Variance of two infinite path length random walk radiosity estimators

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Abstract. The error in an unbiased Monte Carlo method is characterized by the variance. By knowing the variance of different Monte Carlo estimators for Radiosity (and also their cost) we should be able to obtain the most efficient of them. This paper gives the variances for two such estimators, the shooting and gathering infinite path length random walk estimators. This completes a previous work of the author on finite path length estimators.

1 Introduction

We study in this paper the variance of two Monte Carlo discrete random walk estimators for radiosity, shooting and gathering random walk with infinite path length. The study of the variance is important because it gives the expected square error for an unbiased estimator. Thus, by knowing the variance of different Monte Carlo estimators for Radiosity (and also their cost) we should be able to obtain the most efficient of them. Gathering random walk proceeds sending paths from the patches of interest to *gather* energy when a source is hit. Shooting random walk *shoots* paths carrying energy from the sources, to update the visited patches. We consider in both cases that the random walk proceeds according to the discrete Form Factor matrix (that is, patch-to-patch Form Factors). Infinite path length estimators are such that the path never ends. Obviously, in a simulation, one has to cut off the path, obtaining a *biased* estimator. In this case the expected square error is the variance plus the squared bias. However, if we can assure a small bias, the variance will give a good approximation of the square error. The solution obtained with a random walk estimator will converge to the solution of the Radiosity system (see for instance [3, 8] for random walk solutions of a system of equations). Also, when the size of the patches decreases, it will converge to the one found with Particle Tracing [6], which uses point-to-point Form Factors. Path-tracing [4], and even distributed ray-tracing [1, 13] can be considered as the limiting case of gathering random walk for the non-discrete case (without the shadow ray). Bidirectional ray-tracing [12, 5] is a mixture of non-discrete shooting and gathering. [10, 2] can be seen as a breadth-first approach to a shooting random walk estimator, which in turn would be the depth-first approach.

As known to the author, only two papers have adressed to date the variance of random walk estimators for radiosity. [11] gives a bound for a shooting estimator, and [9] gives variances for a wide family of shooting and gathering estimators. The purpose of this

paper is to enlarge the work in [9] adressing the infinite path length estimators.
The structure of this paper is the following. In section 2 we will give previous results on finite path length estimators. In sections 3 and 4 the variance of the shooting and gathering infinite path length estimators will be obtained. In section 5 results will be given to support the theoretical findings, and our conclusions and future research will be presented in section 6.

2 Previous Work

In [11] a bound was given for a shooting finite path length estimator. This bound was intended more to study the complexity than to give a realistic approximation of error. In [9] different shooting and gathering estimators with finite expected path length were studied, and their variances given. Three estimators for shooting, $\frac{\Phi_T}{1-R_i}$, $\frac{\Phi_T}{R_i}$ and Φ_T and three for gathering $\frac{E_s}{1-R_s}$, $\frac{E_s}{R_s}$ and E_s were analyzed. In table 1 the characteristics of the different estimators are given. The best estimators for each case were found to be the Φ_T and E_s, which agrees with intuition, as both update *each* patch in a path (this advantage can be shown to overbalance the positive covariances). Two infinite path length estimators were also characterized, but we were only able to give bounds for their variances. These two estimators correspond in practice to *biased* estimators, because the path is cut off under some criterion (having reached a predetermined length, or being the accumulated reflectivities inferior to some threshold). In the next two sections we will reexamine those estimators and find their variances.

Table 1: Different Random Walk estimators. The meaning of the different quantities is in table 2.

Shooting	**Patch scored**	**Variance**
$\frac{\Phi_T}{1-R_i}$	last	$b_i\left(\frac{\Phi_T R_i}{A_i(1-R_i)} - b_i\right)$
$\frac{\Phi_T}{R_i}$	all but last	$b_i\left(\frac{\Phi_T R_i}{A_i}(\frac{1}{R_i} + 2\xi_i) - b_i\right)$
Φ_T	all	$b_i\left(\frac{R_i\Phi_T}{A_i}(1 + 2R_i\xi_i) - b_i\right)$
Gathering	**Patch scored**	**Variance**
$\frac{E_s}{1-R_s}$	last	$\frac{R_i}{p_i}\sum_s \frac{E_s}{(1-R_s)} b_{is} - {b_i}^2$
$\frac{E_s}{R_s}$	all but last	$\frac{R_i}{p_i}\sum_s \frac{E_s+2b_s}{R_s} b_{is} - {b_i}^2$
E_s	all	$\frac{R_i}{p_i}\sum_s (E_s+2b_s) b_{is} - {b_i}^2$

3 An infinite path length shooting estimator

Let us first consider what the expected value of any unbiased Monte Carlo estimator should be for the incoming power on a patch. Let us suppose that the initial power of source s is Φ_s, ϕ_i is the incoming power on patch i, F_{kl} denotes the Form Factor from patch k to patch l, and R_k denotes the reflectance of patch k. Then we have, by developing the Power system in Neumann series and dropping the zero order term:

$$\phi_i = \sum_s \Phi_s F_{si} + \sum_s \sum_h \Phi_s F_{sh} R_h F_{hi}$$

Table 2: Meaning of the different quantities appearing in table 1. The suffix i means for patch i, suffix s indexes the sources.

E_i	Emissivity
b_i	Reflected radiosity $= B_i - E_i$
Φ_T	Total power in scene
A_i	Area
R_i	Reflectivity
ξ_i	Received power (or radiosity) due to self-emitted unit power (or emittance)
b_{is}	Reflected radiosity on i due to source s
p_i	Probability for a path to begin at i

$$+\sum_s \sum_h \sum_j \Phi_s F_{sh} R_h F_{hj} R_j F_{ji} + \cdots$$

This can be expressed as:

$$\phi_i = \phi_i^{(1)} + \phi_i^{(2)} + \phi_i^{(3)} + \cdots$$

where $\phi_i^{(1)} = \sum_s \Phi_s F_{si}$, $\phi_i^{(2)} = \sum_s \sum_h \Phi_s F_{sh} R_h F_{hi}$, $\phi_i^{(3)} = \sum_s \sum_h \sum_j \Phi_s F_{sh} R_h F_{hj} R_j F_{ji}$ and so on. That is, $\phi_i^{(1)}$ represents the power arrived directly from the sources, $\phi_i^{(2)}$ represents the power arrived after one bounce, and so on. Let us now consider the following simulation. A infinite length path γ starts from source s with probability $p_s = \frac{\Phi_s}{\Phi_T}$ (that is, according to the power of the source), and from here on it evolves according to the transition probabilities given by the Form Factors. For instance, from s it will go to patch j with probability F_{sj}. The random variables $\widehat{\phi}_i^{(1)}, \widehat{\phi}_i^{(2)}, \widehat{\phi}_i^{(3)}, \ldots$ are defined in the following way:
All of those random variables are initially null. If the path γ arrives at patch i at length l, and if $s, h_1, h_2, \ldots, h_{l-1}, i$ is the trajectory the path has followed, then the value of $\widehat{\phi}_i^{(l)}$ is set to $R_{h_1} R_{h_2} \ldots R_{h_{l-1}} \Phi_T$. Let us define also a new random variable $\widehat{\phi}_i$ as:

$$\widehat{\phi}_i = \widehat{\phi}_i^{(1)} + \widehat{\phi}_i^{(2)} + \widehat{\phi}_i^{(3)} + \cdots$$

The expected values are

$$E(\widehat{\phi}_i^{(1)}) = \sum_s \Phi_T \times \frac{\Phi_s}{\Phi_T} \times F_{si} = \phi_i^{(1)}$$

$$E(\widehat{\phi}_i^{(2)}) = \sum_s \sum_h R_h \times \Phi_T \times \frac{\Phi_s}{\Phi_T} \times F_{sh} \times F_{hi} = \phi_i^{(2)}$$

and so on. Then, we have

$$E(\widehat{\phi}_i) = E(\widehat{\phi}_i^{(1)} + \widehat{\phi}_i^{(2)} + \cdots) = E(\widehat{\phi}_i^{(1)}) + E(\widehat{\phi}_i^{(2)}) + \cdots = \phi_i^{(1)} + \phi_i^{(2)} + \cdots = \phi_i$$

Thus the random variable $\widehat{\phi}_i^{(l)}$ is a centered estimator for the power arrived to patch i after l bounces, and the sum of all this family of estimators gives a new centered estimator $\widehat{\phi}_i$ which corresponds to the total incoming power arrived to patch i after any number of bounces. Our aim now is to obtain the variance for this estimator. We can decompose $Var(\widehat{\phi}_i)$ in the following way

$$\begin{aligned} Var(\widehat{\phi}_i) &= Var(\widehat{\phi}_i^{(1)} + \widehat{\phi}_i^{(2)} + \cdots) \\ &= E(\left(\widehat{\phi}_i^{(1)} + \widehat{\phi}_i^{(2)} + \cdots\right)^2) - \left(E(\widehat{\phi}_i)\right)^2 \\ &= E(\widehat{\phi}_i^{(1)2}) + E(\widehat{\phi}_i^{(2)2}) + \cdots \\ &+ 2 \sum_{1 \le n < m} E(\widehat{\phi}_i^{(n)} \widehat{\phi}_i^{(m)}) - {\phi_i}^2 \end{aligned} \tag{1}$$

We find first the cross terms, which are not null because a path can arrive at length n on patch i and again at length m:

$$\begin{aligned} E(\widehat{\phi}_i^{(n)} \widehat{\phi}_i^{(m)}) &= \sum_s \sum_{h_1} \cdots \sum_{h_{n-1}} \sum_{h_{n+1}} \cdots \sum_{h_{m-1}} R_{h_1} \cdots R_{h_{n-1}} \Phi_T R_{h_1} \cdots R_{h_{n-1}} R_i R_{h_{n+1}} \cdots R_{h_{m-1}} \Phi_T \cdot \\ &\quad \frac{\Phi_s}{\Phi_T} F_{sh_1} \cdots F_{h_{n-1}i} F_{ih_{n+1}} \cdots F_{h_{m-1}i} \\ &= \sum_s \sum_{h_1} \cdots \sum_{h_{n-1}} R_{h_1}^2 \cdots R_{h_{n-1}}^2 \Phi_s F_{sh_1} \cdots F_{h_{n-1}i} \cdot \\ &\quad R_i \Phi_T \sum_{h_{n+1}} \cdots \sum_{h_{m-1}} R_{h_{n+1}} \cdots R_{h_{m-1}} \cdots F_{ih_{n+1}} \cdots F_{h_{m-1}i} \cdot \\ &= \psi_i^{(n)} R_i \Phi_T \xi_i^{(m-n)} \end{aligned}$$

where $\psi_i^{(n)}$ is the incoming energy after n bounces in the same environment having changed all the reflectivities by their square and $\xi_i^{(m-n)}$ is the expected value of the incoming power (or radiosity) on patch i after $m-n$ bounces due to a unit power (or unit emittance) on the same patch i. Then

$$2 \sum_{1 \le n} \left(\sum_{n<m} E(\widehat{\phi}_i^{(n)} \widehat{\phi}_i^{(m)}) \right) = 2 R_i \Phi_T \sum_{1 \le n} \psi_i^{(n)} \sum_{n<m} \xi_i^{(m-n)} = 2 R_i \Phi_T \psi_i \xi_i$$

where ψ_i is the incoming energy in the same environment having changed all the reflectivities by their square and ξ_i is the expected value of the incoming power (or radiosity) on patch i due to a unit power (or unit emittance) on the same patch i. ξ_i is is equal to $\frac{I_i - 1}{R_i}$, where I_i is the self-importance of patch i (for a definition of importance, see [7]). Also we have

$$E(\widehat{\phi}_i^{(1)2}) = \sum_s (\Phi_T)^2 \times \frac{\Phi_s}{\Phi_T} \times F_{si} = \Phi_T \phi_i^{(1)} = \Phi_T \psi_i^{(1)}$$

because $\phi_i^{(1)} = \psi_i^{(1)}$.

$$E(\widehat{\phi}_i^{(2)2}) = \sum_s \sum_h (R_h \times \Phi_T)^2 \times \frac{\Phi_s}{\Phi_T} \times F_{sh} \times F_{hi} = \Phi_T \psi_i^{(2)}$$

and so on. Then we finally obtain

$$\begin{aligned} Var(\widehat{\phi}_i) &= \Phi_T(\psi_i^{(1)} + \psi_i^{(2)} + \cdots) + 2 R_i \Phi_T \psi_i \xi_i - {\phi_i}^2 \\ &= \psi_i \Phi_T (1 + 2 R_i \xi_i) - {\phi_i}^2 \end{aligned}$$

where ψ_i is the incoming energy in the same environment having changed all the reflectivities by their square. For the radiosity estimator $\widehat{B}_i = E_i + \frac{R_i}{A_i}\widehat{\phi}_i$ we have

$$Var(\widehat{B}_i) = \frac{R_i^2}{A_i^2}\left(\psi_i \Phi_T (1 + 2R_i\xi_i) - \phi_i^2\right) = \frac{\beta_i \Phi_T}{A_i}(1 + 2R_i\xi_i) - b_i^2 \tag{2}$$

where β_i is the reflected radiosity (or radiosity due to incoming energy) in the same environment having changed all reflectivities by their squares. The variance of this estimator can be shown to be lower than the one for the Φ_T estimator [9]. But this does not necessarily mean that this estimator is better. First, the cost of the infinite estimator is determined by how we cut off the infinite path. This can be done on a predetermined length, or alternatively when the accumulated reflectivity (that is, the product of reflectivities along a path) is less than a preestablished threshold. Second, this procedure imposes a *bias* on the solution which should be taken into account when comparing errors. If we want no bias, we can use *Russian Roulette*, which consists simply in switching to some finite path length estimator (such as the ones in table 1) to distribute the left energy. This will however increase the cost considerably. We can alternatively consider acceptable this small percentage of undistributed energy (the variance accounts for the noise in the image, the bias for the undistributed energy). Also, if this threshold is small enough we can be confident that the variance of the resulting biased estimator is close to the variance of the infinite length estimator.

4 An infinite path length gathering estimator

Let us first consider what the expected value of any unbiased Monte Carlo estimator should be for the radiosity of a patch. Let us suppose that the emittance of source s is E_s, b_i is the reflected radiosity, or radiosity of patch i due to the received power (that is, $b_i = B_i - E_i$, and so for a non-emitter patch, it equals the total radiosity), F_{kl} denotes the Form Factor from patch k to patch l, and R_k denotes the reflectance of patch k. Then we have, by developing the Radiosity system in Neumann series (dropping the zero order term):

$$\begin{aligned} b_i &= R_i \sum_s E_s F_{is} + R_i \sum_h \sum_s E_s F_{ih} R_h F_{hs} \\ &\quad + R_i \sum_h \sum_j \sum_s E_s F_{ih} R_h F_{hj} R_j F_{js} + \cdots \end{aligned}$$

This can be expressed as:

$$b_i = b_i^{(1)} + b_i^{(2)} + b_i^{(3)} + \cdots$$

where $b_i^{(1)} = R_i \sum_s E_s F_{is}$, $b_i^{(2)} = R_i \sum_s \sum_h E_s F_{ih} R_h F_{hs}$, $b_i^{(3)} = R_i \sum_s \sum_h \sum_j E_s F_{ih} R_h F_{hj} R_j F_{js}$ and so on. That is, $b_i^{(1)}$ represents the radiosity due to direct illumination, $b_i^{(2)}$ represents the radiosity after one bounce, and so on. It is also useful to define the following quantities:

$$b_{is} = b_{is}^{(1)} + b_{is}^{(2)} + \cdots$$

$b_{is}^{(1)}$ represents the radiosity due to direct illumination from source s, $b_{is}^{(2)}$ represents the radiosity after one bounce from source s, and so on. It is clear that:

$$b_i = \sum_s b_{is}$$

Let us now consider the following simulation. A infinite length path γ starts from patch i with probability p_i (this probability can be considered as the initial or emitted importance of the patch), and from here on it evolves according to the transition probabilities given by the Form Factors. For instance, from i it will go to patch j with probability F_{ij}. Let us define now the random variables $\widehat{b}_i^{(1)}, \widehat{b}_i^{(2)}, \widehat{b}_i^{(3)}, \ldots$ in the following way:
All of those random variables are initially null. If the path γ happens to arrive at source s at length l, and if $i, h_1, h_2, \ldots, h_{l-1}, s$ is the trajectory the path has followed, then the value of $\widehat{b}_i^{(l)}$ is set to $R_i R_{h_1} R_{h_2} \ldots R_{h_{l-1}} \frac{E_s}{p_i}$. Let us define also a new random variable $\widehat{b}_i$ as:

$$\widehat{b}_i = \widehat{b}_i^{(1)} + \widehat{b}_i^{(2)} + \widehat{b}_i^{(3)} + \cdots$$

Now let us find the expected value of those random variables. Applying the definition of expected value, and remembering that the probability of selecting patch i is p_i, the probability of landing on source s just after leaving patch i is F_{is}, we have

$$E(\widehat{b}_i^{(1)}) = \sum_s R_i \times \frac{E_s}{p_i} \times p_i \times F_{is} = b_i^{(1)}$$

Now, to go from patch i to a source s in a two length path we can pass through any patch h, so we have

$$E(\widehat{b}_i^{(2)}) = \sum_h \sum_s R_i \times R_h \times \frac{E_s}{p_i} \times p_i \times F_{ih} \times F_{hs} = b_i^{(2)}$$

and so on. Then, we have

$$E(\widehat{b}_i) = E(\widehat{b}_i^{(1)} + \widehat{b}_i^{(2)} + \cdots) = E(\widehat{b}_i^{(1)}) + E(\widehat{b}_i^{(2)}) + \cdots = b_i^{(1)} + b_i^{(2)} + \cdots = b_i$$

So it is clear that the random variable $\widehat{b}_i^{(l)}$ is a centered estimator for the radiosity due to the power arrived on patch i after l bounces, and the sum of all this family of estimators gives a new centered estimator $\widehat{b}_i$ which corresponds to the total radiosity of patch i due to the power arrived after any number of bounces. Our aim now is, as before, to obtain the variance for this estimator, which we will do decomposing $Var(\widehat{b}_i)$ in the same way as in formula 1. The terms of the form $E(\widehat{b}_i^{(n)} \widehat{b}_i^{(m)})$ are not null, because if a path arrives at length n on source s it can also arrive later at source s' at length m, and we find them as in the previous section:

$$\begin{aligned}
E(\widehat{b}_i^{(n)} \widehat{b}_i^{(m)}) &= \sum_s \sum_{s'} \sum_{h_1} \cdots \sum_{h_{n-1}} \sum_{h_{n+1}} \cdots \sum_{h_{m-1}} R_i R_{h_1} \cdots R_{h_{n-1}} \frac{E_s}{p_i} \cdot \\
& \quad R_i R_{h_1} \cdots R_{h_{n-1}} R_s R_{h_{n+1}} \cdots R_{h_{m-1}} \frac{E_{s'}}{p_i} \cdot \\
& \quad p_i F_{sh_1} \cdots F_{h_{n-1}s} F_{sh_{n+1}} \cdots F_{h_{m-1}s'} \\
&= \sum_s \sum_{h_1} \cdots \sum_{h_{n-1}} R_i^2 R_{h_1}^2 \cdots R_{h_{n-1}}^2 \frac{E_s}{p_i} p_i F_{sh_1} \cdots F_{h_{n-1}s} \cdot \\
& \quad \sum_{s'} \sum_{h_{n+1}} \sum_{h_{m-1}} R_s R_{h_{n+1}} \cdots R_{h_{m-1}} \frac{E_{s'}}{p_i} \\
&= \frac{1}{p_i} \sum_s \beta_{is}^{(n)} \sum_{s'} b_{ss'}^{(m-n)} = \frac{1}{p_i} \sum_s \beta_{is}^{(n)} b_s^{(m-n)}
\end{aligned}$$

where $\beta_{is}^{(n)}$ is the radiosity due to the incoming energy after n bounces in the same environment having changed all the reflectivities by their square. Then

$$\sum_{1\leq n}\left(\sum_{n<m} E(\widehat{b}_i^{(n)}\widehat{b}_i^{(m)})\right) = \frac{1}{p_i}\sum_s\sum_{1\leq n}\beta_{is}^{(n)}\sum_{n<m} b_s^{(m-n)} = \frac{1}{p_i}\sum_s \beta_{is} b_s$$

and also

$$E(\widehat{b}_i^{(1)2}) = \sum_s \left(R_i \frac{E_s}{p_i}\right)^2 \times p_i \times F_{is} = \sum_s \frac{E_s}{p_i}\beta_{is}^{(1)}$$

$$E(\widehat{b}_i^{(2)2}) = \sum_h\sum_s \left(R_i \times R_h \frac{E_s}{p_i}\right)^2 \times p_i \times F_{ih} \times F_{hs} = \sum_s \frac{E_s}{p_i}\beta_{is}^{(2)}$$

and so on. Then we obtain

$$Var(\widehat{b}_i) = \frac{1}{p_i}\sum_s E_s(\beta_{is}^{(1)} + \beta_{is}^{(2)} + \cdots) + 2\frac{1}{p_i}\sum_s \beta_{is} b_s - {b_i}^2 = \frac{1}{p_i}\sum_s (E_s + 2b_s)\beta_{is} - {b_i}^2$$

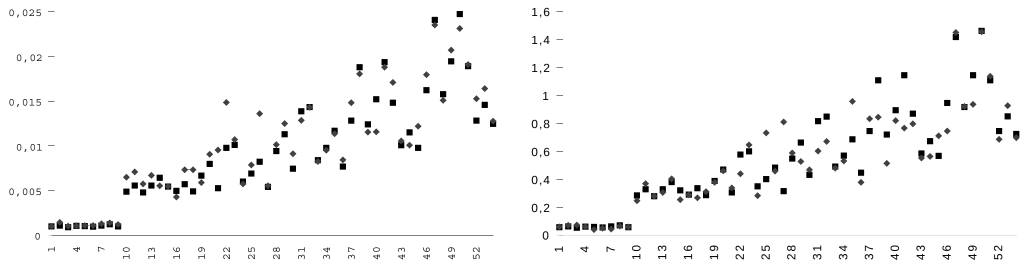
where β_{is} is the radiosity due to the incoming energy in the same environment having changed all the reflectivities by their square. For the radiosity our estimator is simply $\widehat{b}_i + E_i$, and as E_i is a constant we have

$$Var(\widehat{b}_i + E_i) = \frac{1}{p_i}\sum_s (E_s + 2b_s)\beta_{is} - {b_i}^2 \tag{3}$$

This variance can be proven to be less than the one for the E_s estimator [9]. But again here we can repeat the considerations of the previous section on which of both estimators is better.

5 Results

Here we present in figure 1 some experiments performed on a very simple scene, a cubical enclosure with each face divided into nine equal size patches, the reflectivities of the faces being 0.3, 0.4, 0.5, 0.6, 0.7, 0.8 respectively, and a source with emissivity 1 in the middle of the first face, in patch 4. Thus patches 1 to 9 receive no direct lighting and have reflectivity 0.3, patches 10 to 18 reflectivity 0.4, and so on. For this scene we computed a reference solution with a finite path length estimator and 10^8 paths. This provided us with the b_i values. A reference solution was also computed for the same scene with the reflectivities squared, that is, each reflectivity was substituted by its square. This provided us with the β_{is} values. Then 100 runs of 10^4 paths each for both infinite estimator were computed (taking for gathering $p_i = \frac{A_i}{A_T}$, the fraction of total area), and used to obtain the square errors, an thus an estimated value of the variances for a single path. The infinite paths were cut off when the product of reflectivities along the path was less than 0.001. The formulae for the variances are the formulae 2 and 3, with the approximation $\xi_i = 0$. Figure 1 shows that the obtained results are in concordance with the theoretically expected ones. Although the scene used in the test has no occlusions, it should be noted that the variance of a patch radiosity does not depend on whether it is due to direct or indirect illumination.



(a) (b)

Figure 1: Comparison of the expected variances (plotted as square dots) and the experimentally obtained square errors for the shooting (a) and gathering (b) infinite path length estimator, for the 54 equal area patches of a cube (on x axis), with face reflectivities 0.3, 0.4, 0.5, 0.6, 0.7, 0.8. A source with emittance 1 is in the middle of the first face.

6 Conclusions

We have given here the variances for the infinite path length random walk estimators for radiosity. These formulae complete the study in [9] for finite path length estimators. A study of the relative efficiencies of both kind of estimators remains to be done, and also the generalization of the results to the R-G-B case.

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References

[1] R.L. Cook, T. Porter, L. Carpenter, "Distributed Ray Tracing", *Computer Graphics (SIGGRAPH '84 Proc.)*, 18(3), pp. 137-145, July 1984.

[2] M. Feda and W. Purgathofer, "Progressive Ray Refinement for Monte Carlo Radiosity", *Fourth Eurographics Workshop on Rendering*, Eurographics Technical Report Series EG 93 RW, Paris, pp. 15-26, June 1993.

[3] J.M. Hammersley and D.C. Handscomb, *Monte Carlo Methods*, Methuen & Co Ltd., London, 1975.

[4] J.T. Kajiya, "The Rendering Equation", *Computer Graphics (SIGGRAPH '86 Proc.)*, vol. 20, pp. 143-150, August 1986.

[5] E. Lafortune and Y. Willems. "Bi-directional path tracing", *Proceedings of CompuGraphics*, Alvor, Portugal, pp. 145-153, December 1993.

[6] S.N. Pattanaik and S.P. Mudur, "Computation of Global Illumination by Monte Carlo Simulation of the Particle Model of Light", *Third Eurographics Workshop on Rendering*, Bristol, UK, pp. 71-83, May 1992.

[7] S.N. Pattanaik and S.P. Mudur, "The potential equation and importance in illumination computation", *Computer Graphics Forum*, vol. 12, pp. 131-136, 1993.

[8] R.Y. Rubinstein, *Simulation and the Monte Carlo method*, John Wiley & Sons, New York, 1981.

[9] M. Sbert, "Error and complexity of random walk Monte Carlo radiosity", to appear in *IEEE Transactions on Visualization and Computer Graphics*, Vol. 3, Number 1, March 1997.

[10] P. Shirley, "A Ray Tracing Method for Illumination Calculation in Diffuse-Specular Scenes", *Proceedings of Graphics Interface '90*, Canadian Information Processing Society, Toronto, Ontario, pp. 205-212, May 1990.

[11] P. Shirley, "Time Complexity of Monte Carlo Radiosity", *Proceedings of EUROGRAPHICS'91*, Vienna, pp. 459-465, September 1991.

[12] E. Veach and L. Guibas, "Optimally Combining Sampling Rendering Techniques for Monte Carlo Rendering", *Computer Graphics (SIGGRAPH '95 Proc.)*, vol. 29, August 1995.

[13] G.J. Ward, F.M. Rubinstein, R.D. Clear, "A Ray Tracing Solution for Diffuse Interreflection", *Computer Graphics (SIGGRAPH '88 Proc.)*, vol. 22, pp. 85-92, August 1988.