University of West Bohemia

Faculty of Applied Sciences

Department of Mathematics

Diploma Thesis

BVPs for singular/degenerated differential equations - spectral properties, solvability, bifurcation, approximation

Pilsen 2014

Lukáš Kotrla

Západočeská univerzita

Fakulta aplikovaných věd

Katedra matematiky

Diplomová práce

Okrajové úlohy pro singulární a degenerované diferenciální rovnice - spektrální vlastnosti, řešitelnost, bifurkace, aproximace

Plzeň 2014

Lukáš Kotrla

Declaration

I hereby declare that the Diploma Thesis includes either my original work or a proper citation is given.

Pilsen ....

Acknowledgement

I wish to thank my mentor Doc. Ing. Petr Girg, Ph.D. for many useful advices, careful readings of the Thesis and a lot of time, which he devoted to our discussions.

Abstract

Let $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{2,\alpha}$ boundary $\partial\Omega$ for $\alpha \in (0,1)$. In the Thesis we consider the problem

$$\left\{ \begin{array}{rcll} -\Delta_p u & = & h\,(\mu; \mathbf{x}, u, \nabla u) & \text{in } \Omega\,, \\ u & = & 0 & \text{on } \partial\Omega \end{array} \right. \tag{1}$$

for an unknown function $u \in W_0^{1,p}(\Omega)$ and $p > 1$. Here the p-Laplace operator Δ_p is defined by $\Delta_p\, u \stackrel{\text{def}}{=} \operatorname{div}\,\left(|\nabla u|^{p-2}\nabla u\right)$, parameter $\mu \in \mathbb{R}$ and $h : \mathbb{R}\times\Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is given Carathéodory function. We suppose that the reaction term h can be decompose into the $(p-1)$-homogeneous part $\mu|u|^{p-2}u$ and a bounded perturbation $g(\mu; \mathbf{x}, u, \nabla u)$, where $g : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$.

At first we prove Krasnoselskii type necessary condition for (1) under the assumptions that μ is in the neighborhood of the first eigenvalue λ_1 and $g(\mu; \mathbf{x}, s_1, s_2) \in L^{r'}(\Omega)$, where $\frac{1}{r} + \frac{1}{r'} = 1$, $r \in (p, p^*)$ and

$$p^* \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} \frac{Np}{N-p} & \text{if } p < N\,, \\ +\infty & \text{if } p \geq N\,. \end{array} \right.$$

Then we assume one-dimensional case of (1) and $g(\mu; \mathbf{x}, s_1, s_2) \in L^\infty(\Omega)$ and we prove the key estimate for the proof of the analogy of Dancer's Theorem. Let us note that the originality of the work consists in including the gradient (the first derivative) of an unknown function to the source term h. The rest of the Thesis is devoted to briefs comments of my papers written in cooperation with my mentor P. Girg. The first paper is focused on the continuity of $\sin_p(x)$, which is the first eigenfunction of $-\Delta_p$. Moreover we discuss the possibility of the expression of $\sin_p(x)$ as the convergent Maclaurin series on some neighborhood of the origin. In the second paper we generalize $\sin_p(x)$ to complex domain for p be an even integer. Please find these papers included in Appendix A1 and Appendix A2 for more details.

Keywords

p-Laplacian, bifurcations, Krasnoselskii type necessary condition, p-trigonometric functions, differentiability, continuity, complex domain

Contents

List of notation

u	an unknown function in an equation, usually $u \in W_0^{1,p}(\Omega)$
(a, b)	open interval
$[a, b]$	closed interval
Ω	bounded domain in $\mathbb{R}^N$
$\partial\Omega$	boundary of domain Ω
$\bar{\Omega}$	closure of the domain Ω
$\lambda_1, \lambda_2, \lambda_3, \ldots$	eigenvalues of $-\Delta_p$
∇u	gradient of u
$\operatorname{div} \mathbf{s}$	divergence of vector function $\mathbf{s}$
$\Delta_p\, u$	p-Laplace operator $\Delta_p\, u = \operatorname{div} \left(\lvert\nabla u\rvert^{p-2} \nabla u\right)$
$L^p(\Omega)$	Lebesgue space with $1 \le p \le +\infty$
$\lVert u \rVert_{L^p(\Omega)}$	norm of Lebesgue space
$W_0^{1,p}(\Omega)$	Sobolev space with $1 \le p \le +\infty$
$\lVert u \rVert_{W_0^{1,p}(\Omega)}$	norm of Sobolev space
$C(\Omega)$	space of continuous functions
$C^{k,\alpha}(\bar{\Omega})$	Hölder space for $k \in \mathbb{N} \setminus \{0\}$ and $\alpha \in (0, 1)$
$C_0^1[a, b]$	space of continuously differentiable functions u with $u(a) = 0 = u(b)$
$\to$	strong convergence
$\rightharpoonup$	weak convergence
X	abstract space
$X \hookrightarrow Y$	continuous embedding of X into Y
$X \overset{c}{\hookrightarrow} Y$	compact embedding of X into Y
ν	measure
$\mathfrak{S}$	σ-algebra

Chapter 1

Introduction

For $p > 1$, the following problem

$$\begin{cases} -\Delta_p u &= h(\mu; \mathbf{x}, u, \nabla u) \qquad \text{in } \Omega\,, \\ \quad u &= 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega \end{cases} \tag{1.1}$$

is considered for an unknown function $u = u(\mathbf{x})$ in a bounded domain $\Omega \subset \mathbb{R}^N$ with $C^{2,\alpha}$ [1] boundary $\partial\Omega$, where $\alpha \in (0,1)$. Operator Δ_p stands for the p-Laplace operator defined by $\Delta_p u \stackrel{\text{def}}{=} \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right)$ and $h : \mathbb{R}\times\Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a given Carathéodory function (for exact definition for vector function u see Definition 2.1.9). In one dimension, the p-Laplace operator is reduced to $\varphi_p(u')$, where $\varphi_p(s) \stackrel{\text{def}}{=} |s|^{p-2}s$. The dependence of φ_p on u' is shown on the Figure 1.1 for $p = 30$ and $p = \frac{30}{29}$.

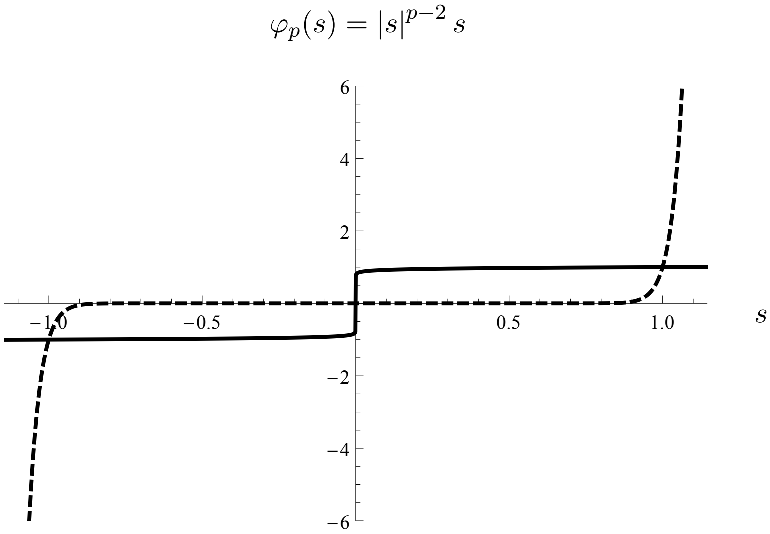

Figure 1.1: Function $\varphi_p(s) = |s|^{p-2}s$ for $p = 30$ (dashed line) and $p = \frac{30}{29}$ (continuous line).

Problem (1.1) can be interpreted as the equation of the steady state of a diffusion equation. Indeed, let u stand for a state variable (e.g. density, concentration, temperature), $\mathbf{j} = \mathbf{j}(\mathbf{x})$ for the diffusion flux, and h for a source term. Then the steady state conservation law has the divergence form

$$\operatorname{div}\,\mathbf{j} = h(\mu; \mathbf{x}, u, \nabla u)\ . \tag{1.2}$$

[1]We say that the boundary is $C^{2,\alpha}$ if it can be decomposed in finite number of parts, such that each can be expressed as a $C^{2,\alpha}$ function in suitable rotated local coordinates. By $C^{2,\alpha}$ function we mean the following. Let $M \subset \mathbb{R}^{N-1}$ and $f \in C(\bar{M})$. Assume $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_{N-1}) \in \bar{M}$ and denote $f_d(\mathbf{x}) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x})$ for $i, j = 1, 2, 3, \ldots, N-1$. We say that $f \in C^{2,\alpha}(\bar{M})$ in the case that $\sup\limits_{\substack{\mathbf{x},\mathbf{y}\in\bar{M} \\ \mathbf{x}\neq\mathbf{y}}} \dfrac{|f_d(\mathbf{x}) - f_d(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha} < +\infty.$

The constitutive relation for diffusion processes (Fick's law) states

$$\mathbf{j} = -D\nabla u\,, \tag{1.3}$$

where $D = D(\mathbf{x})$ is the diffusion coefficient, which depends on the diffusing material (see DRÁBEK-HOLUBOVÁ [20] for more details). In some circumstances the diffusion coefficient depends also on u and/or ∇u. In this thesis we suppose that $D(\mathbf{x}, u, \nabla u) \stackrel{\text{def}}{=} |\nabla u|^{p-2}$ (see Figure 1.2 and compare with Figure 1.1), which appears in many practical situations (see e.g. ARONSSON-EVANS-WU [4] or WU-ZHAO-YIN-LI [55]).

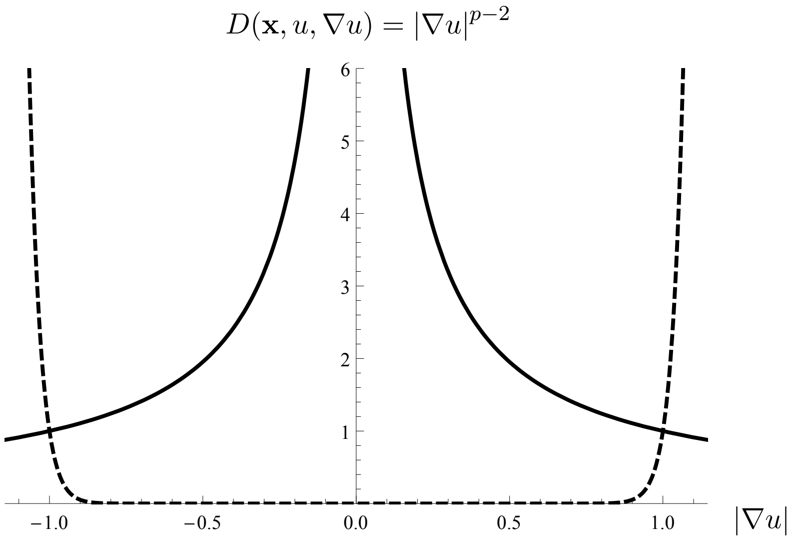

Figure 1.2: Diffusion coefficient $D(\mathbf{x}, u, \nabla u) = |\nabla u|^{p-2}$ restricted to the plain $|\nabla u| \times D$ for $p = 30$ (dashed line) and $p = \frac{30}{29}$ (continuous line).

Combining (1.2) and (1.3) and considering Dirichlet boundary condition, we get the problem (1.1). Let us note that (1.2) is stated in a divergence form as it would be for classical solution $u \in C^2(\Omega)$ under the assumption that u has continuous partial derivatives. However, the existence of classical solution is extremely difficult to establish. Thus (1.1) is understood in the weak sense. Hence by a solution to (1.1) we mean a function $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}\,\nabla u \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega h(\mu; \mathbf{x}, u, \nabla u)\varphi \,\mathrm{d}x \quad \forall \varphi \in W_0^{1,p}(\Omega)\,.$$

Papers [4] and/or EVANS-FELDMAN-GARIEPY [25] are concerned with growing of sandpiles. In a non-stationary case of (1.1), i.e.

$$\left\{\begin{array}{rcll} \frac{\partial u}{\partial t} - \Delta_p u &=& f(x,t) & \text{in } \mathbb{R}^N \times (0, +\infty)\,, \\ u &=& \bar{u} & \text{on } \mathbb{R}^N \times \{t = 0\}\,, \end{array}\right. \tag{1.4}$$

they interpret $u(\mathbf{x}, t)$ as a height of sandpile. Hence ∇u corresponds to the slope of the sandpile. Assume $p \to +\infty$. Then $D \to 0$ within the region $|\nabla u| < 1 - \delta$ and $D \to +\infty$ for $|\nabla u| > 1 + \delta$ for any small $\delta > 0$ (see Figure 1.2). It follows that there is no diffusion until the time, when the slope reaches critical value 1. Then the pile collapses and the slope decreases.

On the other hand, the article by KUIJPER [35] is concerned with image processing using nonlinear diffusion. Here the domain $\Omega \subset \mathbb{R}^2$ (two-dimensional image - e.g. photograph) and $u(\mathbf{x}, t)$ is interpreted as a value on a gray scale at a point $\mathbf{x} \in \Omega$. It turns out that the equation (1.4) for $p \to 1^+$ preserves edges of the image. This is due to the fact that $D \to 0$ within the region $|\nabla u| > 1 - \delta$ and $D \to +\infty$ for $|\nabla u| < 1 + \delta$ for any small $\delta > 0$ (see Figure 1.2). Note that $|\nabla u|$ is large on the edges.

It is natural to ask about the number of solutions of (1.1) for the given source function h. The answer depends on the asymptotic properties of the function h and on properties of the eigenvalue problem (see e.g. ANANE [1], ANANE-TSOULI [2], ČEPIČKA-DRÁBEK-GIRG [13], ELBERT [23] and references therein)

$$\begin{cases} -\Delta_p u = \lambda\, |u|^{p-2}\, u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.5}$$

where $\lambda \in \mathbb{R}$ is an eigenvalue of (1.5) if there is a nonzero function u which satisfies (1.5). Such function is called eigenfunction. In one-dimensional case there is $\lambda_k = k^p(p-1)$ for any $k \in \mathbb{N}$ and corresponding normalized eigenfunction is $\sin_p(kx)$. ELBERT showed in [23] that $\sin_p(x)$ is the unique solution of the problem

$$\begin{cases} -\left(|u'|^{p-2}\, u'\right)' = \lambda\, |u|^{p-2}\, u & \text{in } (0, \pi_p)\,, \\ u(0) = 0\,, \\ u'(0) = 1 \end{cases}$$

and

$$\pi_p = 2\int_0^1 \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s = \frac{2\pi}{p\sin(\pi/p)}\,.$$

In higher dimension, the structure of the spectrum of (1.5) is not fully understood yet, but as ANANE [1] prove there is the first eigenvalue $\lambda_1 > 0$, which is isolated and the corresponding normalized eigenfunction is positive in Ω. Hence the question about the number of solution of (1.1) is a difficult problem in general and for simplicity, we add some assumptions. Firstly, we consider the one-dimensional case (except Section 3.1). Secondly, suppose that the reaction term h can be decomposed into the $(p-1)$-homogeneous part $\mu|u|^{p-2}u$ and a bounded perturbation $g(\mu; \mathbf{x}, u, \nabla u)$. It means that

$$h\left(\mu; \mathbf{x}, u, \nabla u\right) = \mu |u|^{p-2} u + g\left(\mu; \mathbf{x}, u, \nabla u\right)\,, \tag{1.6}$$

where $g : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ satisfies Carathéodory condition and there is $a(\mathbf{x}) \in L^{r'}(\Omega)$ such that

$$|g\left(\mu; \mathbf{x}, s_1, s_2\right)| \leq a\,. \tag{1.7}$$

The allowable values of parameter r' will be specified below. Thirdly, we focus only on such values of the parameters μ which are in the small neighborhood of the first eigenvalue λ_1.

Under this assumptions, we search for the number of solutions using bifurcation theory. There are some papers devoted to this problem where the source term does not depend on ∇u. In DEL PINO-MANÁSEVICH [21] authors deal with the bifurcations from zero. They consider the source term $h = h(\mu; \mathbf{x}, u)$ satisfies (1.6) with $g = g(\mu; \mathbf{x}, u)$. Function g fulfills Carathéodory condition in $\mathbf{x}$ and u,

$$g(\mu; \mathbf{x}, s) = o\left(|s|^{p-1}\right)$$

near $s = 0$, uniformly a.e. in Ω and uniformly with respect to μ on bounded sets. Furthermore, g satisfies the growth condition

$$\lim_{|s|\to+\infty} \frac{|g(\mu;\mathbf{x},s)|}{|s|^{q-1}} = 0$$

uniformly with respect to μ on bounded sets for any $1 < q < p^*$, where p^* depends on p and dimension N as follows:

$$p^* \stackrel{\text{def}}{=} \begin{cases} \frac{Np}{N-p} & \text{if } p < N\,, \\ +\infty & \text{if } p \geq N\,. \end{cases}$$

Also in Girg-Takáč [31], the bifurcations from zero and/or from infinity are studied. In this case, the source term $h = h(\mu;\mathbf{x},u)$ satisfies (1.6) again. In [31], the function $g : \mathbb{R}\times\Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function moreover there is a constant $C \in (0,+\infty)$ such as in the case of the bifurcation from zero

$$|g(\mu;\mathbf{x},s)| \leq C|s|^{p-1}$$

and in the case of the bifurcation from infinity

$$|g(\mu;\mathbf{x},s)| \leq C\left(1 + |s|^{p-1}\right)$$

in both cases for a.e. $\mathbf{x} \in \Omega$ and for all pairs $(\mu, s) \in \mathbb{R}\times\mathbb{R}$, and

$$\frac{g(\mu;\mathbf{x},s)}{|s|^{p-1}} \to 0 \quad \text{as} \quad |s| \to 0 \quad \text{and} \quad \frac{g(\mu;\mathbf{x},s)}{|s|^{p-1}} \to 0 \quad \text{as} \quad |s| \to +\infty$$

uniformly for a.e. $\mathbf{x} \in \Omega$ and in μ from bounded intervals in $\mathbb{R}$, respectively. Note that our assumptions for the function g are far more strict, but we work with the function g depending also on ∇u. Let us mention that [31] continues the paper by Drábek-Girg-Takáč-Ulm [19], where $g = g(\mathbf{x})$ is considered.

This thesis is organized as follows. In Chapter 2 we introduce some useful definitions and we define Carathéodory function. In Chapter 3 we show that the bifurcation point from infinity of the problem (1.1) is also the eigenvalue of (1.5). Here we consider the assumptions (1.6) and (1.7), where $a \in L^{r'}(\Omega)$. In this case, parameter r' satisfies $\frac{1}{r} + \frac{1}{r'} = 1$ and $r \in (p, p^*)$. Then we suppose one-dimensional case of (1.1) and we prove the key estimate for the proof of theorem analogous to Dancer's Theorem under the assumptions (1.6) and (1.7), where $a \in L^\infty(\Omega)$. Chapters 4 and 5 are devoted to brief commentary of my papers "Differentiability properties of p-trigonometric functions" and "Generalized trigonometric functions in complex domain". Both papers were written in collaboration with my mentor P. Girg. In these chapters you can find sections where my contribution is specified. The articles were attached in Appendix A1 and Appendix A2, respectively.

Chapter 2

Preliminaries

The results of this chapter are well known. We include them here for the sake of completeness of the presentation since results are scattered in the literature.

2.1 Measure theoretic preliminaries

Here we introduce some useful well (and/or less) known facts from the measure theory, see e.g. Ambrosetti-Prodi [3], Folland [26], and/or Malý [41]. We assume that the reader is familiar with definitions of *sigma algebra*, *measure space*, *measure*, *complete measure*, *Borel set*, etc. Through out the section we suppose the measure space $(X, \mathfrak{S}, \nu)$ with sigma algebra $\mathfrak{S} \subset 2^X$ and a complete measure ν. We also define $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ such that for any $b \in \mathbb{R}$ holds $-\infty < b < +\infty$.

2.1.1 Basic definitions

Definition 2.1.2 (see, [41], 3.3 Definition, p. 5). *Let $D \in \mathfrak{S}$. We say that function $f : D \to \overline{\mathbb{R}}$ is $\mathfrak{S}$-measurable if for any Borel set $\mathfrak{B} \subset 2^{\overline{\mathbb{R}}}$ is $f^{-1}(\mathfrak{B}) \in \mathfrak{S}$.*

In case no confusion may arise, we use the term *measurable function* instead of the term $\mathfrak{S}$*-measurable function.*

2.1.3 Convergence theorems

Let us introduce the space L^+. It contains all measurable functions that maps from X to $[0, +\infty]$. In other words it is the space of the non-negative measurable functions.

Proposition 2.1.4 (see [26], 2.14 The Levi Monotone Convergence Theorem, p. 50). *If $\{f_n\}_{n=1}^{+\infty}$ is a sequence in L^+ such that $f_j \leq f_{j+1}$ for all $j \in \mathbb{N}$, and*

$$f = \lim_{n\to+\infty} f_n (= \sup_{n\in\mathbb{N}} f_n)\,,$$

then

$$\int_X f = \lim_{n\to+\infty} \int_X f_n\,.$$

Proposition 2.1.5 (see [26], Fatou's Lemma 2.18, p. 52). *If $\{f_n\}_{n=1}^{+\infty}$ is any sequence in L^+, then*

$$\int_X \left(\liminf_{n\to+\infty} f_n\right) \leq \liminf_{n\to+\infty} \int_X f_n\,.$$

Corollary 2.1.6. *Let $\{f_n\}_{n=1}^{+\infty}$ be a sequence of measurable functions. Assume that there exists $h \in L^1(X)$ such that $f_n \leq h$. Then*

$$\limsup_{n\to+\infty} \int_X f_n \leq \int_X \limsup_{n\to+\infty} f_n$$

Proof. Choose any $h \in L^+$ such that $h - f_n \geq 0$ for all $n \in \mathbb{N}$. Hence the sequence $\{h - f_n\}_{n=1}^{+\infty}$ belongs to L^+ and by Proposition 2.1.5

$$\liminf_{n\to+\infty} \int_X (h - f_n) \geq \int_X \liminf_{n\to+\infty} (h - f_n).$$

Since Lebesgue's integral is additive and h does not depend on n we obtain

$$\int_X h - \limsup_{n\to+\infty} \int f_n \geq \int_X h - \int \limsup_{n\to+\infty} f_n \,,$$

which follows

$$\limsup_{n\to+\infty} \int_X f_n \leq \int_X \limsup_{n\to+\infty} f_n \,.$$

∎

Let us recall classical Lebesgue Dominated Convergence Theorem. For more detail see e.g. [41] and/or [26].

Proposition 2.1.7 (see [41], 6.2 Lebesgue Dominated Convergence Theorem, p. 15)**.** *Let $D \in \mathfrak{S}$ and f, f_j, $j \in \mathbb{N}$, are measurable functions on D. Let $\{f_j\}_{j=1}^{+\infty}$ converge to f a.e. in D. Let there exist integrable function such that for every $j \in \mathbb{N}$,*

$$|f_j(x)| \leq g(x), \qquad \text{for } x \in D.$$

Then

$$\int_D f = \lim_{j\to+\infty} \int_D f_j \,.$$

2.1.8 Carathéodory condition

Definition 2.1.9. *Let $g : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$, satisfies*

(i) $g(\mu; \cdot\,, u, \mathbf{v}) : \Omega \to \mathbb{R}$ is measurable for all $\mu, u \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^N$.

(ii) $g(\cdot\,; x, \cdot\,, \cdot\,) : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous for x a.e. in Ω

Then we say that h satisfies Carathéodory condition.

Our aim is to show that Carathéodory condition is sufficient for measurability of function g. It is well known fact, but its proof is difficult to find in the literature. For that reason, we present a proof in one dimension, which can be easily generalized to higher dimension. Before we introduce the proof, we state two useful propositions from [26] and prove a lemma from [41]. Since [41] refers to lecture notes in Czech, the proof is presented below.

Proposition 2.1.10 (see [26], 2.7 Proposition, p. 45)**.** *If $\{f_j\}_{j=1}^{+\infty}$ is a sequence of $\overline{\mathbb{R}}$-valued measurable functions on $(X, \mathfrak{S})$, then the functions*

$$g_1(x) = \sup_{j\in\mathbb{N}} f_j(x), \quad g_3(x) = \limsup_{j\to+\infty} f_j(x),$$
$$g_2(x) = \inf_{j\in\mathbb{N}} f_j(x), \quad g_4(x) = \liminf_{j\to+\infty} f_j(x)$$

are all measurable. If $f(x) = \lim_{j\to+\infty} f_j(x)$ exists for every $x \in X$, then f is measurable.

Proposition 2.1.11 (see [26], 2.10 Theorem, p. 47). *Let $(X, \mathfrak{S})$ be a measure space. If $f : X \to \mathbb{C}$ is measurable, there is a sequence $\{\phi_n\}_{n=1}^{+\infty}$ of simple functions such that $0 \leq |\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$, $\phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.*

Lemma 2.1.12 (see [41] (in Czech), 3.10 Theorem (d), p. 6). *Let functions $f_j(x)$ be measurable on $D \in \mathfrak{S}$ for all $j \in \mathbb{N}$. Then the set D' of all points, where the limit $\lim_{j\to+\infty} f_j(x)$ exists, is measurable and $\lim_{j\to+\infty} f_j(x)$ is measurable on D'.*

Proof. Denote D' set of all points where $\lim_{j\to+\infty} f_j(x)$ exists, i.e., $\limsup_{j\to+\infty} f_j(x) = \liminf_{j\to+\infty} f_j(x)$. It is easily seen that

$$D' = D \setminus \left(\bigcup_{r\in\mathbb{Q}} \left\{ x \in D : \liminf_{j\to+\infty} f_j(x) < r < \limsup_{j\to+\infty} f_j(x) \right\} \right).$$

Since D is measurable, the difference of two measurable sets is a measurable set and countable unification of measurable sets is a measurable set, we get that D' is measurable if we show that the set

$$\left\{ x \in D : \liminf_{j\to+\infty} f_j(x) < r < \limsup_{j\to+\infty} f_j(x) \right\} \tag{2.1}$$

is measurable for all $r \in \mathbb{Q}$. At first we justify that $\{x \in D : \liminf_{j\to+\infty} f_j(x) < r\}$ and $\{x \in D : r < \limsup_{j\to+\infty} f_j(x)\}$ are measurable. Indeed, the functions $\liminf_{j\to+\infty} f_j(x)$ and $\limsup_{j\to+\infty} f_j(x)$ are measurable functions by Proposition 2.1.10 and hence from Definition 2.1.2 of measurable function also the sets must be measurable. Since for given $r \in \mathbb{Q}$ the set (2.1) is the intersection of sets $\{x \in D : \liminf_{j\to+\infty} f_j(x) < r\}$ and $\{x \in D : r < \limsup_{j\to+\infty} f_j(x)\}$, we get desired measurability of the set (2.1). ■

Idea of the following proof is taken from Ambrosetti-Prodi [3].

Theorem 2.1.13. *Let $g : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ fulfill Carathéodory condition. Then $g(\mu; x, u(x), \mathbf{v}(x))$ is measurable function for any measurable functions $u : \Omega \to \mathbb{R}$ and $\mathbf{v} : \Omega \to \mathbb{R}^N$.*

Proof. For simplicity $N = 1$. The idea of proof in higher dimension ($N \geq 2$) is analogous, but it is more technical and it produces lengthly formulas.

Since u and v are measurable there exist nondecreasing sequences $u_k^+ \nearrow u^+$, $u_k^- \nearrow u^-$, $v_k^+ \nearrow v^+$ and $v_k^- \nearrow v^-$ of simple functions such that

$$u_k^{\pm} = \sum_{j=1}^{k^{\pm}} \alpha_j^{\pm} \chi_{E_j^{\pm}} \qquad \text{and} \qquad v_k^{\pm} = \sum_{j=1}^{\bar{k}^{\pm}} \beta_j^{\pm} \chi_{\bar{E}_j^{\pm}}. \tag{2.2}$$

for almost all $x \in \Omega$. It follows from Proposition 2.1.11. Hence sequences $u_k = u_k^+ - u_k^-$ and $v_{\bar{k}} = v_{\bar{k}}^+ - v_{\bar{k}}^-$ converge to u and v, respectively. For any $k, \bar{k} \in \mathbb{N}$ the sums (2.2) are finite and hence

$$u_k = \sum_{j=1}^{l(k)} \alpha_j \chi_{E_j} \qquad \text{and} \qquad v_k = \sum_{j=1}^{\bar{l}(\bar{k})} \beta_j \chi_{\bar{E}_j}.$$

We claim that $g(\mu; x, u_k, v_{\bar{k}})$ is measurable function. To prove this we show that set

$$\{x \in \Omega \,:\, g(\mu; x, u_k, v_{\bar{k}}) > t\} \tag{2.3}$$

is measurable for all $t \in \mathbb{Q}$. We can divide the set (2.3) as follows:

$$\begin{aligned}\{x \in \Omega : g(\mu; x, u_k, v_{\bar{k}}) > t\} \quad &= \quad \Big(\bigcup_{j=1}^{l} \bigcup_{m=1}^{\bar{l}} \{x \in \Omega : g(\mu; x, \alpha_j, \beta_m) > t\ \} \cap E_j \cap \bar{E}_m\Big) \cup \dots \\ &\dots \cup \Big(\{x \in \Omega : g(\mu; x, t, t) > t\} \cap \Big(\Omega \setminus \Big(\bigcup_{j=1}^{l} E_j \cup \bigcup_{m=1}^{\bar{l}} \bar{E}_m\Big)\Big)\Big)\end{aligned}$$

and the problem of measurability of g is falling to the problem of measurability of E_j, $\bar{E}_m$, $\{x \in \Omega : g(\mu; x, \alpha_j, \beta_m) > t\ \}$, and $\{x \in \Omega : g(\mu; x, t, t) > t\}$.

Since $u_k = \sum_{j=1}^{l} \alpha_j \chi_{E_j}$ and $v_{\bar{k}} = \sum_{j=1}^{\bar{l}} \beta_j \chi_{\bar{E}_j}$ are measurable function, the sets E_j and $\bar{E}_m$ are measurable as well.

From the Carathéodory condition (see Definition 2.1.9, *(i)*) the function $g(\mu; \cdot\,, u, v)$ is measurable and using Definition 2.1.2 we get measurability of the sets $\{x \in \Omega : g(\mu; x, \alpha_j, \beta_m) > t\}$ and $\{x \in \Omega : g(\mu; x, t, t) > t\}$.

Due to the continuity of $g(\cdot\,; x, \cdot\,, \cdot)$ (see Definition 2.1.9, *(ii)*), $u_k(x) \to u(x)$, and $v_{\bar{k}}(x) \to v(x)$ for x a.e. in Ω we have $g(\mu; x, u_k, v_{\bar{k}}) \to g(\mu; x, u, v)$ for any $\mu \in \mathbb{R}$. Let us denote $E = \{x \in \Omega : \lim_{k\to+\infty} u_k(x) = u(x) \wedge \lim_{k\to+\infty} v_{\bar{k}}(x) = v(x)\}$. The measure $\mu(E) = \mu(\Omega)$ since u_k and $v_{\bar{k}}$ converge for x a.e. in Ω and so $\mu(\Omega \setminus E) = 0$. Limit function g of the sequence of measurable functions which converge on measurable set D' is also measurable on D' in the sense of Lemma 2.1.12. ■

2.2 Abstract preliminaries

In this section we introduce some properties of the inverse operator to the p-Laplacian. We assume that the reader has basic knowledge of functional analysis. More precisely he/she is familiar with definitions of *compact operator*, *compact set*, *continuous operator*, *functional*, *strong* and *weak convergence* etc. At first let us define some function spaces and the norms on these spaces. More details can be found in Benedikt-Girg [5] or [41].

Definition 2.2.1. *Let $\Omega \subset \mathbb{R}^N$ is domain. The symbol $C^k(\Omega)$ denotes the space of continuously differentiable functions on Ω up to the order $k \in \mathbb{N} \cup \{0\}$. Moreover $C^\infty(\Omega)$ denotes the space of infinitely continuously differentiable functions on Ω.*

Let us note for $k = 0$, we write $C(\Omega)$ instead of $C^0(\Omega)$. Let us also define the support of function $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ as the set

$$\operatorname{supp} f \stackrel{\text{def}}{=} \overline{\{x \in \Omega : f(x) = 0\}}\,,$$

where the closure is considered in euclidean metric.

Definition 2.2.2. *Let $\Omega \subset \mathbb{R}^N$ is domain. By $C_0^\infty(\Omega)$ we denote the space of all functions $f \in C^\infty(\Omega)$ for which*

$$\operatorname{supp} f \subset \Omega$$

and $\operatorname{supp} f$ *is compact set.*

Definition 2.2.3. *Let $[a, b] \subset \mathbb{R}$ is closed interval. The symbol $C_0[a, b]$ (resp. $C_0^1[a, b]$) denotes the space of functions $f \in C(a, b)$ (resp. $f \in C^1(a, b)$) such that $f(a) = 0 = f(b)$.*

Definition 2.2.4 (see [41] 17.2 Definition (L^p-norms), p. 37)**.** *Let $(X, \mathfrak{S}, \nu)$ is a space with a measure. If f is a measurable function on X and $1 \le p < +\infty$ is real parameter, then we define*

$$\|f\|_p \stackrel{\text{def}}{=} \left(\int_X |f|^p \, \mathrm{d}\nu \right)^{\frac{1}{p}} .$$

Moreover we define

$$\|f\|_\infty \stackrel{\text{def}}{=} \inf \{ C \ge 0 : |f| \le C \text{ almost everywhere} \} .$$

Definition 2.2.5 (see [41], Lebesgue space, p. 37)**.** *Let $(X, \mathfrak{S}, \nu)$ is space with measure. Define $L^p(X)$ as the space of ν-measurable functions f on X such that $\|f\|_p$ is finite. Moreover we define*

$$\|f\|_{L^p(X)} \stackrel{\text{def}}{=} \|f\|_p .$$

Let us mention that mapping $\| \cdot \|_p$ is seminorm on the space $L^p(\Omega)$. For the correct definition of norm $\| \cdot \|_{L^p(X)}$ we have to assume that if $f_1 = f_2$ almost everywhere in X, then f_1 and f_2 are the same element of $L^p(X)$. It is realized by the concept of equivalence classes of Lebesgue measurable functions. See [5] and/or [41] for more details.

Definition 2.2.6 (see [5] Definition 3.1, p. 94)**.** *Let $\Omega \subset \mathbb{R}^N$ is domain and $1 < p < +\infty$. Sobolev space $W^{1,p}(\Omega)$ is space of all functions $f \in L^p(\Omega)$ such that for $i = 1, 2, \ldots, N$, there exists function $g_i \in L^p(\Omega)$ satisfying*

$$\int_\Omega f \frac{\partial \phi}{\partial x_i} \, \mathrm{d}x = - \int_\Omega g_i \, \phi \, \mathrm{d}x$$

for all ϕ from the space of test function $C_0^\infty(\Omega)$. Function g_i is called weak partial derivative of the function f with respect to x_i.

Definition 2.2.7 (see [5], Definition 3.9 and Theorem 3.10, p. 95)**.** *For $1 < p < +\infty$ we define the norm $\| \cdot \|_{W^{1,p}(\Omega)} : W^{1,p}(\Omega) \to [0, +\infty)$ as*

$$\|f\|_{W^{1,p}(\Omega)} \stackrel{\text{def}}{=} \left(\|f\|^p_{L^p(\Omega)} + \sum_{i=1}^N \left\| \frac{\partial f}{\partial x_i} \right\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} ,$$

where $\frac{\partial f}{\partial x_i} \in L^p(\Omega)$ are weak derivatives of f.

Definition 2.2.8 (see [5], Definition 3.12, p. 95)**.** *Let $\Omega \subset \mathbb{R}^N$ is domain and $1 < p < +\infty$. Sobolev space $W_0^{1,p}(\Omega)$ is defined as closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ with respect to the norm $\| \cdot \|_{W^{1,p}(\Omega)}$.*

Define $p' \stackrel{\text{def}}{=} \frac{p}{p-1}$ and recall that $p > 1$ and

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \ge N. \end{cases}$$

By Rellich-Kondrachov Compactness Theorem (see Evans [24, Theorem 1, p. 272]), the Sobolev space $W_0^{1,p}(\Omega)$ is compactly embedded to $L^r(\Omega)$ for $1 \le r < p^*$. In particular for $p \ge 2$, we get the following chain of embeddings

$$W_0^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^r(\Omega) \hookrightarrow L^{r'}(\Omega) \overset{c}{\hookrightarrow} W^{-1,p'}(\Omega)\,, \tag{2.4}$$

where $W^{-1,p'}(\Omega)$ denotes the dual space of $W_0^{1,p}(\Omega)$ and $\frac{1}{r} + \frac{1}{r'} = 1$. It is well known fact that the problem

$$\begin{aligned} -\Delta_p u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.5}$$

has the unique weak solution for each $f \in W^{-1,p'}(\Omega)$ by Zeidler [57, Theorem 26.A, p. 557]. In other words, there is the unique $u \in W_0^{1,p}(\Omega)$ which satisfies

$$\int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla v \,\mathrm{d}x = \langle h, v\rangle$$

for all $v \in W_0^{1,p}(\Omega)$. Note that $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$. We denote the unique solution of (2.5) by $\mathcal{R}_p(h)$, which is a continuous operator from $W^{-1,p'}(\Omega)$ to $W_0^{1,p}(\Omega)$ for $p \ge 2$ by [Zeidler, Theorem 26.A (d), p. 557]. By (2.4), the operator $\mathcal{R}_p : L^{r'}(\Omega) \to W_0^{1,p}(\Omega)$ is compact. In the following section we prove the continuity and compactness of $\mathcal{R}_p(h)$ for any $p > 1$, but only in one dimension.

2.2.9 Solution operator for one-dimensional case for $p > 1$

This subsection provides continuity and compactness of the solution operator $\mathcal{R}_p : L^{r'}(0, \pi_p) \to C_0^1[0, \pi_p]$, $h \mapsto u$ of one-dimensional boundary value problem

$$\begin{aligned} -\left(|u'|^{p-2}\, u'\right)' &= h \qquad \text{a.e. in } (0, \pi_p)\,, \\ u(0) &= u(\pi_p) = 0\,, \end{aligned} \tag{2.6}$$

for any $p > 1$. The equation is understood in the weak sense. By regularity (see e.g. Girg [27], where the more general case of φ-Laplacian is considered) for weak solution, it can be shown that $u \in C_0^1[0, \pi_p]$ and the equation is satisfied pointwise a.e. in $(0, \pi_p)$. Note that the results of this section are known, see e.g. [43], and we provide them only for completeness of the presentation. Since the solution $u \in C_0^1[0, \pi_p]$, the function u is absolutely continuous (see e.g. del Pino [16]) and integrating (2.6) we get

$$\left|u'(t)\right|^{p-2} u'(t) - \left|u'(0)\right|^{p-2} u'(0) = -\int_0^t h(s)\,\mathrm{d}s\,. \tag{2.7}$$

Let us define function

$$\varphi_p(\xi) = \begin{cases} |\xi|^{p-2}\,\xi & \xi \ne 0, \\ 0 & \xi = 0, \end{cases} \tag{2.8}$$

which is increasing and continuous for any $p > 1$. It is not difficult to verify that its inverse function is $\varphi_{p'}$ and hence it has the same properties as φ_p does. Using function φ_p we can rewrite (2.7) as follows

$$\varphi_p\left(u'(t)\right) = \left|u'(0)\right|^{p-2} u'(0) - \int_0^t h(s)\,\mathrm{d}s.$$

Applying the monotone continuous function φ_p^{-1} we obtain

$$u'(t) = \varphi_p^{-1}\left(\left|u'(0)\right|^{p-2} u'(0) - \int_0^t h(s)\,\mathrm{d}s\right). \tag{2.9}$$

Integrating (2.9) once more and using homogeneous Dirichlet boundary condition leads us to

$$u(x) = \int_0^x \varphi_p^{-1}\left(\left|u'(0)\right|^{p-2} u'(0) - \int_0^t h(s)\,\mathrm{d}s\right)\,\mathrm{d}x \tag{2.10}$$

and

$$0 = u(\pi_p) = \int_0^{\pi_p} \varphi_p^{-1}\left(\left|u'(0)\right|^{p-2} u'(0) - \int_0^t h(s)\,\mathrm{d}s\right)\,\mathrm{d}x\,.$$

Denoting

$$a \stackrel{\text{def}}{=} \left|u'(0)\right|^{p-2} u'(0) \tag{2.11}$$

we get

$$0 = \int_0^{\pi_p} \varphi_p^{-1}\left(a - \int_0^t h(s)\,\mathrm{d}s\right)\,\mathrm{d}x\,. \tag{2.12}$$

Lemma 2.2.10. *Equation* (2.12) *has unique solution* $a \in \mathbb{R}$.

Proof. Let us define function $F : a \mapsto \int_0^{\pi_p} \varphi_p^{-1}\left(a - \int_0^t h(s)\,\mathrm{d}s\right)\,\mathrm{d}x$. Since the function $\varphi_p^{-1}(s) = \varphi_{p'}(s)$ is continuous and $\int_0^t h(s)\,\mathrm{d}s$ is constant with respect to a we have continuity of F. Next part of the proof naturally falls into two steps.
Step 1. - Existence. Fact that $h \in L^{r'}(0,\pi_p)$ implies

$$\exists K > 0 : \int_0^{\pi_p} |h(s)|\,\mathrm{d}s \le K\,.$$

Choosing $\bar{A} > K$ we get $\forall t \in [0,\pi_p]$

$$\bar{A} - \int_0^t h(s)\,\mathrm{d}s > 0\,, \tag{2.13}$$

and choosing $\underline{A} < -K$ we get $\forall t \in [0,\pi_p]$

$$\underline{A} - \int_0^t h(s)\,\mathrm{d}s < 0\,. \tag{2.14}$$

Combining (2.13) and (2.14) with continuity of F we get that there is at least one solution on $(\underline{A},\bar{A})$ by definition of φ_p due to (2.8).
Step 2. - Uniqueness. We show that the function F is monotone which guarantees the uniqueness of the solution of (2.12). Indeed, let $a_2 > a_1$. Then monotonicity of φ_p^{-1} yields

$$\begin{aligned} \int_0^{\pi_p} \varphi_p^{-1}\left(a_2 - \int_0^t h(s)\,\mathrm{d}s\right)\,\mathrm{d}x - \int_0^{\pi_p} \varphi_p^{-1}\left(a_1 - \int_0^t h(s)\,\mathrm{d}s\right)\,\mathrm{d}x &= \dots \\ \dots = \int_0^{\pi_p} \left[\varphi_p^{-1}\left(a_2 - \int_0^t h(s)\,\mathrm{d}s\right) - \varphi_p^{-1}\left(a_1 - \int_0^t h(s)\,\mathrm{d}s\right)\right]\,\mathrm{d}x &> 0 \end{aligned}$$

■

Let us denote $\alpha : L^{r'}(0,\pi_p) \to \mathbb{R}$, which maps any $h \in L^{r'}(0,\pi)$ to unique a such that a solves the equation

$$\int_0^{\pi_p} \varphi_p^{-1}\left(a - \int_0^t h(s)\,\mathrm{d}s\right)\,\mathrm{d}x = 0\,. \tag{2.15}$$

Lemma 2.2.11. *Functional α maps any bounded set in $L^r(0,\pi_p)$ to the bounded set in $\mathbb{R}$.*

Proof. Let $\|h\|_{L^{r'}(0,\pi_p)} \le K$. Then $|a| \le 2K$ since from (2.13) and (2.14) follows that $a \in (\underline{A}, \bar{A})$ for any $\underline{A} < -K$ and any $\bar{A} > K$. ∎

Lemma 2.2.12. *Functional α is continuous.*

Proof. Let $h_n \to h$ in $L^{r'}(0,\pi_p)$. Hence h_n is bounded in $L^{r'}(0,\pi_p)$ and the real sequence $\alpha(h_n)$ is bounded by Lemma 2.2.11. Then there exists subsequence $\alpha_{h_{n_k}}$ so it converges to some point $a_0 \in \mathbb{R}$. If we prove

$$\int_0^{\pi_p} \varphi_p^{-1}\left(a_0 - \int_0^t h(s)\,\mathrm{d}s\right)\,\mathrm{d}x = 0,$$

the statement of Lemma follows from uniqueness of solution of (2.12). Our current aim is to show

$$0 = \lim_{k\to+\infty}\int_0^{\pi_p} \varphi_p^{-1}\left(\alpha(h_{n_k}) - \int_0^t h_{n_k}(s)\,\mathrm{d}s\right)\,\mathrm{d}x = \int_0^{\pi_p} \varphi_p^{-1}\left(a_0 - \int_0^t h(s)\,\mathrm{d}s\right)\,\mathrm{d}x$$

The first equation follows easily from the definiton of α due to (2.15). The second one follows Proposition 2.1.7 (Lebesgue Dominated Convergence Theorem),

$$\lim_{k\to+\infty}\int_0^t h_{n_k}(s)\,\mathrm{d}s = \lim_{k\to+\infty}\int_0^{\pi_p} h_{n_k}(s)\chi_{[0,t]}(s)\,\mathrm{d}s = \int_0^{\pi_p} h(s)\chi_{[0,t]}(s)\,\mathrm{d}s = \int_0^t h(s)\,\mathrm{d}s \tag{2.16}$$

using the boundedness of h_{n_k} and fact that $h_{n_k} \to h$. The function χ_A is characteristic function of set A which is defined as

$$\chi_A(\xi) \stackrel{\text{def}}{=} \begin{cases} 1 & \xi \in A\,, \\ 0 & \text{otherwise.} \end{cases}$$

Since φ_p^{-1} is defined for any $\xi \in \mathbb{R}$ and it is continuous, boundedness of argument implies boundedness of φ_p^{-1}. Due to the continuity of φ_p^{-1}, the fact that $\alpha(h_{n_k}) \to a_0$, and (2.16), we find

$$\lim_{k\to+\infty}\int_0^{\pi_p} \varphi_p^{-1}\left(a(h_{n_k}) - \int_0^t h_{n_k}(s)\,\mathrm{d}s\right)\,\mathrm{d}x = \int_0^{\pi_p} \varphi_p^{-1}\left(a_0 - \int_0^t h(s)\,\mathrm{d}s\right)\,\mathrm{d}x$$

by Proposition 2.1.7 (Lebesgue Dominated Convergence Theorem) again.
Hence

$$0 = \int_0^{\pi_p} \varphi_p^{-1}\left(a_0 - \int_0^t h(s)\,\mathrm{d}s\right)\,\mathrm{d}x\,,$$

which implies $a_0 = \alpha(h)$ because $\alpha(h)$ is the unique solution of (2.12) by Lemma 2.2.10. ∎

Proposition 2.2.13 (see Yosida [56], Theorem (Eberlein-Shmulyan), p. 141)**.** *A Banach space X is reflexive if and only if it is locally sequentialy weakly compact; that is, X is reflexive if and only if every strongly bounded sequence of X contains a subsequence which converge weakly to an element of X.*

Let us summarize the knowledge of the solution of the problem (2.6). From (2.10) we have

$$u(x) = \int_0^x \left(a(h) - \int_0^t h(s)\,\mathrm{d}s \right) \mathrm{d}x\,,$$

where $\alpha : L^{r'}(0, \pi_p) \to \mathbb{R}$ is continuous and it maps a bounded domain in $L^{r'}(0, \pi_p)$ to a bounded domain in $\mathbb{R}$. Let us consider the mapping $\mathcal{R}_p : L^{r'}(0, \pi_p) \to C_0^1[0, \pi_p]$, which assigns solution u to every function $h \in L^{r'}(0, \pi_p)$.

Theorem 2.2.14. *Operator* $\mathcal{R}_p : L^{r'}(0, \pi_p) \to C_0^1[0, \pi_p]$ *is continuous and compact.*

Proof. Step 1. - Continuity. Let $h_n \to h$ in $L^{r'}(0, \pi_p)$. If we prove that

$$\varphi_p^{-1}\left(\alpha(h_n) - \int_0^t h_n(s)\,\mathrm{d}s \right) \to \varphi_p^{-1}\left(\alpha(h) - \int_0^t h(s)\,\mathrm{d}s \right) \tag{2.17}$$

the proof follows from Proposition 2.1.7 (Lebesgue Dominated Convergence Theorem). The convergence $h_n \to h$ implies the boundedness of h_n in $L^r(0, \pi_p)$. Hence we get the boundedness of

$$\varphi_p^{-1}\left(\alpha(h_n) - \int_0^t h_n(s)\,\mathrm{d}s \right)$$

and

$$\varphi_p^{-1}\left(\alpha(h) - \int_0^t h(s)\,\mathrm{d}s \right)$$

by Lemma 2.2.11 and by the continuity of φ_p^{-1} on $\mathbb{R}$.

It remains to show (2.17). Since α is continuous by Lemma 2.2.12, we get

$$\alpha(h_n) \to \alpha(h)\,. \tag{2.18}$$

Moreover

$$\int_0^t h_n(s)\,\mathrm{d}s \to \int_0^t h(s)\,\mathrm{d}s \tag{2.19}$$

by Proposition 2.1.7 (Lebesgue Dominated Convergence Theorem) due to the boundedness of h_n and the convergence $h_n \to h$. Since φ_p^{-1} is continuous, the convergence (2.17) follows from (2.18) and (2.19).

Step 2. - Compactness. Let B is bounded set in $L^{r'}(0, \pi_p)$ which is reflexive Banach space (we have $1 < r < +\infty$). Then there is $K > 0$ such that $\|h_n\|_{L^{r;}(0,\pi_p)} \leq K$ for any $h_n \in B$ and there exists a subsequence $h_{n_k} \rightharpoonup h \in L^{r'}(0, \pi_p)$ by Proposition 2.2.13 (Eberlein-Shmulyan Theorem).

If we show, that

$$\varphi_p^{-1}\left(\alpha(h_{n_k}) - \int_0^t h_{n_k}(s)\,\mathrm{d}s \right) \to \varphi_p^{-1}\left(\alpha(h) - \int_0^t h(s)\,\mathrm{d}s \right) \tag{2.20}$$

then

$$u_n'(t) = \varphi_p^{-1}\left(\alpha(h_{n_k}) - \int_0^t h_{n_k}(s)\,\mathrm{d}s \right) \to \varphi_p^{-1}\left(\alpha(h) - \int_0^t h(s)\,\mathrm{d}s \right) = u'(t). \tag{2.21}$$

by (2.9). Integrating (2.21) we obtain

$$u_n(x) = \int_0^x \varphi_p^{-1}\left(\alpha(h_{n_k}) - \int_0^t h_{n_k}(s)\,\mathrm{d}s\right)\,\mathrm{d}x \to \int_0^x \varphi_p^{-1}\left(\alpha(h) - \int_0^t h(s)\,\mathrm{d}s\right)\,\mathrm{d}x$$

and $u(x)$ is solution of (2.6) by (2.10), definition of a due to (2.11) and α due to (2.15). Hence $u(x) \in C_0^1[0,\pi_p]$ and $\mathcal{R}_p$ is compact operator.

It remains to obtain (2.20). Due to Lemma 2.2.11 and boundedness of h_{n_k}, there is subsequence of h_{n_k} (still denoted by h_{n_k} for simplicity) such that $\alpha(h_{n_k})$ converges to some $\alpha_0 \in \mathbb{R}$. It follows from the weak convergence of h_{n_k} that

$$\int_0^{\pi_p} h_{n_k}(x)\phi\,\mathrm{d}x \to \int_0^{\pi_p} h(x)\phi\,\mathrm{d}x \quad \forall \phi \in L^{r'}(0,\pi_p)\,.$$

Choosing $\chi_{[0,t]} \in L^\infty(0,\pi_p) \hookrightarrow L^{r'}(0,\pi_p)$ for all $1 \le r' \le +\infty$, we obtain

$$\int_0^t h_{n_k}(s)\,\mathrm{d}s \to \int_0^t h(s)\,\mathrm{d}s$$

using the fact that

$$\int_0^{\pi_p} h(s)\chi_{[0,t]}(s)\,\mathrm{d}s = \int_0^t h(s)\,\mathrm{d}s\,.$$

Hence

$$\int_0^t h_{n_k}(s)\,\mathrm{d}s = \int_0^{\pi_p} h_{n_k}(s)\chi_{[0,t]}(s)\,\mathrm{d}s \to \int_0^{\pi_p} h(s)\chi_{[0,t]}(s)\,\mathrm{d}s$$

and

$$\lim_{k\to+\infty}\int_0^{\pi_p} \varphi_p^{-1}\left(\alpha(h_{n_k}) - \int_0^t h_{n_k}(s)\,\mathrm{d}s\right)\,\mathrm{d}x = \int_0^{\pi_p} \varphi_p^{-1}\left(\alpha_0 - \int_0^t h(s)\,\mathrm{d}s\right)\,\mathrm{d}x \tag{2.22}$$

by Proposition 2.1.7 (Lebesgue Dominated Convergence Theorem) and the continuity of φ_p^{-1}. Due to the fact that

$$\int_0^{\pi_p} \varphi_p^{-1}\left(\alpha(h_{n_k}) - \int_0^t h_{n_k}(s)\,\mathrm{d}s\right)\,\mathrm{d}x = 0$$

for all $k \in \mathbb{N}$ by the definition (2.15) of α, we have

$$\int_0^{\pi_p} \varphi_p^{-1}\left(\alpha_0 - \int_0^t h(s)\,\mathrm{d}s\right)\,\mathrm{d}x = 0$$

from (2.22). It follows that $\alpha_0 = \alpha(h)$ by the uniqueness of the solution (see Lemma 2.2.10). Hence the limit (2.20) holds. ■

Chapter 3

Bifurcations from infinity

3.1 Krasnoselskii type necessary condition

In this section we consider the problem

$$\begin{array}{rcll} -\Delta_p u & = & \mu\,|u|^{p-2}\,u + g(\mu;x,u,\nabla u) & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \tag{3.1}$$

(in the weak sense) where $\mu \in \mathbb{R}$, $\Delta_p \stackrel{\text{def}}{=} \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right)$ denotes p-Laplace operator, the domain $\Omega \in \mathbb{R}^N$ is bounded with $C^{2,\alpha}$-boundary for some $\alpha \in (0,1)$. Let us note that by Section 2.2 , the inverse of p-Laplace operator $\mathcal{R}_p : L^{r'}(\Omega) \to W_0^{1,p}(\Omega)$ is continuous and compact for $p > 1$ in the case $N = 1$ and/or for $p \geq 2$ in the case $N \geq 2$. Let $g : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy Carathéodory condition (see Definition 2.1.9) and the following growth condition:

$$g(\mu;x,u,v) \leq a(x) \tag{3.2}$$

for some $a(x) \in L^{r'}(\Omega)$. The parameter r' satisfies $\frac{1}{r} + \frac{1}{r'} = 1$ and $r \in (p, p^*)$. Our aim is to formulate necessary condition for the bifurcation of solutions from infinity.

Definition 3.1.1. We say that $\bar{\lambda} \in \mathbb{R}$ is a bifurcation point from infinity of the problem (3.1), if there exists a sequence $\{(\mu_n, u_n)\}_{n=1}^{\infty} \in \mathbb{R} \times W_0^{1,p}(\Omega)$ of weak solutions to

$$\begin{array}{rcll} -\Delta_p u_n & = & \mu_n\,|u_n|^{p-2}\,u_n + g(\mu_n;x,u_n,\nabla u_n) & \text{in } \Omega, \\ u_n & = & 0 & \text{on } \partial\Omega, \end{array} \tag{3.3}$$

such that $\mu_n \to \bar{\lambda}$ and $\|u_n\|_{W_0^{1,p}(\Omega)} \to +\infty$

Next Proposition is a slightly modified version of Proposition 2.1 from [21], where the bifurcation is considered from zero and the function on right hand side *does not* depend on ∇u.

Proposition 3.1.2. *Let $p \geq 2$ for $N > 1$ or $p > 1$ for $N = 1$. We assume that $\bar{\lambda}$ is a bifurcation point from infinity of the problem* (3.1)*. Moreover there is $\delta > 0$ such that the sequence μ_n from Definition 3.1.1 satisfies $|\lambda_2 - \lambda_1| > \delta > |\mu_n - \lambda_1|$ the sequence form Definition 3.1.1. Then $\bar{\lambda}$ is an eigenvalue of* (1.5)*.*

Proof. We perform the proof for $p \geq 2$ and $N > 1$; the other case is analogous. Assume that the sequence $(\mu_n, u_n) \in \mathbb{R} \times W_0^{1,p}(\Omega)$ satisfies (3.3), i.e.,

$$-\operatorname{div}\left(|\nabla u_n|^{p-2}\,\nabla u_n\right) = \mu_n\,|u_n|^{p-2}\,u_n + g(\mu_n;x,u_n,\nabla u_n), \tag{3.4}$$

and $\mu_n \to \bar{\lambda}$, $\|u_n\|_{W_0^{1,p}(\Omega)} \to +\infty$ as $n \to +\infty$. Dividing (3.4) by $\|u_n\|_{W_0^{1,p}(\Omega)}^{p-1}$ and substituting

$$w_n = \frac{u_n}{\|u_n\|_{W_0^{1,p}(\Omega)}}$$

into (3.4) we get

$$-\mathrm{div}\left(|\nabla w_n|^{p-2}\nabla w_n\right)=\mu_n|w_n|^{p-2}w_n+\frac{g(\mu_n;x,u_n,\nabla u_n)}{\|u_n\|_{W_0^{1,p}(\Omega)}}.$$

Hence using the inverse operator $\mathcal{R}_p : L^{r'}(\Omega)\to W_0^{1,p}(\Omega)$ of p-Laplacian, we obtain

$$w_n=\mathcal{R}_p\left(\mu_n|w_n|^{p-2}w_n+\frac{g(\mu_n;x,u_n,\nabla u_n)}{\|u_n\|_{W_0^{1,p}(\Omega)}}\right).$$

Now we show that there is $K>0$ such that for all $n\in\mathbb{N}$ holds

$$\left\|\mu_n|w_n|^{p-2}w_n+\frac{g(\mu_n;x,u_n,\nabla u_n)}{\|u_n\|_{W_0^{1,p}(\Omega)}}\right\|_{L^{r'}(\Omega)}\le K. \tag{3.5}$$

Recall that $W_0^{1,p}(\Omega)\hookrightarrow L^q(\Omega)$ for $q\in\left[1,\frac{Np}{N-p}\right]$ with $p<N$ and for any $q\ge 1$ with $N=p$. In the case $p>N$ holds $W_0^{1,p}(\Omega)\hookrightarrow C^{0,1-\frac{N}{p}}(\bar{\Omega})$. Thus $W^{1,p}(\Omega)\hookrightarrow L^r(\Omega)$ for $r\in(p,p^*)$. It is also well known that $L^p(\Omega)\hookrightarrow L^q(\Omega)$ for $p>q$ on bounded domain Ω. With this in hand, we claim that $\|w_n\|_{W_0^{1,p}(\Omega)}=1$ implies $\||w_n|^{p-1}\|_{L^{r'}(\Omega)}<c'$. Indeed from embeddings $W_0^{1,p}(\Omega)\hookrightarrow L^r(\Omega)$ we get that $\|w_n\|_{L^r(\Omega)}\le c$.
Hence

$$c^r\ge\int_\Omega|w_n|^r\,\mathrm{d}x\ge\int_\Omega\left(|w_n|^{p-1}\right)^{\frac{r}{p-1}}\,\mathrm{d}x$$

and

$$\left(\int_\Omega\left(|w_n|^{p-1}\right)^{\frac{r}{p-1}}\,\mathrm{d}x\right)^{\frac{p-1}{r}}\le c^{r\frac{p-1}{r}}.$$

It follows that $\||w_n|^{p-1}\|_{L^{\frac{r}{p-1}}(\Omega)}\le c^{p-1}$. Since for $r\in(p,p*)$ we have $L^{\frac{r}{p-1}}(\Omega)\hookrightarrow L^{r'}(\Omega)$ and there exist $c'\in\mathbb{R}$ such that

$$\||w_n|^{p-1}\|_{L^{r'}(\Omega)}\le c'.$$

Moreover the function $g(\mu;x,u,v)$ is bounded by assumption (3.2) and so inequality (3.5) holds. It follows that there is a subsequence of w_n (still denoted w_n) such that $w_n\rightharpoonup w$ weakly in $W_0^{1,p}(\Omega)$. Hence

$$w=\mathcal{R}_p(\bar{\lambda}|w|^{p-2}w),$$

with $\|w\|_{W_0^{1,p}(\Omega)}=1$ and so $\bar{\lambda}$ is an eigenvalue of (1.5). ■

3.2 The key estimate for the proof of an analogy of Dancer's Theorem

In this section our aim is to prove the key estimate, which will be used to prove an analogy of Dancer's Theorem (see [15], Theorem 2., p. 1071) for the equation (1.1) in one dimension and the bifurcation from infinity. The bifurcations of the positive and negative solutions will be studied in detail in the prepared paper [30]. The process of the proof is identical with the proof given in DANCER [15]. The first fundamental part is to prove that there is the jump in the Leray-Schauder degree. This is done in DEL PINO-MANASEVICH [21]. Let us define

$$T_p^\mu : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$$

such that $T_p^\mu \stackrel{\text{def}}{=} \mathcal{R}_p(\mu\varphi_p(u))$.

Proposition 3.2.1 (see [21] Proposition 2.2, p. 231)**.** *Let $r > 0$, $p > 1$, $B(0, r)$ is the ball in $W_0^{1,p}(\Omega)$ centered at origin and $\mu \in \mathbb{R}$. Then*

$$\deg_{W_0^{1,p}(\Omega)}(I - T_p^\mu, B(0, r), 0) = \begin{cases} 1 & \text{if } \mu < \lambda_1 \\ -1 & \text{if } \lambda_1 < \mu < \lambda_2 \,. \end{cases}$$

Using this proposition we can follow the proof of Dancer up to Lemma 2. The main difficulty is to prove analogy of Lemma 3. It is the aim of this section. More precisely, we consider the pairs $(\mu_n, u_n) \in \mathbb{R} \times W_0^{1,p}(0, \pi_p)$ satisfying problem (3.1) in one dimension, i.e.

$$\begin{cases} -\left(|u_n'|^{p-2}u_n'\right)' - \mu_n|u_n|^{p-2}u_n &= g\left(\mu_n; x, u_n, u_n'\right) \qquad \text{on } (0, \pi_p) \\ u_n(0) &= u_n(\pi_p) = 0\,, \end{cases} \tag{3.6}$$

with $g(\mu_n; x, u_n, u_n') \leq K$ and we show that following Theorem holds.

Theorem 3.2.2. *Let $(\mu_n, u_n) \in \mathbb{R} \times W_0^{1,p}(0, \pi_p)$ is sequence of solution of (3.6) such that $\|u_n\|_{W_0^{1,p}(0,\pi_p)} \to +\infty$. Moreover let there is $\delta \in \mathbb{R}$ such that $|\lambda_2 - \lambda_1| > \delta > |\mu_n - \lambda_1|$ for all $n \in \mathbb{N}$. Then $\mu_n \to \lambda_1$ as $n \to +\infty$ and there is $n_0 \in \mathbb{N}$ such that μ_n does not change a sign for all $n > n_0$.*

We state some auxiliary facts first. The weak formulation of (3.6) leads us to

$$\int_0^{\pi_p} |u_n'|^{p-2}u_n'\phi'\,\mathrm{d}x - \mu_n\int_0^{\pi_p} |u_n|^{p-2}u_n\phi\,\mathrm{d}x = \int_0^{\pi_p} g(\mu_n; x, u_n, u_n')\phi\,\mathrm{d}x \tag{3.7}$$

for all $\phi \in W_0^{1,p}$. Choosing $\phi = u_n$ in (3.7) we obtain

$$\int_0^{\pi_p} |u_n'|^p\,\mathrm{d}x - \mu_n\int_0^{\pi_p} |u_n|^p\,\mathrm{d}x = \int_0^{\pi_p} g(\mu_n; x, u_n, u_n')u_n\,\mathrm{d}x\,. \tag{3.8}$$

Let us recall the variational characterization of the first eigenvalue of p-Laplacian, e.g.

$$\lambda_1 = \inf_{w\in W_0^{1,p}} \frac{\int_0^{\pi_p} |w'|^p\,\mathrm{d}x}{\int_0^{\pi_p} |w|^p\,\mathrm{d}x}$$

and hence

$$\lambda_1 \leq \frac{\int_0^{\pi_p} |w'|^p\,\mathrm{d}x}{\int_0^{\pi_p} |w|^p\,\mathrm{d}x} \tag{3.9}$$

for all $w \in W_0^{1,p}$. It follows that

$$0 \leq \int_0^{\pi_p} |w'|^p\,\mathrm{d}x - \lambda_1\int_0^{\pi_p} |w|^p\,\mathrm{d}x\,. \tag{3.10}$$

From (3.8) we obtain

$$\int_0^{\pi_p} g(\mu_n; x, u_n, u_n')u_n \,\mathrm{d}x = \int_0^{\pi_p} |u_n'|^p \,\mathrm{d}x - \lambda_1 \int_0^{\pi_p} |u_n|^p \,\mathrm{d}x + \lambda_1 \int_0^{\pi_p} |u_n|^p \,\mathrm{d}x - \mu_n \int_0^{\pi_p} |u_n|^p \,\mathrm{d}x$$

and using (3.10) we get

$$\int_0^{\pi_p} |u_n'|^p \,\mathrm{d}x - \lambda_1 \int_0^{\pi_p} |u_n|^p \,\mathrm{d}x + \lambda_1 \int_0^{\pi_p} |u_n|^p \,\mathrm{d}x - \mu_n \int_0^{\pi_p} |u_n|^p \,\mathrm{d}x \geq (\lambda_1 - \mu_n) \int_0^{\pi_p} |u_n|^p \,\mathrm{d}x \,.$$

Hence

$$\frac{\int_0^{\pi_p} g(\mu_n; x, u_n, u_n')u_n \,\mathrm{d}x}{\int_0^{\pi_p} |u_n|^p \,\mathrm{d}x} \geq \lambda_1 - \mu_n \,. \tag{3.11}$$

Lemma 3.2.3. *Let $\{w_n\}_{n=1}^{+\infty}$ satisfies (3.8) with $g(\mu_n; x, w_n, w_n') \in L^\infty(0, \pi_p)$ for all $n \in \mathbb{N}$. Moreover let there is $\delta \in \mathbb{R}$ such that $|\lambda_2 - \lambda_1| > \delta > |\mu_n - \lambda_1|$ for all $n \in \mathbb{N}$. Then $\|w_n\|_{W_0^{1,p}(0,\pi_p)}$ is bounded if and only if $\|w_n\|_{L^p(0,\pi_p)}$ is bounded.*

Proof. The fact that there exists $K \geq 0$ such that $\|w_n\|_{L^p(0,\pi_p)} \leq K\|w_n\|_{W_0^{1,p}(0,\pi_p)}$ follows from $W_0^{1,p}(0, \pi_p) \hookrightarrow L^p(0, \pi_p)$.

Conversely assume that $\|w_n\|_{L^p(0,\pi_p)}$ is bounded sequence and there is $a \geq 0$ such that $g(\mu_n; x, w_n, w_n') \leq a$ by the assumption of the Thesis. Then from (3.8) it follows

$$\int_0^{\pi_p} |w_n'|^p \,\mathrm{d}x = \mu_n \int_0^{\pi_p} |w_n|^p \,\mathrm{d}x + \int_0^{\pi_p} g(\mu_n; x, w_n, w_n')w_n \,\mathrm{d}x \leq \mu_n \|w_n\|^p_{L^p(0,\pi_p)} + a \int_0^{\pi_p} |w_n| \,\mathrm{d}x \,.$$

Since $p > 1$ we obtain

$$\mu_n \|w_n\|^p_{L^p(0,\pi_p)} + a \int_0^{\pi_p} |w_n| \,\mathrm{d}x \leq \mu_n \|w_n\|^p_{L^p(0,\pi_p)} + a \int_0^{\pi_p} |w_n|^p \,\mathrm{d}x$$

and hence

$$\int_0^{\pi_p} |w_n'|^p \,\mathrm{d}x \leq (\mu_n + a)\|w_n\|^p \,.$$

Since μ_n is bounded by assumption, the statement of Lemma 3.2.3 follows. ■

Lemma 3.2.4. *Let $(\mu_n, w_n) \in \mathbb{R} \times W_0^{1,p}(0, \pi_p)$ fulfills (3.8), $\|w_n\|_{W_0^{1,p}(0,\pi_p)} \to \infty$ as $n \to +\infty$, and there is $\delta > 0$ such that $|\lambda_2 - \lambda_1| > \delta > |\mu_n - \lambda_1|$ for all $n \in \mathbb{N}$. Then $\mu_n \to \lambda_1$.*

Proof. Applying Hölder's inequality we can rewrite (3.11) as

$$\lambda_1 - \mu_n \leq \frac{\left(\int_0^{\pi_p} |g(\mu_n, x, u_n, u_n')|^{p'} \,\mathrm{d}x\right)^{1/p'} \left(\int_0^{\pi_p} |u_n|^p \,\mathrm{d}x\right)^{1/p}}{\int_0^{\pi_p} |u_n|^p \,\mathrm{d}x} = \frac{\|g(\mu_n, x, u_n, u_n')\|_{L^{p'}(0,\pi_p)}}{\|u_n\|^{p-1}_{L^p(0,\pi_p)}} \,.$$

Since $L^\infty(0, \pi_p) \hookrightarrow L^{p'}(0, \pi_p)$ for all $p' \in [1, +\infty]$ and $g(\mu_n, x, u_n, u_n') \in L^\infty(0, \pi_p)$, there is $0 < C < +\infty$ such that $\|g(\mu_n, x, u_n, u_n')\|_{L^{p'}(0,\pi_p)} \leq C$. Moreover $\|u_n\|_{L^p(0,\pi_p)} \to +\infty$ by Lemma 3.2.3 and hence

$$\lim_{n\to+\infty} (\lambda_1 - \mu_n) \leq 0 \,.$$

It remains to prove that there is no $\underline{\lambda} > 0$ such that $\mu_n \to \underline{\lambda}$. On the contrary, suppose that there is such $\underline{\lambda}$. Hence $\underline{\lambda}$ is a bifurcation point from infinity by Definition 3.1.1. Since $\lambda_1 < \lim_{n\to+\infty} \mu_n < \lambda_2 - \delta$, we get a contradiction with Proposition 3.1.2. The same argument follows that any subsequence of μ_n converge to λ_1 and thus $\lim_{n\to+\infty} \mu_n = \lambda_1$. ■

Proof of Theorem 3.2.2. The proof is based on the following substitution

$$u_n = t_n^{-1}(\varphi_1 + v_n^\top), \tag{3.12}$$

where $t_n \in \mathbb{R} \setminus \{0\}$ and $v_n^\top \in W_0^{1,p}(0,\pi_p)$ such that scalar product $(\varphi_1, v_n^\top)_{W_0^{1,p}(0,\pi_p)} = 0$. We have also $v_n^\top \to 0$. Indeed, $\mu_n \to \lambda_1$ by Lemma 3.2.4 and all eigenfunctions corresponding to the first eigenvalue has form $\kappa\varphi_1$ for any $\kappa \in \mathbb{R}$. Therefore, the case that there is $\lim_{n\to+\infty} v_n^\top \neq 0$ contradicts that λ_1 is the first eigenvalue. Hence $v_n^\top \to 0$ and since $\|u_n\|_{W_0^{1,p}(0,\pi_p)} \to +\infty$, it is obvious that $t_n \to 0$.

Substituting (3.12) into (3.11) gives

$$\lambda_1 - \mu_n \le \frac{\int_0^{\pi_p} g(\mu_n, x, u_n, u_n') t_n^{-1}(\varphi_1 + v_n^\top)\,\mathrm{d}x}{\int_0^{\pi_p} \left|t_n^{-1}(\varphi_1 + v_n^\top)\right|^p\,\mathrm{d}x} = \frac{t_n^{-1}}{|t_n|^{-p}} \frac{\int_0^{\pi_p} g(\mu_n, x, u_n, u_n')(\varphi_1 + v_n^\top)\,\mathrm{d}x}{\int_0^{\pi_p} |(\varphi_1 + v_n^\top)|^p\,\mathrm{d}x}$$

and thus

$$\frac{\int_0^{\pi_p} g(\mu_n, x, u_n, u_n')(\varphi_1 + v_n^\top)\,\mathrm{d}x}{\int_0^{\pi_p} |(\varphi_1 + v_n^\top)|^p\,\mathrm{d}x} \ge \frac{\lambda_1 - \mu_n}{|t_n|^{p-2}\,t_n}\,. \tag{3.13}$$

Due to the fact that $v_n^\top \to 0$ as $n \to +\infty$, there is n_0 such that for all $n > n_0$ holds $\varphi_1 + v_n^\top > \frac{1}{2}\varphi$. Furthermore in one dimension $\varphi_1 = \sin_p(x)$, which is positive function on the open interval $(0, \pi_p)$. Consequently

$$\varphi_1 + v_n^\top > 0 \tag{3.14}$$

for any $n > n_0$ here.

Our goal is to show that $\lambda_1 - \mu_n$ does not change a sign for $t \to 0^+$ and $t \to 0^-$, respectively. We give the proof only for $t \to 0^+$, because the second case is similar. In this case we are proving $\lambda_1 - \mu_n < 0$ for all $n > n_0$. Combining (3.13) with (3.14) we find that it is sufficient to obtain

$$\int_0^{\pi_p} g(\mu_n, x, u_n, u_n')(\varphi_1 + v_n^\top)\,\mathrm{d}x < 0\,. \tag{3.15}$$

For this purpose let us introduce the sequence of functions

$$g(\mu_n, x, u_n, u_n')(\varphi_1 + v_n^\top) - P_m\left(t_n^{-1}\right)\varphi_1^{p-1}(x)\,, \tag{3.16}$$

where

$$P_m(s) = \begin{cases} 0 & 0 \le s \le m\,, \\ \frac{C_k}{m}s - C_k & m < 2 < 2m\,, \\ C_k & 2m \le s < +\infty \end{cases}$$

with $C_k > 0$, which will be specified below. Replacing $g(\mu_n, x, u_n, u_n')$ in (3.15) by (3.16) we get for fixed $m \in \mathbb{N}$ in the limit case

$$\lim_{k\to+\infty} \sup_{n\ge k} \int_0^{\pi_p} \left[g(\mu_n, x, u_n, u_n') - P_m\left(t_n^{-1}\right)\varphi_1^{p-1}\right](\varphi_1 + v_n^\top)\,\mathrm{d}x\,.$$

By the definition of limes superior we get

$$\limsup_{k\to+\infty} \int_0^{\pi_p} \left[g(\mu_n, x, u_n, u_n') - P_m\left(t_n^{-1}\right)\varphi_1^{p-1}\right](\varphi_1 + v_n^\top)\,\mathrm{d}x \tag{3.17}$$

and by straightforward rearrangement we obtain that (3.17) is lower or equal to

$$\limsup_{k\to+\infty}\int_0^{\pi_p} g(\mu_n,x,u_n,u_n')(\varphi_1+v_n^\top)\,\mathrm{d}x+\limsup_{k\to+\infty}\int_0^{\pi_p}\left[-P_m\left(t_n^{-1}\right)\right](\varphi_1^p+\varphi_1^{p-1}v_n^\top)\,\mathrm{d}x\,. \tag{3.18}$$

Since

$$\begin{array}{rcl} g(\mu_n,x,u_n,u_n') & \in & L^\infty(0,\pi_p)\,,\\ |\sin_p(x)| & \le & 1\,,\\ P_m & \le & C_k\,,\\ v_n^\top & \to & 0\,, \end{array} \tag{3.19}$$

these functions are measurable and there is a function $f\in L^\infty(0,\pi_p)$ such that

$$|g(\mu_n,x,u_n,u_n')(\varphi_1+v_n^\top)|\le f \qquad \text{and} \qquad |P_m\left(t_n^{-1}\right)(\varphi_1^p+\varphi_1^{p-1}v_n^\top)|\le f\,.$$

and we can apply Corollary 2.1.6 we obtain that

$$\int_0^{\pi_p}\limsup_{k\to+\infty} g(\mu_n,x,u_n,u_n')(\varphi_1+v_n^\top)\,\mathrm{d}x+\int_0^{\pi_p}\limsup_{k\to+\infty}\left[-P_m\left(t_n^{-1}\right)\right](\varphi_1^p+\varphi_1^{p-1}v_n^\top)\,\mathrm{d}x$$

is greater or equal to (3.18). Using (3.19) again we have

$$\int_0^{\pi_p}\limsup_{k\to+\infty} g(\mu_n,x,u_n,u_n')(\varphi_1+v_n^\top)\,\mathrm{d}x\le K\int_0^{\pi_p}\varphi_1\,\mathrm{d}x$$

and

$$\int_0^{\pi_p}\limsup_{k\to+\infty}\left[-P_m\left(t_n^{-1}\right)\right](\varphi_1^p+\varphi_1^{p-1}v_n^\top)\,\mathrm{d}x\le -C_k\int_0^{\pi_p}\varphi_1^p\,\mathrm{d}x\,.$$

Hence there is $n_1\in\mathbb{N}$ such that

$$\int_0^{\pi_p}\left[g(\mu_n,x,u_n,u_n')-P_m\left(t^{-1}\right)\varphi_1^{p-1}\right](\varphi_1+v_n^\top)\,\mathrm{d}x<0 \tag{3.20}$$

for

$$C_k\ge K\frac{\int_0^{\pi_p}\varphi_1\,\mathrm{d}x}{\int_0^{\pi_p}\varphi_1^p\,\mathrm{d}x}\,,$$

any $m\in\mathbb{N}$, and $n>n_1$. Since

$$\lim_{m\to+\infty}\left[g(\mu_n,x,u_n,u_n')-P_m\left(t_n^{-1}\right)\varphi_1^{p-1}\right](\varphi_1+v_n^\top)=g(\mu_n,x,u_n,u_n')\,,$$

all function are bounded by (3.19) again, and there is integrable function $f(x)=\max\{2K,2C_k\}$ such that

$$|g(\mu_n,x,u_n,u_n')(\varphi_1+v_n^\top)|\le f \text{ and } \forall m\in\mathbb{N}: |g(\mu_n,x,u_n,u_n')-P_m\left(t_n^{-1}\right)(\varphi_1^p+\varphi_1^{p-1}v_n^\top)|\le f\,,$$

for fixed $n>\max\{n_0,n_1\}$ holds that

$$\int_0^{\pi_p} g(\mu_n,x,u_n,u_n')(\varphi_1+v_n^\top)\,\mathrm{d}x=\lim_{m\to+\infty}\int_0^{\pi_p}\left[g(\mu_n,x,u_n,u_n')-P_m\left(t_n^{-1}\right)\varphi_1^{p-1}\right](\varphi_1+v_n^\top)\,\mathrm{d}x$$

by Proposition 2.1.7 (Lebesgue Dominated Convergence Theorem). Consequently

$$\int_0^{\pi_p} g(\mu_n,x,u_n,u_n')(\varphi_1+v_n^\top)\,\mathrm{d}x<0\,.$$

■

Chapter 4

Differentiability of $\sin_p$

4.1 Introduction

This chapter is a short summary of results obtained jointly with my mentor P. Girg in the paper "Differentiability properties of p-trigonometric functions" published in the proceedings of Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems, see [28]. The published version of the paper [28] is included in the Appendix A1. Since the paper was a joint work, I briefly comment on my personal contribution to the paper in Section 4.3.

The p-trigonometric functions arise from the study of the eigenvalue problem for the one-dimensional p-Laplacian. Recently, the p-trigonometric functions have attracted attention of many researchers; see, e.g., [8, 10, 12, 13, 22, 23, 25, 36, 38, 39, 51], and references therein. We assume $p > 1$ and say, that $\lambda \in \mathbb{R}$ is an eigenvalue of

$$\begin{gathered} -(|u'|^{p-2}u')' - \lambda |u|^{p-2}u = 0 \quad \text{in } (0, \pi_p)\,, \\ u(0) = u(\pi_p) = 0\,, \end{gathered} \tag{4.1}$$

if there is a nonzero function $u \in W_0^{1,p}(0, \pi_p)$ that satisfies (4.1) in a weak sense. Here

$$\pi_p = 2 \int_0^1 \frac{1}{(1 - s^p)^{1/p}} \, \mathrm{d}s = \frac{2\pi}{p \sin(\pi/p)}\,. \tag{4.2}$$

Let us note, that the problem can be considered on any bounded open interval, but the choice $(0, \pi_p)$ simplifies the calculations. The discreetness of the spectrum of this eigenvalue problem was established by Nečas [47]. This eigenvalue problem was later studied by the means of the initial-value problem

$$\begin{gathered} -(|u'|^{p-2}u')' - \lambda |u|^{p-2}u = 0 \quad \text{in } (0, \infty)\,, \\ u(0) = 0, \quad u'(0) = 1\,; \end{gathered} \tag{4.3}$$

see Elbert [23] for initial work in this direction. Later it was independently studied by del Pino-Elgueta-Manasevich [17], Ôtani [48] and Lindqvist [37].

Let $\sin_p(x)$ denote the solution of (4.3) with $\lambda = (p-1)$. It follows from [23] that $\sin_p(x)$ is positive on $(0, \pi_p)$ and satisfies an identity

$$|\sin_p(x)|^p + |\sin_p'(x)|^p = 1 \quad \forall x \in \mathbb{R}\,, \tag{4.4}$$

which for $p = 2$ becomes the familiar identity for sine and cosine. This suggests the definition $\cos_p(x) := \sin_p'(x)$ and justifies the notation $\sin_p(x)$ and $\cos_p(x)$. The identity (4.4) is called p-trigonometric identity. It also follows from [23] that the eigenvalues of (4.3) form a sequence $\lambda_k = k^p(p-1), k \in \mathbb{N}$ and corresponding eigenfunctions are functions $\sin_p(kx), k \in \mathbb{N}$. Thus all the eigenfunctions are determined by the function $\sin_p(x)$. It comes as no surprise that the properties of the function $\sin_p(x)$ were studied extensively in the previous 30 years. As was shown in [23] that $\sin_p(x)$ can be expressed on $[0, \pi_p/2]$ (the p-trigonometric identity (4.4) can be thought of as the first integral of (4.3)) as the inverse of

$$\arcsin_p(x) = \int_0^x \frac{1}{(1-s^p)^{1/p}}\, \mathrm{d}s\,, \quad x \in [0,1]\,, \tag{4.5}$$

which is extended to $[0, \pi_p]$ by reflection $\sin_p(x) = \sin_p(\pi_p - x)$ and to $[-\pi_p, \pi_p]$ as the odd function. Finally, it is extended to $\mathbb{R}$ as the $2\pi_p$-periodic function. The function $\arcsin_p(x)$ from (4.5) is extended to $[-1, 1]$ as an odd function. Then

$$\sin_p(\arcsin_p(x)) = x \quad \forall x \in [-1,1]\,. \tag{4.6}$$

Note that for $p = 2$, we obtain classical arcsine and sine from this definition.

In our article [28] we focus on the differentiability and analyticity properties of p-trigonometric functions. One can immediately see from (4.2), (4.5), and (4.6) that $\sin_p(0) = 0$ and $\sin_p(\pi_p/2) = 1$ for all $p > 1$. From (4.4) and the definition of $\cos_p(x)$, we obtain $\cos_p(0) = 1$ and $\cos_p(\pi_p/2) = 0$. It follows from the results in [23, 38, 48] that the possible differentiability issues are located at $x = 0$ and $x = \pi_p/2$. There are several results concerning differentiability and asymptotic behaviour of $\sin_p(x)$ at $x = 0$ and $x = \pi_p/2$ in Manásevich-Takáč [44] and Benedikt-Girg-Takáč [6]. In Peetre [51], generalized formal Maclaurin series for $\sin_p(x)$ were studied and their convergence was conjectured on $(-\pi_p/2, \pi_p/2)$. The local convergence of the generalized Taylor series (and/or the generalized Maclaurin series) for $\sin_p(x)$ follows from Paredes-Uchiyama [50]. Taking into account that the point $x = 0$ is often considered as the center for the Taylor (i.e. the Maclaurin) series or the generalized Taylor (i.e. the generalized Maclaurin) series for $\sin_p(x)$, we decided to provide a detailed study of the convergence of these series towards $\sin_p(x)$ on $(-\pi_p/2, \pi_p/2)$. We were also motivated by work of Ôtani [49], where he studies properties of the solutions of

$$\begin{gathered} (|u'|^{p-2}u')' + |u|^{q-2}u = 0 \quad \text{in } (a,b)\,, \\ u(a) = u(b) = 0\,, \end{gathered} \tag{4.7}$$

for general exponents $p, q \in (1, +\infty)$ with $p \neq q$. Among other properties he proved that for $p = \frac{2m+2}{2m+1}$, $m \in \{0\} \cup \mathbb{N}$ and for q even, any solution of (4.7) belongs to $C^\infty(a, b)$. In our case, $p = q$ we find that $\sin_p(x)$ belongs to $C^\infty(-\pi_p/2, \pi_p/2)$ if and only if p is even. Let us also remark that local analytic solutions of the radial variant of (4.7) were studied in Bognár [9].

Our main result provides convergence of these partial sums. We treat two cases separately, $p > 2$ is an even integer and $p > 2$ is an odd integer. Namely, for the particular case $\sin_{2(m+1)}(x)$, $m \in \mathbb{N}$, $x \in (-\pi_p/2, \pi_p/2)$, we show that the Maclaurin series converges towards the values $\sin_{2(m+1)}(x)$ on the interval $(-\pi_p/2, \pi_p/2)$. Conversely, we show that the Maclaurin series converge towards $\sin_{2m+1}(x)$, $m \in \mathbb{N}$, for $x \in (0, \pi_p/2)$ and does not for $x \in (-\pi_p/2, 0)$. More precisely, the Maclaurin series converges on $x \in (-\pi_p/2, \pi_p/2)$, but not towards values of $\sin_{2m+1}(x)$, $m \in \mathbb{N}$ for $x \in (-\pi_p/2, 0)$.

4.2 Main Results of [28]

Our main results concern derivatives of $\sin_p(x)$ for $p \in \mathbb{N}$, $p > 2$ on the interval $x \in (-\pi_p/2, \pi_p/2)$. We distinguish two cases p is even, i.e., $p = 2(m + 1)$ and $m \in \mathbb{N}$, and p

is not an even integer, i.e., $p = \mathbb{R} \setminus \{2m\}$ and $m \in \mathbb{N}$. In the first case $p = 2(m+1)$, the p-trigonometric identity (4.4) takes form

$$(\sin_{2(m+1)}(x))^{2(m+1)} + (\cos_{2(m+1)(x)})^{2(m+1)} = 1\,, \tag{4.8}$$

which is valid for any $x \in \mathbb{R}$ and hence on $(-\pi_p/2, \pi_p/2)$. Note that there is no absolute value, since there are even powers.

In the second case assume $p = 2k+1$ for clarity. We have to distinguish two subcases. For $0 < x < \frac{\pi_p}{2}$, the p-trigonometric identity takes form

$$(\sin_{2m+1}(x))^{2m+1} + (\cos_{2m+1}(x))^{2m+1} = 1\,. \tag{4.9}$$

On the other hand, for $-\pi_p/2 < x < 0$, the p-trigonometric identity takes form

$$-(\sin_{2m+1}(x))^{2m+1} + (\cos_{2m+1}(x))^{2m+1} = 1\,. \tag{4.10}$$

Since there is only one identity (4.8) for $p = 2(m+1)$, this case has nice smoothness properties on $(-\pi_p/2, \pi_p/2)$ and we obtain a rather surprising result concerning smoothness of function $\sin_p(x)$ for even p.

Theorem 4.2.1 (see [28], Thm. 3.1, p. 105)**.** *Let $p = 2(m+1)$, $m \in \mathbb{N}$. Then*

$$\sin_{2(m+1)}(x) \in C^\infty\Big(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2}\Big)\,.$$

Conversely, for $p = 2m+1$, we have to distinguish two subcases (4.9) and (4.10), which has damaging effect on the differentiability of $\sin_p(x)$. Thus the smoothness is lost when p is odd. The smoothness is also lost if p is not an integer.

Theorem 4.2.2 (see [28], Thm. 3.2, p. 105)**.** *Let $p \in \mathbb{R} \setminus \{2m\}$, $m \in \mathbb{N}$, $p > 1$. Then*

$$\sin_p(x) \in C^{\lceil p \rceil}(-\pi_p/2, \pi_p/2)\,,$$

but

$$\sin_p(x) \notin C^{\lceil p \rceil + 1}(-\pi_p/2, \pi_p/2)\,.$$

Here $\lceil p \rceil := \min\{k \in \mathbb{N} : k \geq p\}$.

Our last result gives an explicit radius of convergence of the Maclaurin series for even $p > 2$. To the best of our knowledge, all previous results concerning convergence of series for $\sin_p(x)$ were only local; see, e.g., [50].

Theorem 4.2.3 (see [28], Thm. 3.3, p. 106)**.** *Let $p = 2(m+1)$ for $m \in \mathbb{N}$. Then the Maclaurin series of $\sin_{2(m+1)}(x)$ converges on $(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$.*

Theorem 4.2.4 (see [28], Thm. 3.4, p. 106)**.** *Let $p = 2m+1$, $m \in \mathbb{N}$. Then the formal Maclaurin series of $\sin_{2m+1}(x)$ converges on $(-\frac{\pi_{2m+1}}{2}, \frac{\pi_{2m+1}}{2})$. Moreover, the formal Maclaurin series of $\sin_p(x)$ converges towards $\sin_{2m+1}(x)$ on $[0, \frac{\pi_{2m+1}}{2})$, but does not converge towards $\sin_{2m+1}(x)$ on $(-\frac{\pi_{2m+1}}{2}, 0)$.*

The convergence of Maclaurin series for p even/odd is illustrated on Figures 5–8 in [28], p. 123–124.

4.3 My contribution to [28]

Due to the purpose of this thesis I would like to devote this paragraph to specifying of my contribution to [28]. Let me note that all my ideas were formalized and improved during discussion with my mentor P. Girg. My Bachelor Thesis [34], where I proved Lemma 4.3.2, was a starting point for this research. Lemma 4.3.2 yields following formula

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{q_{k,n}}(x) \cos_p^{1-q_{k,n}}(x) \tag{4.11}$$

on $\left(0, \frac{\pi_p}{2}\right)$. Here $\sin_p^{(n)}(x)$ denotes the n-th derivative of the function $\sin_p(x)$. Formula (4.11) is essential for the proofs of Theorems 4.2.1 and 4.2.2 which were the final results of my Bachelor Thesis [34]. During the work on [28] I proved Lemmas 4.3.3 and 4.3.4. Lemma 4.3.4 states that $\sin_p^{(n)} \le 0$ for $p \ge 3$ be an integer and $n \ge 2$. It was used by P. Girg in a proof of Theorems 4.2.3 and 4.2.4. Since the proof of Lemma 4.3.4 was very technical, it is not included in this thesis and the reader is invited to read it in Appendix A1.

In the sequel of the Section 4.3 the Lemmas, which was mentioned above, are stated for convenience of the reader as well as and some definitions from [28]. Following 'symbolic' operators (rewriting rules) are defined on expressions of the form

$$a \cdot \sin_p^q(x) \cdot \cos_p^{1-q}(x) \quad \text{ with } a, q \in \mathbb{R} \tag{4.12}$$

as follows

$$\mathrm{D}_s\, a \cdot \sin_p^q(x) \cdot \cos_p^{1-q}(x) \stackrel{\text{def}}{=} \begin{cases} a \cdot q \cdot \sin_p^{q-1}(x) \cdot \cos_p^{1-(q-1)}(x) & q \neq 0\,, \\ 0 & q = 0\,. \end{cases} \tag{4.13}$$

$$\mathrm{D}_c\, a \cdot \sin_p^q(x) \cdot \cos_p^{1-q}(x) \stackrel{\text{def}}{=} \begin{cases} -a \cdot (1-q) \cdot \sin_p^{q+p-1}(x) \cdot \cos_p^{1-(q+p-1)}(x) & q \neq 1\,, \\ 0 & q = 1\,. \end{cases} \tag{4.14}$$

Let us observe that the results of application D_s and D_c have the form (4.12). Hence they are also in the domain of definition of D_s and D_c. Thus we can consider compositions of D_c and D_s of arbitrary length. The first derivative of $\sin_p^q(x) \cdot \cos_p^{1-q}(x)$ (here $a = 1$) can be written using these symbolic operators as follows (see [28] for details)

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x} \sin_p^q(x) \cdot \cos_p^{1-q}(x) \\ &= \mathrm{D}_s \sin_p^q(x) \cdot \cos_p^{1-q}(x) + \mathrm{D}_c \sin_p^q(x) \cdot \cos_p^{1-q}(x)\,. \end{aligned}$$

In fact, there are three cases $q \in \mathbb{R} \setminus \{0,1\}$, $q = 1$, and $q = 0$.

Case $q \in \mathbb{R} \setminus \{0,1\}$**.** Here

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x} \sin_p^q(x) \cdot \cos_p^{1-q}(x) \\ &= \mathrm{D}_s \sin_p^q(x) \cdot \cos_p^{1-q}(x) + \mathrm{D}_c \sin_p^q(x) \cdot \cos_p^{1-q}(x)\,. \end{aligned}$$

Note that the distance between the exponents of $\sin_p(x)$ in the resulting terms, i.e., $\sin_p^{q_0-1}(x) \cdot \cos_p^{2-q_0}(x)$ and $\sin_p^{q_0+p-1} \cdot \cos_p^{2-p-q_0}(x)$, is exactly p. This is the fundamental fact of the proof of Lemma 4.3.2 below because in a sum of the type

$$c_0 \sin_p^{q_0}(x) \cdot \cos_p^{1-q_0}(x) + c_1 \sin_p^{q_0+p}(x) \cdot \cos_p^{1-(q_0+p)}(x)$$

the terms combine together as in the diagram depicted on Figure 4.1

Case $q = 1$**.** In this case the term $\sin_p^q(x) \cdot \cos_p^{1-q}(x) = \sin_p(x)$ and hence the derivative of this term is the *single* term $\cos_p(x) = \mathrm{D}_s \sin_p(x) + \mathrm{D}_c \sin_p(x)$. The fact $\mathrm{D}_c \sin_p(x) = 0$ (by Definition (4.14)) will be reflected in our diagrams by omitting the 'right-down' edge departing from this node, see Figure 4.2.

Case $q = 0$**.** This case corresponds to $\sin_p^q(x) \cdot \cos_p^{1-q}(x) = \cos_p(x)$. Thus the derivative of this term is the *single* term $-\sin_p^{p-1}(x) \cos_p^{1-(p-1)}(x) = \mathrm{D}_s \cos_p(x) + \mathrm{D}_c \cos_p(x)$ by the Definitions (4.13) and (4.14) of D_s and D_c, respectively. The fact $\mathrm{D}_s \cos_p(x) = 0$ will be reflected in our diagrams by omitting 'left-down' edge departing from this node, see Figure 4.3. Note that since in our diagrams we write powers only, the node corresponding to $-\sin_p^{p-1}(x) \cos_p^{1-(p-1)}(x)$ is labeled by $s_p^{p-1}\, c_p^{1-(p-1)}$.

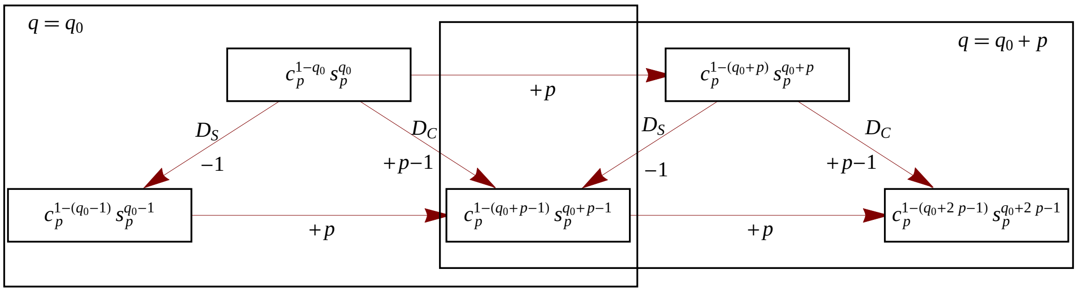

Figure 4.1: Rewriting diagram of the first derivative of $c_0 \sin_p^{q_0}(x) \cdot \cos_p^{1-q_0}(x) + c_1 \sin_p^{q_0+p}(x) \cdot \cos_p^{1-(q_0+p)}(x)$. For the lack of space, we *do not* write the coefficients standing in front of these terms and *use abbreviations*, i.e., we write s_p^q instead of $\sin_p^q(x)$ and c_p^{1-q} instead of $\cos_p^{1-q}(x)$.

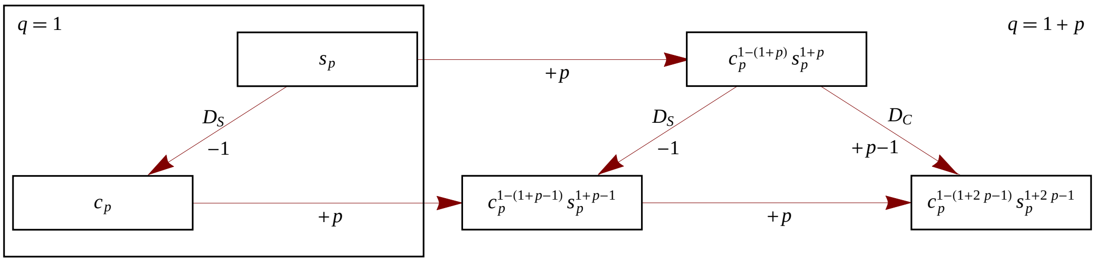

Figure 4.2: Rewriting diagram of the case $q = 1$. Recall that we write s_p^q instead of $\sin_p^q(x)$ and c_p^{1-q} instead of $\cos_p^{1-q}(x)$ and do not write the coefficients.

The higher order derivatives are obtained in the same way, thus, e.g., the second derivative

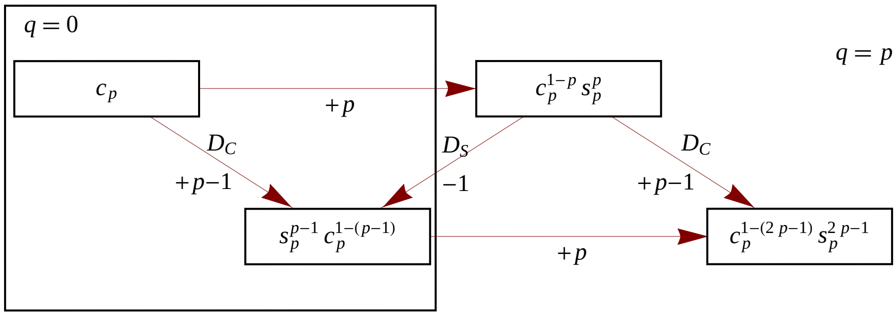

Figure 4.3: Rewriting diagram of the case $q = 0$. Recall that we write s_p^q instead of $\sin_p^q(x)$ and c_p^{1-q} instead of $\cos_p^{1-q}(x)$ and do not write the coefficients.

of $\sin_p^q(x) \cdot \cos_p^{1-q}(x)$ (here $a = 1$) can be written as

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}x^2} \sin_p^q(x) \cdot \cos_p^{1-q}(x) \\
&= (\mathrm{D}_s \circ \mathrm{D}_s) \sin_p^q(x) \cdot \cos_p^{1-q}(x) + (\mathrm{D}_c \circ \mathrm{D}_s) \sin_p^q(x) \cdot \cos_p^{1-q}(x) \\
&\quad + (\mathrm{D}_s \circ \mathrm{D}_c) \sin_p^q(x) \cdot \cos_p^{1-q}(x) + (\mathrm{D}_c \circ \mathrm{D}_c) \sin_p^q(x) \cdot \cos_p^{1-q}(x).
\end{aligned}$$

Let us recall the sum (4.11). The k-th term of this sum for n-th derivative can be derived using composition of the symbolic operators D_s and D_c, which acts on the $\sin_p''(x)$. Before we introduce the composition of the operators D_s and D_c, let us recall some notation from formal languages.

Definition 4.3.1. (Salomaa-Soittola [52, I.2, p. 4,], and/or Manna [42, p. 2–3, p. 47, p. 78]) An *alphabet* (denoted by V) is a finite nonempty set of letters. A *word* (denoted by w) over an alphabet V is a finite string of zero or more letters from the alphabet V. The word consisting of zero letters is called the *empty word*. The set of all words over an alphabet V is denoted by V^* and the set of all nonempty words over an alphabet V is denoted by V^+. For strings w_1 and w_2 over V, their juxtaposition w_1w_2 is called *catenation* of w_1 and w_2, in operator notation $\mathrm{cat} : V^* \times V^* \to V^*$ and $\mathrm{cat}(w_1, w_2) = w_1w_2$. We also define the length of the word w, in operator notation $\mathrm{len} : V^* \to \{0\} \cup \mathbb{N}$, which for a given word w yields the number of letters in w when each letter is counted as many times as it occurs in w. We also use the *reverse function* $\mathrm{rev} : V^* \to V^*$ which reverses the order of the letters in any word w (see [42, p. 47, p. 78]).

For our purposes here, we consider the alphabet $V = \{0, 1\}$ and the set of all nonempty words V^+. Thus words in V^+ are, e.g.,

$$\text{“0”}, \text{“1”}, \text{“01”}, \text{“10”}, \text{“11”} \ldots .$$

For instance, $\mathrm{cat}(\text{“1110”}, \text{“011”}) = \text{“1110011”}$, and

$$\begin{aligned}
\mathrm{rev}(\text{“010011000”}) &= \text{“000110010”}\,, \\
\mathrm{len}(\text{“010011000”}) &= 9\,.
\end{aligned}$$

Let $n \in \mathbb{N}$, $k \in \{0\} \cup \mathbb{N}, 0 \leq k \leq 2^{n-2} - 1$ and $(k)_{2,n-2}$ be the string of bits of the length $n-2$ which represents a binary expansion of k (it means, e.g., for $k = 3$ and $n = 5$, $(3)_{2,5-2} =$ "`011`"). Now we are ready to define $D_{k,n}$ in two steps as follows.

Step 1 We create an ordered $n-2$-tuple $d_{k,n-2} \in \{\mathrm{D}_s, \mathrm{D}_c\}^{n-2}$ (cartesian product of sets $\{\mathrm{D}_s, \mathrm{D}_c\}$ of length $n-2$) from $\mathrm{rev}((k)_{2,n-2})$ such that for $1 \leq i \leq n-2$, $d_{k,n-2}$ contains D_s on the i-th position if $\mathrm{rev}((k)_{2,n-2})$ contains "`0`" on the i-th position, and $d_{k,n}$ contains D_c on the i-th position if $\mathrm{rev}((k)_{2,n-2})$ contains "`1`" on the i-th position (it means, e.g., for $k = 3$, and $n = 5$, we obtain $d_{3,5-2} = (\mathrm{D}_c, \mathrm{D}_c, \mathrm{D}_s)$).

Step 2 We define $D_{k,n}$ as the composition of operators $\mathrm{D}_s, \mathrm{D}_c$ in the order they appear in the ordered n-tuple $d_{k,n-2}$ (it means, e.g., for $k = 3$, and $n = 5$, we obtain $D_{3,5} = (\mathrm{D}_c \circ \mathrm{D}_c \circ \mathrm{D}_s)$).

With this notation in the hand, we can state chain of Lemmas 4.3.2 – 4.3.4.

Lemma 4.3.2 (see [28], Lemma 4.5, p. 110)**.** *Let $p \in \mathbb{R}$, $p > 1$, $n \in \mathbb{N}$. Then $\sin_p^{(n)}(x)$ exists on $(0, \pi_p/2)$ and it is continuous. Moreover,*

$$\text{for } n = 1: \quad \sin_p'(x) = \cos_p(x)\,, \tag{4.15}$$

$$\text{for } n = 2: \quad \sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)\,, \tag{4.16}$$

and for $n = 3, 4, 5, \ldots$, $k = 0, 1, 2, 3, \ldots, 2^{n-2} - 1$ there exists $a_{k,n} \in \mathbb{R}$, $l_{k,n}, m_{k,n} \in \mathbb{Z}$ such that

$$\mathrm{D}_{k,n} \sin_p''(x) = a_{k,n} \cdot \sin_p^{p \cdot l_{k,n} + m_{k,n}}(x) \cdot \cos_p^{1 - p \cdot l_{k,n} - m_{k,n}}(x)\,, \tag{4.17}$$

and

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_p^{p \cdot l_{k,n} + m_{k,n}}(x) \cdot \cos_p^{1 - p \cdot l_{k,n} - m_{k,n}}(x)\,. \tag{4.18}$$

Moreover, let $j(k) \in \{0\} \cup \mathbb{N}$ be the digit sum of the binary expansion of $k = 0, 1, 2, \ldots, 2^{n-2} - 1$ (thus $j(k)$ is the number of occurrences of D_c in $\mathrm{D}_{k,n}$) and let $\mathrm{D}_{k,n} \sin_p''(x) \not\equiv 0$. Then, for $k = 0, 1, 2, \ldots, 2^{n-2} - 1$, the exponents

$$q_{k,n} := p \cdot l_{k,n} + m_{k,n} \tag{4.19}$$

satisfy

$$q_{k,n} = j(k)(p-1) + (n - 2 - j(k))(-1) + p - 1\,. \tag{4.20}$$

Lemma 4.3.3 (see [28], Lemma 4.6, p. 113)**.** *Let $p \in \mathbb{N}$, $p > 1$, and for all $n \in \mathbb{N}$, $n \geq 2$*

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{q_{k,n}}(x) \cdot \cos_p^{1 - q_{k,n}}(x)\,. \tag{4.21}$$

Then for all $n \in \mathbb{N}$, $n \geq 2$, and all $k \in \{0\} \cup \mathbb{N}$, $k \leq 2^{n-2} - 1$

$$q_{k,n} \in \{0\} \cup \mathbb{N}\,. \tag{4.22}$$

Lemma 4.3.4 (see [28], Lemma 4.7, p. 114)**.** *Let $p \in \mathbb{N}$, $p \geq 3$. Then for all $n \in \mathbb{N}$, $n \geq 2$*

$$\sin_p^{(n)}(x) \leq 0 \quad \text{on} \quad (0, \frac{\pi_p}{2})\,.$$

Chapter 5

Generalization of $\sin_p$ in complex domain

5.1 Introduction

This chapter is a short summary of results obtained jointly with my mentor P. GIRG in a paper "Generalized trigonometric functions in complex domain" accepted in Mathematica Bohemica, special issue dedicated to Equadiff 13, see [29]. Please find this paper included in Appendix A2. Since the paper was written in cooperation, I briefly comment my contribution to the paper in Section 5.3.

The paper [29] extends the results of [28] to complex domain. The research on [29] was stimulated by an interesting question of O. DOŠÝ, which was posed during my talk at an international conference "Nonlinear Analysis Plzeň 2013". Recall the most surprising result of [28], i.e. $\sin_p(x)$ is a real analytic function on $(-\pi_p/2, \pi_p/2)$ for $p = 4, 6, 8 \ldots$. In other words, $\sin_p(x)$ equals to its Maclaurin on $(-\pi_p/2, \pi_p/2)$ for $p = 4, 6, 8 \ldots$. This approach naturally allows to extend $\sin_p(x)$ for $p = 4, 6, 8 \ldots$ to an open disk

$$\{z \in \mathbb{C} : |z| < \pi_p/2\}$$

in the complex domain using power series (cf. [38], where the convergence of the series is conjectured without proof). O. DOŠLÝ in his question inquired whether this extension satisfies (4.3) in the sense of differential equations in complex domain. The paper [29] addresses his question. For p being an even integer the initial value problem (4.3) in $\mathbb{R}$ is equivalent to

$$\begin{cases} -(u')^{p-2}\,u'' - u^{p-1} &= 0\,, \\ u(0) &= 0\,, \\ u'(0) &= 1\,. \end{cases} \tag{5.1}$$

Note that for $p > 1$ real not being an even positive integer, we cannot get rid off the absolute values in (4.3). Thus the equation (4.3) does not make sense for general $p > 1$ in the complex domain. In this paper we consider the (5.1) in complex domain for integer $p > 2$. The complex valued ordinary differential equations are studied by means of power series (mostly by Maclaurin series). Note that, by Theorem 4.2.2 (i.e. [28, Theorem 3.2 on p. 5]), $\sin_p^{(n)}(0)$ exists for $1 < n \leq p$, but $\sin_p^{(n)}(0)$ *does not exist when* $p \geq 3$ is odd integer and $n > p$. Thus, by the formal Maclaurin series of $\sin_p(x)$, we mean a series calculated from the limits of the derivatives $\lim_{x\to 0+} \sin_p^{(n)}(x)$, which were shown to exist in [28] for any $n \in \mathbb{N}$ and $p \geq 3$ odd integer.

In Chapter 5, the independent variable z stands for a complex number and the independent variable x stands for a real variable. In the same spirit, $\sin_p(z)$ stands for a complex valued function and $\sin_p(x)$ stands for a function of one real variable.

5.2 Main results of [29]

In this section we summarize all main results of the paper [29] regardless of my contribution, which will be specified in Section 5.3.

Let $M_{\sin_p}(x)$ denotes formal Maclaurin series of $\sin_p(x)$ for $p \geq 3$ being an integer. It is proved in [28] that

$$M_{\sin_p}(x) = \sum_{k=0}^{+\infty} \alpha_k x^{kp+1}\,, \tag{5.2}$$

where $\alpha_0 > 0$ and $\alpha_k \leq 0$ (all other coefficients are zero). The first result of [29] answers the question by O. Došlý affirmatively.

Theorem 5.2.1 (see [29], Theorem 2.1, p. 4)**.** *Let $p = 4, 6, 8, \dots$, then the unique solution of the initial value problem (5.1) on $|z| < \pi_p/2$ is the Maclaurin series (5.2).*

Lindqvist [38] suggested alternative definition of $\sin_p(z)$ in complex domain as the solution of the equation equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(w'\right)^{p-1} + (p-1)w^{p-1} = 0\,, \quad w(0) = 0\,, \quad w'(0) = 1\,. \tag{5.3}$$

This definition works formally for $p > 1$. Lindquist also warned that $\sin_p(z)$ defined in this way in complex domain may be different from $\sin_p(x)$ on $\mathbb{R}$ (see [29, Section 3, p. 6] for more details). The following Theorem 5.2.2 confirms the legitimacy of the warning.

Theorem 5.2.2 (see [29], Theorem 3.1, p. 7)**.** *Let $p = 3, 5, 7 \dots$. Then the unique solution $u(z)$ of the complex initial value problem* (5.1) *differs from the solution* $\sin_p(x)$ *of the Cauchy problem* (4.3) *for $z = x \in (-\pi_p/2, 0)$.*

The next result describes an interesting relationship between real and imaginary part of $\sin_p(z)$ for $p = 4,\ 8,\ 12,\ \dots$.

Theorem 5.2.3 (see [29], Theorem 4.1, p. 7)**.** *Let $p = 4, 8, 12, \dots$. Then*

$$\Re[\sin_p(z)] = \Im[\sin_p(\mathrm{i} \cdot z)]$$

for all $z \in \mathbb{C}$: $|z| < \pi_p/2$.

However, there is no such relationship for $p = 2, 6, 10, 14 \dots$. Note, that this case includes the classical sine function.

Theorem 5.2.4 (see [29], Theorem 4.2, p. 8)**.** *Let $p = 2, 6, 10, 14 \dots$. Then for all $\varphi \in [0, 2\pi)$ there exists $z \in \mathbb{C}$: $|z| < \pi_p/2$ such that*

$$\Re[\sin_p(z)] \neq \Im[\sin_p(\mathrm{e}^{\mathrm{i}\varphi} \cdot z)]\,.$$

5.3 My contribution to [29]

This section is devoted to specifying of my contribution to [29]. This paper was mainly created during numerous and intensive discussions among the both authors. Hence it is very difficult to clearly separate my contribution. As was mentioned earlier, the paper [29] was motivated by the question of O. Došlý (see Section 5.1) which was answered in Theorem 5.2.1. The proof of Theorem 5.2.1 followed my ideas however the final version of the proof was joint work. The proof contained the following auxiliary Lemma 5.3.1.

Lemma 5.3.1 (see [29], Lemma 2.1, p. 5)**.** *There is* $\delta > 0$ *such that in* $U_0 \stackrel{\text{def}}{=} \{z \in \mathbb{C}\colon |z| < \delta\}$ *the initial value problem* (5.1) *has the unique solution* $u(z)$ *which is an analytic function in* U_0*.*

The result of Section 3 (Theorem 5.2.2) followed the same ideas as the proof of Theorem 5.2.1 and this theorem was proved during the discussion. Theorem 5.2.3 and Theorem 5.2.4 were based on the observations of P. Girg (see Figure 1 in [29], p. 12–13) and the proofs were created together. Conversely, Sections 5 and 6 were worked by P. Girg. Section 5 contains an interesting link between p-trigonometric identity (4.4) and complex analysis. Section 6 is devoted to the visualization of $\sin_p(z)$.

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Appendix A1

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DIFFERENTIABILITY PROPERTIES OF p-TRIGONOMETRIC FUNCTIONS

PETR GIRG, LUKÁŠ KOTRLA

Abstract. p-trigonometric functions are generalizations of the trigonometric functions. They appear in context of nonlinear differential equations and also in analytical geometry of the p-circle in the plain. The most important p-trigonometric function is $\sin_p(x)$. For $p > 1$, this function is defined as the unique solution of the initial-value problem

$$(|u'(x)|^{p-2}u'(x))' = (p-1)|u(x)|^{p-2}u(x), \quad u(0) = 0,\ u'(0) = 1\,,$$

for any $x \in \mathbb{R}$. We prove that the n-th derivative of $\sin_p(x)$ can be expressed in the form

$$\sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{q_{k,n}}(x) \cos_p^{1-q_{k,n}}(x)\,,$$

on $(0, \pi_p/2)$, where $\pi_p = \int_0^1 (1-s^p)^{-1/p} \mathrm{d}s$, and $\cos_p(x) = \sin_p'(x)$. Using this formula, we proved the order of differentiability of the function $\sin_p(x)$. The most surprising (least expected) result is that $\sin_p(x) \in C^\infty(-\pi_p/2, \pi_p/2)$ if p is an even integer. This result was essentially used in the proof of theorem, which says that the Maclaurin series of $\sin_p(x)$ converges on $(-\pi_p/2, \pi_p/2)$ if p is an even integer. This completes previous results that were known e.g. by Lindqvist and Peetre where this convergence was conjectured.

1. Introduction

In the previous two decades, p-trigonometric functions have attracted attention of many researchers; see, e.g., [1, 5, 6, 7, 10, 11, 12, 13, 15, 16, 25], and references therein. The p-trigonometric functions arise from the study of the eigenvalue problem for the one-dimensional p-Laplacian. We assume $p > 1$ and say, that $\lambda \in \mathbb{R}$ is an eigenvalue of

$$\begin{gathered} -(|u'|^{p-2}u')' - \lambda |u|^{p-2}u = 0 \quad \text{in } (0, \pi_p)\,, \\ u(0) = u(\pi_p) = 0\,, \end{gathered} \tag{1.1}$$

if there is a nonzero function $u \in W^{1,p}(0, \pi_p)$ that satisfy (1.1) in a weak sense. Here

$$\pi_p = 2\int_0^1 \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s = \frac{2\pi}{p\sin(\pi/p)}\,. \tag{1.2}$$

2000 *Mathematics Subject Classification.* 34L10, 33E30, 33F05.
Key words and phrases. p-Laplacian; p-trigonometry; analytic functions; approximation.

Published February 10, 2014.

Let us note, that the problem can be considered on any bounded open interval, but the choice $(0, \pi_p)$ significantly simplifies the calculations. The discreetness of the spectrum of this eigenvalue problem was established already by Nečas [21]. This eigenvalue problem was later studied by means of the initial-value problem

$$\begin{gathered} -(|u'|^{p-2}u')' - \lambda|u|^{p-2}u = 0 \quad \text{in } (0,\infty)\,, \\ u(0) = 0, \quad u'(0) = 1\,; \end{gathered} \tag{1.3}$$

see Elbert [11] for initial work in this direction. Later it was independently studied by del Pino-Elgueta-Manasevich [8], Ôtani [22] and Lindqvist [14].

Let $\sin_p(x)$ denote the solution of (1.3) with $\lambda = (p-1)$. It follows from [11] that $\sin_p(x)$ is positive on $(0, \pi_p)$ and satisfies an identity

$$|\sin_p(x)|^p + |\sin_p'(x)|^p = 1 \quad \forall x \in \mathbb{R}\,, \tag{1.4}$$

which for $p = 2$ becomes the familiar identity for sine and cosine. This suggest the definition $\cos_p(x) := \sin_p'(x)$ and justifies the notation $\sin_p(x)$ and $\cos_p(x)$. The identity (1.4) is called p-trigonometric identity. It also follows from [11] that the eigenvalues of (1.3) form a sequence $\lambda_k = k^p(p-1), k \in \mathbb{N}$ and corresponding eigenfunctions are functions $\sin_p(kx), k \in \mathbb{N}$. Thus all the eigenfunctions are determined by the function $\sin_p(x)$. It comes as no surprise that the properties of the function $\sin_p(x)$ were studied extensively in the previous 30 years. It was shown in [11] that $\sin_p(x)$ can be expressed on $[0, \pi_p/2]$ (the p-trigonometric identity (1.4) can be thought of as the first integral of (1.3)) as the inverse of

$$\operatorname{arcsin}_p(x) = \int_0^x \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s\,, \quad x \in [0,1]\,, \tag{1.5}$$

which is extended to $[0, \pi_p]$ by reflection $\sin_p(x) = \sin_p(\pi_p - x)$ and to $[-\pi_p, \pi_p]$ as the odd function. Finally, it is extended to $\mathbb{R}$ as the $2\pi_p$-periodic function. The function $\operatorname{arcsin}_p(x)$ from (1.5) is extended to $[-1, 1]$ as an odd function. Then

$$\sin_p(\operatorname{arcsin}_p(x)) = x \quad \forall x \in [-1,1]\,. \tag{1.6}$$

Note that for $p = 2$, we obtain classical arcsine and sine from this definition. The (now familiar) notation $\sin_p$ appears in [8] for the first time, where the authors studied homotopic deformation along p to calculate the degree of trivial solutions of (1.1) in order to establish existence results for the nonlinear problem $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$, $T > 0$. The homotopy result from [8] initiated development of bifurcation theory for quasilinear bifurcations.

As a historical remark, let us mention that generalizations of arcsine similar to (1.5) were studied in a very different context by Lundberg [17] in 1879. It is interesting to mention that the p-trigonometric functions satisfy certain relations to geometrical objects such as arclength and area of a circle in a noneuclidean metric; see Elbert [11], and Lindqvist [15]. The p-trigonometric functions also possesses some approximation properties in certain function spaces; see, e.g., Binding-Boulton-Čepička-Drábek-Girg [1], Lang-Edmunds [13] for theoretical research, and Boulton-Lord [6] for a very interesting computational application in evolutionary PDEs. In Wood [27], the particular case $p = 4$ was studied and "p-polar" coordinates in the xy-plane were proposed.

In this article we focus on the differentiability and analyticity properties of p-trigonometric functions. One can immediately see from (1.2), (1.5), and (1.6) that $\sin_p(0) = 0$ and $\sin_p(\pi_p/2) = 1$ for all $p > 1$. From (1.4) and the definition of

$\cos_p(x)$, we obtain $\cos_p(0) = 1$ and $\cos_p(\pi_p/2) = 0$. It follows from the results in [11, 15, 22] that the possible differentiability issues are located at $x = 0$ and $x = \pi_p/2$. There are several results concerning differentiability and asymptotic behaviour of $\sin_p(x)$ at $x = 0$ and $x = \pi_p/2$ in Manásevich-Takáč [19] and Benedikt-Girg-Takáč [2]. In Peetre [25], generalized formal Maclaurin series for $\sin_p(x)$ were studied and their convergence was conjectured on $(-\pi_p/2, \pi_p/2)$. The local convergence of the generalized Taylor series (and/or the generalized Maclaurin series) for $\sin_p(x)$ follows from Paredes-Uchiyama [24]. Taking into account that the point $x = 0$ is often considered as the center for the Taylor (i.e. the Maclaurin) series or the generalized Taylor (i.e. the generalized Maclaurin) series for $\sin_p(x)$, we decided to provide detailed study of the convergence of these series towards $\sin_p(x)$ on $(-\pi_p/2, \pi_p/2)$. We were also motivated by work of Ôtani [23], where he studies properties of the solutions of

$$\begin{gathered} (|u'|^{p-2}u')' + |u|^{q-2}u = 0 \quad \text{in } (a,b)\,, \\ u(a) = u(b) = 0\,, \end{gathered} \tag{1.7}$$

for general exponents $p, q \in (1, +\infty)$ with $p \neq q$. Among other properties he proved that for $p = \frac{2m+2}{2m+1}, m \in \{0\} \cup \mathbb{N}$ and for q even, any solution of (1.7) belongs to $C^\infty(a, b)$. In our case, $p = q$ we find that $\sin_p(x)$ belongs to $C^\infty(-\pi_p/2, \pi_p/2)$ if and only if p is even. Let us also remark that local analytic solutions of the radial variant of (1.7) were studied in Bognár [4].

Though we are aware that our methods are elementary mathematics, we are sure that our results will help to better understand the behavior of $\sin_p(x)$ and its derivatives in the vicinity of 0. This behavior is crucial in establishing asymptotic estimates such as those in the proof of the Fredholm alternative for the p-Laplacian in the degenerate case Benedikt-Girg-Takáč [2, 3]. Moreover, knowledge of the convergence/nonconvergence of the Taylor and/or the Maclaurin series is very important in the development of numerical methods for calculating approximations of function values of p-trigonometric functions. Recently, Marichev [20] from the Wolfram Research, Inc., pointed out to the first author of this paper in a personal communication that Mathematica from version 8.0 has a capability to effectively compute coefficients for $\sin_p(x)$ for formal generalized Maclaurin power series by means of the Bell Polynomials. With few lines of Mathematica code one can obtain partial sums of generalized Maclaurin series for $\sin_p(x)$ of large order in a couple of minutes. Thus the question of the convergence of the partial sums of the Maclaurin series is becoming quite urgent. This was our main motivation to address this topic.

Our main result provides convergence of these partial sums. We treat two cases separately, $p > 2$ is an even integer and $p > 2$ is an odd integer. Namely, for the particular case $\sin_{2(m+1)}(x)$, $m \in \mathbb{N}$, $x \in (-\pi_p/2, \pi_p/2)$, we show that the Maclaurin series converges towards the values $\sin_{2(m+1)}(x)$ on the interval $(-\pi_p/2, \pi_p/2)$. On the other hand, we show that the Maclaurin series converge towards $\sin_{2m+1}(x)$, $m \in \mathbb{N}$, for $x \in (0, \pi_p/2)$ and does not for $x \in (-\pi_p/2, 0)$. More precisely, the Maclaurin series converges on $x \in (-\pi_p/2, \pi_p/2)$, but not towards values of $\sin_{2m+1}(x)$, $m \in \mathbb{N}$ for $x \in (-\pi_p/2, 0)$.

The article is organized as follows. In Section 2, we give a definition of the function $\sin_p(x)$ by means of a differential equation and also introduce other useful notation. In Section 3, we state and discuss our main results concerning differentiability and/or non-differentiability of $\sin_p(x)$ and convergence of Maclaurin series of

$\sin_p(x)$. In Section 4, we express higher derivatives of $\sin_p(x)$ by means of powers of $\sin_p(x)$ and $\cos_p(x)$. Finally, in Section 5, we prove our main results using formulas for higher derivatives of $\sin_p(x)$ from Section 4. In Section 6, we conclude with remarks and open problems.

2. Definitions of p-trigonometric functions

Proposition 2.1. *The initial-value problem*

$$\begin{gathered} -(|u'|^{p-2}u')' - (p-1)|u|^{p-2}u = 0 \\ u(0) = 0, \quad u'(0) = 1\,, \end{gathered} \tag{2.1}$$

has the unique local solution and moreover any local solution to (2.1) *can be continued to* $(-\infty, +\infty)$.

For uniqueness of the solution see [8, Sect. 3], and for the existence of global solutions see [9, Lemma A.1].

Definition 2.2. The function $\sin_p(x)$ is defined as the unique solution of the initial-value problem (2.1) on $\mathbb{R}$.

For any $q > 1$ and $z \in \mathbb{R}$ we define

$$\varphi_q(z) = \begin{cases} |z|^{q-2}z & \text{for} z \neq 0\,, \\ 0 & \text{for } z = 0\,. \end{cases} \tag{2.2}$$

Note that $\varphi_{p'}(\varphi_p(z)) = \varphi_p(\varphi_{p'}(z)) = z$ provided $p > 1$ and $1/p + 1/p' = 1$. With this notation, we can rewrite the initial-value problem (2.1) as an equivalent first-order system

$$\begin{gathered} u'(x) = \varphi_{p'}(v(x))\,, \\ v'(x) = -(p-1)\varphi_p(u(x))\,, \\ u(0) = 0, \quad v(0) = 1\,. \end{gathered} \tag{2.3}$$

Clearly, from the definition of Carathéodory solution, it follows that $u(x) = \sin_p(x)$ and $v(x) = \varphi_p(\sin_p'(x))$ must be absolutely continuous on any compact interval $[-K, K]$, $K > 0$. Thus $\sin_p'(x) = \varphi_{p'}(v(x))$ is continuous on any $[-K, K]$, $K > 0$, which entails that $\sin_p'(x) = \varphi_{p'}(v(x))$ is continuous on $(-\infty, +\infty)$. Thus the following definition makes sense.

Definition 2.3. For $x \in \mathbb{R}$, we define $\cos_p(x) = \sin_p'(x)$.

Since $\cos_p(0) = \sin_p'(0) = 1$ and $\cos_p(x)$ is continuous, there exists an interval $(-c, c)$ such that $\cos_p(x) > 0$ on $(-c, c)$, $c > 0$. Moreover, since $\sin_p'(0) = 1$ and $\sin_p \in C^1(\mathbb{R})$, there exists an interval $[0, s)$, $s > 0$, such that $\sin_p(x) \geq 0$ on $[0, s)$.

Definition 2.4. For $p > 1$, let π_p denote

$$2 \sup\{s > 0 : \forall x \in (0, s) \text{ holds } \sin_p(x) > 0 \wedge \cos_p(x) > 0\}\,.$$

It was shown in [11], that

$$\pi_p = 2\int_0^1 \frac{1}{(1 - x^p)^{1/p}}\,\mathrm{d}x = \frac{2\pi}{p \cdot \sin(\pi/p)}\,,$$

for $p > 1$. It was also shown in [11], that $\sin_p(x)$ can be expressed on $[0, \pi_p/2]$ as the inverse of

$$\operatorname{arcsin}_p(x) = \int_0^x \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s \quad x \in [0,1]\,, \tag{2.4}$$

and, moreover, it extends to $[0, \pi_p]$ by reflection $\sin_p(x) = \sin_p(\pi_p - x)$ and to $[-\pi_p, \pi_p]$ as the odd function. Finally, it extends to $\mathbb{R}$ as the $2\pi_p$-periodic function.

Remark 2.5. In the following text, formulas containing higher order derivatives and powers of $\sin_p(x)$ and $\cos_p(x)$ appear. We try to keep our notation as close as possible to the usual notation for classical trigonometric functions. Thus the derivatives are denoted by, e.g., $\sin_p'(x), \dots, \sin_p'''(x), \sin_p^{(\mathrm{iv})}(x)$ (primes and roman numerals) and/or, e.g., $\sin_p^{(n)}(x)$, $\sin_p^{(2n-1)}$ and $\sin_p^{(2n)}$ for $n \in \mathbb{N}$. On the other hand, the powers are denoted by $\sin_p^2(x), \sin_p^3(x), \sin_p^q(x)$, $q \in \mathbb{R}$. Where a confusion may happen, we denote the powers by, e.g., $(\sin_p(x))^m$, $m \in \mathbb{N}$, to distinguish them clearly from derivatives. For the convenience of the reader, we write the values of p as explicit as possible, with a few exceptions such as in the proofs of Theorems 3.3 and 3.4, where this approach would produce very lengthy formulas.

3. Main results

In the sequel, we study derivatives of $\sin_p(x)$ for $p \in \mathbb{N}$, $p > 2$ on the interval $x \in (-\pi_p/2, \pi_p/2)$. We distinguish two cases p is even, i.e., $p = 2(m+1)$ and $m \in \mathbb{N}$, and p is odd; i.e., $p = 2m+1$ and $m \in \mathbb{N}$. In the first case $p = 2(m+1)$, the p-trigonometric identity (1.4) takes form

$$(\sin_{2(m+1)}(x))^{2(m+1)} + (\cos_{2(m+1)(x)})^{2(m+1)} = 1\,, \tag{3.1}$$

which is valid for any $x \in \mathbb{R}$ and hence on $(-\pi_p/2, \pi_p/2)$. Note that there is no absolute value, since there are even powers.

In the second case $p = 2k+1$, we have to distinguish two subcases. For $0 < x < \frac{\pi_p}{2}$, the p-trigonometric identity takes form

$$(\sin_{2m+1}(x))^{2m+1} + (\cos_{2m+1}(x))^{2m+1} = 1\,. \tag{3.2}$$

On the other hand, for $-\pi_p/2 < x < 0$, the p-trigonometric identity takes form

$$-(\sin_{2m+1}(x))^{2m+1} + (\cos_{2m+1}(x))^{2m+1} = 1\,. \tag{3.3}$$

Since there is only one identity (3.1) for $p = 2(m+1)$, this case has nice smoothness properties on $(-\pi_p/2, \pi_p/2)$ and we obtain a rather surprising result concerning smoothness of function $\sin_p(x)$ for even p.

Theorem 3.1. *Let $p = 2(m+1)$, $m \in \mathbb{N}$. Then*

$$\sin_{2(m+1)}(x) \in C^\infty\Big(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2}\Big)\,.$$

On the other hand, for $p = 2m+1$, we have to distinguish two subcases (3.2) and (3.3), which has damaging effect on the differentiability of $\sin_p(x)$. Thus the smoothness is lost when p is odd. The smoothness is also lost if p is not an integer.

Theorem 3.2. *Let $p \in \mathbb{R} \setminus \{2m\}$, $m \in \mathbb{N}$, $p > 1$. Then*

$$\sin_p(x) \in C^{\lceil p \rceil}(-\pi_p/2, \pi_p/2)\,,$$

but

$$\sin_p(x) \notin C^{\lceil p \rceil + 1}(-\pi_p/2, \pi_p/2)\,.$$

Here $\lceil p \rceil := \min\{k \in \mathbb{N} : k \geq p\}$.

Our last result gives an explicit radius of convergence of the Maclaurin series for even $p > 2$. To the best of our knowledge, all previous results concerning convergence of series for $\sin_p(x)$ were only local; see, e.g., [24].

Theorem 3.3. *Let $p = 2(m+1)$ for $m \in \mathbb{N}$. Then the Maclaurin series of $\sin_{2(m+1)}(x)$ converges on $(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$.*

Theorem 3.4. *Let $p = 2m+1$, $m \in \mathbb{N}$. Then the formal Maclaurin series of $\sin_{2m+1}(x)$ converges on $(-\frac{\pi_{2m+1}}{2}, \frac{\pi_{2m+1}}{2})$. Moreover, the formal Maclaurin series of $\sin_p(x)$ converges towards $\sin_{2m+1}(x)$ on $[0, \frac{\pi_{2m+1}}{2})$, but does not converge towards $\sin_{2m+1}(x)$ on $(-\frac{\pi_{2m+1}}{2}, 0)$.*

The proofs of Theorems 3.1–3.4 are postponed to Section 5.

4. Derivatives of $\sin_p(x)$

The following lemma summarizes basic properties of $\sin_p(x)$ and $\cos_p(x)$.

Lemma 4.1. *Let $p \in \mathbb{R}, p > 1$. Functions $\sin_p(x)$ and $\cos_p(x)$ have the following basic properties.*

1. *$\sin_p(x) > 0$ on $(0, \pi_p)$, $\sin_p(0) = 0$, $\sin_p(x) = \sin_p(\pi_p - x)$ for $x \in (\frac{\pi_p}{2}, \pi_p)$, and $\sin_p(x) = -\sin_p(-x)$ on $(-\pi_p, 0)$. The function $\sin_p(x)$ extends to $\mathbb{R}$ as $2\pi_p$-periodic function.*
2. *$\sin_p(x)$ is strictly increasing on $(-\pi_p/2, \pi_p/2)$.*
3. *$\cos_p(x) > 0$ on $(-\pi_p/2, \pi_p/2)$, $\cos_p(-\frac{\pi_p}{2}) = \cos_p(\frac{\pi_p}{2}) = 0$ and $\cos_p(x) < 0$ on $\left[-\pi_p, -\frac{\pi_p}{2}\right) \cup \left(\frac{\pi_p}{2}, \pi_p\right]$.*
4. *For all $n \in \mathbb{N}$, if $\sin_p^{(2n-1)}(x)$ exists on $(-\pi_p/2, \pi_p/2)$, then it is even function on $(-\pi_p/2, \pi_p/2)$.*
5. *For all $n \in \mathbb{N}$, if $\sin_p^{(2n)}(x)$ exists on $(-\pi_p/2, \pi_p/2)$, then it is odd function on $(-\pi_p/2, \pi_p/2)$.*

Statements 1–3 follows from [11]. Statements 4, and 5 are trivial consequence of statement 1.

Lemma 4.2. *For all $p \in \mathbb{R}, p > 1$*

$$\sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) \quad \text{for } x \in (0, \pi_p/2)\,, \tag{4.1}$$

$$\sin_p''(x) = \sin_p^{p-1}(-x) \cdot \cos_p^{2-p}(x) \quad \text{for } x \in (-\pi_p/2, 0)\,. \tag{4.2}$$

Proof. The identity (4.1) is obtained by a straightforward calculation; see, e.g., [13]. For $x \in (-\pi_p/2, 0)$, we obtain from Lemma 4.1 statement 1 and 3 and the identity (1.4)

$$\sin_p^p(-x) + \cos_p^p(x) = |-\sin_p(-x)|^p + |\cos_p(x)|^p = |\sin_p(x)|^p + |\cos_p(x)|^p = 1. \tag{4.3}$$

Taking

$$\sin_p^p(-x) + \cos_p^p(x) = 1 \tag{4.4}$$

into derivative we obtain

$$-p \cdot \sin_p^{p-1}(-x) \cdot \cos_p(-x) + p \cdot \cos_p^{p-1}(x) \cdot \sin_p''(x) = 0\,. \tag{4.5}$$

From Lemma 4.1, statements 3 and 4, we obtain

$$\sin_p^{p-1}(-x) \cdot \cos_p(x) = \cos_p^{p-1}(x) \cdot \sin_p''(x)$$

which yields

$$\sin_p''(x) = \sin_p^{p-1}(-x) \cdot \cos_p^{2-p}(x)\,.$$

□

Lemma 4.3. *Let $p \in \mathbb{R} \setminus \{2\}$ such that $p > 1$.*

(1) *If $p > 2$, then the function $\sin_p(x) \in C^1(\mathbb{R})$ and $\sin_p(x) \notin C^2(\mathbb{R})$.*
(2) *If $p \in (1, 2)$, then the function $\sin_p(x) \in C^2(\mathbb{R})$ and $\sin_p(x) \notin C^3(\mathbb{R})$.*

Proof. By the definition of $\cos_p(x)$, $\sin_p'(x) = \cos_p(x)$. The function $\cos_p(x) \in C(\mathbb{R})$, for all $p > 1$. Thus $\sin_p(x) \in C^1(\mathbb{R})$. By Lemma 4.2,

$$\sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) \quad \text{for } x \in (0, \pi_p/2)\,.$$

Taking into account that

$$\lim_{x \to \frac{\pi_p}{2}-} \sin_p^{p-1}(x) = 1 \quad \text{and} \quad \lim_{x \to \frac{\pi_p}{2}-} \cos_p^{2-p}(x) = +\infty \quad \text{for } p > 2\,,$$

we find that

$$\lim_{x \to \frac{\pi_p}{2}-} \sin_p''(x) = -\infty\,.$$

Thus the continuity of $\sin_p''(x)$ fails at $x = \pi_p/2$ for $p > 2$ and the statement 1 of Lemma 4.3 follows.

From (2.3), we find that the function $v'(x) = -(p-1)\varphi_p(\sin_p(x))$ is continuous on $\mathbb{R}$ as $\sin_p(x)$ is continuous on $\mathbb{R}$. We also find that $\cos_p(x) = \varphi_{p'}(v(x))$ from (2.3). Taking into account that $\varphi_{p'} \in C^1(\mathbb{R})$ for $p \in (1, 2)$ (observe that $p' = \frac{p}{p-1} > 2$ in this case), we infer that $\cos_p'(x) = \varphi_{p'}'(v(x)) \cdot v'(x)$ is continuous on $\mathbb{R}$. Thus $\sin_p(x)$ is two times continuously differentiable on $\mathbb{R}$ for $p \in (1, 2)$. On the other hand, taking

$$\sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) \quad \text{on } (0, \frac{\pi_p}{2})$$

into derivative, we obtain

$$\sin_p'''(x) = -(p-1)\sin_p^{p-2}(x) \cdot \cos_p^{3-p}(x) - (2-p) \cdot \sin_p^{p-1}(x) \cdot \cos_p^{1-p}(x) \cdot \sin_p''(x)\,.$$

Substituting for $\sin_p''(x)$ from the later equation into the former, we have

$$\sin_p'''(x) = -(p-1)\sin_p^{p-2}(x) \cdot \cos_p^{3-p}(x) + (2-p) \cdot \sin_p^{2p-2}(x) \cdot \cos_p^{3-2p}(x)\,.$$

Since $\lim_{x \to 0+} \sin_p(x) = 0$ and $\lim_{x \to 0+} \cos_p(x) = 1$, we obtain

$$\lim_{x \to 0+} \sin_p'''(x) = -\infty$$

for $p \in (1, 2)$. This concludes the proof of statement 2 of Lemma 4.3. □

Let us define the following 'symbolic' operators (rewriting rules) defined on expressions of the form

$$a \cdot \sin_p^q(x) \cdot \cos_p^{1-q}(x) \qquad \text{with } a, q \in \mathbb{R} \tag{4.6}$$

as follows

$$\mathrm{D}_s\, a \cdot \sin_p^q(x) \cdot \cos_p^{1-q}(x) := \begin{cases} a \cdot q \cdot \sin_p^{q-1}(x) \cdot \cos_p^{1-(q-1)}(x) & q \neq 0\,, \\ 0 & q = 0\,. \end{cases} \tag{4.7}$$

$$\mathrm{D}_c\, a \cdot \sin_p^q(x) \cdot \cos_p^{1-q}(x) := \begin{cases} -a \cdot (1-q) \cdot \sin_p^{q+p-1}(x) \cdot \cos_p^{1-(q+p-1)}(x) & q \neq 1\,, \\ 0 & q = 1\,. \end{cases} \tag{4.8}$$

Let us observe that the results of application D_s and D_c have the form (4.6). Hence they are also in the domain of definition of D_s and D_c. Thus we can consider compositions of D_c and D_s of arbitrary length. We will show that the first derivative of $\sin_p^q(x) \cdot \cos_p^{1-q}(x)$ (here $a = 1$) can be written using these symbolic operators as follows

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x} \sin_p^q(x) \cdot \cos_p^{1-q}(x) \\ &= \mathrm{D}_s \sin_p^q(x) \cdot \cos_p^{1-q}(x) + \mathrm{D}_c \sin_p^q(x) \cdot \cos_p^{1-q}(x)\,. \end{aligned}$$

To show this, we have to distinguish three cases $q \in \mathbb{R} \setminus \{0, 1\}$, $q = 1$, and $q = 0$.
Case $q \in \mathbb{R} \setminus \{0, 1\}$. Here

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x} \sin_p^q(x) \cdot \cos_p^{1-q}(x) \\ &= q \sin_p^{q-1}(x) \cdot \cos_p^{1-(q-1)}(x) - (1-q) \sin_p^{q+p-1}(x) \cdot \cos_p^{1-(q+p-1)}(x) \\ &= \mathrm{D}_s \sin_p^q(x) \cdot \cos_p^{1-q}(x) + \mathrm{D}_c \sin_p^q(x) \cdot \cos_p^{1-q}(x)\,. \end{aligned}$$

Note that the distance between the exponents of $\sin_p(x)$ in the resulting terms, i.e., $\sin_p^{q_0-1}(x) \cdot \cos_p^{2-q_0}(x)$ and $\sin_p^{q_0+p-1} \cdot \cos_p^{2-p-q_0}(x)$, is exactly p. This is crucial in the sequel of the paper, because in a sum of the type

$$c_0 \sin_p^{q_0}(x) \cdot \cos_p^{1-q_0}(x) + c_1 \sin_p^{q_0+p}(x) \cdot \cos_p^{1-(q_0+p)}(x)$$

the terms combine together as in the diagram depicted on Fig. 1
Case $q = 1$. In this case the term $\sin_p^q(x) \cdot \cos_p^{1-q}(x) = \sin_p(x)$. Thus the derivative of this term is the *single* term $\cos_p(x)$. By the definitions of $\mathrm{D}_s, \mathrm{D}_c$, we find that $\mathrm{D}_s \sin_p(x) = \cos_p(x)$ and $\mathrm{D}_c \sin_p(x) = 0$. Thus $\frac{\mathrm{d}}{\mathrm{d}x} \sin_p(x) = \mathrm{D}_s \sin_p(x) + \mathrm{D}_c \sin_p(x)$. The fact $\mathrm{D}_c \sin_p(x) = 0$ will be reflected in our diagrams by omitting 'right-down' edge departing from this node, see Figure 2.
Case $q = 0$. This case corresponds to $\sin_p^q(x) \cdot \cos_p^{1-q}(x) = \cos_p(x)$. Thus the derivative of this term is the *single* term $-\sin_p^{p-1}(x) \cos_p^{1-(p-1)}(x)$. By the definitions of $\mathrm{D}_s, \mathrm{D}_c$, we find that $\mathrm{D}_s \cos_p(x) = 0$ and

$$\mathrm{D}_c \cos_p(x) = -\sin_p^{p-1}(x) \cos_p^{1-(p-1)}(x)\,.$$

Thus $\frac{\mathrm{d}}{\mathrm{d}x} \cos_p(x) = \mathrm{D}_s \cos_p(x) + \mathrm{D}_c \cos_p(x)$. The fact $\mathrm{D}_s \cos_p(x) = 0$ will be reflected in our diagrams by omitting 'left-down' edge departing from this node, see Figure 3. Note that since in our diagrams we write powers only, the node corresponding to $-\sin_p^{p-1}(x) \cos_p^{1-(p-1)}(x)$ is labeled by $s_p^{p-1}\, c_p^{1-(p-1)}$.

In the same way, we can express higher order derivatives, thus, e.g., the second derivative of $\sin_p^q(x) \cdot \cos_p^{1-q}(x)$ (here $a = 1$) can be written as

$$\begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}x^2} \sin_p^q(x) \cdot \cos_p^{1-q}(x) \\ &= (\mathrm{D}_s \circ \mathrm{D}_s) \sin_p^q(x) \cdot \cos_p^{1-q}(x) + (\mathrm{D}_c \circ \mathrm{D}_s) \sin_p^q(x) \cdot \cos_p^{1-q}(x) \\ &\quad + (\mathrm{D}_s \circ \mathrm{D}_c) \sin_p^q(x) \cdot \cos_p^{1-q}(x) + (\mathrm{D}_c \circ \mathrm{D}_c) \sin_p^q(x) \cdot \cos_p^{1-q}(x). \end{aligned}$$

To better understand our methods of proof, it is good to have in mind the diagrams Figures 1–3.

The way how the term in the n-th derivative on the k-th position was derived from $\sin_p''(x)$ can be recovered from n and k as follows. First let us recall some notation from formal languages.

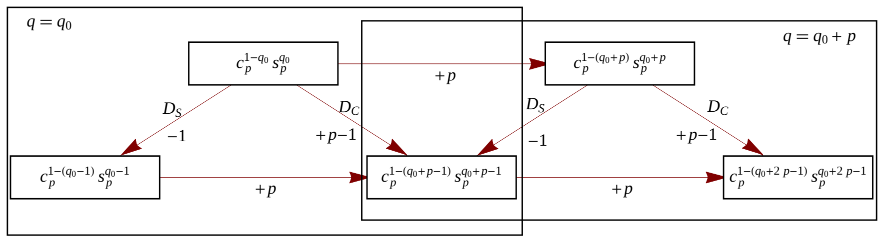

FIGURE 1. Rewriting diagram of the first derivative of $c_0 \sin_p^{q_0}(x) \cdot \cos_p^{1-q_0}(x) + c_1 \sin_p^{q_0+p}(x) \cdot \cos_p^{1-(q_0+p)}(x)$. For the lack of space, we *do not* write the coefficients standing in front of these terms and *use short-cuts*, i.e., we write s_p^q instead of $\sin_p^q(x)$ and c_p^{1-q} instead of $\cos_p^{1-q}(x)$

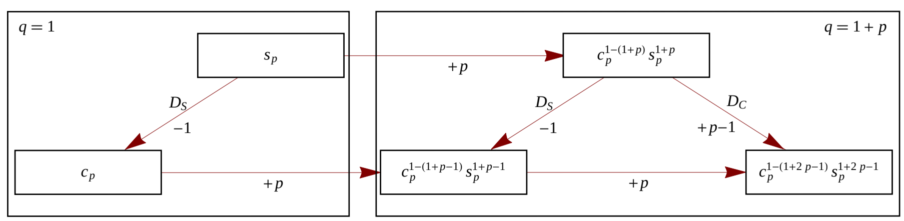

FIGURE 2. Rewriting diagram of the case $q = 1$. Recall that we write s_p^q instead of $\sin_p^q(x)$ and c_p^{1-q} instead of $\cos_p^{1-q}(x)$ and do not write the coefficients

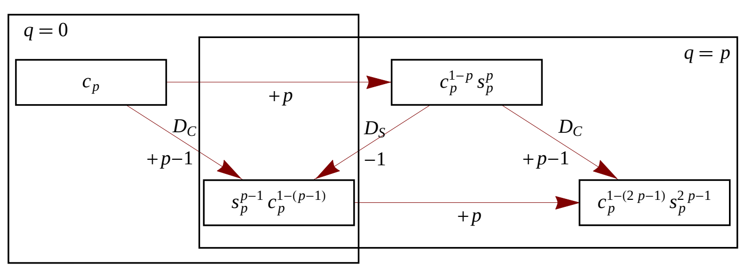

FIGURE 3. Rewriting diagram of the case $q = 0$. Recall that we write s_p^q instead of $\sin_p^q(x)$ and c_p^{1-q} instead of $\cos_p^{1-q}(x)$ and do not write the coefficients

Definition 4.4. (Salomaa-Soittola [26, I.2, p. 4,], and/or Manna [18, p. 2–3, p. 47, p. 78]) An *alphabet* (denoted by V) is a finite nonempty set of letters. A *word* (denoted by w) over an alphabet V is a finite string of zero or more letters from the alphabet V. The word consisting of zero letters is called the *empty word*. The set of all words over an alphabet V is denoted by V^* and the set of all nonempty words over an alphabet V is denoted by V^+. For strings w_1 and w_2 over V, their juxtaposition w_1w_2 is called *catenation* of w_1 and w_2, in operator notation $\mathrm{cat} : V^* \times V^* \to V^*$ and $\mathrm{cat}(w_1, w_2) = w_1w_2$. We also define the length of the word w, in operator notation $\mathrm{len} : V^* \to \{0\} \cup \mathbb{N}$, which for a given word w yields the number of letters in w when each letter is counted as many times as it occurs in w. We also use *reverse function* $\mathrm{rev} : V^* \to V^*$ which reverses the order of the letters in any word w (see [18, p. 47, p. 78]).

For our purposes here, we consider the alphabet $V = \{0, 1\}$ and the set of all nonempty words V^+. Thus words in V^+ are, e.g.,

$$\text{“}\mathtt{0}\text{”}, \text{“}\mathtt{1}\text{”}, \text{“}\mathtt{01}\text{”}, \text{“}\mathtt{10}\text{”}, \text{“}\mathtt{11}\text{”} \dots .$$

For instance, $\mathrm{cat}(\text{“}\mathtt{1110}\text{”}, \text{“}\mathtt{011}\text{”}) = \text{“}\mathtt{1110011}\text{”}$, and

$$\mathrm{rev}(\text{“}\mathtt{010011000}\text{”}) = \text{“}\mathtt{000110010}\text{”}\,,$$
$$\mathrm{len}(\text{“}\mathtt{010011000}\text{”}) = 9\,.$$

Let $n \in \mathbb{N}$, $k \in \{0\} \cup \mathbb{N}, 0 \le k \le 2^{n-2} - 1$ and $(k)_{2,n-2}$ be the string of bits of the length $n - 2$ which represents binary expansion of k (it means, e.g., for $k = 3$ and $n = 5$, $(3)_{2,5-2} = \text{“}\mathtt{011}\text{”}$). Now we are ready to define $D_{k,n}$ in two steps as follows.

- Step 1 We create an ordered $n-2$-tuple $d_{k,n-2} \in \{\mathrm{D}_s, \mathrm{D}_c\}^{n-2}$ (cartesian product of sets $\{\mathrm{D}_s, \mathrm{D}_c\}$ of length $n-2$) from $\mathrm{rev}((k)_{2,n-2})$ such that for $1 \le i \le n-2$, $d_{k,n-2}$ contains D_s on the i-th position if $\mathrm{rev}((k)_{2,n-2})$ contains “$\mathtt{0}$” on the i-th position, and $d_{k,n}$ contains D_c on the i-th position if $\mathrm{rev}((k)_{2,n-2})$ contains “$\mathtt{1}$” on the i-th position (it means, e.g., for $k = 3$, and $n = 5$, we obtain $d_{3,5-2} = (\mathrm{D}_c, \mathrm{D}_c, \mathrm{D}_s)$).
- Step 2 We define $D_{k,n}$ as the composition of operators $\mathrm{D}_s, \mathrm{D}_c$ in the order they appear in the ordered n-tuple $d_{k,n-2}$ (it means, e.g., for $k = 3$, and $n = 5$, we obtain $D_{3,5} = (\mathrm{D}_c \circ \mathrm{D}_c \circ \mathrm{D}_s)$).

The following Lemma implies that

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} D_{k,n} \sin_p''(x) \tag{4.9}$$

for all $x \in (0, \pi_p/2)$.

Lemma 4.5. *Let $p \in \mathbb{R}$, $p > 1$, $n \in \mathbb{N}$. Then $\sin_p^{(n)}(x)$ exists on $(0, \pi_p/2)$ and it is continuous. Moreover,*

$$\textit{for } n = 1: \quad \sin_p'(x) = \cos_p(x)\,, \tag{4.10}$$

$$\textit{for } n = 2: \quad \sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)\,, \tag{4.11}$$

and for $n = 3, 4, 5, \dots$, $k = 0, 1, 2, 3, \dots, 2^{n-2} - 1$ there exists $a_{k,n} \in \mathbb{R}$, $l_{k,n}, m_{k,n} \in \mathbb{Z}$ such that

$$\mathrm{D}_{k,n} \sin_p''(x) = a_{k,n} \cdot \sin_p^{p \cdot l_{k,n} + m_{k,n}}(x) \cdot \cos_p^{1-p \cdot l_{k,n} - m_{k,n}}(x)\,, \tag{4.12}$$

and

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_p^{p \cdot l_{k,n} + m_{k,n}}(x) \cdot \cos_p^{1-p \cdot l_{k,n} - m_{k,n}}(x)\,. \tag{4.13}$$

Moreover, let $j(k) \in \{0\} \cup \mathbb{N}$ be the digit sum of the binary expansion of $k = 0, 1, 2, \dots, 2^{n-2} - 1$ (thus $j(k)$ is the number of occurrences of D_c in $\mathrm{D}_{k,n}$) and let $\mathrm{D}_{k,n} \sin_p''(x) \not\equiv 0$. Then, for $k = 0, 1, 2, \dots, 2^{n-2} - 1$, the exponents

$$q_{k,n} := p \cdot l_{k,n} + m_{k,n} \tag{4.14}$$

satisfy

$$q_{k,n} = j(k)(p-1) + (n-2-j(k))(-1) + p - 1\,. \tag{4.15}$$

Proof. The cases $n = 1$ and $n = 2$ follows immediately from the definition of $\cos_p(x)$ and from Lemma 4.2.

We proceed by induction to prove the validity of the statement for $n = 3, 4, 5, \dots$.

Step 1. Taking (4.11) into derivative, we obtain

$$\sin_p'''(x) = -(p-1) \cdot \sin_p^{p-2}(x) \cdot \cos_p^{3-p}(x) + (2-p) \cdot \sin_p^{2p-2}(x) \cdot \cos_p^{3-2p}(x)\,.$$

For $k = 0, 1$ we obtain $a_{0,3} = -(p-1)$, $a_{1,3} = (2-p)$, $l_{0,3} = 1$, $l_{1,3} = 2$ $m_{0,3} = -2$, and $m_{1,3} = -2$. Hence

$$\sin_p'''(x) = \sum_{k=0}^{1} a_{k,3} \cdot \sin_p^{p \cdot l_{k,3} + m_{k,3}}(x) \cdot \cos_p^{1 - p \cdot l_{k,3} - m_{k,3}}(x)\,.$$

Since we assume $p > 1$ we obtain $p - 1 \neq 0$ and thus by the definition of D_s and $\mathrm{D}_{k,n}$

$$\begin{aligned} \mathrm{D}_{0,3} \sin_p''(x) &= \mathrm{D}_s(-\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)) \\ &= -(p-1) \cdot \sin_p^{p-2}(x) \cdot \cos_p^{3-p}(x) \\ &= a_{0,3} \cdot \sin_p^{p \cdot l_{0,3} + m_{0,3}}(x) \cdot \cos_p^{1 - p \cdot l_{0,3} - m_{0,3}}(x). \end{aligned}$$

Analogously, by the definition of D_c and $\mathrm{D}_{k,n}$ for $p \neq 2$, we find

$$\begin{aligned} \mathrm{D}_{1,3} \sin_p''(x) &= \mathrm{D}_c(-\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)) \\ &= -(-1) \cdot (2-p) \cdot \sin_p^{2p-2}(x) \cdot \cos_p^{3-2p}(x) \\ &= a_{1,3} \cdot \sin_p^{p \cdot l_{1,3} + m_{1,3}}(x) \cdot \cos_p^{1 - p \cdot l_{1,3} - m_{1,3}}(x)\,, \end{aligned}$$

and for $p = 2$, we obtain

$$\mathrm{D}_{1,3} \sin_p''(x) = \mathrm{D}_c(-\sin_2(x) \cdot \cos_2^0(x)) = 0\,.$$

Hence,

$$\begin{aligned} \sin_p'''(x) &= \mathrm{D}_s \sin_p''(x) + \mathrm{D}_c \sin_p''(x) \\ &= \mathrm{D}_{0,3} \sin_p''(x) + \mathrm{D}_{1,3} \sin_p''(x) \\ &= \sum_{k=0}^{1} \mathrm{D}_{k,3} \sin_p''(x)\,. \end{aligned}$$

Step 2. Let us assume that $\sin_p^{(n)}(x)$ exists, it is continuous on $(0, \pi_p/2)$, and for all $k = 0, 1, 2, \dots, 2^{n-2} - 1$ there exist $a_{k,n} \in \mathbb{R}$, $l_{k,n}, m_{k,n} \in \mathbb{Z}$ such that

$$\mathrm{D}_{k,n} \sin_p^{(n)}(x) = a_{k,n} \cdot \sin_p^{p \cdot l_{k,n} + m_{k,n}}(x) \cdot \cos_p^{1 - p \cdot l_{k,n} - m_{k,n}}(x)\,, \tag{4.16}$$

and

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_p^{p \cdot l_{k,n} + m_{k,n}}(x) \cdot \cos_p^{1 - p \cdot l_{k,n} - m_{k,n}}(x)\,. \tag{4.17}$$

By the additivity rule of the derivative, we find that

$$\begin{aligned}\sin_p^{(n+1)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{2^{n-2}-1} a_{k,n}\cdot\sin_p^{p\cdot l_{k,n}+m_{k,n}}(x)\cdot\cos_p^{1-p\cdot l_{k,n}-m_{k,n}}(x)\\ &= \sum_{k=0}^{2^{n-2}-1}\frac{\mathrm{d}}{\mathrm{d}x}\big(a_{k,n}\cdot\sin_p^{p\cdot l_{k,n}+m_{k,n}}(x)\cdot\cos_p^{1-p\cdot l_{k,n}-m_{k,n}}(x)\big)\,.\end{aligned}\tag{4.18}$$

For all $k = 0, 1, 2, \dots, 2^{n-2}-1$, we find

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}(a_{k,n}\cdot\sin_p^{p\cdot l_{k,n}+m_{k,n}}(x)\cdot\cos_p^{1-p\cdot l_{k,n}-m_{k,n}}(x))\\ &= a_{k,n}\cdot(p\cdot l_{k,n}+m_{k,n})\cdot\sin_p^{p\cdot l_{k,n}+m_{k,n}-1}(x)\cdot\cos_p^{1-(p\cdot l_{k,n}+m_{k,n}-1)}(x)\\ &\quad + a_{k,n}(1-p\cdot l_{k,n}-m_{k,n})\cdot\sin_p^{p\cdot l_{k,n}+m_{k,n}}(x)\cdot\cos_p^{1-p\cdot l_{k,n}-m_{k,n}-1}(x)\sin_p''(x)\\ &= a_{k,n}\cdot(p\cdot l_{k,n}+m_{k,n})\cdot\sin_p^{p\cdot l_{k,n}+m_{k,n}-1}(x)\cdot\cos_p^{1-(p\cdot l_{k,n}+m_{k,n}-1)}(x)\\ &\quad - a_{k,n}(1-p\cdot l_{k,n}-m_{k,n})\cdot\sin_p^{p\cdot(l_{k,n}+1)+m_{k,n}-1}(x)\cdot\cos_p^{1-(p\cdot(l_{k,n}+1)+m_{k,n}-1)}(x).\end{aligned}\tag{4.19}$$

For $k = 0, 1, 2, \dots, 2^{n-2}-1$, we denote

$$a_{2k,n+1} := a_{k,n}\cdot(p\cdot l_{k,n}+m_{k,n})\,,\tag{4.20}$$

$$a_{2k+1,n+1} := -a_{k,n}\cdot(1-p\cdot l_{k,n}-m_{k,n})\,,\tag{4.21}$$

$$l_{2k,n+1} := l_{k,n}\,,\tag{4.22}$$

$$m_{2k,n+1} := m_{k,n}-1\,,\tag{4.23}$$

$$l_{2k+1,n+1} := l_{k,n}+1\,,\tag{4.24}$$

$$m_{2k+1,n+1} := m_{k,n}-1\,.\tag{4.25}$$

Hence from (4.18), (4.19), and (4.20)–(4.25) we obtain

$$\sin_p^{(n+1)}(x) = \sum_{k'=0}^{2^{n-1}-1} a_{k',n+1}\cdot\sin_p^{p\cdot l_{k',n+1}+m_{k',n+1}}(x)\cdot\cos_p^{1-p\cdot l_{k',n+1}-m_{k',n+1}}(x).\tag{4.26}$$

Note that $\sin_p(x) > 0$ and $\cos_p(x) > 0$ for $x \in (0, \pi_p/2)$ by Lemma 4.1, statements 1 and 3, and continuous by Lemma 4.3. Moreover, the function $z \mapsto z^q$, defined for $z > 0$ and $q \in \mathbb{R}$ belongs to $C^\infty(0,+\infty)$. Thus the function on the right-hand side of (4.26) is continuous for $x \in (0, \pi_p/2)$ which implies the continuity of $\sin_p^{(n+1)}(x)$ for $x \in (0, \pi_p/2)$.

Now, we show that for all $k' = 0, 1, 2, \dots, 2^{n-2}-1$: $a_{k',n+1} \in \mathbb{R}$, $l_{k',n+1}, m_{k',n+1} \in \mathbb{Z}$ and, moreover,

$$\mathrm{D}_{k',n+1}\sin_p^{(n)}(x) = a_{k',n+1}\cdot\sin_p^{p\cdot l_{k',n+1}+m_{k',n+1}}(x)\cdot\cos_p^{1-p\cdot l_{k',n+1}-m_{k',n+1}}(x)\,.\tag{4.27}$$

Let us set

$$\mathrm{D}_{2k,n+1} := \mathrm{D}_s\circ\mathrm{D}_{k,n}\,,\tag{4.28}$$

$$\mathrm{D}_{2k+1,n+1} := \mathrm{D}_c\circ\mathrm{D}_{k,n}\,.\tag{4.29}$$

Then it follows easily from corresponding binary expansion of k and $2k$ that

$$(2k)_{2,n-1} = \mathrm{cat}((k)_{2,n-2}, \text{“}\mathtt{0}\text{”}),$$

$$(2k+1)_{2,n-1} = \mathrm{cat}((k)_{2,n-2}, \text{“1”})$$

and also that (4.28), (4.29) cover all 2^{n-1} of $k' = 0, 1, \dots 2^{n-1} - 1$. Thus our definitions (4.28) and (4.29) conform the relation between binary expansion of $k' = 2k$ and/or $k' = 2k+1$ and order of compositions of $\mathrm{D}_s, \mathrm{D}_c$ in $\mathrm{D}_{k',n+1}$.

For $k' = 0, 2, 4, \dots, 2^{n-1} - 2$ even,

$$\mathrm{D}_{k',n+1} \sin_p''(x) = \mathrm{D}_{2k,n+1} \sin_p''(x) = \mathrm{D}_s \circ \mathrm{D}_{k,n} \sin_p''(x) . \tag{4.30}$$

From the induction assumption (4.16), the definition of D_s (4.7) and (4.20), (4.22), (4.23), we find

$$\begin{aligned}
&\mathrm{D}_s(\mathrm{D}_{k,n} \sin_p''(x)) \\
&= \mathrm{D}_s(a_{k,n} \cdot \sin_p^{p\cdot l_{k,n}+m_{k,n}}(x) \cdot \cos_p^{1-p\cdot l_{k,n}-m_{k,n}}(x)) \\
&= a_{k,n} \cdot (p \cdot l_{k,n} + m_{k,n}) \cdot \sin_p^{p\cdot l_{k,n}+m_{k,n}-1}(x) \cdot \cos_p^{1-(p\cdot l_{k,n}+m_{k,n}-1)}(x) \\
&= a_{2k,n+1} \cdot \sin_p^{p\cdot l_{2k,n+1}+m_{2k,n+1}}(x) \cdot \cos_p^{1-p\cdot l_{2k,n+1}-m_{2k,n+1}}(x).
\end{aligned}$$

We can treat $k' = 1, 3, 5, \dots, 2^{n-1} - 1$ in the same way using D_c instead of D_s and (4.8) and (4.21), (4.24), (4.25). This concludes the proof by induction.

It remains to show (4.15). In fact, from the definition (4.8) of D_c, each occurrence of the symbolic operator D_c in $\mathrm{D}_{k,n}$ increases the exponent q of $\sin_p^q(x)$ by $p-1$. Analogously, from the definition of (4.7) of D_s, each occurrence of the symbolic operator D_s in $\mathrm{D}_{k,n}$ decreases the exponent q of $\sin_p^q(x)$ by 1. Taking into account these facts and also that $q_{1,2} = p-1$, the formula (4.15) follows. This concludes the proof of Lemma 4.5. $\square$

Lemma 4.6. *Let $p \in \mathbb{N}$, $p > 1$, and for all $n \in \mathbb{N}$, $n \geq 2$*

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{q_{k,n}}(x) \cdot \cos_p^{1-q_{k,n}}(x) . \tag{4.31}$$

Then for all $n \in \mathbb{N}$, $n \geq 2$, and all $k \in \{0\} \cup \mathbb{N}$, $k \leq 2^{n-2} - 1$

$$q_{k,n} \in \{0\} \cup \mathbb{N} . \tag{4.32}$$

Proof. From the definitions (4.7) and (4.8),

$$\begin{aligned}
q_{2k,n+1} &= q_{k,n} - 1 \quad \text{(we applied } \mathrm{D}_S \text{ on the expression)} \\
q_{2k+1,n+1} &= q_{k,n} + p - 1 \quad \text{(we applied } \mathrm{D}_\mathrm{C} \text{ on the expression)}
\end{aligned} \tag{4.33}$$

The proof proceeds by induction in n.

Step 1. From Lemma 4.2, for $\sin_p''(x)$ on $(0, \pi_p/2)$ we obtain the formula

$$\sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) .$$

Thus $q_{1,2} = p - 1$. By assumption $p \in \mathbb{N}$, $p > 1$ we find $q_{1,2} \in \mathbb{N}$.

Step 2. We distinguish two cases, $q_{k,n} \in \mathbb{N}$ and $q_{k,n} = 0$. Let $q_{k,n} \in \mathbb{N}$. Then from (4.33), $p \in \mathbb{N}$, $p > 1$, we obtain

$$\begin{gathered}
q_{2k,n+1} = q_{k,n} - 1 \in \{0\} \cup \mathbb{N}, \\
q_{2k+1,n+1} = q_{k,n} + p - 1 \in \mathbb{N},
\end{gathered}$$

which satisfies (4.32). Let $q_{k,n} = 0$. Then the corresponding term in (4.31) has form

$$a_{k,n} \cdot \cos_p(x) , \tag{4.34}$$

since $\sin_p^0(x) \equiv 1$ for $x \in (0, \pi_p/2)$. Taking (4.34) into derivative, we find
$$a_{k,n} \cdot \cos_p'(x) = -a_{k,n} \cdot \sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)$$
and $q_{2k+1,n+1} = p - 1 \in \mathbb{N}$, because $p \in \mathbb{N}$, $p > 1$. This concludes the proof by induction. □

Lemma 4.7. *Let $p \in \mathbb{N}$, $p \geq 3$. Then for all $n \in \mathbb{N}$, $n \geq 2$*
$$\sin_p^{(n)}(x) \leq 0 \quad \text{on} \quad (0, \frac{\pi_p}{2}) .$$

Proof. By Lemma 4.5 and substitution (4.14), we have
$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_p^{q_{k,n}}(x) \cdot \cos_p^{1-q_{k,n}}(x) . \tag{4.35}$$
Let Q_n denote the set of all values of $q_{k,n}$ attained in the previous expression (this is to handle possible multiplicities), i.e.,
$$Q_n = \{q_{k,n} : k = 0, \dots, 2^{n-2} - 1\} . \tag{4.36}$$
By Lemma 4.6 for all $n \geq 2$ and for all $k \leq 2^{n-2} - 1$, we have $q_{k,n} \in \{0\} \cup \mathbb{N}$. Clearly, $Q_n \subset \{0\} \cup \mathbb{N}$ has at most 2^{n-2} elements and thus there exists $i_0 \in \mathbb{N} : 0 < i_0 \leq 2^{n-2} - 1$ and bijective mapping
$$\overline{q}_n : \{0, 1, 2, \dots i_0\} = Q_n \tag{4.37}$$
satisfying the order condition
$$\forall i, j = 0, 1, \dots, i_0 : i < j \Rightarrow \overline{q}_i < \overline{q}_j . \tag{4.38}$$
In the sequel, $\overline{q}_{i,n}$ stands for $\overline{q}_n(i)$. With this at hand, we add together the coefficients in (4.35) corresponding to the same value of powers $q_{k,n}$ and for any $i = 0, 1, \dots, i_0$ define
$$c_{i,n} := \sum_{\substack{k=0,1,\dots 2^{n-2}-1 \\ q_{k,n} = \overline{q}_{i,n}}} a_{k,n} . \tag{4.39}$$
Now, we rewrite (4.35) using coefficients $c_{i,n}$:
$$\sin_p^{(n)}(x) = \sum_{i=0}^{i_0} c_{i,n} \cdot \sin_p^{\overline{q}_{i,n}}(x) \cdot \cos_p^{1-\overline{q}_{i,n}}(x) . \tag{4.40}$$
Later, we will prove by induction that
$$\forall i = 0, 1, \dots, i_0 : c_{i,n} \leq 0 . \tag{4.41}$$
By Lemma 4.1 statements 1 and 3, $\sin_p(x) > 0$ and $\cos_p(x) > 0$ on $(0, \frac{\pi_p}{2})$, which implies that for all $q, r \in \{0\} \cup \mathbb{N}$ and $x \in (0, \pi_p/2)$
$$\sin_p^q(x) \cdot \cos_p^r(x) > 0 . \tag{4.42}$$
Thus from (4.40)–(4.42) the statement of Lemma 4.7 follows.

Now it remains to prove by induction in n that (4.41) holds.

Step 1. By Lemma 4.2 we find that
$$\sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) \tag{4.43}$$
for all $x \in (0, \pi_p/2)$ and so $c_{0,2} = -1 < 0$.

Taking the derivative of (4.43) (and after some straightforward rearrangements),
$$\sin_p'''(x) = -(p-1) \cdot \sin_p^{p-2}(x) \cdot \cos_p^{3-p}(x) + (2-p) \cdot \sin_p^{2p-2}(x) \cdot \cos_p^{3-2p}(x) \tag{4.44}$$

for $x \in (0, \pi_p/2)$. Since $p \geq 3$, we have $c_{0,3} = -(p-1) \leq -2 \leq 0$ and $c_{1,3} = (2-p) \leq -1 \leq 0$ as desired. Taking the derivative (4.44),

$$\begin{aligned}
\sin_p^{(\mathrm{iv})} &= -(p-1)\cdot(p-2)\cdot\sin_p^{p-3}(x)\cdot\cos_p^{4-p}(x)+ \\
&\quad + (p-1)\cdot(3-p)\cdot\sin_p^{2p-3}(x)\cdot\cos_p^{4-2p}(x) \\
&\quad + (2-p)\cdot(2p-2)\cdot\sin_p^{2p-3}(x)\cdot\cos_p^{4-2p}(x) \\
&\quad - (2-p)\cdot(3-2p)\cdot\sin_p^{3p-3}(x)\cdot\cos_p^{4-3p}(x) \qquad (4.45)\\
&= -(p-1)\cdot(p-2)\cdot\sin_p^{p-3}(x)\cdot\cos_p^{4-p}(x) \\
&\quad + ((p-1)\cdot(3-p)+(2-p)\cdot(2p-2))\cdot\sin_p^{2p-3}(x)\cdot\cos_p^{4-2p}(x) \\
&\quad - (2-p)\cdot(3-2p)\cdot\sin_p^{3p-3}(x)\cdot\cos_p^{4-3p}(x)
\end{aligned}$$

for all $x \in (0, \pi_p/2)$. Since $p \geq 3$ we have $c_{0,4} = -(p-1)\cdot(p-2) \leq -2 \leq 0$, $c_{1,4} = (p-1)\cdot(3-p)+(2-p)\cdot(2p-2) \leq -4 \leq 0$, and $c_{2,4} = -(2-p)\cdot(3-2p) \leq -3 \leq 0$

Step 2. Let us assume that $\sin_p^{(n)}(x)$ for $n \geq 4$ can be written in the form (4.40) and

$$\forall i \leq i_0 \,:\, c_{i,n} \leq 0\,. \qquad (4.46)$$

The proof falls naturally into two parts.

Case 1. If

$$\overline{q}_{i,n} \geq 1\,, \qquad (4.47)$$

then taking the i-th term of (4.40), which is

$$c_{i,n}\cdot\sin_p^{\overline{q}_{i,n}}(x)\cdot\cos_p^{1-\overline{q}_{i,n}}(x)\,, \qquad (4.48)$$

into derivative we obtain

$$\begin{aligned}
&c_{i,n}\cdot\overline{q}_{i,n}\cdot\sin_p^{\overline{q}_{i,n}-1}(x)\cdot\cos_p^{1-\overline{q}_{i,n}+1}(x) \\
&+ c_{i,n}\cdot(1-\overline{q}_{i,n})\cdot\sin_p^{\overline{q}_{i,n}}(x)\cdot\cos_p^{1-\overline{q}_{i,n}-1}(x)\cdot\sin_p''(x)\,.
\end{aligned}$$

Substituting (4.43) for $\sin_p''(x)$ into the previous expression, we obtain

$$\begin{aligned}
&c_{i,n}\cdot\overline{q}_{i,n}\cdot\sin_p^{\overline{q}_{i,n}-1}(x)\cdot\cos_p^{2-\overline{q}_{i,n}}(x) \\
&- c_{i,n}\cdot(1-\overline{q}_{i,n})\cdot\sin_p^{\overline{q}_{i,n}+p-1}(x)\cdot\cos_p^{-\overline{q}_{i,n}-p+2}(x)\,.
\end{aligned}$$

Let us denote

$$\begin{aligned}
a'_{2i-1,n+1} &:= c_{i,n}\cdot\overline{q}_{i,n}\,, \\
a'_{2i,n+1} &:= c_{i,n}\cdot(\overline{q}_{i,n}-1)\,.
\end{aligned}$$

By the induction assumption (4.46) and assumption of Case 1 (4.47), we have $a'_{2i-1,n+1}, a'_{2i,n+1} \leq 0$.

Case 2. If $\overline{q}_{i,n} = 0$, then $i = 0$ (by the ordering) and the corresponding term of (4.40) is

$$c_{0,n}\cdot\sin_p^0(x)\cdot\cos_p^1(x). \qquad (4.49)$$

Taking derivatives in (4.49) we find

$$-\,c_{0,n}\cdot\sin_p^{p-1}(x)\cdot\cos_p^{2-p}(x)\,. \qquad (4.50)$$

Denote $a'_{1,n+1} := -c_{0,n}$ which is clearly nonnegative by the induction assumption (4.46). We consider the second term of (4.40) ($i = 1$) and take the derivative,

$$\frac{\mathrm{d}}{\mathrm{d}x}c_{1,n}\cdot\sin_p^{\overline{q}_{1,n}}(x)\cdot\cos_p^{1-\overline{q}_{1,n}}(x)$$

$$= \mathrm{D}_s\, c_{1,n} \cdot \sin_p^{\overline{q}_{1,n}}(x) \cdot \cos_p^{1-\overline{q}_{1,n}}(x) + \mathrm{D}_c\, c_{1,n} \cdot \sin_p^{\overline{q}_{1,n}}(x) \cdot \cos_p^{1-\overline{q}_{1,n}}(x)\,.$$

Since $\overline{q}_{0,n} = 0$, $\overline{q}_{1,n} = p$ (see Figure 4). Note that the right-hand side of

$$\mathrm{D}_s\, c_{1,n} \sin_p^p(x) \cdot \cos_p^{1-p}(x) = p \cdot c_{1,n} \cdot \sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) \tag{4.51}$$

has the same exponent $q = p-1$ as (4.50) has. It remains to prove that $p \cdot c_{1,n} - c_{0,n} \leq 0$. Using $(n-2)$-th derivative of $\sin_p(x)$ we obtain (4.50),

$$\begin{aligned}(\mathrm{D}_c \circ \mathrm{D}_s \circ \mathrm{D}_s) c_{0,n-2} \cdot \sin_p^2(x) \cos_p^{-1}(x) &= (\mathrm{D}_c \circ \mathrm{D}_s) 2 \cdot c_{0,n-2} \cdot \sin_p(x) \\ &= \mathrm{D}_c\, 2 \cdot c_{0,n-2} \cdot \cos_p(x) \\ &= -2 \cdot c_{0,n-2} \cdot \sin_p^{p-1}(x) \cos_p^{2-p}(x)\end{aligned} \tag{4.52}$$

and (4.51),

$$\begin{aligned}&(\mathrm{D}_s \circ \mathrm{D}_s \circ \mathrm{D}_c) c_{0,n-2} \cdot \sin_p^2(x) \cos_p^{-1}(x) \\ &= (\mathrm{D}_s \circ \mathrm{D}_s) c_{0,n-2} \cdot \sin_p^{1+p}(x) \cos_p^{-p}(x) \\ &= \mathrm{D}_s (1+p) \cdot c_{0,n-2} \cdot \sin_p^p(x) \cdot \cos_p^{1-p}(x) \\ &= p \cdot (1+p) \cdot c_{0,n-2} \cdot \sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)\,.\end{aligned} \tag{4.53}$$

Comparing (4.52) with (4.50), we find that

$$-c_{0,n} = -2 \cdot c_{0,n-2}\,.$$

In addition, comparing (4.51) and (4.53), we find

$$p \cdot c_{1,n} = p \cdot (p+1) \cdot c_{0,n-2}\,.$$

From the induction assumption, $c_{0,n-2} \leq 0$ and for $p \geq 3$, we easily find

$$p \cdot c_{1,n} - c_{0,n} = (p \cdot (p+1) - 2) c_{0,n-2} \leq 0$$

by adding the previous two identities.

In the definition (4.39) of $c_{i,n}$, we are adding coefficients

$$a'_{k,n}\,, \quad k = 0, 1, \dots, 2(i_0 + 1)$$

corresponding to the same value of exponent $\overline{q}$. From the both cases, we obtain $c_{i,n+1} \leq 0$ for all $i \in \mathbb{N}$, $i \leq i_1$, $0 < i_1 \leq 2^{n-1} - 1$. This concludes the proof by induction. $\square$

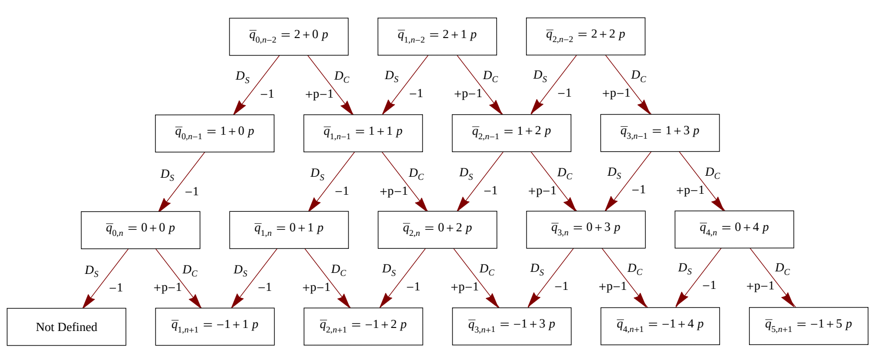

FIGURE 4. Rewriting diagram - starting with $\overline{q}_{0,n-2}, \overline{q}_{1,n-2}, \overline{q}_{2,n-2}$

5. Proofs of main results

Proof of Theorem 3.1. By Lemma 4.5 and substitution (4.14), we can write

$$\sin_{2(m+1)}^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_{2(m+1)}^{q_{k,n}}(x) \cdot \cos_{2(m+1)}^{1-q_{k,n}}(x)\,,$$

where

$$q_{k,n} = (2(m+1)-1)\cdot j(k) + (n-j(k)-2) + 2(m+1) - 1\,,$$

and $j(k)$ has the same meaning as in Lemma 4.5. Thus $a_{k,n} \in \mathbb{Z}$.

From Lemma 4.1, statement 4 and 5, we also know that $\sin_{2(m+1)}^{(n)}(x)$ is even function for n odd and $\sin_{2(m+1)}^{(n)}(x)$ is odd function for n even. It follows that for $x \in (-\frac{\pi_{2(m+1)}}{2}, 0)$

$$\sin_{2(m+1)}^{(n)}(x) = \begin{cases} -\sin_{2(m+1)}^{(n)}(-x) & \text{for } n \text{ even}\,, \\ \sin_{2(m+1)}^{(n)}(-x) & \text{for } n \text{ odd}\,. \end{cases} \tag{5.1}$$

Now we assume $p = 2(m+1)$, $m \in \mathbb{N}$, and

$$\begin{aligned} q_{k,n} &= (2(m+1)-1)j(k) + (n-j(k)-2) + 2(m+1) - 1 \\ &= (2(m+1)-1)(j(k)+1) + j(k) + 2 - n \\ &= 2(m+1)(j(k)+1) - n + 1 \end{aligned}$$

which implies $q_{k,n}$ is *odd* for n even. Thus we obtain

$$\begin{aligned} -\sin_{2(m+1)}^{(n)}(-x) &= -\sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_{2(m+1)}^{q_{k,n}}(-x) \cdot \cos_{2(m+1)}^{1-q_{k,n}}(-x) \\ &= \sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_{2(m+1)}^{q_{k,n}}(x) \cdot \cos_{2(m+1)}^{1-q_{k,n}}(x)\,. \end{aligned} \tag{5.2}$$

Analogously, $q_{k,n}$ is *even* for n odd and

$$\begin{aligned} \sin_{2(m+1)}^{(n)}(-x) &= \sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_{2(m+1)}^{q_{k,n}}(-x) \cdot \cos_{2(m+1)}^{1-q_{k,n}}(-x) \\ &= \sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_{2(m+1)}^{q_{k,n}}(x) \cdot \cos_{2(m+1)}^{1-q_{k,n}}(x)\,. \end{aligned} \tag{5.3}$$

Hence from (5.2), (5.3), we obtain

$$\sin_{2(m+1)}^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_{2(m+1)}^{q_{k,n}}(x) \cdot \cos_{2(m+1)}^{1-q_{k,n}}(x) \tag{5.4}$$

for all $x \in (-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2}) \setminus \{0\}$.

Now, we prove the continuity of $\sin_{2(m+1)}^{(n)}(x)$ for all $x \in (-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$ by induction in n.

Step 1. For $x \in (-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$ the function

$$v(x) = \varphi_{2(m+1)}(\cos_{2(m+1)}(x)) > 0$$

and so we can take the first equation in (2.3) into its derivative and obtain

$$u''(x) = \varphi'_{p'}(v(x))v'(x)\,, \text{ where } p' = \frac{2m+1}{2m}\,.$$

Since v' is continuous and $\varphi_{p'} \in C^1(0,+\infty)$ ($\varphi_{p'}(z) = z^{p-1}$ for $z > 0$), we obtain continuity of $\sin^{(n)}_{2(m+1)}(x)$ for $n = 2$.

Step 2. Let us assume that $\sin^{(n)}_{2(m+1)}(x)$ is continuous on $(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$. From Lemma 4.5 we know that $\sin^{(n+1)}_{2(m+1)}(x)$ is continuous on $(0, \frac{\pi_{2(m+1)}}{2})$. Now we distinguish two cases: $n+1$ is odd then $\sin^{(n+1)}_{2(m+1)}(x)$ is even by Lemma 4.1, statement 4, and $n+1$ is even then $\sin^{(n+1)}_{2(m+1)}(x)$ is odd by Lemma 4.1, statement 5. In both cases, $\sin^{(n+1)}_{2(m+1)}(x) \in C(0, \frac{\pi_{2(m+1)}}{2})$ implies $\sin^{(n+1)}_{2(m+1)}(x) \in C(-\frac{\pi_{2(m+1)}}{2}, 0)$. It remains to prove the continuity at $x = 0$. From (5.4) we know that

$$\lim_{x\to 0-} \sin^{(n+1)}_{2(m+1)}(x) = \lim_{x\to 0+} \sin^{(n+1)}_{2(m+1)}(x)\,. \tag{5.5}$$

At the end we compute the derivative of $\sin^{(n)}_{2(m+1)}(0)$ from its definition:

$$\sin^{(n+1)}_{2(m+1)}(0) = \lim_{h\to 0} \frac{\sin^{(n)}_{2(m+1)}(h) - \sin^{(n)}_{2(m+1)}(0)}{h}\,.$$

It is a limit of the type "0/0". Since the limit $\lim_{h\to 0} \sin^{(n+1)}_{2(m+1)}(h)$ exists, we obtain $\sin^{(n+1)}_{2(m+1)}(0) = \lim_{h\to 0} \sin^{(n+1)}_{2(m+1)}(h)$ by L'Hôspital's rule. Note that by Lemma 4.6, $q_{k,n} \ge 0$ for all $n \in \mathbb{N}$, $n \ge 2$, and all $k \in \{0\} \cup \mathbb{N}$, $k \le 2^{n-2} - 1$, these limits are finite and we obtain continuity. This proves the continuity of $\sin^{(n+1)}_{2(m+1)}(x)$ for all $x \in (-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$. $\square$

Proof of Theorem 3.2. By Lemma 4.5 and substitution (4.14), we have

$$\sin^{(n)}_p(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_p^{q_{k,n}}(x) \cdot \cos_p^{1-q_{k,n}}(x) \quad \text{on } (0, \frac{\pi_p}{2})\,.$$

Moreover, by Lemma 4.1, statement 4 and 5, we obtain

$$\sin^{(n)}_p(x) = \begin{cases} -\sin^{(n)}_p(-x) & \text{for } n \text{ even}\,, \\ \sin^{(n)}_p(-x) & \text{for } n \text{ odd}\,, \end{cases} \tag{5.6}$$

for $x \in (-\pi_p/2, 0)$. Since $\sin^{(n)}_p(x)$ is continuous for $x \in (0, \pi_p/2)$, it is also continuous on $x \in (-\pi_p/2, 0)$ by (5.6). Thanks to (5.6) it is enough to study the behavior of $\sin_p(x)$ in the right neighborhood of 0. From Lemma 4.5, we have that

$$q_{k,n} = j(k)\cdot(p-1) + (-1)\cdot(n-2-j) + p - 1 = p\cdot(j(k)+1) + 1 - n\,. \tag{5.7}$$

for all $n \in \mathbb{N}$, $n \ge 2$ and all $k \in \{0\} \cup \mathbb{N}$, $k \le 2^{n-2} - 1$. Since $j(k) \in \{0\} \cup \mathbb{N}$ we find that

$$q_{k,n} \ge p + 1 - n\,.$$

Then, for $n < p + 1$, we have $q_{k,n} > 0$ for all $k \in \{0\} \cup \mathbb{N}$, $k \le 2^{n-2} - 1$. And so using the theorem of the algebra of the limits from any classical analysis textbook, we find that

$$\lim_{x\to 0+} \sin^{(n)}_p(x) = 0\,.$$

From (5.6),

$$\lim_{x\to 0-} \sin_p^{(n)}(x) = \begin{cases} -\lim_{x\to 0+} \sin_p^{(n)}(x) = 0 & \text{for } n \text{ even}\,,\\ \lim_{x\to 0+} \sin_p^{(n)}(x) = 0 & \text{for } n \text{ odd}\,. \end{cases}$$

The continuity at $x = 0$ follows from L'Hôspital's rule used recurrently from $n = 2$ to $n = \lceil p \rceil$.

By Lemma 4.5, $\sin_{2(m+1)}^{(2m+2)}(x)$ satisfies

$$\sin_p^{(\lceil p\rceil+1)}(x) = \sum_{k=0}^{2^{\lceil p\rceil-1}-1} D_{k,\lceil p\rceil+1} \sin_p''(x) \quad \text{on } (0, \frac{\pi_p}{2})\,.$$

Since $q_{k,n} > 0$ for all $n < \lceil p\rceil$ and all $k \in \{0\} \cup \mathbb{N}$, $k < 2^{\lceil p\rceil} - 1$, the function $\mathrm{D}_S\, a_{k,n} \cdot \sin_p^{q_{k,n}}(x) \cdot \cos_p^{1-q_{k,n}}(x)$ does not vanish identically. Thus $a_{0,\lceil p\rceil+1} \neq 0$. Since $a_{0,\lceil p\rceil+1} \neq 0$, we can apply (5.7) for $j(0) = 0$ which gives

$$q_{0,\lceil p\rceil+1} = p - \lceil p\rceil \le 0\,.$$

From the fact that $j(k) > j(0)$ for all $k \in \{0\} \cup \mathbb{N}$, $k \le 2^{\lceil p\rceil-1} - 1$ and from (5.7) we know that

$$q_{k,\lceil p\rceil+1} > q_{0,\lceil p\rceil+1}\,.$$

Moreover from (5.7),

$$q_{k,\lceil p\rceil+1} = (j(k)+1)\cdot p + 1 - (\lceil p\rceil + 1) = (j(k)+1)\cdot p - \lceil p\rceil > 0$$

for $j(k) \ge 1$ and $p > 1$. Since, for all $q_{k,n} > 0$,

$$\lim_{x\to 0} a_{k,n} \cdot \sin_p^{q_{k,n}}(x) \cdot \cos_p^{1-q_{k,n}}(x) = 0\,,$$

we obtain

$$\begin{aligned} \lim_{x\to 0+} \sin_p^{(\lceil p\rceil+1)}(x) &= \lim_{x\to 0+} a_{0,\lceil p\rceil+1} \cdot \sin_p^{p-\lceil p\rceil}(x) \cdot \cos_p^{1-p+\lceil p\rceil}(x)\\ &\quad + \sum_{k=1}^{2^{\lceil p\rceil-1}-1} a_{k,\lceil p\rceil+1} \cdot \sin_p^{q_{k,\lceil p\rceil+1}}(x) \cdot \cos_p^{1-q_{k,\lceil p\rceil+1}}(x)\\ &= \lim_{x\to 0+} a_{0,\lceil p\rceil+1} \cdot \sin_p^{p-\lceil p\rceil}(x) \cdot \cos_p^{1-p+\lceil p\rceil}(x) \end{aligned} \tag{5.8}$$

by the theorem of the algebra of the limits.

Now the proof falls into two cases, $p = 2m+1$ and $p \in \mathbb{R} \setminus \mathbb{N}$, $p > 1$.

Case 1. For $p = 2m+1$, we have by (5.8)

$$\lim_{x\to 0+} \sin_{2m+1}^{(2m+2)}(x) = \lim_{x\to 0+} a_{0,2m+2} \cdot \cos_p(x) = a_{0,2m+2} \neq 0\,.$$

Since $2m+2$ is even, $\sin_{2m+1}^{(2m+2)}(x)$ is odd function by Lemma 4.1, statement 5. Thus

$$\lim_{x\to 0-} \sin_{2m+1}^{(2m+2)}(x) = -a_{0,2m+2}\,.$$

Hence $\sin_{2m+1}^{(2m+2)}(x)$ is not continuous at $x = 0$.

Case 2. Since for $p \in \mathbb{R} \setminus \mathbb{N}$, $p > 1$, we have

$$\lim_{x\to 0+} \sin_p^{(\lceil p\rceil+1)}(x) = \lim_{x\to 0+} a_{0,\lceil p\rceil+1} \cdot \sin_p^{p-\lceil p\rceil}(x) \cdot \cos_p^{1-p+\lceil p\rceil}(x) = +\infty$$

from (5.8). Hence $\sin_p^{\lceil p\rceil+1}(x)$ is discontinuous at $x = 0$. This concludes the proof. □

Proof of Theorem 3.3. It follows from [24, Thm. 1.1, consider $p = q$ and $\sigma = 0$] that there exists a unique analytic function $F(z)$ near origin such that the unique solution $u(x) = \sin_p(x)$ of the initial value problem (2.1); i.e.,

$$-(|u'|^{p-2}u')' - (p-1)|u|^{p-2}u = 0$$
$$u(0) = 0, \quad u'(0) = 1\,,$$

takes the form $\sin_p(x) = u(x) = x \cdot F(|x|^p)$. Note that for $p = 2(m+1)$ and $m \in \mathbb{N}$,

$$\sin_p(x) = x \cdot F(|x|^p) = x \cdot F(x^p) = \sum_{l=0}^{+\infty} \alpha_l \cdot x^{l \cdot p+1}, \quad \text{where } F(z) = \sum_{l=0}^{+\infty} \alpha_l z^l\,,$$

which is also an analytic function in a neighborhood of $x = 0$. In the sequel of this proof $p = 2(m + 1)$, $m \in \mathbb{N}$. By the uniqueness of the Maclaurin series of analytic function, we see that

$$\sum_{l=0}^{+\infty} \alpha_l \cdot x^{l\cdot p+1} = \sum_{l=0}^{+\infty} \frac{\sin_p^{(l\cdot p+1)}(0)}{(l \cdot p + 1)!} \cdot x^{l\cdot p+1}\,,$$

where the right-hand side also converges to $\sin_p(x)$ on some neigbourhood of $x = 0$. Note that $\sin_p^{(k)}(0) = 0$ for any $k \in \mathbb{N}$ such that

$$\forall l \in \{0\} \cup \mathbb{N} : k \neq l \cdot p + 1$$

as it follows from Lemma 4.5 and Lemma 4.6.

Since the restriction of $\sin_p(x)$ to $[-\frac{\pi_p}{2}, \frac{\pi_p}{2}]$ is the inverse function of $\arcsin_p(x)$, by the identity (1.6); i.e.,

$$\forall x \in [-1, 1] : \sin_p(\arcsin_p(x)) = x\,.$$

It is well known see, e.g., [13] that

$$\begin{aligned} \arcsin_p(x) &= \int_0^x (1 - s^p)^{-\frac{1}{p}}\,\mathrm{d}s \\ &= \frac{s \cdot {}_2F_1(1, \frac{1}{p}; 1 + \frac{1}{p}; s^p)}{p} \\ &= \sum_{n=0}^{+\infty} \frac{\Gamma(n + \frac{1}{p})}{\Gamma(\frac{1}{p})(n \cdot p + 1)} \cdot \frac{1}{n!} \cdot x^{n\cdot p+1} \end{aligned} \tag{5.9}$$

for $x \in (0, 1)$. Observe that for our special case $p = 2(m + 1)$ with $m \in \mathbb{N}$, this formula is valid on $[-1, 1]$. Note also that in our special case, (5.9) is in fact the Maclaurin series for $\arcsin_p(x)$ and, moreover, all coefficients are nonnegative (the explicitly written coefficients are positive, the other ones are zero).

To apply the formula for composite formal power series, we need to consider series for $\sin_p(x)$ and $\arcsin_p(x)$ including the zero terms. For this reason, we define for all $j \in \mathbb{N}$

$$\alpha'_j := \sin_p^{(j)}(0)/j! = \begin{cases} \alpha_i & \text{if } j = ip + 1 \text{ for some } i \in \{0\} \cup \mathbb{N}\,, \\ 0 & \text{otherwise} \end{cases} \tag{5.10}$$

and

$$\beta'_j := \begin{cases} \frac{\Gamma(n+\frac{1}{p})}{\Gamma(\frac{1}{p})(n\cdot p+1)} \cdot \frac{1}{n!} & \text{if } j = ip + 1 \text{ for some } i \in \{0\} \cup \mathbb{N}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{5.11}$$

Thus by well-known composite formal power series formula

$$\sin_p(\arcsin_p(x)) = \sum_{n=1}^{+\infty} c_n x^n \,, \tag{5.12}$$

where

$$c_n = \sum_{\substack{k\in\mathbb{N},\ j_1,j_2,\dots,j_k\in\mathbb{N}\\ j_1+j_2+\dots+j_k=n}} \alpha'_k\cdot\beta'_{j_1}\cdot\beta'_{j_2}\cdot\dots\cdot\beta'_{j_k}\,. \tag{5.13}$$

Since both functions $\sin_p(x)$ and $\arcsin_p(x)$ are analytic in some neighborhood of $x=0$, the series from (5.12) with coefficients given by (5.13) is convergent towards the identity $x\mapsto x$ on some neighborhood of $x=0$. From this fact, we infer that $c_1=1$ and $c_n=0$ for all $n\in\mathbb{N}, n\ge 2$. Thus for any $x\in\mathbb{R}$

$$x = \sum_{n=1}^{+\infty} x^n \sum_{\substack{k\in\mathbb{N},\ j_1,j_2,\dots,j_k\in\mathbb{N}\\ j_1+j_2+\dots+j_k=n}} \alpha'_k\cdot\beta'_{j_1}\cdot\beta'_{j_2}\cdot\dots\cdot\beta'_{j_k} \tag{5.14}$$

and in particular

$$1 = \sum_{n=1}^{+\infty} \sum_{\substack{k\in\mathbb{N},\ j_1,j_2,\dots,j_k\in\mathbb{N}\\ j_1+j_2+\dots+j_k=n}} \alpha'_k\cdot\beta'_{j_1}\cdot\beta'_{j_2}\cdot\dots\cdot\beta'_{j_k}\,. \tag{5.15}$$

Now we show that also

$$\sum_{n=1}^{+\infty} \sum_{\substack{k\in\mathbb{N},\ j_1,j_2,\dots,j_k\in\mathbb{N}\\ j_1+j_2+\dots+j_k=n}} |\alpha'_k\cdot\beta'_{j_1}\cdot\beta'_{j_2}\cdot\dots\cdot\beta'_{j_k}| \tag{5.16}$$

is convergent. By Lemma 4.7 and (5.10) we see that $\alpha'_j\le 0$ for all $j\in\mathbb{N}, j\ge 2$ and $\alpha'_1=\cos_p(0)=1$. Moreover, from (5.11) it follows that $\beta'_j\ge 0$ for all $j\in\mathbb{N}$. Thus the product $\alpha'_k\cdot\beta'_{j_1}\cdot\beta'_{j_2}\cdot\dots\cdot\beta'_{j_k}$ is positive if and only if $k=1$. All positive terms can be written as $\alpha'_1\cdot\beta'_n=\beta'_n$ for $n\in\mathbb{N}$ (if $k=1$ then $j_1=n$ is the only decomposition of n). Since the sum of all positive terms in (5.15) is $\sum_{n=1}^{+\infty}\beta'_n=\arcsin_p(1)=\frac{\pi_p}{2}<+\infty$, the sum of all negative terms must be finite too and equals $1-\frac{\pi_p}{2}$. Thus (5.16) converges. This means that the series (5.15) converges absolutely to 1 and any rearrangement of this series must converge. Also any subseries of any rearrangement of this series must converge absolutely. Let $s_M=\sum_{m=1}^{M}\beta'_m$. Then the series $\sum_{k=1}^{+\infty}\alpha'_k\cdot(s_M)^k$ is a subseries of one of the rearrangements of (5.15) and it is convergent. Observe that s_M is nondecreasing and converging to $\sum_{m=1}^{+\infty}\beta'_m=\pi_p/2$ as $M\to+\infty$. Thus the Maclaurin series for $\sin_p(x)=\sum_{k=1}^{+\infty}\alpha'_k\cdot x^k$ is convergent for any $x\in(-\pi_p/2,\pi_p/2)$ to some analytic function.

Now it remains to show that it converges towards $\sin_p(x)$ on $(-\pi_p/2,\pi_p/2)$. This last step follows from the formal identity (5.14), which on the established range of convergence holds also analytically and the fact that the function $\sin_p(x)$ is the only function that satisfies the identity (1.6). $\square$

Proof of Theorem 3.4. From [24, Thm. 1.1, consider $p = q$ and $\sigma = 0$] it follows that, for any $p > 1$, there exists a unique analytic function $F(z)$ near origin such that $\sin_p(x) = x \cdot F(|x|^p)$; thus we have

$$\sin_p(x) = x \cdot F(|x|^p) = \sum_{l=0}^{+\infty} \alpha_l \cdot x \cdot |x|^{l \cdot p} \quad , \text{ where } F(z) = \sum_{l=0}^{+\infty} \alpha_l \cdot z^l \, .$$

Note that for $p = 2m + 1$, $m \in \mathbb{N}$, the series

$$\sum_{l=0}^{+\infty} \alpha_l \cdot x^{l \cdot p+1} \tag{5.17}$$

defines an analytic function $G(x)$ in a neighborhood of $x = 0$ and also that

$$\sin_p(x) = \sum_{l=0}^{+\infty} \alpha_l \cdot x^{l \cdot p+1} = G(x) \quad \text{for } x > 0 \tag{5.18}$$

on a neighborhood of 0. Our aim is to show that the radius of convergence of (5.17) is $\pi_p/2$ for $p = 2m + 1$, $m \in \mathbb{N}$. By (5.18), the following derivatives are equal

$$\sin_p^{(n)}(x) = G^{(n)}(x) = \sum_{l=\lceil \frac{n-1}{p} \rceil}^{+\infty} \alpha_l \cdot \frac{(l \cdot p + 1)!}{(l \cdot p + 1 - n)!} x^{l \cdot p - n + 1}$$

for $x > 0$ on the neighborhood of 0 where the series converges. Now take a one-sided limit from the right in the previous equation

$$\lim_{x \to 0+} \sin_p^{(n)}(x) = \lim_{x \to 0+} \sum_{l=\lceil \frac{n-1}{p} \rceil}^{+\infty} \alpha_l \cdot \frac{(l \cdot p + 1)!}{(l \cdot p + 1 - n)!} x^{l \cdot p - n + 1}.$$

For $j := \frac{n-1}{p} \in \{0\} \cup \mathbb{N}$, we obtain

$$\lim_{x \to 0+} \sum_{l=j}^{+\infty} \alpha_l \cdot \frac{(l \cdot p + 1)!}{(l \cdot p + 1 - n)!} x^{l \cdot p - n + 1} = \alpha_j \cdot \frac{(j \cdot p + 1)!}{(j \cdot p + 1 - n)!} \, .$$

Thus

$$\lim_{x \to 0+} \sin_p^{(n)}(x) = \alpha_j \cdot \frac{(j \cdot p + 1)!}{(j \cdot p + 1 - n)!}$$

for $j \in \{0\} \cup \mathbb{N}$. By Lemma 4.7, $\lim_{x \to 0+} \sin_p^{(n)}(x) \le 0$ for $n \ge 2$, $p \in \mathbb{N}$ and $p \ge 3$. Thus $\alpha_j \le 0$ for $j \in \mathbb{N}, j > 1$.

The rest of the proof of the theorem is identical to the proof of Theorem 3.3 and we find that the convergence radius of the series (5.17) is $\frac{\pi_p}{2}$ for $p = 2m + 1$, $m \in \mathbb{N}$. The only difference against the proof of Theorem 3.3 is that the series (5.17) converges towards $\sin_p(x)$ only on $(0, \pi_p/2)$ for $p = 2m + 1$, $m \in \mathbb{N}$. Note that the series is still convergent on $(-\pi_p/2, 0)$ towards $G(x) \ne \sin_p(x)$ for $x < 0$. The changes in the proof are obvious and are left to the reader. $\square$

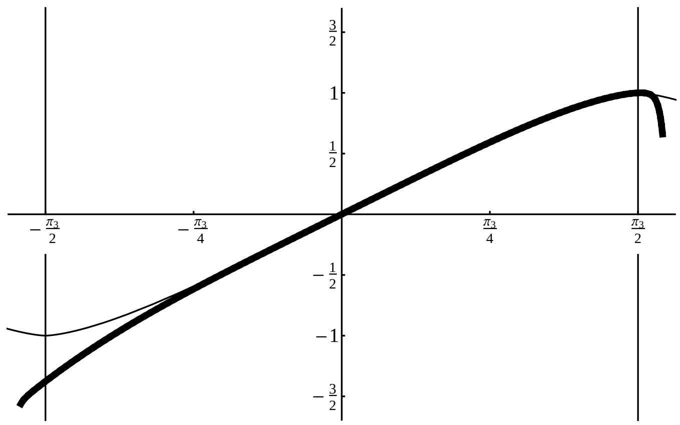

FIGURE 5. Graph of $\sin_3(x)$ obtained by high-precision numerical integration of (1.3) (thin line) versus graph of partial sum of the Maclaurin series for $\sin_3(x)$ up to the power x^{100} (thick line). Notice that the Maclaurin series does not converge to $\sin_3(x)$ for $x < 0$ and $x > \frac{\pi_3}{2}$

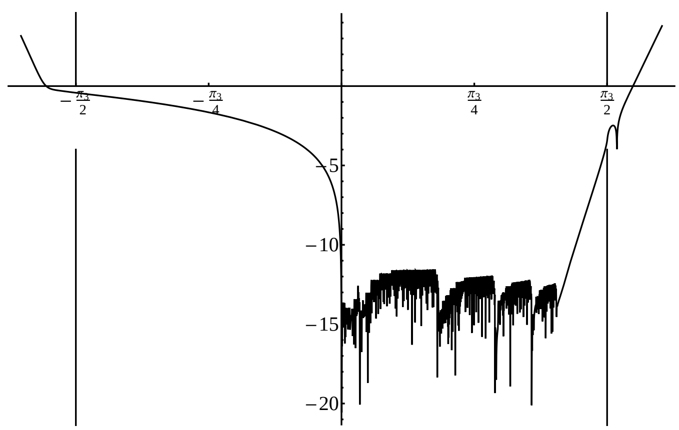

FIGURE 6. Graph of the function $\log_{10} |\sin_3(x) - \sum_{n=1}^{100} \alpha'_n x^n|$ where $\sum_{n=1}^{100} \alpha'_n x^n$ is the partial sum of the Maclaurin series of $\sin_3(x)$. The values of $\sin_3(x)$ were obtained by high-precision numerical integration of (1.3) using Mathematica command `NDSolve` with option `WorkingPrecision->50` which sets internal computations to be done up to 50-digit decadic precision. Notice that the Maclaurin series does not converge to $\sin_3(x)$ for $x < 0$ and $x > \pi_3/2$

6. Concluding remarks and open problems

As it was mentioned in the proofs of Theorems 3.3 and 3.4, it follows from [24, Thm. 1.1, consider $p = q$ and $\sigma = 0$] that, for any $p > 1$, there exists a unique

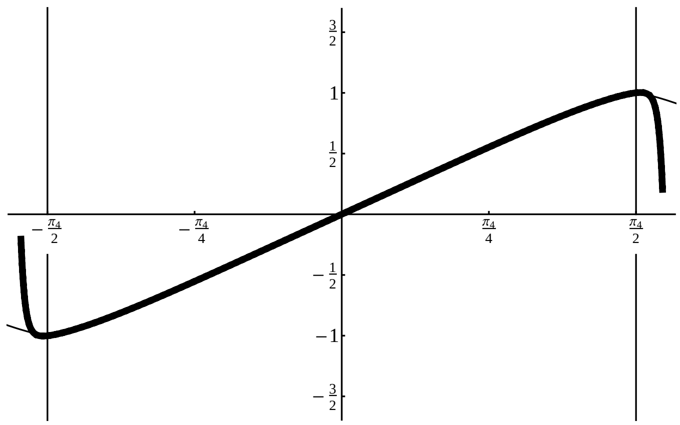

FIGURE 7. Graph of $\sin_4(x)$ obtained by high-precision numerical integration of (1.3) (thin line) versus graph of partial sum of the Maclaurin series for $\sin_4(x)$ up to the power x^{100} (thick line)

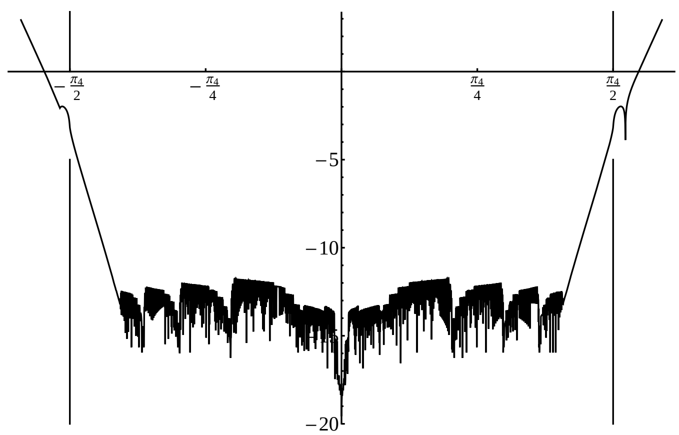

FIGURE 8. Graph of the function $\log_{10}|\sin_4(x) - \sum_{n=1}^{100} \alpha'_n x^n|$ where $\sum_{n=1}^{100} \alpha'_n x^n$ is the partial sum of the Maclaurin series of $\sin_4(x)$. The values of $\sin_4(x)$ were obtained by high-precision numerical integration of (1.3) using Mathematica command `NDSolve` with option `WorkingPrecision->50` which sets internal computations to be done up to 50-digit decadic precision. Notice that the Maclaurin series does not converge to $\sin_4(x)$ for $|x| > \pi_4/2$

analytic function $F(z)$ near origin such that

$$\sin_p(x) = x \cdot F(|x|^p)\,.$$

Thus the function $\sin_p(x)$ can be expanded into generalized Maclaurin series near the origin:

$$\sin_p(x) = x \cdot F(|x|^p) = \sum_{l=0}^{+\infty} \alpha_l \cdot x \cdot |x|^{l \cdot p}, \quad \text{where } F(z) = \sum_{l=0}^{+\infty} \alpha_l \cdot z^l\,.$$

Remark 6.1. (Convergence of generalized Maclaurin series) Let $p = 2m+1$ for $m \in \mathbb{N}$. It follows from the symmetry of the function $\sin_{2m+1}(x)$ with respect to the origin and from the proof of Theorem 3.4 that the generalized Maclaurin series $\sum_{l=0}^{+\infty} \alpha_l \cdot x \cdot |x|^{l\cdot(2m+1)}$ converges towards the values of $\sin_{2m+1}(x)$ on $(-\frac{\pi_{2m+1}}{2}, \frac{\pi_{2m+1}}{2})$.

Remark 6.2 (Complex argument for p even)**.** Let $p = 2(m+1)$ for $m \in \mathbb{N}$. It follows from the proof of Theorem 3.3 that the Maclaurin series $\sum_{l=0}^{+\infty} \alpha_l \cdot x^{l\cdot 2(m+1)+1}$ converges towards the values of $\sin_{2(m+1)}(x)$ on $(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$ absolutely. This enables us to extend the range of definition of the function $\sin_{2(m+1)}(x)$ to the complex open disc

$$B_m = \{z \in \mathbb{C} : |z| < \frac{\pi_{2(m+1)}}{2}\}$$

by setting $\sin_{2(m+1)}(z) := \sum_{l=0}^{+\infty} \alpha_l \cdot z^{l\cdot 2(m+1)+1}$. Since all the powers of z are of positive-integer order $l \cdot 2(m+1)+1$, the function $\sin_{2(m+1)}(z)$ is an analytic complex function on B_m and thus is single-valued. Unfortunately, this easy approach works only for $p = 2(m+1)$ with $m \in \mathbb{N}$; cf [15].

Our methods for proving convergence of the Maclaurin or generalized Maclaurin series are based on the fact that p is an integer. Thus a natural question appears.

Open Problem 6.3 (Convergence for $p > 1$ not integer)**.** Consider $p > 1$, $p \notin \mathbb{N}$. Prove (or find a counterexample) that the generalized Maclaurin series corresponding to $\sin_p(x)$ 'suggests the convergence' on $(-\pi_p/2, \pi_p/2)$ towards the values of $\sin_p(x)$.

For the sake of completeness, we remark that [15] claims the convergence of the generalized Maclaurin series on $(-\pi_p/2, \pi_p/2)$ for any $p > 1$, but there is no proof nor any indication for the proof of this claim.

Moreover, we are not able to decide about the convergence at the endpoints. This is another open question.

Open Problem 6.4 (Endpoints of the interval)**.** Consider $p > 1$. Prove (or find a counterexample) that the generalized Maclaurin series of $\sin_p(x)$ converge at $-\frac{\pi_p}{2}$ and/or $\frac{\pi_p}{2}$.

Remark 6.5 (Function $\cos_p$ for p even)**.** Let $p = 2(m+1)$ for $m \in \mathbb{N}$. Since $\cos_p(x) = \sin_p'(x)$ by definition, the Maclaurin series for $\cos_{2(m+1)}(x)$ can be obtained by taking into derivative the Maclaurin series for $\sin_{2(m+1)}(x)$ term by term. The Maclaurin series for $\cos_{2(m+1)}(x)$ then converges towards the value $\cos_{2(m+1)}(x)$ for any $x \in (-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$.

Remark 6.6 (Function $\cos_p$ for p odd)**.** Let $p = 2m+1$ for $m \in \mathbb{N}$. In this case the Maclaurin series for $\cos_{2m+1}(x)$ can also be obtained by taking into derivative the Maclaurin series for $\sin_{2m+1}(x)$ term by term. This Maclaurin series then converges for $x \in (-\frac{\pi_{2m+1}}{2}, \frac{\pi_{2m+1}}{2})$. However, the Maclaurin series for $\cos_{2m+1}(x)$ converges towards the value $\cos_{2m+1}(x)$ for $x \in [0, \frac{\pi_{2m+1}}{2})$, but it does not converge towards the value $\cos_{2m+1}(x)$ for any $x \in (-\frac{\pi_{2m+1}}{2}, 0)$.

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Petr Girg
Department of Mathematics, University of West Bohemia, Univerzitní 22, 30614, Plzeň, Czech Republic
E-mail address: pgirg@kma.zcu.cz

Lukáš Kotrla
Department of Mathematics, University of West Bohemia, Univerzitní 22, 30614, Plzeň, Czech Republic
E-mail address: kotrla@students.zcu.cz

Appendix A2

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GENERALIZED TRIGONOMETRIC FUNCTIONS IN COMPLEX DOMAIN

Petr Girg, Plzeň, Lukáš Kotrla, Plzeň

(Received October, 2013)

Abstract. In this paper we study extension of p-trigonometric functions $\sin_p$ and $\cos_p$ to complex domain. For $p = 4, 6, 8, \dots$, the function $\sin_p$ satisfies initial value problem which is equivalent to

$$(*) \qquad \begin{cases} -\left(u'\right)^{p-2} u'' - u^{p-1} &= 0\,, \\ u(0) &= 0\,, \\ u'(0) &= 1 \end{cases}$$

in $\mathbb{R}$. In our recent paper [2], we showed that $\sin_p(x)$ is a real analytic function for $p = 4, 6, 8, \dots$ on $(-\pi_p/2, \pi_p/2)$, where $\pi_p/2 = \int_0^1 (1-s^p)^{-1/p}$. This allows us to extend $\sin_p$ to complex domain by its Maclaurin series convergent on disc $\{z \in \mathbb{C}\colon |z| < \pi_p/2\}$. The question is whether this extensions $\sin_p(z)$ satisfies (*) in the sense of differential equations in complex domain. This interesting question was posed by Došlý and we show that the answer is affirmative. We also discuss difficulties concerning extension of $\sin_p$ to complex domain for $p = 3, 5, 7 \dots$. Moreover, we show that the structure of the complex valued initial value problem (*) does not allow entire solutions for any $p \in \mathbb{N}$, $p > 2$. Finally, we provide some graphs of real and imaginary parts of $\sin_p(z)$ and suggest some new conjectures.

Keywords: p-Laplacian, differential equations in complex domain, extension of $\sin_p$.

MSC 2010: 33E30, 34B15, 34M05, 34M99

1. Introduction

The initial value problem

$$(1.1) \qquad \begin{cases} -\left(|u'|^{p-2}u'\right)' - (p-1)|u|^{p-2}u &= 0\,, \\ u(0) &= 0\,, \\ u'(0) &= 1 \end{cases}$$

The research has been supported by the Grant Agency of the Czech Republic, project no. 13-00863S.

arises in connection with nonlinear boundary value problems for $p > 1$ (see e.g. [4, 5, 7]). The solution of (1.1) is known as $\sin_p$, see e.g. [4], and $\cos_p \overset{\text{def}}{=} \sin_p'$. Since the functions $\sin_p$ and $\cos_p$ satisfy well-known p-trigonometric identity, see e.g. [5],

$$|\sin_p(x)|^p + |\cos_p(x)|^p = 1\,, \tag{1.2}$$

they are also known as the p-trigonometric and/or generalized trigonometric functions. Note that (1.2) is in fact the so-called first integral of (1.1) (see e.g. [5]). It follows from this identity (see e.g. [5]) that

$$\int_0^{\sin_p(x)} (1-s^p)^{-1/p}\mathrm{d}s = x$$

for $0 \le x \le \pi_p/2$, where $\sin_p(x) \ge 0$ and $\cos_p(x) \ge 0$ and where

$$\pi_p \overset{\text{def}}{=} 2\int_0^1 (1-s^p)^{-1/p}\mathrm{d}s\,.$$

Thus it is natural to define

$$\operatorname{arcsin}_p(x) \overset{\text{def}}{=} \int_0^x (1-s^p)^{-1/p}\mathrm{d}s = \text{ for } 0 \le x \le 1\,, \tag{1.3}$$

and extend it to $[-1,1]$ as an odd function. The function $\sin_p$ is the inverse function to $\operatorname{arcsin}_p(x)$ on $[-\pi_p/2, \pi_p/2]$. Moreover, $\sin_p(x) = \sin_p(\pi_p - x)$ for $x \in (\pi_p/2, \pi_p]$ and $\sin_p(x) = -\sin_p(-x)$ for $x \in [-\pi_p, 0]$. Finally, $\sin_p(x) = \sin_p(x + 2\pi_p)$ for all $x \in \mathbb{R}$ (see [5] for details).

Smoothness of $\sin_p$ on $(-\pi_p/2, \pi_p/2)$ for $p > 1$ was studied in [2]. The most surprising result of [2] is that $\sin_p$ is a real analytic function on $(-\pi_p/2, \pi_p/2)$ for $p = 4, 6, 8 \ldots$, i.e., $\sin_p(x)$ equals to its Maclaurin on $(-\pi_p/2, \pi_p/2)$ for $p = 4, 6, 8 \ldots$. This approach naturally allows to extend $\sin_p$ for $p = 4, 6, 8 \ldots$ to an open disk

$$\{z \in \mathbb{C} : |z| < \pi_p/2\}$$

in the complex domain using power series (cf. [7], where the convergence of the series is conjectured without proof). When our recent result was presented in the conference "Nonlinear Analysis Plzeň 2013", O. Došlý posed an interesting question if this extension satisfies (1.1) in the sense of differential equations in complex domain. This paper addresses his question. For $p = 4, 6, 8, \ldots$, the initial value problem (1.1) in $\mathbb{R}$ is equivalent to

$$\begin{cases} -(u')^{p-2}\,u'' - u^{p-1} & = & 0\,, \\ u(0) & = & 0\,, \\ u'(0) & = & 1\,. \end{cases} \tag{1.4}$$

Note that for $p > 1$ real not being an even positive integer, we cannot get rid off the absolute values in (1.1). Thus the equation (1.1) does not make sense for general $p > 1$ in the complex domain. In this paper we consider the (1.4) in complex domain for integer $p > 2$. The complex valued ordinary differential equations are studied by means of power series (mostly by Maclaurin series). Note that, by [2, Theorem 3.2 on p. 5], $\sin_p^{(n)}(0)$ exists for $1 < n \le p$, but $\sin_p^{(n)}(0)$ *does not exist* when $p \ge 3$ is odd integer and $n > p$. Thus, by the formal Maclaurin series of $\sin_p(x)$, we mean a series calculated from the limits of the derivatives $\lim_{x\to 0+} \sin_p^{(n)}(x)$, which were shown to exist in [2] for any $n \in \mathbb{N}$ and $p \ge 3$ odd integer.

In Section 2, we prove that, for $p = 4, 6, 8, \dots$, the function $\sin_p$ can be extended by its Maclaurin series to the disc $\{z \in \mathbb{C}\colon |z| < \pi_p/2\}$ and that this series solves the ordinary differential equation (1.4) in the sense of differential equations in the complex domain. On the other hand, in Section 3, we show that the complex valued formal Maclaurin series $M_{\sin_p}(z)$ of the real function $\sin_p(x)$ does not satisfy (1.4) in the sense of differential equations in the complex domain for odd powers $p = 3, 5, 7, \dots$. In Section 4, we explain relations between the real and imaginary components of the complex valued function $\sin_p(z)$ for $p = 2, 6, 10, \dots$ and $p = 4, 8, 12, \dots$, and also the complex valued formal Maclaurin series $M_{\sin_p}(z)$ of the real function $\sin_p(x)$ for $p = 3, 5, 7, \dots$. In Section 5, we show that the fact that the function $\sin_p(z)$ cannot be extended as an entire function follows from an interesting connection between p-trigonometric identity and some classical results from complex analysis. Finally, in Section 6, we visualize some of our result.

In the whole paper, the independent variable z stands for a complex number and the independent variable x stands for a real number. In the same spirit, $\sin_p(z)$ stands for a complex valued function and $\sin_p(x)$ stands for a function of one real variable.

2. Extension of $\sin_p$ for $p = 4, 6, 8 \dots$ to complex domain.

We assume that $p = 4, 6, 8 \dots$ throughout this section unless specified differently. In [2, Thm. 3.3] we proved the following result.

Proposition 2.1 ([2], Theorem 3.3 on p. 6)**.** *Let $p = 4, 6, 8, \dots$. Then the Maclaurin series of $\sin_p(x)$ converges on $(-\pi_p/2, \pi_p/2)$.*

Let $M_{\sin_p}(x)$ denotes the formal Maclaurin series of $\sin_p(x)$, $p = 3, 4, 5, 6, \dots$ (any integer greater than 2). We also proved in the paper [2] that this Maclaurin series has the following particular structure

$$M_{\sin_p}(x) = \sum_{k=0}^{+\infty} \alpha_k x^{kp+1}\,, \tag{2.1}$$

where $\alpha_0 > 0$ and $\alpha_k \leq 0$ (all other coefficients are zero).

The following result answers the question by O. Došlý in the positive way.

Theorem 2.1. *Let $p = 4, 6, 8, \ldots$, then the unique solution of the initial value problem (1.4) on $|z| < \pi_p/2$ is the Maclaurin series (2.1).*

In order to prove this result, we need to state several auxilliar results. First of all, let us note that the equation (1.4) is of second order. In order to apply known theory, we rewrite (1.4) as an equivalent system. Using the substitution $u' = v$, we get the following first order system

$$\left\{ \begin{array}{rcl} u' &=& v\,, \\ v' &=& -u^{p-1}/v^{p-2}\,, \\ u(0) &=& 0\,, \\ v(0) &=& 1\,. \end{array} \right. \tag{2.2}$$

To study systems of equations like (2.2) in complex domain, we need to use complex functions of several variables. We will often make use of the following result.

Proposition 2.2 ([6], Theorem 16 on p. 33)**.** *Let f and g be holomorphic functions in open set $M \subset \mathbb{C}^r$, $r \in \mathbb{N}$. Then the functions $f+g$, $f-g$ and fg are holomorphic in M. Moreover if $g(z) \neq 0$ for all $z \in M$, then $\frac{f}{g}$ is holomorphic on M.*

Let us consider first order ODE system

$$\left\{ \begin{array}{rcl} \mathbf{y}' &=& \mathbf{f}(z, \mathbf{y})\,, \\ \mathbf{y}(z_0) &=& \mathbf{y}_0\,, \end{array} \right. \tag{2.3}$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in \mathbb{C}^n$ and $\mathbf{f} = (f_1(z, \mathbf{y}), f_2(z, \mathbf{y}), \ldots, f_n(z, \mathbf{y}))^{\mathrm{T}} \in \mathbb{C}^n$ and the function $\mathbf{f} : \mathbb{C}^{n+1} \to \mathbb{C}^n$ is analytic complex function of $n+1$ complex variables. The folowing result concerning existence and uniqueness of the initial values problem in the complex domain is crucial in our proofs.

Proposition 2.3 ([3], Theorem 9.1 on p. 76)**.** *Let function $\mathbf{f}\colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ be analytic and bounded in the region*

$$R\colon |z - z_0| < \alpha\,, \qquad \|\mathbf{w} - \mathbf{w_0}\| < \beta\,,$$

where $\alpha > 0$, $\beta > 0$, and let

$$\mu \overset{\text{def}}{=} \sup_{(z,\mathbf{w}) \in R} \|\mathbf{f}(z, \mathbf{w})\|\,, \qquad \gamma \overset{\text{def}}{=} \min\left(\alpha, \frac{\beta}{\mu}\right)\,.$$

Then there exists in the disk $D_0\colon |z - z_0| < \gamma$ a unique analytic function $\mathbf{w}\colon \mathbb{C} \to \mathbb{C}^n$ which is the solution of (2.3).

Lemma 2.1. *There is $\delta > 0$ such that in $U_0 \stackrel{\text{def}}{=} \{z \in \mathbb{C}\colon |z| < \delta\}$ the initial value problem* (1.4) *has the unique solution $u(z)$ which is an analytic function in U_0.*

Proof. Consider (2.2) in complex domain. Let us denote

$$f_1(z, \xi, \eta) \stackrel{\text{def}}{=} \eta$$

and (recall $p = 4, 6, 8, \dots$ by assumption of this section)

$$f_2(z, \xi, \eta) \stackrel{\text{def}}{=} -\frac{\xi^{2m+1}}{\eta^{2m}}\,, \text{ where } z,\ \xi,\ \eta \in \mathbb{C} \text{ and } m \in \mathbb{N}\,.$$

Naturally, the functions $f = \xi$ and $g = \eta$ are holomorphic in entire complex plane. Thus by Proposition 2.2, functions $f_1(z, \xi, \eta)$ and $f_2(z, \xi, \eta)$ are holomorphic on some neighborhood of $[0, 0, 1]$. Let R denote the maximal closed subset of this neigborhood. Then the functions f_1 and f_2 are holomorphic on the closed domain R and so they are continuous on R. Hence they are bounded on R (see [6], p. 37). Therefore, the system (2.2) has unique solution by Proposition 2.3. □

The previous lemma yields local solution $u(z)$ of (1.4) in a small neighborhood U_0 of 0 in $\mathbb{C}$. Since $u(z)$ is analytic in U_0, it can be written as a power series $u(z) = \sum_{k=0}^{\infty} a_k z^k$, where this power series converges towards $u(z)$ for all $z \in U_0$. Our next aim is to show that the series corresponding to $u(z)$ has the same coefficients as the series corresponding to $\sin_p(x)$, which is the unique solution to the real-valued initial value problem (1.1). For this purpose, we will use the following result concerning sum of two powers series.

Proposition 2.4 ([9], Theorem 16.6 on p. 352)**.** *If the sum of two power series in the variable $z - z_0$ coincide on a set of points E for which z_0 is a limit point and $z_0 \notin E$, then identical powers of $z - z_0$ have identical coefficients, i.e., there is a unique power series in the variable $z - z_0$ which has given sum on the set E.*

Now we are ready to prove the main result of this section.

Proof of Theorem 2.1. By Lemma 2.1, $u(z) = \sum_{k=0}^{\infty} a_k z^k$ is the unique solution of (1.4) in any point $z \in U_0$. Observe that the solution $u(z) = \sum_{k=0}^{\infty} a_k z^k$ solves also the real-valued Cauchy problem (1.4) in the sense of real analysis. On the other hand, $\sin_p$ is the unique solution of the real-valued Cauchy problem (1.4). Since the Maclaurin series (2.1) of $\sin_p$ converges towards $\sin_p$ in $(-\pi_p/2, \pi_p/2)$ under the assumption of this section, we find that (2.1) satisfies (1.4) in $(-\pi_p/2, \pi_p/2)$. Moreover, convergence of (2.1) on $(-\pi_p/2, \pi_p/2)$ implies convergence of $\sum_{k=0}^{\infty} \alpha_k z^{kp+1}$ for all $z \in \mathbb{C}$: $|z| < \pi_p/2$. Therefore,

$$\sum_{j=0}^{+\infty} a_j z^j = \sum_{k=0}^{+\infty} \alpha_k z^{kp+1} \text{ for all } z \in U_0 \cap (-\pi_p/2, \pi_p/2) \ .$$

Now we consider the set of points $z_n = \frac{\delta}{n+1}$, $n \in \mathbb{N}$. From the previous equation, we have

$$\sum_{j=0}^{+\infty} a_j z_n^j - \sum_{k=0}^{+\infty} \alpha_k z_n^{kp+1} = 0 = \sum_{j=0}^{+\infty} 0 \cdot z_n^j \, .$$

By Proposition 2.4, we find that these two series must coincide on U_0. Hence the Maclaurin series (2.1) satisfies (1.4) on U_0. Let u be given by the series (2.1). Then $u'', (u')^{p-2}, u^{p-1}$ have the radius of convergence $\pi_p/2$ for $p > 2, p \in \mathbb{N}$. Since any power series converges absolutely within the radius of its convergence, we see from (1.4) that

$$-\left[\left(\sum_{k=0}^{+\infty} \alpha_k z_n^{kp+1}\right)'\right]^{p-2} \left(\sum_{k=0}^{+\infty} \alpha_k z_n^{kp+1}\right)'' - \left(\sum_{k=0}^{+\infty} \alpha_k z_n^{kp+1}\right)^{p-1} = 0 = \sum_{j=0}^{+\infty} 0 \cdot z_n^j$$

for all $z_n = \frac{\delta}{n+1}$, $n \in \mathbb{N}$. Thus, by Proposition 2.4, u given by the series (2.1) is the solution of (1.4) on the disc $D = \{z \in \mathbb{C} \colon |z| < \pi_p/2\}$. $\square$

3. Obstacles for extension of $\sin_p$ for $p = 3, 5, 7\dots$ to complex domain.

Lindqvist [7] proposed alternative definition of $\sin_p$ as the solution of

$$\frac{\mathrm{d}}{\mathrm{d}z} (w')^{p-1} + w^{p-1} = 0\,, \quad w(0) = 0\,, \quad w'(0) = 1 \tag{3.1}$$

in complex domain for $p > 1$ (considered only formally). In [7, Section 7], he conjectures the possibility that solutions to (3.1) and real Cauchy problem

$$\left(|u'|^{p-2}\, u'\right)' + |u|^{p-2}\, u\,, \quad u(0) = 0\,, \quad u'(0) = 1 \tag{3.2}$$

could produce different solutions on $\mathbb{R}$. We address this question in this section. However, we have different definition of π_p and $\sin_p$ in this paper than in [7]. Turning to our definitions of π_p and $\sin_p$, we get an equation corresponding to (3.1):

$$\frac{\mathrm{d}}{\mathrm{d}z} (w')^{p-1} + (p-1) w^{p-1} = 0\,, \quad w(0) = 0\,, \quad w'(0) = 1 \tag{3.3}$$

which is equivalent to (1.4), which is equivalent to (2.2). Since the $p-1$-th power is multivalued complex function, we will limit ourselves to $p \in \mathbb{N}, p > 1$, in order to be able to perform rigorous analysis. The question is whether (3.3) produces a solution which is different from solution (1.1) on $\mathbb{R}$. In the previous section we proved that for $p = 4, 6, 8, \ldots$ (and of course for $p = 2$) the solutions of (3.3) and (1.1) are identical. Now we show that for $p = 3, 5, 7, \ldots$ the solutions are different for negative arguments.

This proposition is crucial for the proof of the main result of this section.

Proposition 3.1 ([2], Theorem 3.4 on p. 6)**.** *Let $p = 3, 5, 7, \ldots$. Then the formal Maclaurin series of $\sin_p(x)$ (the solution of the Cauchy problem (1.1)) converges on $(-\pi_p/2, \pi_p/2)$. Moreover, the formal Maclaurin series of $\sin_p(x)$ converges towards $\sin_p(x)$ on $[0, \pi_p/2)$, but does not converge towards $\sin_p(x)$ on $(-\pi_p/2, 0)$.*

Now we are ready to formulate main result of this section.

Theorem 3.1. *Let $p = 3, 5, 7 \ldots$. Then the unique solution $u(z)$ of the complex initial value problem (1.4) differs from the solution $\sin_p(x)$ of the Cauchy problem (1.1) for $z = x \in (-\pi_p/2, 0)$.*

Proof. Let us recall that (3.3) is equivalent to (2.2). There exists unique solution of (2.2) on some nonempty open disc in $\mathbb{C}$ containing 0 by Proposition 2.3. In the same way as in the proof of Theorem 2.1 (with obvious modifications), it follows that $M_{\sin_p}(z)$ solves (3.3) on the open disc $|z| < \pi_p/2$ and it is the unique solution on this disc. Since $\sin_p(x)$ is the unique solution of (1.1), $\sin_p(x) \neq M_{\sin_p}(x)$ for $x \in (-\pi_p/2, 0)$ by Proposition 3.1, we see that (1.1) and (3.3) produce different solutions on $\mathbb{R}$. $\square$

4. Relations between real and imaginary parts

Let us mention an interesting relationship between real and imaginary part of $\sin_p(z)$ for $p = 4, 8, 12, \ldots$. One can see in the Figure 1, that the graph of the imaginary part of $\sin_4(z)$ is the graph of the real part, rotated by $-\pi/2$.

Theorem 4.1. *Let $p = 4, 8, 12, \ldots$. Then*

$$\Re[\sin_p(z)] = \Im[\sin_p(\mathrm{i} \cdot z)]$$

for all $z \in \mathbb{C}$: $|z| < \pi_p/2$.

Proof. Note that by (2.1)

$$\sin_p(z) = \sum_{k=0}^{+\infty} \alpha_k z^{kp+1} = z \sum_{k=0}^{+\infty} \alpha_k z^{kp}$$

for $z \in \mathbb{C}\colon |z| < \pi_p/2$. We assume $p = 4l$ where $l = 1, 2, 3\dots$ and thus

$$\sin_p(z) = z\sum_{k=0}^{+\infty} \alpha_k z^{4kl}\,.$$

Substituting $\mathrm{i}\cdot z$ into this formula we find

$$\sin_p(\mathrm{i}\cdot z) = \mathrm{i}\cdot z\sum_{k=0}^{+\infty} \alpha_k (\mathrm{i}\cdot z)^{4kl} = \mathrm{i}\cdot\sum_{k=0}^{+\infty} \alpha_k z^{4kl+1} = \mathrm{i}\cdot \sin_p(z)\,.$$

Now the result easily follows from comparison of the real and imaginary parts of $\sin_p(z)$ and $\mathrm{i}\cdot \dot{\sin}_p(\dot{z})$. This ends the proof. □

Theorem 4.2. *Let $p = 2, 6, 10, 14\dots.$ Then for all $\varphi \in [0, 2\pi)$ there exists $z \in \mathbb{C}\colon |z| < \pi_p/2$ such that*

$$\Re[\sin_p(z)] \neq \Im[\sin_p(\mathrm{e}^{\mathrm{i}\varphi}\ \cdot\ z)]\,.$$

Proof. It is known from [2], that the series $M_{\sin_p}(z)$ has the form

$$M_{\sin_p}(z) = \sum_{k=0}^{+\infty} \alpha_k z^{k\,p+1}\,,$$

where the other coefficients are known to be zero. At first we show that $\alpha_0 = 1$ and $\alpha_1 = -\frac{1}{p(p+1)} < 0$ (cf e.g. [7]). In fact, evaluating the integral in (1.3), we see that

$$\operatorname{arcsin}_p(x) = \int_0^x (1-s^p)^{-1/p}\mathrm{d}s = {}_2\mathrm{F}_1\left(\frac{1}{p}, \frac{1}{p}, 1+\frac{1}{p}, x^p\right)x \qquad \text{for } 0 \le x \le 1\,,$$

where ${}_2\mathrm{F}_1$ is the Gauss's hypergeometric function. Using the known series

$${}_2\mathrm{F}_1(a, b, c, z) = \sum_{k=0}^{+\infty} \frac{(a)_k\,(b)_k\,z^k}{(c)_k\,k!} \qquad \text{for} |z| < 1\,,$$

where $(a)_k = \prod_{j=0}^{k}(a + k - 1)$ for any $a \in \mathbb{R}$ stands for the rising factorial, we find

$$\operatorname{arcsin}_p(w) = w\sum_{k=0}^{\infty} \frac{\left(\frac{1}{p}\right)_k^2 w^{kp}}{\left(1+\frac{1}{p}\right)_k k!} \quad \text{for } 0 < w < 1\,.$$

Hence

$$\operatorname{arcsin}_p(w) = w + \frac{1}{p(p+1)}w^{p+1} + O\left(w^{2p+1}\right) \quad \text{for } 0 < w < 1\,.$$

Denoting $w = \sin_p(x)$, we find

$$x = w + \frac{1}{p(p+1)} w^{p+1} + O\left(w^{2p+1}\right) ,$$

which yields

$$w = x - \frac{1}{p(p+1)} w^{p+1} + O\left(w^{2p+1}\right) . \tag{4.1}$$

Substituting (4.1) into itself we obtain

$$w = x - \frac{1}{p(p+1)} \left(x - \frac{1}{p(p+1)} + O\left(w^{2p+1}\right) \right)^{p+1} + O\left(w^{2p+1}\right) .$$

Hence

$$\sin_p(x) = x - \frac{1}{p(p+1)} x^{p+1} + O\left(w^{2p+1}\right) , \tag{4.2}$$

which gives desired formulas for $\alpha_1 = 1$ and $\alpha_2 = -\frac{1}{p(p+1)}$. With this at hand, we can write

$$\begin{aligned} M_{\sin_p}(z) &= z - \frac{1}{p(p+1)} z^{p+1} + \sum_{m=2}^{+\infty} \alpha_m z^{m\,p+1} = \\ &= z - \frac{z^{p+1}}{p(p+1)} - z^{2p+1} \sum_{m=0}^{+\infty} \alpha_{m+2} z^{m\,p} . \end{aligned} \tag{4.3}$$

Let $z = a$, $a \in \mathbb{R}\colon 0 < a < \pi_p/2$ for simplicity. Then $\phi_0 = \pi/2$ is the unique angle in $[0, 2\pi)$ such that $\Re[z] = \Im[\mathrm{e}^{\mathrm{i}\phi_0}\, z]$. Assumption on p of this theorem is that there exists $l \in \mathbb{N} \cup \{0\}$ such that $p = 4l + 2$. Thus $\Re[z^{p+1}] = \Re[z^{4l+3}] = \Re[a^{4l+3}]$. On the other hand, $\Im[(\mathrm{e}^{\mathrm{i}\phi_0}\, z)^{p+1}] = \Im[(\mathrm{i}a)^{4l+3}] = -a^{4l+3}$ for $\phi_0 = \pi/2$. Pluging $z = a$ and $z = \mathrm{i}a$ into (4.3), taking real and imaginary part, respectively, and subtracting, we get

$$\begin{aligned} &\Re\left[M_{\sin_p}(a)\right] - \Im[M_{\sin_p}(\mathrm{i}a)\right] = \\ &\quad = -\frac{2a^{p+1}}{p(p+1)} + a^{2p+1} \left(\Re\left[\sum_{m=0}^{+\infty} \alpha_{m+2} a^{m\,p} \right] - \Im\left[i^{2p+1} \sum_{m=0}^{+\infty} \alpha_{m+2} (\mathrm{i}a)^{m\,p} \right] \right) \end{aligned} \tag{4.4}$$

Since the series on the right-hand side are convergent on disc $\{z \in \mathbb{C}\colon |z| < \pi_p/2\}$, then

$$A \overset{\text{def}}{=} \max_{\{z\in\colon\, |z| \le \pi_p/4\}} \left| \left(\Re\left[\sum_{m=0}^{+\infty} \alpha_{m+2} z^{m\,p} \right] - \Im\left[i^{2p+1} \sum_{m=0}^{+\infty} \alpha_{m+2} (\mathrm{i}z)^{m\,p} \right] \right) \right| < +\infty$$

exists and from (4.4) we find

$$\left| \left(\Re \left[M_{\sin_p}(a) \right] - \Im [M_{\sin_p}(\mathrm{i}a)] \right) / a^{p+1} - \frac{2}{p(p+1)} \right| \leq A a^p \,.$$

Taking $0 < a < \min \left\{ \pi_p/4, \left(\frac{1}{Ap(p+1)} \right)^{1/p} \right\}$, we see that $\Re \left[M_{\sin_p}(a) \right] - \Im [M_{\sin_p}(\mathrm{i}a)] \neq 0$. This concludes the proof. $\square$

5. Consequence of complex p-trigonometric identity

As it was mentioned earlier, the maximal possible radius of convergence for the (formal) Maclaurin series for functions $\sin_p$ and $\cos_p$ is $\pi_p/2$. This fact was anticipated in [7] and studied in detail in [2]. In this section we explain that there was no hope for these series to have their radius of convergence infinite for $p = 3, 4, 5, 6, \ldots$. To the contrary what one would think, we will show that it is not the absolute value in (1.1) that produces the main difficulty. It is a complex analogy of the p-trigonometric identity that produces the impossibility of $\sin_p$ to be an entire complex functions for $p = 3, 4, 5, 6, \ldots$.

Let us reconsider (1.4), i.e.,

$$\begin{cases} -(u')^{p-2}\, u'' - u^{p-1} & = & 0 \,, \\ u(0) & = & 0 \,, \\ u'(0) & = & 1 \,, \end{cases}$$

now for any $p = 3, 4, 5, 6, \ldots$ in a complex domain. Let us assume that u is a solution which is a holomorphic function on some neighborhood U_0 of 0. Multiplying the equation of (1.4) by u' and integrating from 0 to $z \in U_0$, we obtain

$$(u'(z))^p - (u'(0))^p + (u(z))^p - (u(0))^p = 0 \,.$$

Now using the initial conditions of (1.4) we get

$$(u'(z))^p + (u(z))^p = 1 \,, \tag{5.1}$$

which is the first integral of (1.4) and we can think of it as complex p-trigonometric identity for holomorphic solutions of (1.4) for $p = 3, 4, 5, 6, \ldots$.

Now we state the very classical result from complex analysis.

Proposition 5.1 ([1], Theorem 12.20 on p. 433)**.** *Let f and g be entire functions and for some positive integer satisfy identity*

$$f^n + g^n = 1 \,.$$

(i) *If* $n = 2$*, then there is an entire function* h *such that* $f = \cos \circ h$, $g = \sin \circ h$.

(ii) *If* $n > 2$*, then* f *and* g *are each constant.*

It follows from this result that holomorphic solution u of (1.4) cannot be entire function for any $p = 3, 4, 5, 6, \ldots,$ since the derivative of entire function is entire function as well and u and u' must satisfy (5.1). Thus by Proposition 5.1 u and u' are constant which contradicts $u'(0) = 1$.

In particular for $p = 4, 6, 8, \ldots,$ with $u(z) = \sin_p(z)$ and $u'(z) = \cos_p(z)$ this becomes

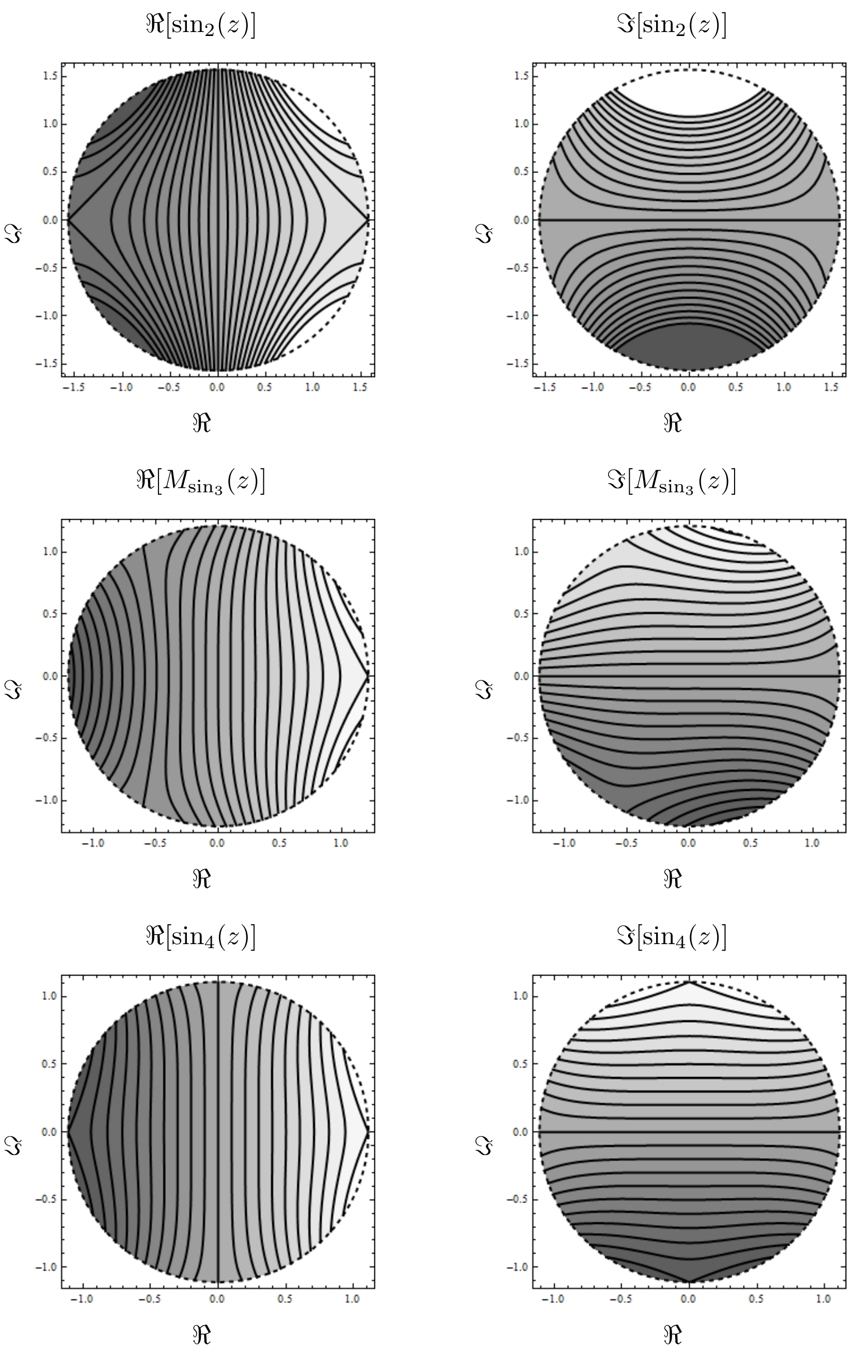
$$\cos_p^p(z) + \sin_p^p(z) = 1$$

and we see that $\sin_p$ and $\cos_p$ cannot be entire functions.

Note that it was an interesting internet discussion [11] that called our attention towards this connection between complex analysis (including the classical reference [1, Thm. 12.20]) and p-trigonometric functions. It seems to us that this connection was overlooked by the 'p-trigonometric community'.

6. Visualization of $\sin_p(z)$ and their Maclaurin series

In this section we visualize graphs of extension of $\sin_p(z)$ by its Maclaurin series for $p = 4, 6$ and the formal Maclaurin series for $p = 3, 5, 7$ and compare it to the classical result $\sin_p(z) = \sin_2(z)$. To the best of our knowledge, these figures in complex domain are new and we believe that they will help to stimulate discussion on this topic. We also formulate some conjectures in the caption of Figure 3. The authors would like to thank to O. Marichev [8] from Wolfram Research, for his valuable advices concerning series representation of functions and their inverses in the software package *Mathematica*®.



$\Re[\sin_2(z)]$
$\Im[\sin_2(z)]$
$\Re[M_{\sin_3}(z)]$
$\Im[M_{\sin_3}(z)]$
$\Re[\sin_4(z)]$
$\Im[\sin_4(z)]$
$\Re$
$\Im$

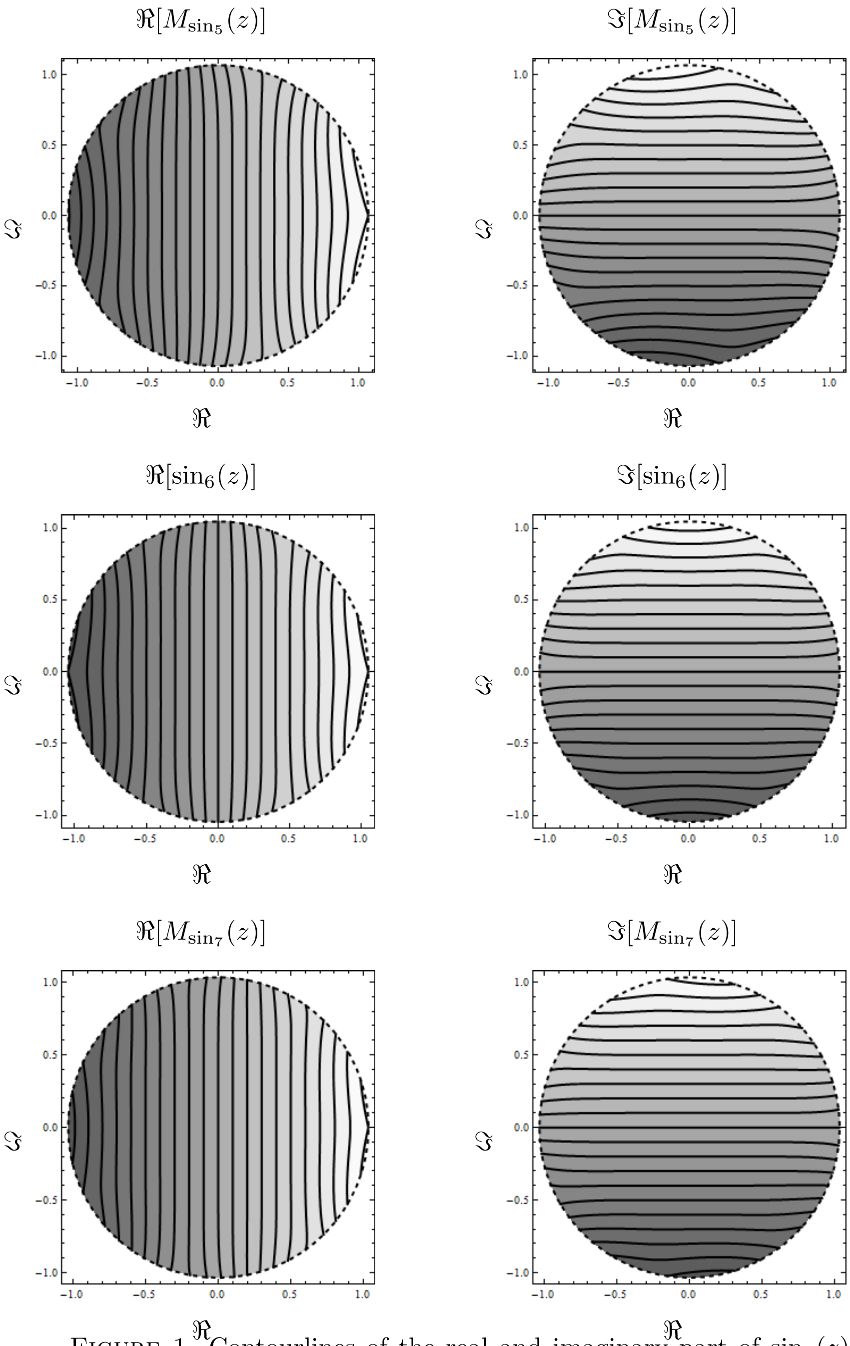


FIGURE 1. Contourlines of the real and imaginary part of $\sin_p(z)$ for $p = 2, 4, 6$ and $M_{\sin_p}(z)$ for $p = 3, 5, 6$. Note that imaginary part of $\sin_4(z)$ is its real part rotated by $-\pi/2$.

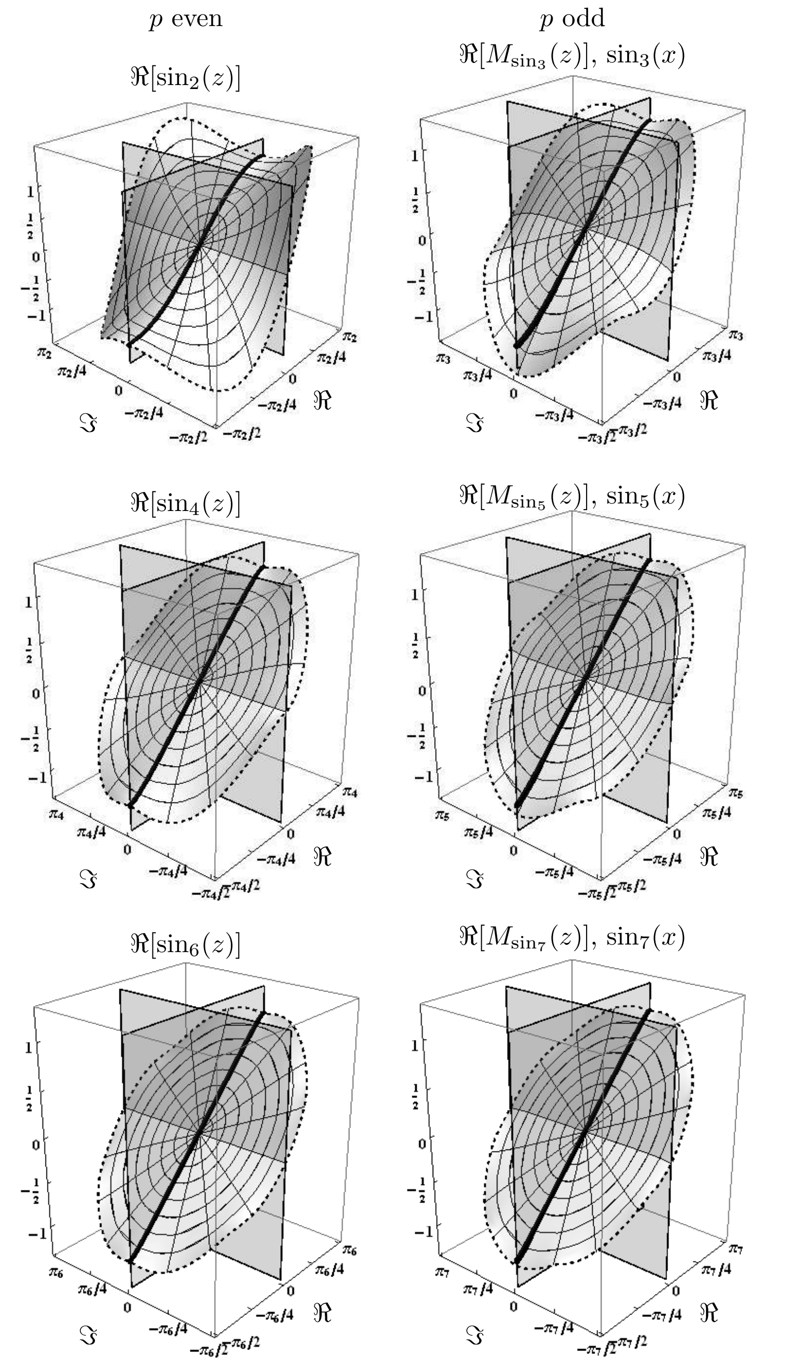


FIGURE 2. Comparison of real parts of $\sin_p(z)$ for p even (extended by the Maclaurin series) and the real parts of the formal Maclaurin series $M_{\sin_p}(z)$ and the real function $\sin_p(x)$ for p odd.

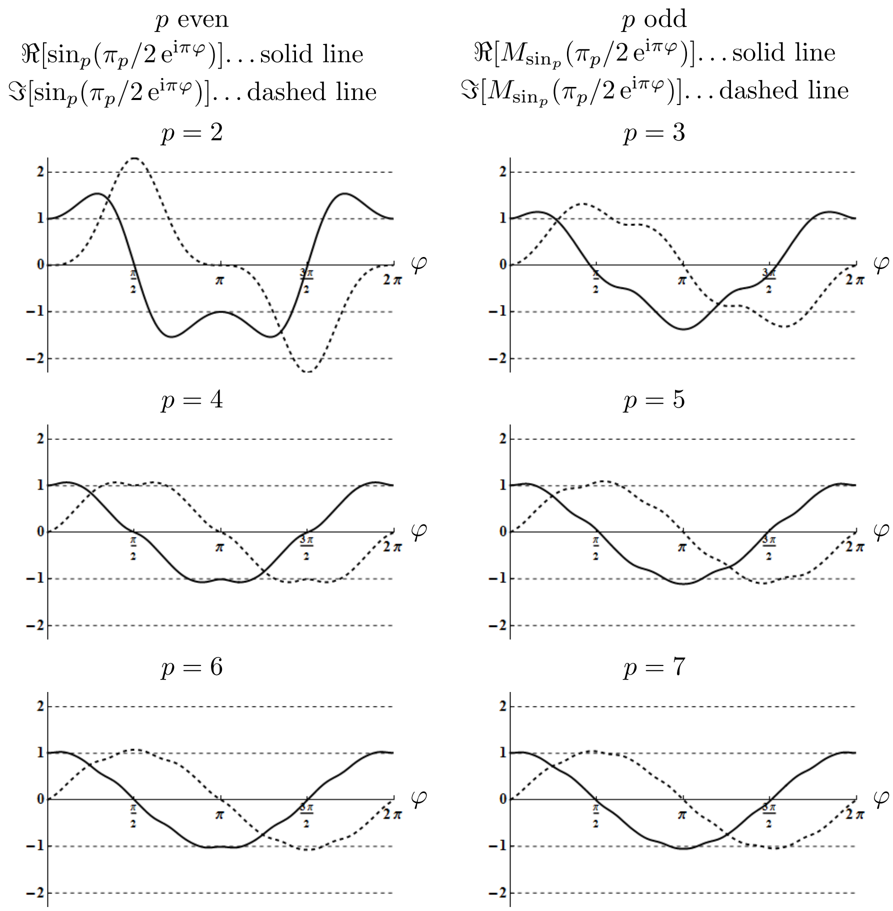


FIGURE 3. Numerical comparison of the real and the imaginary parts of $\sin_p(\pi_p/2\,\mathrm{e}^{\mathrm{i}\pi\varphi})$ for $p = 2, 4, 6$ (extended by Maclaurin series) and the real and the imaginary parts of $M_{\sin_p}(\pi_p/2\,\mathrm{e}^{\mathrm{i}\pi\varphi})$ for $p = 3, 5, 7$. Note that these graphs are only an illustration, because we do not know about the convergence of the series for $z \in \mathbb{C}\colon |z| = \pi_p/2$. From these pictures we conjecture this convergence. It is interesting to note at these pictures that for larger p, the graph of real part is a small perturbation of $\pi_p/2\cos\phi$ and the graph of imaginary part is a small perturbation of $\pi_p/2\sin\phi$. We conjecture that this phenomena occurs due to the fact that the Maclaurin series $M_{\sin_p}(z) = z - \frac{1}{p(p+1)}z^{p+1} + O(z^{2p+1})$ and for large p the higher order terms are negligible. Moreover, $\lim_{p\to+\infty}\pi_p/2 = 1$. Thus we conjecture that these graphs tend to graphs of $\sin\phi$ and $\cos\phi$ for $p \to +\infty$, respectively.

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Authors' addresses: *Petr Girg*, Department of Mathematics and NTIS, University of West Bohemia, Plzeň, Czech Republic e-mail: `pgirg@kma.zcu.cz`. *Lukáš Kotrla*, Department of Mathematics and NTIS, University of West Bohemia, Plzeň, Czech Republic e-mail: `kotrla@ntis.zcu.cz`.