25])	$f(z) = \frac{h}{\pi}\sin\frac{\pi z}{h}$
Model 5	(Soldatos [22])	$f(z) = \left[z\cosh\left(\frac{1}{2}\right) - h\sinh\left(\frac{z}{h}\right)\right]$
Model 6	(Karama et al. [14])	$f(z) = z\exp\left[-2\left(\frac{z}{h}\right)^2\right]$
Model 7	(Akavci [1])	$f(z) = \frac{3\pi}{2}\left[h\tanh\left(\frac{z}{h}\right) - z\sec^2\mathrm{h}\left(\frac{1}{2}\right)\right]$

2.3. Necessity of refined theories

The shear deformation effects are more pronounced in the thick beams than in the slender beams. These effects are neglected in elementary theory of beam (ETB) bending. In order to describe the correct bending behavior of thick beams including shear deformation effects and the associated cross sectional warping, shear deformation theories are required. This can be accomplished by selection of proper kinematics and constitutive models. The functions $f(z)$ is included in the displacement field of higher order theories to take into account effect of transverse shear deformation and to get the zero shear stress conditions at top and bottom surfaces of the beam.

2.4. Strain-displacement relationship

Normal strain and transverse shear strain for beam are given by

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} + f(z)\frac{\partial \phi}{\partial x}, \tag{4}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = f^{'}(z)\phi. \tag{5}$$

2.5. Stress-Strain relationship

According to one dimensional constitutive law, the axial stress/normal bending stress and transverse shear stress are given by

$$\sigma_x = E\varepsilon_x = E\left[-z\frac{\partial^2 w}{\partial x^2} + f(z)\frac{\partial \phi}{\partial x}\right], \tag{6}$$

$$\tau_{zx} = G\gamma_{zx} = Gf'(z)\phi. \tag{7}$$

3. Governing equations and boundary conditions

Using Eqns. (4) through (7) and the principle of virtual work, variationally consistent governing differential equations and boundary conditions for the beam under consideration can be obtained. The principle of virtual work when applied to the beam leads to

$$\begin{aligned} &\int_0^L \int_{-h/2}^{+h/2} (\sigma_x \delta\varepsilon_x + \tau_{zx}\delta\gamma_{zx})\,\mathrm{d}z\,\mathrm{d}x + \\ \rho \int_0^L \int_{z-h/2}^{+h/2} &\left(\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 w}{\partial t^2}\delta w\right)\mathrm{d}z\,\mathrm{d}x - \int_0^L q\delta w\,\mathrm{d}x = 0, \end{aligned} \tag{8}$$

where the symbol δ denotes the variational operator. Integrating the preceding equations by parts, and collecting the coefficients of δw and $\delta\phi$, the governing equations in terms of displacement variables are obtained as follows

$$A_0\frac{\partial^4 w}{\partial x^4} - B_0\frac{\partial^3 \phi}{\partial x^3} - \frac{\rho A_0}{E}\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho B_0}{E}\frac{\partial^3 \phi}{\partial x \partial t^2} + \rho h\frac{\partial^2 w}{\partial t^2} = q, \tag{9}$$

$$B_0\frac{\partial^3 w}{\partial x^3} - C_0\frac{\partial^2 \phi}{\partial x^2} + D_0\phi - \frac{\rho B_0}{E}\frac{\partial^3 w}{\partial x \partial t^2} - \frac{\rho C_0}{E}\frac{\partial^2 \phi}{\partial t^2} = 0 \tag{10}$$

and the associated boundary conditions obtained are of following form

$$-A_0\frac{\partial^3 w}{\partial x^3} + B_0\frac{\partial^2 \phi}{\partial x^2} + \frac{\rho A_0}{E}\frac{\partial^3 w}{\partial x \partial t^2} - \frac{\rho B_0}{E}\frac{\partial^2 \phi}{\partial t^2} = 0 \quad \text{or } w \text{ is prescribed} \tag{11}$$

$$A_0\frac{\partial^2 w}{\partial x^2} - B_0\frac{\partial \phi}{\partial x} = 0 \quad \text{or } \frac{\mathrm{d}w}{\mathrm{d}x} \text{ is prescribed} \tag{12}$$

$$-B_0\frac{\partial^2 w}{\partial x^2} + C_0\frac{\partial \phi}{\partial x} = 0 \quad \text{or } \phi \text{ is prescribed} \tag{13}$$

where A_0, B_0, C_0 and D_0 are the stiffness coefficients given as follows

$$A_0 = E\int_{-h/2}^{+h/2} z^2\,\mathrm{d}z, \quad B_0 = E\int_{-h/2}^{+h/2} zf(z)\,\mathrm{d}z, \tag{14}$$

$$C_0 = E\int_{-h/2}^{+h/2} f^2(z)\,\mathrm{d}z, \quad D_0 = G\int_{-h/2}^{+h/2} [f'(z)]^2\,\mathrm{d}z.$$

3.1. Illustrative examples

In order to prove the efficacy of the present theories, the following numerical examples are considered. The following material properties for beam are used

$$E = 210\ \text{GPa}, \qquad \mu = 0.3, \qquad G = \frac{E}{2(1+\mu)} \text{ and } \rho = 7\,800\ \text{kg/m}^3, \tag{15}$$

where E is the Young's modulus, ρ is the density, and μ is the Poisson's ratio of beam material.

Example 1: Bending analysis of beam

A simply supported uniform beam shown in Fig. 2 subjected to uniformly distributed load $q(x) = \sum_{m=1}^{m=\infty} q_m \sin\left(\frac{m\pi x}{L}\right)$ acting in the z-direction, where q_m is the coefficient of single Fourier expansion of load. The value of q_m for uniformly distributed load given as follows

$$\begin{aligned} q_m &= \frac{4q_0}{m\pi}\ , \quad m = 1, 3, 5, \ldots, \\ q_m &= 0 \quad\ , \quad m = 2, 4, 6, \ldots, \end{aligned} \tag{16}$$

where q_0 is the intensity of uniformly distributed load.

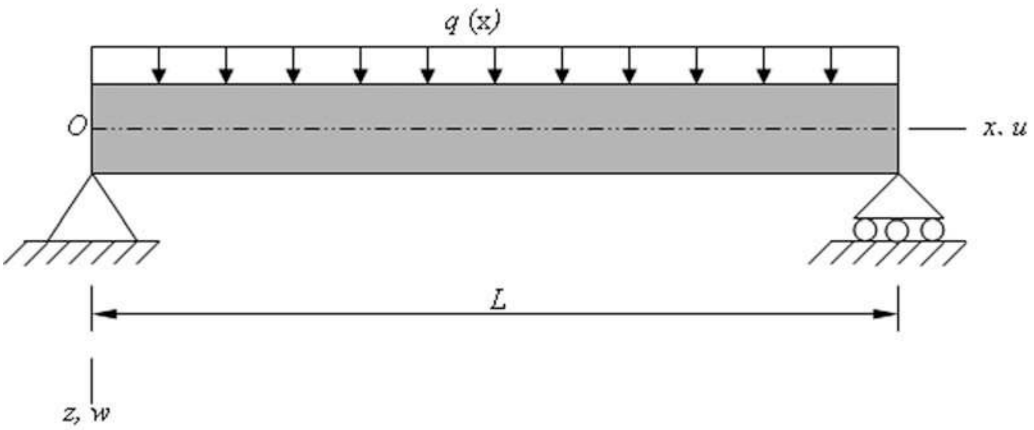

Fig. 2. Simply supported beam subjected to uniformly distributed load

The governing equations for bending analysis of beam (static flexure), discarding all the terms containing time derivatives become

$$A_0 \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - B_0 \frac{\mathrm{d}^3 \phi}{\mathrm{d}x^3} = q, \tag{17}$$

$$B_0 \frac{\mathrm{d}^3 w}{\mathrm{d}x^3} - C_0 \frac{\mathrm{d}^2 \phi}{\mathrm{d}x^2} + D_0 \phi = 0. \tag{18}$$

The following is the solution form assumed for $w(x)$ and $\phi(x)$ which satisfies the boundary conditions exactly

$$w(x) = \sum_{m=1}^{\infty} w_m \sin \frac{m\pi x}{L}, \qquad \phi(x) = \sum_{m=1}^{\infty} \phi_m \cos \frac{m\pi x}{L}, \tag{19}$$

where w_m and ϕ_m are the unknown coefficients of the respective Fourier expansion and m is the positive integer. Substituting this form of solution and the load $q(x)$ into governing equations, yields the following two algebraic simultaneous equations

$$\left(A_0\frac{m^4\pi^4}{L^4}\right)w_m - \left(B_0\frac{m^3\pi^3}{L^3}\right)\phi_m = q_m, \tag{20}$$

$$-\left(B_0\frac{m^3\pi^3}{L^3}\right)w_m + \left(C_0\frac{m^2\pi^2}{L^2} + D_0\right)\phi_m = 0. \tag{21}$$

Solving Eqns. (20) and (21) simultenously to determine unknowns w_m and ϕ_m

$$w_m = \frac{q_m\left(C_0\frac{m^2\pi^2}{L^2} + D_0\right)}{\left(C_0\frac{m^2\pi^2}{L^2} + D_0\right)\left(A_0\frac{m^4\pi^4}{L^4}\right) - \left(B_0\frac{m^3\pi^3}{L^3}\right)\left(B_0\frac{m^3\pi^3}{L^3}\right)}, \tag{22}$$

$$\phi_m = \frac{q_m\left(B_0\frac{m^3\pi^3}{L^3}\right)}{\left(C_0\frac{m^2\pi^2}{L^2} + D_0\right)\left(A_0\frac{m^4\pi^4}{L^4}\right) - \left(B_0\frac{m^3\pi^3}{L^3}\right)\left(B_0\frac{m^3\pi^3}{L^3}\right)}. \tag{23}$$

Using Eqns. (22) and (23) substitute Eqn. (19) in the displacement field [Eqns. (2) and (3)] and stress-strain relationships [Eqns. (6) and (7)] to obtain expressions for axial displacement, transverse displacement, axial bending stress and transverse shear stress

Axial displacement: $$u = \left[-z\frac{m\pi}{L}w_m + f(z)\phi_m\right]\cos\frac{m\pi x}{L}. \tag{24}$$

Transverse displacement: $$w = w_m \sin\frac{m\pi x}{L}. \tag{25}$$

Axial bending stress: $$\sigma_x = E\left[z\frac{m^2\pi^2}{L^2}w_m - f(z)\frac{m\pi}{L}\phi_m\right]\sin\frac{m\pi x}{L}. \tag{26}$$

Transverse shear stress: $$\tau_{zx} = Gf'(z)\phi_m\cos\frac{m\pi x}{L}. \tag{27}$$

Example 2: Free flexural vibration of beam

The governing equations for free flexural vibration of simply supported beam can be obtained by setting the applied transverse load equal to zero in Eqns. (9) and (10). A solution to resulting governing equations, which satisfies the associated initial conditions, is of the form

$$w = w_m \sin\frac{m\pi x}{L}\sin\omega_m t, \tag{28}$$

$$\phi = \phi_m \cos\frac{m\pi x}{L}\sin\omega_m t, \tag{29}$$

where w_m and ϕ_m are the amplitudes of translation and rotation respectively, and ω_m is the natural frequency of the m^{th} mode of vibration. Substitution of this solution form into the governing equations of free vibration of beam results in following algebraic equations

$$\left[\left(A_0\frac{m^4\pi^4}{L^4}\right)w_m - \left(B_0\frac{m^3\pi^3}{L^3}\right)\phi_m\right] - \omega^2\left[\left(\frac{\rho A_0}{E}\frac{m^2\pi^2}{L^2} + \rho h\right)w_m - \frac{\rho B_0}{E}\frac{m\pi}{L}\phi_m\right] = 0, \tag{30}$$

$$\left[-B_0\frac{m^3\pi^3}{L^3}w_m + \left(C_0\frac{m^2\pi^2}{L^2} + D_0\right)\phi_m\right] - \omega^2\left[-\left(\frac{\rho B_0}{E}\frac{m\pi}{L}\right)w_m + \frac{\rho C_0}{E}\phi_m\right] = 0. \tag{31}$$

The Eqns. (30) and (31) can be written in the following matrix form

$$\left(\begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} - \omega^2 \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \right) \begin{Bmatrix} w_m \\ \phi_m \end{Bmatrix} = 0. \tag{32}$$

Above Eqn. (32) can be written in following more compact form

$$([K] - \omega_m^2 [M])\{\Delta\} = 0, \tag{33}$$

where $\{\Delta\}$ denotes the vector, $\{\Delta\}^T = \{W_m, \phi_m\}$. The $[K]$ and $[M]$ are symmetric matrices. The elements of the coefficient matrix $[K]$ are given by

$$K_{11} = \left(A_0 \frac{m^4 \pi^4}{L^4} \right), \quad K_{12} = K_{21} = -\left(B_0 \frac{m^3 \pi^3}{L^3} \right), \quad K_{22} = \left(C_0 \frac{m^2 \pi^2}{L^2} + D_0 \right). \tag{34}$$

The elements of the coefficient matrix $[M]$ are given by

$$M_{11} = \left(\frac{\rho A_0}{E} \frac{m^2 \pi^2}{L^2} + \rho h \right), \qquad M_{12} = M_{21} = -\frac{\rho B_0}{E} \frac{m\pi}{L}, \qquad M_{22} = \frac{\rho C_0}{E}. \tag{35}$$

For nontrivial solution of Eqn. (33), $\{\Delta\} \neq 0$, the condition expressed by

$$([K] - \omega_m^2 [M]) = 0, \tag{36}$$

yields the eigen-frequencies ω_m. From this solution natural frequencies of beam for various modes of vibration can be obtained.

4. Numerical results

The results for transverse displacement (w), axial bending stress (σ_x), transverse shear stress (τ_{zx}) and fundamental frequency ω_m are presented in the following non-dimensional form

$$\bar{w} = \frac{10Ebh^3 w}{q_0 L^4}, \quad \bar{\sigma}_x = \frac{b\sigma_x}{q_0}, \quad \bar{\tau}_{zx} = \frac{b\tau_{zx}}{q_0}, \quad \bar{\omega} = \omega_m \left(\frac{L^2}{h} \right) \sqrt{\frac{\rho}{E}}, \quad S = \frac{L}{h}, \tag{37}$$

where S is the aspect ratio.

The percentage error in results obtained by theories/models of various researchers with respect to the corresponding results obtained by theory of elasticity is calculated as follows

$$\text{error} = \left(\frac{\text{value by a particular model} - \text{value by exact elasticity solution}}{\text{value by exact elasticity solution}} \right) \times 100\,\%. \tag{38}$$

The results obtained for the above examples (static and dynamics) solved in this paper are presented in Tables 2 through 5.

5. Discussion of results

The results obtained from the present theories are compared with the elementary theory of beam (ETB), first order shear deformation theory (FSDT) of Timoshenko [24], higher order shear deformation theories of Heyliger and Reddy [12], Ghugal [10] and exact elasticity solutions given by Timoshenko and Goodier [23] and Cowper [7]. The value of dynamic shear correction factor is compared with its exact value given by Lamb [16]:

a) **Transverse Displacement ($\bar{w}$):** The comparison of maximum transverse displacement for the simply supported thick isotropic beams subjected to uniformly distributed load is presented in Table 2. The maximum transverse displacement predicted by models 5 and 6 is in excellent agreement with the exact solution for all the aspect ratios whereas the error in predicting transverse displacement by other models decreases with increase in aspect ratio. The FSDT overestimates the maximum transverse displacement whereas ETB underestimates the same for all the aspect ratios as compared to that of exact solution.

b) **Axial Bending Stress ($\bar{\sigma}_x$):** Table 2 shows the comparison of axial bending stress for the simply supported thick isotropic beams subjected to uniformly distributed load. Among all the models, model 6 overestimates the value of axial bending stress for all the aspect ratios as compared to that of exact solution whereas axial bending stress predicted by rest of the models is in excellent agreement with that of exact solution. The values of axial

Table 2. Comparison of transverse displacement $\bar{w}$ at ($x = L/2$, $z = 0$), axial bending stress $\bar{\sigma}_x$ at ($x = L/2$, $z = \pm h/2$) and transverse shear stress $\bar{\tau}_{zx}$ at ($x = 0$, $z = 0$) for isotropic beam subjected to uniformly distributed load

S	Theory	$\bar{w}$	% Error	$\bar{\sigma}_x$	% Error	$\bar{\tau}_{zx}$	% Error
2	Model 1 [2]	2.357	−3.913	3.210	0.312	1.156	−22.93
	Model 2 [15]	2.515	2.527	3.261	1.906	1.333	−11.13
	Model 3 [18]	2.532	3.220	3.261	1.906	1.415	−5.667
	Model 4 [25]	2.529	3.098	3.278	2.437	1.451	−3.267
	Model 5 [22]	2.513	2.445	3.206	0.187	1.442	−3.866
	Model 6 [14]	2.510	2.323	3.322	3.812	1.430	−4.667
	Model 7 [1]	2.523	2.853	3.253	1.656	1.397	−6.866
	Timoshenko [FSDT] [24]	2.538	3.465	3.000	−6.250	0.984	−34.40
	Bernoulli-Euler [ETB]	1.563	−3.628	3.000	−6.250	—	—
	Timoshenko and Goodier [Exact] [23]	2.453	0.000	3.200	0.000	1.500	0.000
4	Model 1 [2]	1.762	−1.288	12.212	0.098	2.389	−20.36
	Model 2 [15]	1.805	1.120	12.262	0.508	2.836	−5.466
	Model 3 [18]	1.806	1.176	12.263	0.516	2.908	−3.066
	Model 4 [25]	1.805	1.120	12.280	0.655	2.993	−0.233
	Model 5 [22]	1.802	0.952	12.207	0.057	2.982	−0.600
	Model 6 [14]	1.801	0.896	12.324	1.016	2.957	−1.433
	Model 7 [1]	1.804	1.064	12.254	0.442	2.882	−3.933
	Timoshenko [FSDT] [24]	1.806	1.176	12.000	−1.639	1.969	−34.36
	Bernoulli-Euler [ETB]	1.563	−12.43	12.000	−1.639	—	—
	Timoshenko and Goodier [Exact] [23]	1.785	0.000	12.200	0.000	3.000	0.000
10	Model 1 [2]	1.595	−0.187	75.216	0.021	6.066	−19.12
	Model 2 [15]	1.602	0.250	75.266	0.087	7.328	−2.293
	Model 3 [18]	1.602	0.250	75.268	0.090	7.361	−1.853
	Model 4 [25]	1.601	0.187	75.284	0.111	7.591	1.213
	Model 5 [22]	1.601	0.187	75.211	0.014	7.576	1.013
	Model 6 [14]	1.601	0.187	75.330	0.172	7.513	0.173
	Model 7 [1]	1.601	0.187	75.259	0.078	7.312	−2.506
	Timoshenko [FSDT] [24]	1.602	0.250	75.000	−0.265	4.922	−34.37
	Bernoulli-Euler [ETB]	1.563	−2.190	75.000	−0.265	—	—
	Timoshenko and Goodier [Exact] [23]	1.598	0.000	75.200	0.000	7.500	0.000

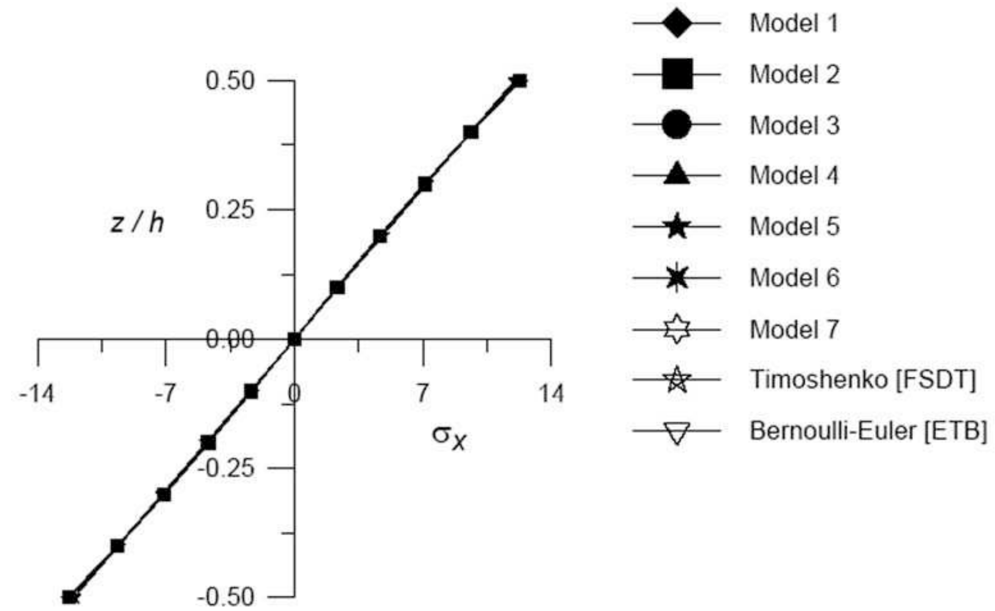

Fig. 3. Variation of axial bending stress ($\bar{\sigma}_x$) through thickness of beam subjected to uniformly distributed load for aspect ratio 4 at ($x = L/2$, z)

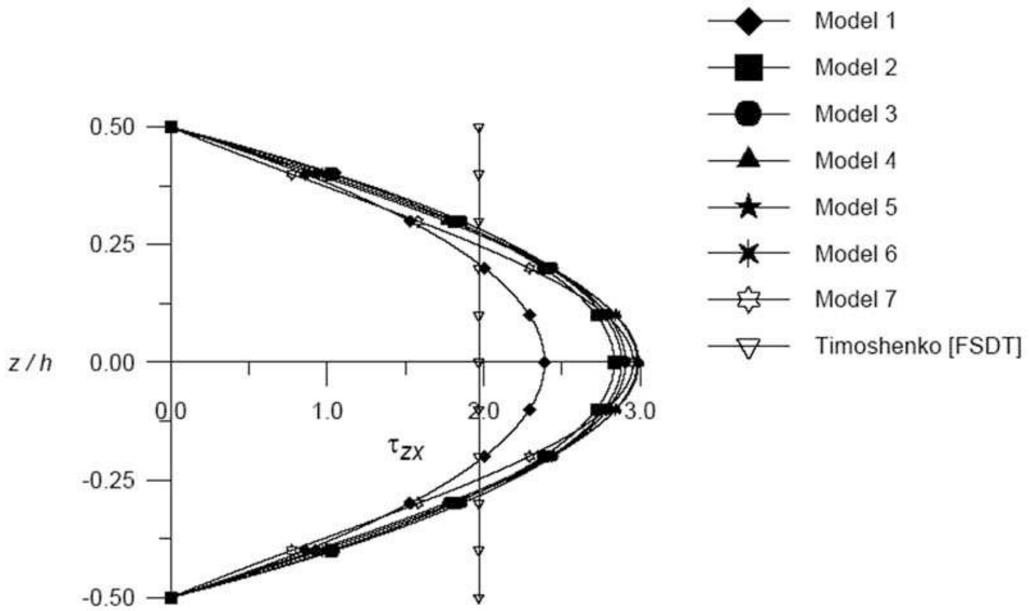

Fig. 4. Variation of transverse stress ($\bar{\tau}_{zx}$) through thickness of beam subjected to uniformly distributed load for aspect ratio 4 at ($x = 0$, z)

bending stress predicted by FSDT and ETB are identical for all the aspect ratios. The through thickness variation of axial bending stress is non-linear in nature as shown in Fig. 3.

c) **Transverse Shear Stress ($\bar{\tau}_{zx}$):** The comparison of maximum transverse shear stress for the simply supported thick isotropic beams subjected to uniformly distributed load is presented in Table 2. The transverse shear stress is obtained using constitutive relation. Examination of Table 2 reveals that model 1 overestimates the value of transverse shear

Table 3. Comparison of non-dimensional fundamental ($m = 1$) flexural and thickness shear mode frequencies of the isotropic beam

Model	$S = 4$			$S = 10$		
	$\bar{\omega}_w$	% Error	$\bar{\omega}_\phi$	$\bar{\omega}_w$	% Error	$\bar{\omega}_\phi$
Model 1 [2]	2.625	0.884	37.237	2.808	0.143	217.439
Model 2 [15]	2.597	−0.192	33.704	2.802	−0.071	194.752
Model 3 [18]	2.596	−0.230	34.259	2.802	−0.071	198.109
Model 4 [25]	2.596	−0.230	34.238	2.802	−0.071	198.109
Model 5 [22]	2.596	−0.230	34.263	2.802	−0.071	198.258
Model 6 [14]	2.608	0.230	34.711	2.805	0.036	201.290
Model 7 [1]	2.598	−0.154	33.748	2.803	−0.036	195.055
Bernoulli-Euler [ETB]	2.779	6.802	—	2.838	1.212	—
Timoshenko [FSDT] [24]	2.624	0.845	34.320	2.808	0.143	198.616
Ghugal [10]	2.602	0.000	34.135	2.804	0.000	198.105
Heyliger and Reddy [12]	2.596	−0.230	34.250	2.802	−0.071	198.235
Cowper [7]	2.602	0.000	—	2.804	0.000	—

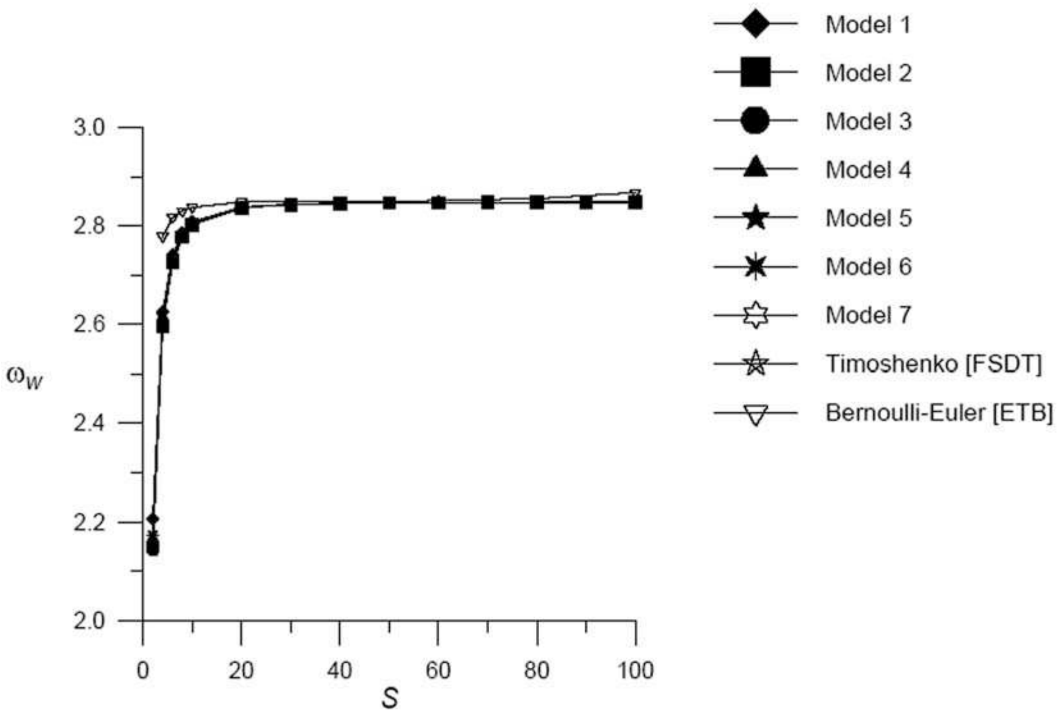

Fig. 5. Variation of fundamental bending frequency ($\bar{\omega}_w$) of beam with aspect ratio

stress whereas it is in excellent agreement when predicted by models 3 through 7 as compared to that of exact solution for all the aspect ratios. The transverse shear stress is overpredicted by models 1 and 2. Fig. 4 shows the through thickness variation of transverse shear stress for the thick isotropic beam subjected to uniformly distributed load for aspect ratio 4.

d) **Fundamental Flexural mode frequency ($\bar{\omega}_w$):** The comparison of lowest natural frequency in flexural mode is shown in Table 3. Observation of Table 2 shows that, Model 1

Table 4. Comparison of non-dimensional flexural frequency $(\bar{\omega}_w)$ of the isotropic beam for various modes of vibration

S	Model	Modes of vibration				
		$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
4	Model 1 [2]	2.625	8.823	16.491	24.713	33.165
	Model 2 [15]	2.597	8.598	15.957	23.923	32.304
	Model 3 [18]	2.596	8.569	15.793	23.435	31.240
	Model 4 [25]	2.596	8.573	15.811	23.483	31.339
	Model 5 [22]	2.596	8.569	15.791	23.429	31.228
	Model 6 [14]	2.608	8.691	16.202	24.357	32.935
	Model 7 [1]	2.598	8.612	16.004	24.027	32.493
	Cowper [7]	2.602	—	—	—	—
10	Model 1 [2]	2.808	10.791	22.903	37.999	55.142
	Model 2 [15]	2.802	10.711	22.582	37.228	53.740
	Model 3 [18]	2.802	10.709	22.566	37.164	53.557
	Model 4 [25]	2.802	10.710	22.570	37.175	53.583
	Model 5 [22]	2.802	10.709	22.566	37.163	53.554
	Model 6 [14]	2.805	10.742	22.708	37.537	54.317
	Model 7 [1]	2.803	10.715	22.598	37.271	53.827
	Cowper [7]	2.804	—	—	—	—

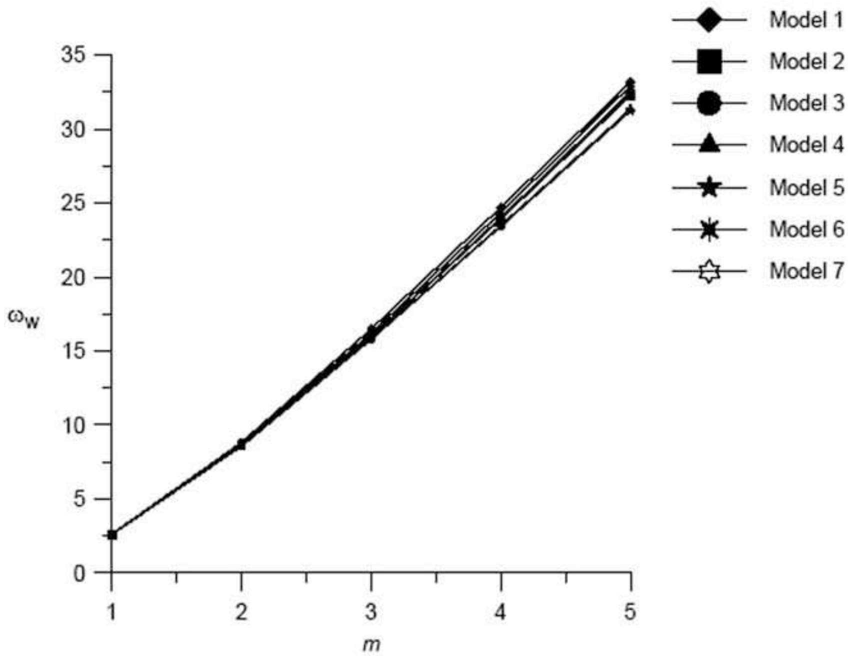

Fig. 6. Variation of fundamental bending frequency $(\bar{\omega}_w)$ of beam with various modes of vibration (m)

overestimates the lowest natural frequencies, in flexural mode by 0.884 % and 0.143 % for aspect ratios 4 and 10 respectively. The fundamental frequencies, in flexural mode predicted by models 2 through 6 is identical and in excellent agreement with that of exact solution Ghugal [10] yields the exact value of lowest natural frequencies, in flexural mode for aspect ratios 4 and 10. FSDT of Timoshenko overestimates the flexural mode

Table 5. Comparison of non-dimensional fundamental frequency of thickness shear mode ($\bar{\omega}_\phi$) of the isotropic beam for various modes of vibrations

S	Model	Modes of vibration				
		$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
4	Model 1 [2]	37.237	44.378	53.547	63.736	74.521
	Model 2 [15]	33.704	41.042	50.402	60.787	71.772
	Model 3 [18]	34.259	41.593	50.941	61.302	72.257
	Model 4 [25]	34.238	41.571	50.917	61.279	72.235
	Model 5 [22]	34.263	41.597	50.945	61.306	72.261
	Model 6 [14]	34.711	41.968	51.251	61.562	72.478
	Model 7 [1]	33.748	41.078	50.431	60.811	71.792
10	Model 1 [2]	217.439	226.391	240.105	257.416	277.363
	Model 2 [15]	194.752	204.080	218.272	236.080	256.514
	Model 3 [18]	198.235	207.555	221.739	239.539	259.959
	Model 4 [25]	198.109	207.425	221.606	239.401	259.819
	Model 5 [22]	198.258	207.578	221.763	239.563	259.984
	Model 6 [14]	201.290	210.468	224.467	242.071	262.302
	Model 7 [1]	195.055	204.368	218.539	236.327	256.740

frequency by 0.845 % and 0.143 % for aspect ratios 4 and 10 respectively whereas ETB overestimates the same by 6.802 % and 1.212 % due to neglect of shear deformation in the theory. The variation of lowest natural frequency in flexural mode with the aspect ratios is shown in Fig. 5. The comparison of flexural frequency for various modes of vibration (m) is shown in Table 4. The examination of Table 4 reveals that, the flexural frequencies obtained by various models are in excellent agreement with each other. The variation of flexural frequencies with various modes of vibration (m) is shown in Fig. 6.

e) **Fundamental frequency ($\bar{\omega}_\phi$):** Table 3 shows comparison of lowest natural frequency in thickness shear mode. Exact solution for the lowest natural frequency in thickness shear mode is not available in the literature. From the Table 3 it is observed that, thickness shear mode frequencies predicted by models 2 through 6 are in excellent agreement with each other whereas model 1 overestimates the same. Table 5 shows comparison of thickness shear mode frequencies for various modes of vibration and found in good agreement with each other. The solution for the circular frequency of thickness shear mode ($m = 0$) for thin rectangular beam is given by

$$\omega_\phi = \sqrt{\frac{K_{22}}{M_{22}}} = \sqrt{K_d \frac{GA}{\rho I}}, \tag{39}$$

where K_d is dynamic shear correction factor.

Table 6. Dynamic shear correction factors

Model	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Exact
K_d	0.995	0.794	0.824	0.822	0.824	0.850	0.797	0.822
% Error	21.046	−3.406	0.243	0.000	0.243	3.406	−3.041	0.000

Dynamic shear correction predicted by model 4 is same as the exact solution given by Lamb [16]. The corresponding values of shear factor for $m = 0$ according to models 3 and 5 is identical. The model 1 yields the higher value of dynamic shear correction factor whereas model 7 shows lower value for the same, Table 6.

6. Conclusions

From the study of comparison of various shear deformation theories for the bending and free vibration analysis of thick isotropic beams following conclusions are drawn.

1. The maximum transverse displacement predicted by all the models is in excellent agreement as compared to that of exact solution.
2. The axial bending stress predicted by the models 1 through 5 and 7 is in tune with exact solution whereas model 6 overestimates it for all the aspect ratios.
3. Through thickness variation of axial bending stress is non-linear in nature.
4. The maximum transverse shear stress predicted by models 3 through 7 is in excellent agreement as compared to that of exact solution whereas model 1 and 2 overestimates the value of transverse shear stress for all the aspect ratios.
5. Results of lowest natural frequencies for flexural mode predicted by models 3 through 5 are identical and in excellent agreement with that of exact solution. Model 1 overestimates the flexural mode frequency as compared to that of exact solution. Flexural mode frequencies predicted by models 2 and 7 are in tune with the exact solution.
6. The results of thickness shear mode frequencies are in excellent agreement with each other for all modes of vibration.
7. Model 4 yields the exact value of dynamic shear correction factor and it is in excellent agreement when predicted by models 3 and 5.

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