G^2 continuous β spline curves defined on the interval <0, 1)

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In this paper the general and special matrices forms of the cubic β-spline curve defined on the interval $\langle 0,1)$ are described.

Geometric continuous, β-spline curves and B-spline curves

Given a control polygon $V_0, V_1, \dots, V_n$ and a set of shape parameter values $\bar{\beta}1 = (\beta_{1,0}, \dots, \beta_{1,n})$ and $\bar{\beta}2 = (\beta_{2,0}, \dots, \beta_{2,n})$. The i-th segment $Q_i(t)$ of a G^2 cubic Beta-spline takes form

/1/ $$Q_i(t) = \sum_{j=-3}^{0} V_{i+j} b_{i+j}(\bar{\beta}1, \bar{\beta}2; t), \qquad t \in \langle 0,1), \ i=3,4,..,n.$$

where the functions $b_{i+j,j}(\bar{\beta}1, \bar{\beta}2; t)$, called the G^2 Beta-spline blending functions, are cubic polynomial functions constructed so that

/2/ $$\begin{aligned} Q_{i+1}(0) &= Q_i(1) \\ Q_{i+1}^{(1)}(0) &= \beta_{1,i} Q_i^{(1)}(1) \\ Q_{i+1}^{(2)}(0) &= \beta_{1,i}^2 Q_i^{(2)}(1) + \beta_{2,i} Q_i^{(1)}(1) \end{aligned} \qquad , \quad i=3,4,..,n$$

Rather than construct the basis functions directly, we follow the approach of Farin and Boehm to construct the Bézier polygons of each of the segments. Let $W_{i,0}, W_{i,1}, W_{i,2}, W_{i,3}$ denote the Bézier polygon of the i-th segment, $i=3,4,..,n$. Then the i-th segment in a Bézier form is expressed as follows

/3/ $$Q_i(t) = \sum_{j=0}^{3} W_{i,j} B_j^3(t), \quad t \in \langle 0,1), \qquad i=3,4,..,n$$

where $B_j^3(t)$ are the Berstein polynomials. The control polygon $W_{i,0}, W_{i,1}, W_{i,2}, W_{i,3}$ is constructed from a control vertices $V_{i-3} V_{i-2} V_{i-1} V_i$ and shapes parameters $\beta_{j,i-3}, \beta_{j,i-2}, \beta_{j,i-1}, \beta_{j,i}$ $j=1,2$ by the next construction.

The Farin-Boehm construction for G^2 β-splines.

1. For $i=0,..n$, compute γ_i from $\beta_{1,i}$ and $\beta_{2,i}$

/4/ $$\gamma_i = \frac{2(1+\beta_{1,i})}{\beta_{2,i} + 2\beta_{1,i}(1+\beta_{1,i})},$$

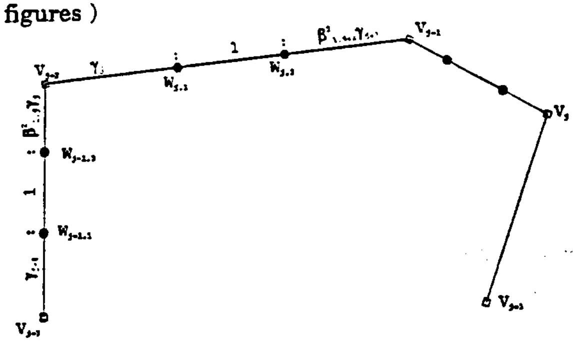
2. For $i=0,..n-1$, compute the interior Bézier vertices

/5/ $$\begin{aligned} W_{i,1} &= \frac{(1+\beta_{1,i+1}^2 \gamma_{i+1}) V_i + \gamma_i V_{i+1}}{1+\gamma_i + \beta_{1,i+1}^2 \gamma_{i+1}}, \\ W_{i,2} &= \frac{\beta_{1,i+1}^2 \gamma_{i+1} V_i + (1+\gamma_i) V_{i+1}}{1+\gamma_i + \beta_{1,i+1}^2 \gamma_{i+1}} \end{aligned}$$

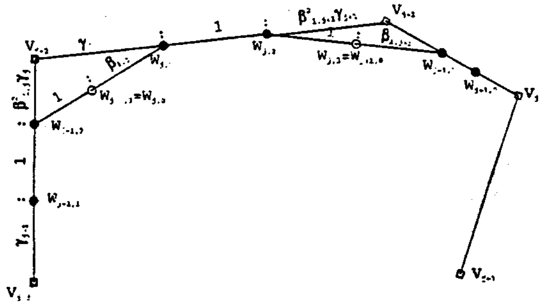
3. For $i=0,..n-2$, compute the exterior (junction) Bézier vertices

/6/ $$\begin{aligned} W_{i,0} &= \frac{\beta_{1,i} W_{i-1,2} + W_{i,1}}{1+\beta_{1,i}}, \\ W_{i,3} = W_{i+1,0} &= \frac{\beta_{1,i+1} W_{i,2} + W_{i+1,1}}{1+\beta_{1,i+1}} \end{aligned}$$

(See next figures)



(a)



(b)

Figure 1 : The Farin-Boehm construction for G^2 Beta-splines.

(a) The construction for the interior Bézier vertices

(b) The construction of the junction vertices

For next considerations, we express the Bézier curve /3/ in a matrix form .

/7/ $Q_i(t) = \mathbf{T}\mathbf{M}^4_{\text{Bz}}\mathbf{W}^T$, $t \in \langle 0,1 \rangle$, $i=3,4,\ldots,n$

/7.a/ $$Q_i(t) = \mathbf{T}\mathbf{M}^4_{\text{Bz}}\mathbf{W}^T = (1 \quad t \quad t^2 \quad t^3)\begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}\begin{pmatrix} W_{i,0} \\ W_{i,1} \\ W_{i,2} \\ W_{i,3} \end{pmatrix},$$

$t \in \langle 0,1 \rangle$, $i=3,4,\ldots,n$

where the Bézier vertices W_{ij}, $j=0,1,..,3$ are described by the construction above.

We concentrate on the expressions of the Bézier vertices W_{ij} now. These vertices are depended on the β-spline control vertices V_i and the shape parameters $\bar{\beta}1$ and $\bar{\beta}2$. Thus, we can express the vector W as

/8/ $\mathbf{W} = \mathbf{M}^4_{\text{Bz-}\beta}\mathbf{V}$

where $\mathbf{V} = (V_{i-3} \quad V_{i-2} \quad V_{i-1} \quad V_i)^T$.

We obtain the general matrix $\mathbf{M}^4_\beta$ of the cubic β-spline easily from the expressions /7/ and /8/.

/9/ $\mathbf{M}^4_\beta = \mathbf{M}^4_{\text{Bz}}\mathbf{M}^4_{\text{Bz-}\beta}$.

The MATHEMATICA algebraic manipulation language was used to obtain this matrix.

/10/

$$\mathbf{M}^4_\beta = \begin{pmatrix} m_{11} & \dfrac{\alpha_i + \tau_{i-1}}{\sigma_{i-1}} & \varphi_i & 0 \\ -3m_{11} & 3\dfrac{\beta_{1,i}\tau_{i-1} - \alpha_i}{\sigma_{i-1}} & 3\beta_{1,i}\varphi_i & 0 \\ 3m_{11} & 3\dfrac{\alpha_i - \beta_{1,i}\tau_{i-1} - \phi_{i-1}}{\sigma_{i-1}} & 3\dfrac{\chi - \beta_{1,i}\gamma_i}{\phi_i} & 0 \\ -m_{11} & \dfrac{\delta_{i-1}(3\chi + \gamma_{i+1}\beta^3_{1,i}) - \alpha_i - \tau_{i-1}}{\sigma_{i-1}} & m_{43} & \varphi_{i+1} \end{pmatrix}$$

kde

$\chi = (1 + \beta_{1,i})$

$\gamma_i = \dfrac{2(1 + \beta_{1,i})}{\beta_{2,i} + 2\beta_{1,i}(1 + \beta_{1,i})}$

$\delta_i = 1 + \gamma_i + \beta^2_{1,i+1}\gamma_{i+1}$

$\alpha_i = \beta_{1,i}\delta_i(1 + \gamma_{i-1})$

$\sigma_{i-1} = (1 + \beta_{1,i})\delta_{i-1}\delta_i$

$\tau_{i-1} = \delta_{i-1}(1 + \beta^2_{1,i+1}\gamma_{i+1})$

$\phi_i = \chi\delta_i$

$\varphi_i = \dfrac{\gamma_i}{\phi_i}$

a

$m_{11} = \dfrac{\beta^3_{1,i}\gamma_i}{\phi_{i-1}}$, $\quad m_{43} = \dfrac{\beta_{1,i}\delta_{i+1}(-2 + \gamma_i) - \delta_{i+1}(3 + \gamma_i) + \tau_i}{\sigma_i}$

Explicit expressions for the G^2 β-spline blending functions $b_{i+j,j}(\bar{\beta}1, \bar{\beta}2; t)$ can be obtained by multiplying of the vector $\mathbf{T} = [1 \quad t \quad \cdots \quad t^3]$ by matrix $\mathbf{M}^4_\beta$.

/11/ $(b_{-3}(t) \quad b_{-2}(t) \quad b_{-1}(t) \quad b_0(t)) = \mathbf{T}\mathbf{M}^4_\beta$ or

$$b_{-3}(t)=m_{11}-3tm_{11}+3t^2m_{11}-t^3m_{11}$$

$$b_{-2}(t)=\frac{\alpha_i+\tau_{i-1}}{\sigma_{i-1}}+3t\frac{\beta_{1,i}\tau_{i-1}-\alpha_i}{\sigma_{i-1}}+3t^2\frac{\alpha_i-\beta_{1,i}\tau_{i-1}-\phi_{i-1}}{\sigma_{i-1}}+3t^3\frac{\delta_{i-1}\left(3\chi+\gamma_{i+1}\beta_{1,i}^3\right)-\alpha_i-\tau_{i-1}}{\sigma_{i-1}}$$

$$b_{-1}(t)=\varphi_i+3t\beta_{1,i}\varphi_i+3t^2\frac{\chi-\beta_{1,i}\gamma_i}{\phi_i}+t^3m_{43}$$

$$b_0(t)=\varphi_{i+1}t^3$$

Then, the i-th segment of the cubic β-spline curve is written as

/12/ $\quad Q(t)=\mathbf{T}\mathbf{M}_{\beta}^{4}\mathbf{V}^{T}, \qquad t\in\langle 0,1), \; i=3,..,n$

Notice : The shape parameters control the shape of the curve *locally*. It obvious from Farin - Boehm construction, that the i-th segment depends on the shape parameters $\beta_{j,i-3},\beta_{j,i-2},\beta_{j,i-1},\beta_{j,i}$ $j=1,2$. And the parameter $\beta_{j,i}$ $j=1,2$ have influence only on the segments $Q_{i-3}(t),..,Q_i(t)$.

Let all β-parameters are corresponding. It means that $\beta_{1,i}=\beta_1$ a $\beta_{2,i}=\beta_2, \forall i,\ \beta_1\geq 1, \beta_2\geq 0$. Then these shape parameters have a *global* influence on the shape of the curve.

In this case, we mark the matrix $\mathbf{M}_{\beta}^{4}$ as $\mathbf{M}_{\beta,C}^{4}$ and write its elements.

/13/
$$\mathbf{M}_{\beta,C}^{4}=\frac{1}{\delta}\begin{pmatrix} 2\beta_1^3 & 4\beta_1+4\beta_1^2+\beta_2 & 2 & 0\\ -6\beta_1^3 & 6\left(\beta_1^3-\beta_1\right) & 6\beta_1 & 0\\ 6\beta_1^3 & -6\beta_1^2(1+\beta_1)-3\beta_2 & 3\left(2\beta_1^2+\beta_2\right) & 0\\ -2\beta_1^3 & 2\beta_1\left(1+\beta_1+\beta_1^2\right)+2\beta_2 & -2\left(1+\beta_1+\beta_1^2+\beta_2\right) & 2 \end{pmatrix}$$

$$\delta=2+4\beta_1+4\beta_1^2+2\beta_1^3+\beta_2$$

In the other case, let all shape parameters $\beta_{1,i}$ are equal to one ($\beta_{1,i}=1$) and the shape parameters $\beta_{2,i}$ are arbitrary. These curves are called β_2***-spline curves***. The coefficients of its blending functions are expressed by the following matrix

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$$\mathbf{M}_{\beta_2}^{4}=$$

$$\begin{pmatrix} \frac{2}{\varphi_i\delta_{i-1}} & \frac{1+\alpha_{i-1}}{2\delta_{i-1}}+\frac{1+\alpha_{i+1}}{2\delta_i} & \frac{2}{\varphi_i\delta_i} & 0\\ \frac{-6}{\varphi_{i-1}\delta_{i-1}} & \frac{24\varphi_i\left(\beta_{2,i-1}-\beta_{2,i+1}\right)}{\mu_{i-1}\mu_i} & \frac{6}{\alpha_i\delta_i} & 0\\ \frac{6}{\alpha_i\delta_{i-1}} & \frac{3\left(-2\left(1+\alpha_{i-1}+\alpha_i\right)+\frac{4\alpha_{i-1}}{\varphi_i}-\frac{4\alpha_i}{\varphi_{i+1}}\right)}{2\delta_{i-1}\delta_i} & \frac{3\left(2-\alpha_{i-1}\right)}{2\delta_i} & 0\\ \frac{-2}{\varphi_{i-1}\delta_{i-1}} & \frac{4\delta_{i-1}-\frac{4\alpha_{i-1}}{\varphi_{i-1}}-\alpha_{i+1}-\frac{4\alpha_{i-1}}{\varphi_{i+1}}}{2\delta_{i-1}\delta_i} & \frac{-5}{2\delta_i}+\frac{1}{2\delta_{i+1}}+\frac{2}{\varphi_{i+2}\delta_{i+1}} & \frac{2}{\varphi_{i+1}\delta_{i+1}} \end{pmatrix}$$

$$\varphi_{i-1}=4+\beta_{2,i-1}$$

$$\alpha_{i-1}=\frac{4}{\varphi_{i-1}}$$

$$\delta_{i-1}=\left(1+\alpha_{i-1}+\alpha_i\right)$$

$$\mu_{i-1}=48+8\beta_{2,i-1}+8\beta_{2,i}+\beta_{2,i-1}\beta_{2,i}$$

Now, let $\beta_{2,i}=\beta_2, \forall i, \beta_2\geq 0, \beta_1=1$. Then the matrix $\mathbf{M}_{\beta_2}^{4}$ has the next form

/15/
$$\mathbf{M}_{\beta_2,C}^{4}=\frac{1}{12+\beta_2}\begin{pmatrix} 2 & 8+\beta_2 & 2 & 0\\ -6 & 0 & 6 & 0\\ 6 & -3(4+\beta_2) & 3(2+\beta_2) & 0\\ -2 & 2(3+\beta_2) & -2(3+\beta_2) & 2 \end{pmatrix}$$

Finally, let $\beta_{1,i}=\beta_1=1$ and $\beta_{2,i}=\beta_2=0$ for each i. The conditions /2/ guarantee only parametric continuous, in this case. Because of , the next matrix is the B-spline matrix and we mark it as $\mathbf{M}_{B}^{4}$.

/16/
$$\mathbf{M}_{B}^{4}=\frac{1}{6}\begin{pmatrix} 1 & 4 & 1 & 0\\ -3 & 0 & 3 & 0\\ 3 & -6 & 3 & 0\\ -1 & 3 & -3 & 1 \end{pmatrix}.$$

In this paper, the segments of the cubic β–spline curves are described in the matrix form. These curves are determined by control polygon $V_0, V_1, \ldots, V_n$ and by two sets of shape parameter values $\bar{\beta}1 = (\beta_{1,0}, \ldots, \beta_{1,n})$ and $\bar{\beta}2 = (\beta_{2,0}, \ldots, \beta_{2,n})$. The general matrix of the β-spline curves was founded and the special causes were distinguished according to choice of the value of the shape parameters. The matrix expressions are suitable especially for hardware generation β-splines curves.

References

[1] Bartels, R.H., Beautty, J.C., and Barsky, B.A.: An Introduction to Splines for Use in Computer Graphics and Geometric Modeling. Morgan-Kaufmann, Los Altos, Calif.,1987

[2] DeRose T.D., Barsky, B.A. : Geometric Continuity, Shape Parameters, and Geometric Constructions ~~for~~ Catmull-Rom Splines. ACM Transaction on Graphics, Vol.7, No. 1, January 1988, Pages 1-41

Computer Graphic Hardware
(Conditions and Experience from Teaching Courses)

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Selected courses on Computer Graphic Hardware at our Department of Computers, Electrical Faculty of Czech Technical University in Prague are attended by 15-20 computer graphics students every year. Sometimes also students interested in other fields take a part in this class.

The course was taught in the winter semester of the 5th study year in the past; since 1993/94 school year it is taught earlier, in the summer semester of the 4th year of study. The idea behind is to give students information how to implement algorithms of computer graphics, including parallel ones, more efficiently. So the main orientation is not towards designing the HW.

Considering the fact, that implementation tools are relatively complex and there is only limited amount of them available, it is not possible to organize the classes in the laboratories for a whole study group. The solution for this problem is, that students take a part in particular course projects, depending on their interests and orientations. The results are published in course colloquias.

As it will follow from the overview below, our HW implementation tools do not cover all the course fields. There is a lot of topics which are taught only theoreticaly. Our opinion is, that it is neither possible nor expedient to let the students experiment during the course time with all the equipment and