SURFACE FLATTENING BASED ON CONSTRAINT GLOBAL OPTIMIZATION

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ABSTRACT

In this paper, the problem of generating a planar development of arbitrary three-dimensional surfaces is addressed. A new method based on a global optimization process under constraints is proposed. In this method an initial planar development is derived which is refined in order to satisfy certain criteria and constraints. The refinement is formulated as a global minimization problem. Using the proposed technique it is not required to predetermine a mapping from the three-dimensional surface to the plane in order to generate the planar development and it is possible to control the local accuracy in the derived planar development. Indicative applications are presented to illustrate the effectiveness of the proposed technique.

Keywords: planar development, doubly-curved surfaces, computer-aided design, constrained optimization.

1. INTRODUCTION

The generation of planar developments of arbitrarily curved surfaces is a well-known problem in the Computer-Aided Design community. A variety of methods are proposed for the flattening of arbitrarily curved surfaces which depend on the kind of the surface, the properties of the material, and whether external forces are required to form the 3D shape. For single-curved surfaces (zero Gaussian curvature), i.e. cylinders, cones or tangential developables, techniques from analytical and constructive differential geometry are usually utilized to generate their planar developments. On the other hand, when the surface is doubly-curved (non-zero Gaussian curvature) several factors, such as Gaussian or geodesic curvature, are taken into consideration for generating the corresponding planar developments.

One category of methods[Elber95-Bodduluri94] for generating planar developments of doubly-curved surfaces is based on the decomposition of the surfaces into simpler parts that are approximated by developable surfaces which are then unrolled into a plane. The initial surface can be fabricated by assembling the sets of developable surfaces which have been cut from planar sheets and rolled back to their Euclidean locations.

Other approaches [Hinds91-Azariadis97] for generating planar developments take into consideration certain inner geometrical properties of the three-dimensional surface. Hinds et al. [Hinds91] proposed a technique for generating radial patterns suitable for woven fabrics based on the concept of Gaussian curvature. A technique for refining planar developments suitable for creating surfaces using composite laminate materials is proposed by Parida and Mudur [Parida93]. In their algorithm, an approximate planar development is obtained where the gaps and overlaps are reoriented to satisfy the orientation constraints. A class of mappings based on an isometric tree is investigated by Manning [Manning80] in order to derive an optimal mapping for flattening curved surfaces. A generalized method

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for generating planar developments of arbitrarily curved surfaces in three-stages was proposed by Azariadis and Aspragathos [Azariadis97]. The first two stages include the generation of a proper initial development, which is refined in the third stage by reducing the gaps and the overlaps of the planar development.

Another category of methods [Maillot93-Bennis91] for generating planar developments of three-dimensional surfaces is based on the formulation and minimization of an objective function usually called energy function. Maillot el al. [Maillot93] proposed a surface flattening algorithm taking into consideration the difference of the edges and the signed area between the surface triangles and the corresponding triangles in the planar development. Their method is an extension of the method proposed by Ma and Lin [Ma88]. Another method [Bennis91] was proposed by Bennis et al., where an iterative procedure is used for minimizing an energy function that expresses the difference of the geodesic curvature between the vertices of the three-dimensional surface and the corresponding vertices of the planar development.

In this paper, a new method for generating a planar development of three-dimensional doubly curved surface is presented. The proposed method is based on the formulation and minimization of an energy function. An initial planar development is generated using the techniques proposed in the first two stages of the method proposed by Azariadis and Aspragathos [Azariadis97] or using the methods for surface flattening proposed in [Hinds91,Parida93]. Then, a global minimization problem is resolved in order to eliminate the gaps and overlaps of the planar development keeping the deformation under a predefined tolerance using certain constraints. In the proposed technique it is not required to predetermine a mapping from the three-dimensional surface to the plane in order to generate the planar development and it is possible to control the local accuracy in the derived planar development. This is the main advantage of the proposed method compared to other methods based on optimization. Furthermore, the introduced method improves the techniques presented in [Azariadis97] for the refinement of the initial planar development. These techniques reduce the gaps and overlaps in the planar development in a local manner where the results of the previous minimization affect the accuracy in the next minimization and thus it is not ensured that the final result will be satisfying. Furthermore, these techniques do not provide the means to handle the local accuracy in the planar development.

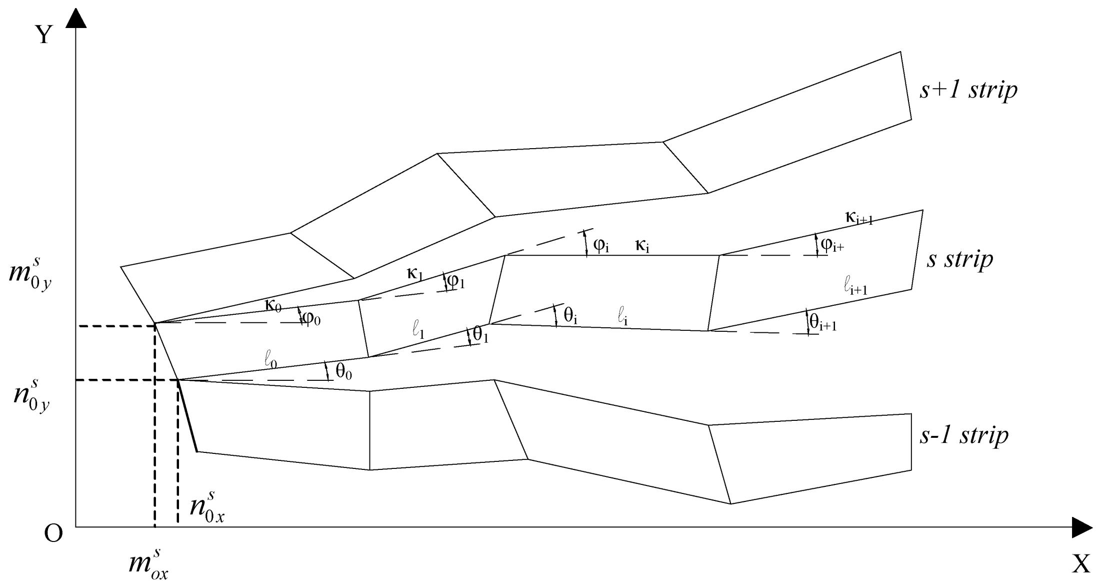
In the rest of this paper, it is considered that the initial three-dimensional surface is approximated by a homogeneous grid of triangular elements. This means that the surface is approximated by a grid of $n \times m$ vertices where each vertex is defined by the intersection of a u-isoparametric curve and a v-isoparametric curve of the $\mathbf{x} = \mathbf{x}(u, v)$ surface, $u, v \in \Re$. A surface strip is a list of triangles defined along two adjacent u or v isoparametric curves. Thus, each surface is divided in $(n-1)$ and $(m-1)$ strips of triangles. Finally, the Euclidean norm of a vector $\mathbf{x}$ is denoted as $\|\mathbf{x}\|$.

2. MATHEMATICAL FORMULATION

2.1. The Energy Function

Using the three-stage method [Azariadis97] a strip of the three-dimensional surface is selected to be flattened first into a plane. This strip is called guide-strip. The surface strips of the other parametric direction are transferred into the plane using the previously flattened guide-strip. The result of this process is a planar development having the same number of strips with the three-dimensional surface, but due to the surface curvature these flattened strips present gaps and overlaps between them. A measure of the size of these gaps and overlaps is expressed using an energy function, where its minimization leads to a planar development with reduced gaps and overlaps.

Lets consider three adjacent flattened strips of a planar development as it is shown in Fig.1. Without loss of generality, the figure assumes the strips do not overlap. All strips have the same number of triangles, which means that each strip has a constant number of vertices $(\mathbf{n}_\mathrm{i}^\mathrm{s}, \mathbf{m}_\mathrm{i}^\mathrm{s})$, where s denotes the strip number, and $i, 0 \le i < N$ the i-vertex of the strip. For each strip the pair $(\mathbf{n}_0^\mathrm{s}, \mathbf{m}_0^\mathrm{s})$ is fixed. Let ℓ_i^s, κ_i^s and θ_i^s, φ_i^s, $0 \le j < N-1$ the length and the angle of the line segments defined between two sequential $\mathbf{n}_\mathrm{i}^\mathrm{s}$ and $\mathbf{m}_\mathrm{i}^\mathrm{s}$ vertices, respectively (see Fig.1). Given such a configuration, a pair of points $(\mathbf{n}_\mathrm{i}^\mathrm{s}, \mathbf{m}_\mathrm{i}^\mathrm{s})$ is obtained by,



Three adjacent flattened strips of a doubly-curved surface.
Figure 1.

$$n_{i,x}^{s} = n_{0,x}^{s} + \sum_{j=0}^{i-1} \ell_{j}^{s} \cos(\sum_{\rho=0}^{j} \theta_{\rho}^{s} + \omega_{\rho}^{s})$$
$$n_{i,y}^{s} = n_{0,y}^{s} + \sum_{j=0}^{i-1} \ell_{j}^{s} \sin(\sum_{\rho=0}^{j} \theta_{\rho}^{s} + \omega_{\rho}^{s}) \quad (1)$$

and

$$m_{i,x}^{s} = m_{0,x}^{s} + \sum_{j=0}^{i-1} \kappa_{j}^{s} \cos(\sum_{\rho=0}^{j} \varphi_{\rho}^{s} + \omega_{\rho}^{s})$$
$$m_{i,y}^{s} = m_{0,y}^{s} + \sum_{j=0}^{i-1} \kappa_{j}^{s} \sin(\sum_{\rho=0}^{j} \varphi_{\rho}^{s} + \omega_{\rho}^{s}) \quad (2)$$

where, $\mathbf{n}_{\mathbf{i}}^{\mathbf{s}} = (n_{i,x}^{s}, n_{i,y}^{s})$, $\mathbf{m}_{\mathbf{i}}^{\mathbf{s}} = (m_{i,x}^{s}, m_{i,y}^{s})$, ω_{ρ}^{s} is an angular displacement (see below) and $i = 1 \ldots N-1$.

An index of the size of the gaps and overlaps between the s-strip and its two neighbor strips is proposed through the following formula:

$$f_{s} = \sum_{i=1}^{N} \left[\left\| \mathbf{n}_{\mathbf{i}}^{\mathbf{s}} - \mathbf{m}_{\mathbf{i}}^{\mathbf{s-1}} \right\|^{2} + \left\| \mathbf{m}_{\mathbf{i}}^{\mathbf{s}} - \mathbf{n}_{\mathbf{i}}^{\mathbf{s+1}} \right\|^{2} \right] \quad (3)$$

The summary of f_s for all the strips of the planar development indicates the size of the gaps and overlaps between the flattened strips of the surface planar development and is given by,

$$\mathrm{E}(\boldsymbol{\omega}) = \sum_{s} f_{s} \quad (4)$$

Since θ_{i}^{s}, φ_{i}^{s} are known fixed angles, it is clear that E is a function of the angular displacements ω_{i}^{s} of each s-strip respectively. The actual number of the angular displacements is,

$$N_{p} = \begin{cases} (n - numFixed - 1) \times (m-2), \\ \text{if the u - parametric direction is} \\ \text{choosen for flattening} \\ \qquad or \\ (n-2) \times (m - numFixed - 1), \\ \text{if the v - parametric direction is} \\ \text{choosen for flattening} \end{cases} \quad (5)$$

where, numFixed is the number of the strips that remain invariant during the optimization process (also referred as fixed strips), and $0 \le numFixed < n$ or $0 \le numFixed < m$.

The unknown angles ω_{i}^{s} are expressed using a N_p-dimensional vector $\boldsymbol{\omega}$, such as

$$\boldsymbol{\omega} = \left[\underbrace{\ldots \; [\omega_{0}^{s} \;\; \omega_{1}^{s} \;\; \ldots \;\; \omega_{m-2}^{s}] \;\; [\omega_{0}^{s+1} \;\; \omega_{1}^{s+1} \;\; \ldots \;\; \omega_{m-2}^{s+1}] \;\; \ldots}_{n-numFixed-1} \right] \quad (6)$$

The gradient of E is computed through the gradient of f_s. The gradient of f_s with respect to ω_ρ^s ($0 \le \rho \le N-1$) is given by,

$$\frac{\partial f_s}{\partial \omega_\rho^s} = 2\sum_{i=1}^{N}\left[\begin{array}{l}\left(n_{i,x}^s - m_{i,x}^{s-1}\right)\frac{\partial n_{i,x}^s}{\partial \omega_\rho^s} + \left(n_{i,y}^s - m_{i,y}^{s-1}\right)\frac{\partial n_{i,y}^s}{\partial \omega_\rho^s} + \\ \left(m_{i,x}^s - n_{i,x}^{s+1}\right)\frac{\partial m_{i,x}^s}{\partial \omega_\rho^s} + \left(m_{i,y}^s - n_{i,y}^{s+1}\right)\frac{\partial m_{i,y}^s}{\partial \omega_\rho^s}\end{array}\right] \quad (7)$$

where

$$\begin{aligned}\frac{\partial n_{i,x}^s}{\partial \omega_\rho^s} &= \sum_{j=\rho}^{i-1}\left[-\ell_j \sin\left(\sum_{\lambda=0}^{j}(\theta_\lambda^s + \omega_\lambda^s)\right)\right] \\ \frac{\partial n_{i,y}^s}{\partial \omega_\rho^s} &= \sum_{j=\rho}^{i-1}\left[\ell_j \cos\left(\sum_{\lambda=0}^{j}(\theta_\lambda^s + \omega_\lambda^s)\right)\right]\end{aligned} \quad (8)$$

and

$$\begin{aligned}\frac{\partial m_{i,x}^s}{\partial \omega_\rho^s} &= \sum_{j=\rho}^{i-1}\left[-\kappa_j \sin\left(\sum_{\lambda=0}^{j}(\varphi_\lambda^s + \omega_\lambda^s)\right)\right] \\ \frac{\partial m_{i,y}^s}{\partial \omega_\rho^s} &= \sum_{j=\rho}^{i-1}\left[\kappa_j \cos\left(\sum_{\lambda=0}^{j}(\varphi_\lambda^s + \omega_\lambda^s)\right)\right]\end{aligned} \quad (9)$$

The minimization of the energy function expressed by Eq.(4) with respect to the angular displacements $\boldsymbol{\omega}$ leads to a planar development with the minimal gaps and overlaps. Using the proposed energy function the length of the planar development strips are kept invariant. This means that one of the two characteristics of all the isomorphic [Parida93] developments that can be derived from a surface is preserved.

2.2. Definition of the constraints

Using the energy function introduced in the previous section it is possible to generate a planar development without gaps and overlaps among its strips. However, during the minimization process each triangle in the planar development suffers a deformation. This deformation appears in the diagonal edge $\boldsymbol{\delta}_\mathbf{i}^\mathbf{s} = \mathbf{n}_\mathbf{i}^\mathbf{s}\mathbf{m}_\mathbf{i+1}^\mathbf{s}$ ($i \le 0 \le N-2$) of each triangle, since the length of the edges defined between two successive $\mathbf{n}_\mathbf{i}^\mathbf{s}$ and $\mathbf{m}_\mathbf{i}^\mathbf{s}$ points is kept invariant. Thus, it is possible to control the deformation of the two triangles that share this common edge by keeping the deformation of $\boldsymbol{\delta}_\mathbf{i}^\mathbf{s}$ under a predefined threshold. Let $\delta_i^s = \left\|\boldsymbol{\delta}_\mathbf{i}^\mathbf{s}\right\|^2$ and $\Delta_i^s = \left\|\mathbf{N}_\mathbf{i}^\mathbf{s}\mathbf{M}_\mathbf{i+1}^\mathbf{s}\right\|^2$ be the square length of the diagonal $\boldsymbol{\delta}_\mathbf{i}$ and its corresponding diagonal $\boldsymbol{\Delta}_\mathbf{i}^\mathbf{s} = \mathbf{N}_\mathbf{i}^\mathbf{s}\mathbf{M}_\mathbf{i+1}^\mathbf{s}$ in the three-dimensional surface. The overall deformation of the s-strip is given by,

$$g_s = g_s(\boldsymbol{\omega}) = \sum_{i=0}^{N-1}\frac{\left(\Delta_i^s - \delta_i^s\right)^2}{\Delta_i^s} \quad (10)$$

Let $\varepsilon_i^s \in \Re$ be the maximum allowable deformation of the diagonal $\boldsymbol{\delta}_\mathbf{i}^\mathbf{s}$. Then it holds,

$$\frac{\sqrt{\Delta_i^s} - \sqrt{\delta_i^s}}{\sqrt{\Delta_i^s}} \le \varepsilon_i^s \Leftrightarrow \frac{\Delta_i^s - \delta_i^s}{\sqrt{\Delta_i^s}\left(\sqrt{\Delta_i^s} + \sqrt{\delta_i^s}\right)} \le \varepsilon_i^s \Leftrightarrow$$

$$\frac{\left(\Delta_i^s - \delta_i^s\right)^2}{\Delta_i^s} \le \left(\sqrt{\Delta_i^s} + \sqrt{\delta_i^s}\right)^2\left(\varepsilon_i^s\right)^2 \Leftrightarrow$$

$$\sum_{i=0}^{N-1}\left[\left(\sqrt{\Delta_i^s} + \sqrt{\delta_i^s}\right)^2\left(\varepsilon_i^s\right)^2\right] - g_s(\boldsymbol{\omega}) \ge 0 \quad (11)$$

Thus, given the tolerances $\varepsilon_i^s \in \Re$ the constraint which corresponds to the s-strip is given by,

$$h_s(\boldsymbol{\omega}) = \sum_{i=0}^{N-1}\left[\left(\sqrt{\Delta_i^s} + \sqrt{\delta_i^s}\right)^2\left(\varepsilon_i^s\right)^2\right] - g_s(\boldsymbol{\omega}) \quad (12)$$

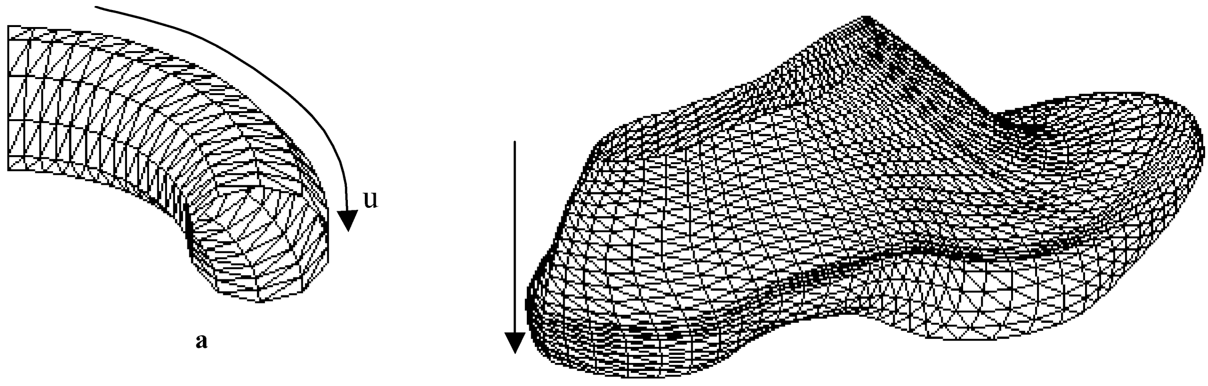
Therefore, the overall optimization problem is written as,

$$\min_{\boldsymbol{\omega} \in R^{N_p}} E(\boldsymbol{\omega}) \quad (13)$$

subject to,

$$h_s(\boldsymbol{\omega}) \ge 0,\ s = 1,\ldots,N_s \quad (14)$$

Where N_s is the number of the strips. This is a non-linear minimization problem with non-linear constraints. This problem is solved using the successive quadratic programming method with very satisfying results. This method is based on the



(a) The triangulated surface of a torus quadrant and (b) of the shoe last.
Figure 2

iterative formulation and solution of quadratic programming subproblems obtained using a quadratic approximation of the Lagrangian and by linearizing the constraints [Schittkowski86].

Generally, the constraints imposed by Eq.(14) do not ensure that the i-triangle's deformation shall be under ε_i^s after the minimization process. However, these constraints ensure that the overall strip deformation $g_s(\boldsymbol{\omega})$ will be under a tolerance ε^s. On the other hand, the determination of independent constraints for each triangle results in a number of constraints equal to the number of the unknown variables hindering significantly the solution of the overall problem. Generally, experimenting in various surfaces it is noticed that given a tolerance $\varepsilon_i^s = 1\%$ the maximum deformation of the triangles did not exceed 5%.

It is also possible to incorporate the constraints within the actual objective function and perform a more efficient optimization using the conjugate gradient method. In such a case the objective function is given by,

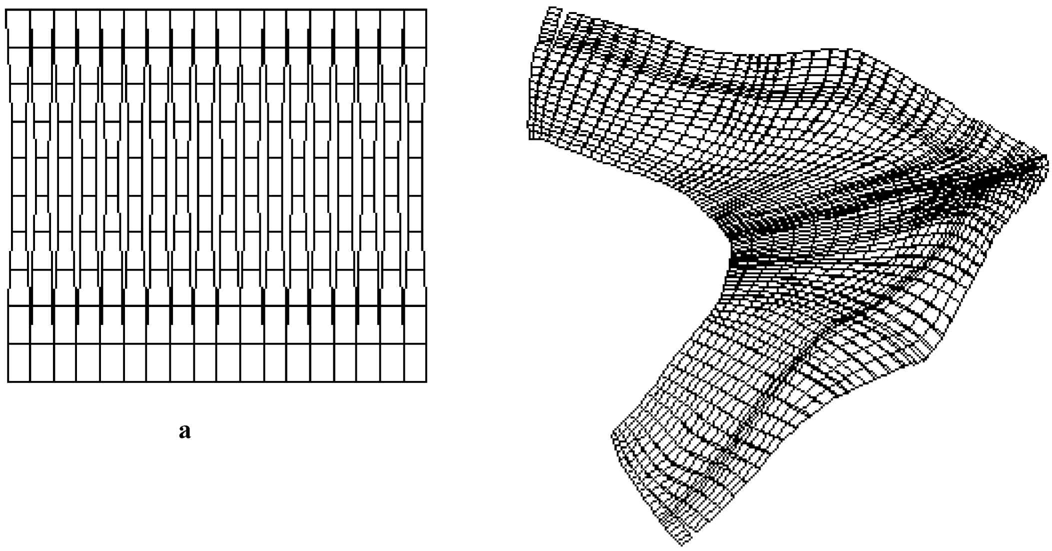
$$\mathrm{E}'(\boldsymbol{\omega}) = \sum_s f_s + \sum_s (g_s)^2 \tag{15}$$

The minimization of Eq.(15) does not ensure that the deformation of each i-triangle or each s-strip is held under a predefined tolerance threshold. However, during the experiments made using Eq.(15) it has been noticed that the maximum triangle deformation was not high but the reduction of the gaps and overlaps was not as satisfactory as the reduction achieved through the minimization of Eq.(13) under the constraints of Eq.(14). Thus, if one wishes to compromise with less accuracy but better computational time may use Eq.(15) for the reduction of the gaps and overlaps of a planar development under constraints. Indicative applications of the proposed technique for generating planar developments of arbitrarily curved surfaces are presented in the next section.

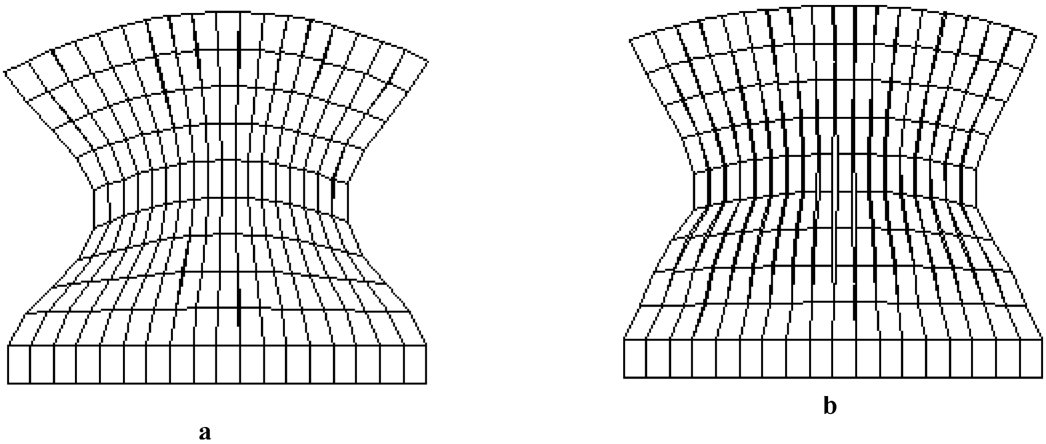
3. EXPERIMENTAL RESULTS AND DISCUSSION

Two indicative applications of the proposed technique are presented. The first one is the flattening of the classical test case of a torus quadrant (fig.2a) and the second one is the surface of a shoe last (fig.2b). The torus surface is approximated by a 19×11 grid of vertices and the surface of the shoe last by a 22×74 grid. The corresponding initial planar developments are shown in fig.3a,b respectively. In order to derive initial planar developments with minimal gaps and/or overlaps, the surface of the torus is flattened using as guide-strip the strip which includes one of the two geodesic isoparametric curves along the u-parametric direction, and the surface of the shoe last is flattened using the first strip in the u-parametric direction, since the absolute Gaussian curvature of the strips in the u-parametric direction is significantly lower than the absolute Gaussian curvature of the strips in the v-parametric direction.

The result of eliminating the gaps and overlaps in the torus planar development without constraints is shown in Fig.4a. The middle strip is kept invariant and thus numFixed = 1 which implies that the actual number of the unknown variables is $N_p = 17 \times 9 = 153$. Obviously the gaps and overlaps are eliminated with high accuracy, but the



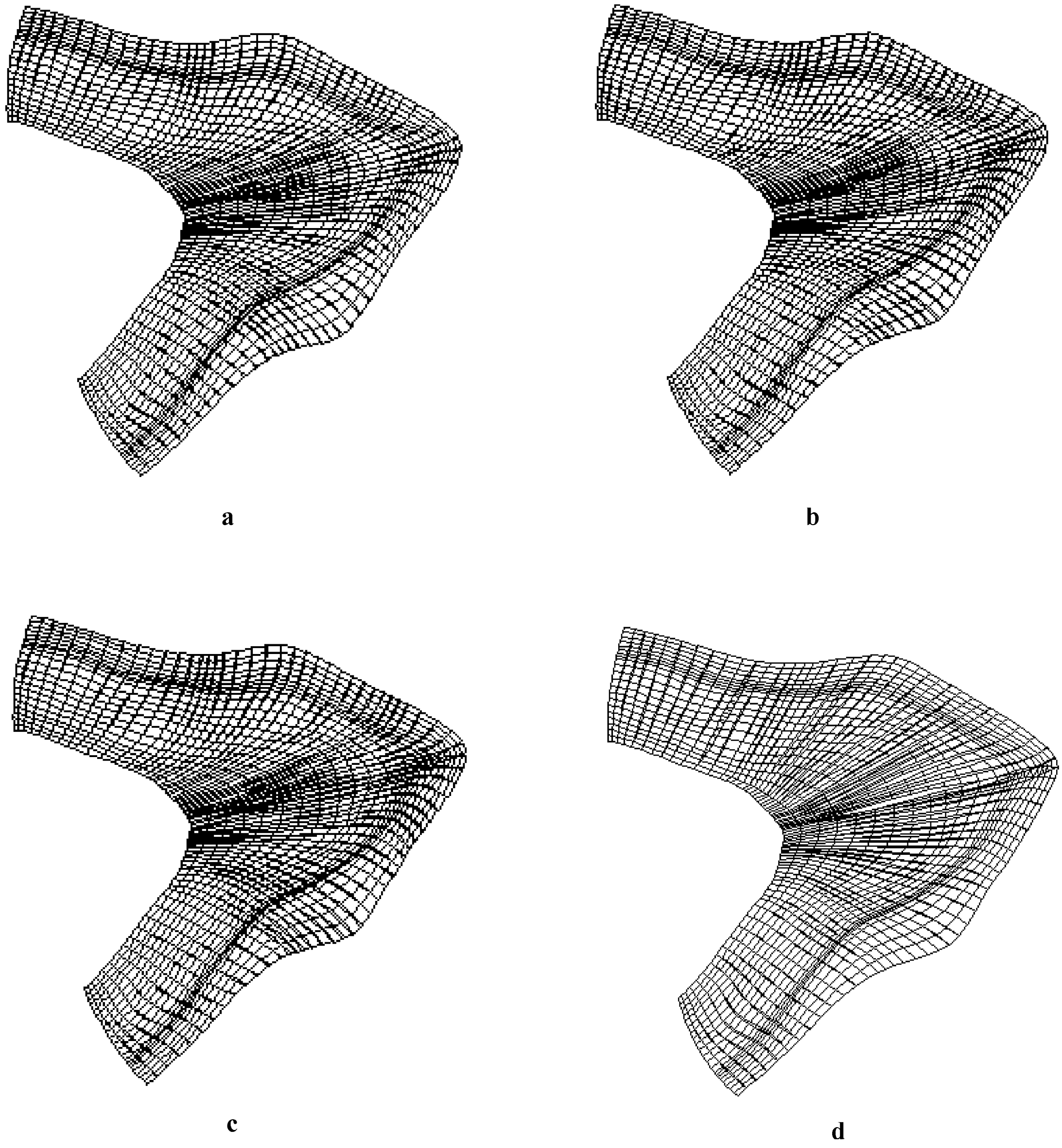
The initial planar developments of (a) the surface of the torus and (b) the surface of the shoe last.
Figure 3



(a) The result of the minimization of Eq.(13) without constraints for the case of the torus. (b) The result of the minimization of Eq.(13) with the constraints of Eq.(14) for the case of the torus.
Figure 4

maximum triangle deformation is about 35%. It is obvious that such surfaces should be decomposed into smaller parts to avoid high deformation. Taking the above into consideration it is chosen to set a maximum deformation for each triangle equal to 15%. The result of the minimization of Eq.(13) under the constraints of Eq.(14) is illustrated in Fig.4b. In this case the maximum triangle deformation was kept under 20% but elimination of the gaps and overlaps in the final planar development is not achieved with high accuracy.

The result of eliminating the gaps and overlaps in the shoe last planar development without constraints is shown in Fig.5a. In this case it is chosen to keep invariant the first and last strip along the u-parametric direction. Thus, numFixed = 2 and the actual number of the unknown variables is $N_p = 20 \times 71 = 1420$. The maximum triangle deformation is about 9%. The result of the minimization of Eq.(13) under the constraints of



The final planar developmens of the last derived through (a) the minimization of Eq.(13) without constraints, (b) the minimization of Eq.(13) with the constraints of Eq.(14), (c) the minimization of Eq.(15), (d) solving an inverse kinematics problem using pseudo-inverses.
Figure 5

Eq.(14) is illustrated in Fig.5b. In this case, the maximum triangle deformation was kept under 6% and the elimination of the gaps and overlaps in the final planar development is achieved with high accuracy. The result of generating a planar development through the minimization of Eq.(15) it is shown in Fig.5c. The triangle deformation was kept under 6.5% but the elimination of the gaps and overlaps in the final planar development is not very satisfactory. Finally, the result of using the technique based on an inverse kinematics problem solution using pseudo-inverses [Azariadis97] is given in Fig.5d. It is clear that the gaps and overlaps in the inner of the planar development are not reduced properly, contrarily to the gaps and overlaps of the boundary which are eliminated with high accuracy. This is a characteristic of the method proposed in [Azariadis97].

4. CONCLUSIONS

In this paper, the problem of generating a planar development of arbitrary three-dimensional surfaces has been addressed. A new method based on a global optimization process under constraints is proposed. Using this method an initial planar development is derived which is refined in order to satisfy certain constraints. The last stage is formulated as a global minimization problem with constraints. With the proposed technique it is possible to control the local accuracy in the planar development. Furthermore, the formulation of the technique allows to keep invariant several parts of the planar development which means that the potential user is allowed to select the portions of the planar development where it is desired to reduce the gaps and overlaps. This feature is especially useful in the woven fabrics where several "cuts" are made during the manufacturing of cloths. It is also possible to keep invariant parts of the planar development that follow a particular direction. This feature is useful when the surface is composed by composite laminates.

Finally, a topic for future research is the investigation of the relationship between the surface parameterization and the quality of its planar development. Obviously, the size of the gaps and overlaps of a planar development is influenced by the surface parameterization since the guide-strip is chosen among the parametric curves of the surface. A criterion for choosing the guide-strip is proposed in [Azariadis97]. However, the overall problem, which is related to the surface parameterization, is still not completely addressed.

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