University of West Bohemia
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Diploma Thesis

Methods for solving hyperbolic partial differential equations with generalized flux functions

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Metody pro řešení parciálních diferenciálních rovnic hyperbolického typu se zobecněnými tokovými funkcemi

Plzeň, 2015 **Tereza Dadáková**

Declaration

I declare that I worked on this thesis on my own and that I used only sources mentioned in the Bibliography section.

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Tereza Dadáková

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Annotation

This paper analyses solutions to scalar conservation law with generalized flux function. For non-convex flux functions the nonclassical shock wave may arise and this solution violates the classical Oleinik entropy condition, however we show, that also this solution is admissible. We also introduce some numerical methods, that are able to detect these nonclassical shocks.

Contents

Preface

The aim of this paper is to introduce the theory for nonlinear conservation laws with generalized flux functions. The conservation law appears in many areas of physics, basically anywhere, where some balance laws are formulated. Solutions of these hyperbolic conservation laws may contain singularities, which appear in finite time even from smooth initial data. Such weak solutions are not unique and we have to impose some additional condition to choose one of them. These conditions are usually called entropy conditions. These fundamental concepts are given in Chapter 1.

In Chapter 2 we impose the Oleinik entropy condition on solutions and define classical solutions of the Riemann problem for conservation law, first for convex flux functions and later we generalize the classical solutions also for nonconvex fluxes. We will also show, that for convex flux is the classical solution the only one admissible.

In the rest of the text we will concentrate on nonconvex flux functions, specifically on concave-convex flux functions with one inflection point. We will define the entropy dissipation function and based on its properties we will show, that for nonconvex flux function there exists also another admissible solution and we will call it the nonclassical solution. Existence and properties of these nonclassical solutions are the subject of Chapter 3.

The Chapter 4 is devoted to applying numerical schemes to our Riemann problem in order to detect nonclassical behaviour.

Chapter 1

Introduction

1.1 Hyperbolic System

Definition 1.1 (Hyperbolic system of equations)**.** Consider the system

$$\boldsymbol{u}_t + \mathbb{A}\boldsymbol{u}_x = 0, \quad \text{in} \quad \mathbb{R} \times (0, +\infty), \tag{1.1}$$

where $\boldsymbol{u} : \mathbb{R} \times (0, +\infty) \to \mathbb{R}^m$ is a vector of unknown functions and $\mathbb{A}$ is a $m \times m$ matrix function of $\boldsymbol{u}$, x and t. This system is called hyperbolic, if $\mathbb{A} = \mathbb{A}(\boldsymbol{u}, x, t)$ is diagonalizable matrix with real eigenvalues $\lambda_1, \ldots, \lambda_m$ for each x, t and $\boldsymbol{u}$.

By adding the initial condition

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

to system (1.1), where $\boldsymbol{u}_0 : \mathbb{R} \to \mathbb{R}^m$ is a given function, we get the Cauchy problem, i.e. a problem to find a solution $\boldsymbol{u} \in C^1$, which satisfies the system (1.1) in $\mathbb{R} \times (0, +\infty)$ and the initial condition in $\mathbb{R}$.

It can be shown, that system (1.1) with initial condition (1.2) has a unique solution in the class C^1 in some finite time interval $t \in [0, T]$, see [1].

For nonlinear systems the solution does not have to exist for all time in classical sense, i.e. after some time, as the solution evolves, discontinuities may appear. In further text we will focus our attention on conservation laws, which are integral relations. Hence we don't need solutions, which are continuous or even differentiable. This will lead us to weak formulation and generalized solutions, which may exist for all time. However, we will see, that for one initial condition there can be many weak solutions, but not all physically relevant.

1.2 Conservation laws

Let $u(\boldsymbol{x}, t)$ be some quantity with a flux $f(u(\boldsymbol{x}, t))$. The quantity is conserved in a bounded fixed domain Ω with a Lipschitz-continuous boundary $\partial\Omega$, if total mass of this quantity changes inside the domain in time only due to the flux through the boundary $\partial\Omega$, thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} u(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} = -\int_{\partial\Omega} f(u)\boldsymbol{n}\mathrm{d}S, \tag{1.3}$$

where $\boldsymbol{n}$ is the outward normal and $\mathrm{d}S$ the surface element on $\partial\Omega$. If we suppose, that u and f are sufficiently smooth functions, than by applying the divergence theorem and by taking into account, that the domain Ω is arbitrary, we can rewrite the equation 1.3 as

$$u_t + \mathrm{div} f = 0, \tag{1.4}$$

which is the differential form of the conservation law.

1.2.1 Weak solution

Let u be a classical solution of the problem 1.4 with initial condition u_0 on $\mathbb{R}^N$ and consider $\phi \in C_0^\infty$, where C_0^∞ denotes the space of all infinitely differentiable functions with compact support in $D = \mathbb{R}^N \times [0,\infty)$. Multiplying the equation 1.4 by ϕ, integrating over D and using Green's theorem we obtain

$$\int_0^\infty \int_{\mathbb{R}^N} (u\phi_t + f \cdot \mathrm{grad}\phi)\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_{\mathbb{R}^N} u_0\phi(\boldsymbol{x},0)\mathrm{d}\boldsymbol{x} = 0. \tag{1.5}$$

In this formulation we no longer demand existence of derivatives of u, so 1.5 makes sense even if u is discontinuous. If the function u satisfies 1.5 for all $\phi \in C_0^\infty$, we call it a weak solution of 1.4. In the following text we will suppose the solution to be the weak one.

1.2.2 Scalar conservation law

Further we will consider only scalar case of conservation law, thus equation of the form

$$u_t(x,t) + f(u(x,t))_x = 0. \tag{1.6}$$

If we suppose, that $f(u)$ is differentiable, than by using chain rule, we can express (1.6) in quasilinear form

$$u_t + f'(u)u_x = 0. \tag{1.7}$$

Curves $x = x(t)$, which are given as solutions of ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f'(u(x,t)), \tag{1.8}$$

are called characteristics of the equation (1.7). It's evident, that the solution is constant along this characteristics,since

$$\frac{\mathrm{d}u(x,t)}{\mathrm{d}t} = u_t + u_x\frac{\mathrm{d}x}{\mathrm{d}t} = u_t + f'u_x = 0. \tag{1.9}$$

This fact together with (1.8) implies, that characteristics are straight lines determined by

$$x = \xi + f'(u_0(\xi))t, \quad \xi \in \mathbb{R}. \tag{1.10}$$

The solution is than given by

$$u = u_0(\xi). \tag{1.11}$$

Let us assume, that the flux function is strictly convex (or concave), i.e. f'' doesn't change sign, and that (1.7) is genuinely nonlinear, i.e. $f'' \neq 0$. Than, for $f'' > 0$, we may consider two cases:

- ${u_0}' > 0$ Since the slope of characteristics can be expressed as $1/f'(u_0)$, it's obvious, that because both f' and u_0 are increasing functions, the slope is nonincreasing function in x. It's form is shown in Figure **??**. Hence, characteristics cover whole upper half-plane $t \geq 0$ and the solution exist for every time in classical sense.

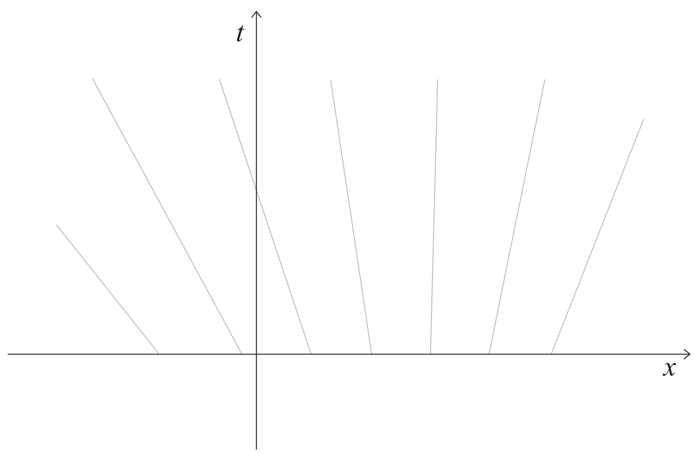


Figure 1.1: Characteristics for ${u_0}' > 0$.

- ${u_0}' < 0$ In this case the slope of characteristics is an increasing function in x. This causes, that for every two points x_1 and x_2 characteristics issuing from these points have to intersect in some time. At the point of intersection two values of u meet and discontinuity appears. It's form is shown in Figure 1.2. This also means that at this point the derivative

$$u_x = \frac{u_0'}{1 + f'' u_0' t} \tag{1.12}$$

becomes unbounded. The first time when this happens is

$$T = \frac{-1}{\min_{x \in \mathbb{R}}(f''(u_0)u_0')}. \tag{1.13}$$

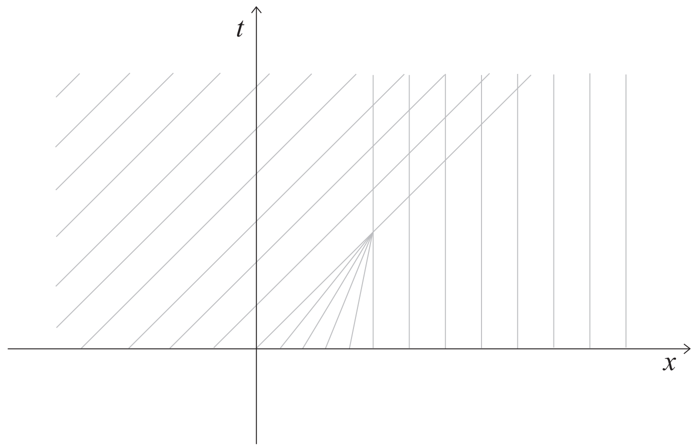


Figure 1.2: Characteristics for ${u_0}' < 0$.

So the classical solution exists on time interval $[0, T)$, where time T is given by (1.13). This shows that also for smooth initial condition discontinuities may rise, but as we suggested, the conservation law is in first place an integral relation and we don't need solution to be continuous. So let's take a further look at weak solutions. Is the resulting discontinuity arbitrary? Is the weak solution unique?

1.2.3 Rankine-Hugoniot jump condition

Let u be a weak solution of (1.7) with single discontinuity. This discontinuity will propagate along some curve $x = y(t)$ with speed

$$s(t) = \frac{\mathrm{d}y}{\mathrm{d}t}. \tag{1.14}$$

The solution satisfies the equation (1.7) on each side of this curve in classical sense and we will denote by u_l and u_r values of u on left and right side of the discontinuity. Let us consider x from interval $[a, b]$ so that it crosses the curve y in some time t, see Figure (1.3). Over this interval the conservation law (1.3) has to hold,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x = f(u(a,t)) - f(u(b,t)). \tag{1.15}$$

The left side of this equation may be expressed as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^{y(t)} u(x,t)\mathrm{d}x + \int_{y(t)}^b u(x,t)\mathrm{d}x\right)\\ &= \int_a^{y(t)} u_t\mathrm{d}x + y'u_l + \int_{y(t)}^b u_t\mathrm{d}x - y'u_r. \end{aligned} \tag{1.16}$$

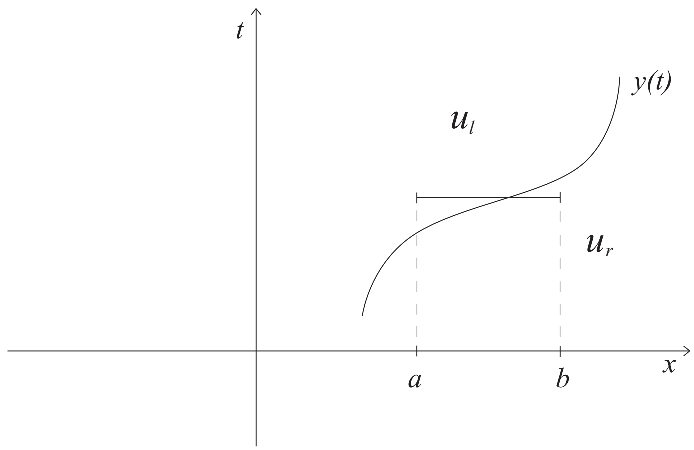


Figure 1.3: The curve, along which the discontinuity propagates.

Since on both sides of discontinuity also holds the conservation law, moreover in differential form, we may set $u_t = -f_x$ and last expression may be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x = -f(u_l) + f(u(a,t)) + su_l + -f(u(b,t)) + f(u_r) - su_r, \tag{1.17}$$

where notation (1.14) was used. This, together with (1.15), gives the relation

$$s(u_r - u_l) = f_r - f_l, \tag{1.18}$$

which is called Rankine-Hugoniot jump condition. Hence, every weak solution has to satisfy this jump condition and on the other hand, every piecewise function, which satisfies (1.6) and this condition, is a weak solution.

1.2.4 Riemann problem

The simplest weak solution of (1.6) is a piecewise constant function with a single discontinuity. The problem

$$u_t + f_x = 0 \tag{1.19}$$

$$u_0 = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0 \end{cases} \tag{1.20}$$

is called the Riemann problem. Since the solution of this problem is homogeneous, i.e. if $u(x,t)$ is a solution of the problem (1.19)-(1.20), than the function $u(kx,kt)$, $k \in \mathbb{R}$ is also its solution. Therefore it is reasonable to look for solution in form

$$u(x,t) = \tilde{u}(x/t), \tag{1.21}$$

which is also called similarity solution. Denoting

$$\xi = \frac{x}{t} \tag{1.22}$$

we obtain

$$u_t = \tilde{u}_\xi(-\frac{x}{t^2}), \quad u_x = \tilde{u}_\xi\frac{1}{t}. \tag{1.23}$$

By inserting this into (1.19), we get

$$\tilde{u}_\xi(f' - \frac{x}{t}) = 0. \tag{1.24}$$

Hence we find that either $\tilde{u}_\xi = 0$, i.e. the solution is constant, or that $f' = \frac{x}{t}$. Since we suppose that $f'' > 0$, the function f' is increasing on interval $[u_l, u_r]$ for $u_l < u_r$ and hence the inverse function $(f')^-1$ is well-defined on $[f(u_l), f(u_r)]$ and we can explicitly determine the solution of (1.19)-(1.20) as

$$u(x,t) = \begin{cases} u_l, & x/t < f'(u_l), \\ (f')^{-1}(x/t) & f'(u_l) < x/t < f'(u_r), \\ u_r, & x/t > f'(u_r). \end{cases} \tag{1.25}$$

This solution is called centered rarefaction wave, for characteristics see Figure 1.4.

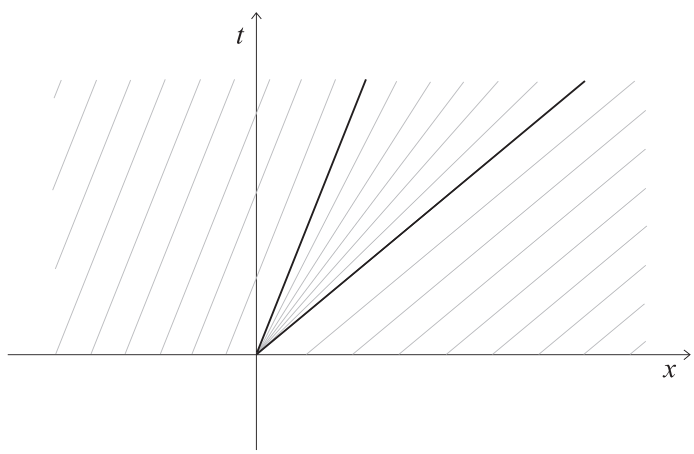


Figure 1.4: Characteristics of rarefaction wave.

However, the function

$$u = \begin{cases} u_l, & x < st, \\ u_r, & x > st, \end{cases} \tag{1.26}$$

where s is the speed of discontinuity determined by Rankine-Hugoniot jump condition, is also a solution of problem (1.19)-(1.20) with $u_l < u_r$. Its characteristics are shown in Figure 1.5.

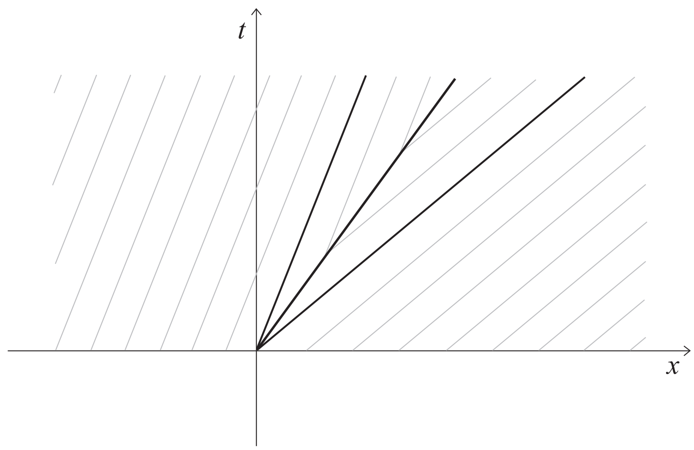


Figure 1.5: Characteristics for the solution 1.26.

We see, that the solution is not necessarily unique, but we are looking for the one with physical meaning. To eliminate the non-uniqueness of solution, we need to impose some additional condition to choose the solution which is physically admissible. These conditions are called admissibility conditions, or entropy conditions.

1.2.5 Entropy

The notion of entropy is here motivated by thermodynamic considerations. In order to keep the theory consistent with the second law of thermodynamics, we suppose, that the entropy function is strictly convex. Interesting remarks about the convexity property of the entropy are given in [15].

Definition 1.2 (Entropy function, entropy flux)**.** Let $\mathcal{U}$ be a space of continuously differentiable solutions of scalar conservation law (1.6). The strictly convex smooth function $U : \mathcal{U} \to \mathbb{R}$ is called entropy and the smooth function $F : \mathcal{U} \to \mathbb{R}$ is called entropy flux, if they satisfy the conservation law

$$U(u)_t + F(u)_x = 0. \tag{1.27}$$

If we multiply the scalar conservation law (1.6) by U', than by comparing with (1.27) we receive the relation for entropy flux

$$F' = U'f'. \tag{1.28}$$

For scalar conservation laws there is no other restriction on entropy function, so any strictly convex function U is an entropy with flux given by

$$F(u) = F_a + \int_a^u U'(v)f'(v), \tag{1.29}$$

where $a \in \mathcal{U}$ is fixed and F_a is arbitrary. The situation is different for systems of conservation laws, see for example [12].

However, we are interested in solutions, which contain discontinuity. The sharp discontinuities are rather mathematical model of some situations than reality. In the real world are these jump more like smooth transitions over narrow regions. This can be expressed by adding some regularization R^ϵ on the right side of the conservation law, which corresponds to viscosity, capillarity, oscillation, etc. depending on physical model. So we restrict our attention to smooth solutions of the equation

$$u_t^\epsilon + f_x(u^\epsilon) = R^\epsilon, \tag{1.30}$$

where the regularization R^ϵ may depend on $u^\epsilon, \epsilon u_x^\epsilon, \epsilon^2 u_{xx}^\epsilon, \ldots$ and it is vanishing for $\epsilon \to 0$, so the solution u of the conservation law (1.6) is the limit

$$u = \lim_{\epsilon \to 0} u^\epsilon. \tag{1.31}$$

Precisely, let us suppose, that the solution u^ϵ is bounded in L^∞ norm and the limit (1.31) holds almost everywhere. Hence, we need the limit u to be a weak solution of (1.6). For every $\phi \in C_0^\infty$ we have

$$\begin{aligned} \int_\Omega (u\phi_t + f\phi_x)\mathrm{d}x\mathrm{d}t &= \lim_{\epsilon \to 0} \int_\Omega (u^\epsilon \phi_t + f(u^\epsilon)\phi_x)\mathrm{d}x\mathrm{d}t = \\ &= \lim_{\epsilon \to 0} \int_\Omega R^\epsilon \phi \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{1.32}$$

which has to be equal to zero. This gives us the first condition on R^ϵ, see Definition 1.3.

The second condition we get by the same manipulations as we used previously to derive the conservation law for entropy function: multiplying the equation (1.30) by U' we obtain

$$U_t(u^\epsilon) + F_x(u^\epsilon) = U'(u^\epsilon)R^\epsilon. \tag{1.33}$$

Due to (1.31) the left-hand side of (1.33) converges in the weak sense to $U_t(u) + F_x(u)$. The second condition will than arrive from the physical demand on the equation, to be entropy dissipative.

Definition 1.3 (Properties of Regularization)**.** The regularization R^ϵ introduced in (1.30) is said to be

- conservative, if

$$\lim_{\epsilon \to 0} \int_\Omega R^\epsilon \phi \mathrm{d}x\mathrm{d}t = 0, \quad \phi \in C_0^\infty, \tag{1.34}$$

  and

- entropy dissipative, if

$$\limsup_{\epsilon\to 0}\int_{\Omega} U'(u^{\epsilon})R^{\epsilon}\phi \mathrm{d}x\mathrm{d}t \le 0, \quad \phi \in C_0^{\infty}, \phi \ge 0. \tag{1.35}$$

Now we can finally introduce the important entropy inequality.

Theorem 1.4 (Entropy inequality). *Let u^{ϵ} be a solution of (1.30), which remains bounded in L^{∞} norm and converges almost everywhere towards the limit u as $\epsilon \to 0$. Also suppose, that the regularization R^{ϵ} on right-hand side of (1.30) is conservative and entropy dissipative for some strictly convex entropy function U of (1.6) with flux F. Than u is a weak solution of (1.6) and satisfies*

$$\int_{\Omega} (U(u)\phi_t + F(u)\phi_x)\mathrm{d}x\mathrm{d}t \ge 0, \quad \phi \in C_0^{\infty}, \phi \ge 0, \tag{1.36}$$

which means in weak sense

$$U_t(u) + F_x(u) \le 0. \tag{1.37}$$

It turns out, that for genuinely nonlinear flux functions ($f'' > 0$), the entropy inequality (1.37) is sufficiently restrictive to choose the unique solution of (1.6), which will be described in next chapter. In this case is the solution even independent on regularization R^{ϵ}. We will refer to these solutions as classical entropy solutions. However, many physical models fail to be genuinely nonlinear and for such systems this does not have to be the truth, as wee will also see later. But first, let us pay the attention to important properties of the entropy inequality and classical entropy solutions for Riemann problem, after one last important relation:

Theorem 1.5 (Jump condition for the entropy inequality). *Let u be a piecewise smooth function with discontinuity satisfying the Rankine-Hugoniot jump condition (1.18) with speed s, than the entropy inequality (1.37) is equivalent to*

$$-s(U_r - U_l) + F_r - F_l \le 0, \tag{1.38}$$

where we used the notation

$$U_r := U(u_r), \qquad F_r := F(u_r), \tag{1.39}$$

$$U_l := U(u_l), \qquad F_l := F(u_l). \tag{1.40}$$

Proof of this theorem is analogous to derivation of the Rankine-Hugoniot jump condition.

Chapter 2

Classical solution

2.1 Entropy condition

Consider again the Riemann problem

$$u_t + f_x = 0 \tag{2.1}$$

$$u_0 = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0, \end{cases} \tag{2.2}$$

where $f(u)$ is smooth function and u_l, u_r are constants. Recall, that we seek for weak solutions satisfying the Rankine-Hugoniot jump condition and the entropy inequality.

Theorem 2.1 (Diffusive regularization). *Let u^ϵ be a solution of the diffusion (viscous) equation*

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon, \quad \epsilon > 0, \tag{2.3}$$

which is bounded in L^∞ norm and the derivative u^{ϵ_x} tends to zero at infinity. Than the regularization $R^\epsilon = \epsilon u_{xx}^\epsilon$ is conservative and entropy dissipative for any strictly convex entropy U with entropy flux $F(u) = \int^u U'(v) f'(v) \mathrm{d}v$.

Proof. First we will show, that the regularization is conservative. We have

$$\begin{aligned} \left| \int_{\mathbb{R}\times\mathbb{R}^+} R^\epsilon \phi \mathrm{d}x\mathrm{d}t \right| = \left| \int_{\mathbb{R}\times\mathbb{R}^+} \epsilon u_{xx}^\epsilon \phi \mathrm{d}x\mathrm{d}t \right| &\leq \int_{\mathbb{R}\times\mathbb{R}^+} \epsilon |u^\epsilon||\phi_{xx}| \mathrm{d}x\mathrm{d}t \\ &\leq \epsilon \|u^\epsilon\|_{L^\infty} \|\phi_{xx}\|_{L^1} \\ &\leq C\epsilon, \end{aligned} \tag{2.4}$$

which tends to zero as $\epsilon \to 0$, for all $\phi \in C_0^\infty$, hence the regularization R^ϵ is conservative. Further, by multiplying the equation (2.3) by U' we find

$$U_t(u^\epsilon) + F_x(u^\epsilon) = \epsilon U_x x(u^\epsilon) - \epsilon U''(u_x^\epsilon)^2, \tag{2.5}$$

since $U'u^\epsilon_{xx} = U_x x(u^\epsilon) - U''(u^\epsilon_x)^2$. We have now

$$\int_{\mathbb{R}\times\mathbb{R}^+} U'(u^\epsilon)R^\epsilon\phi \mathrm{d}x\mathrm{d}t \leq \epsilon\int_{\mathbb{R}\times\mathbb{R}^+} |U(u^\epsilon)||\phi_{xx}|\mathrm{d}x\mathrm{d}t - \epsilon\int_{\mathbb{R}\times\mathbb{R}^+} U''(u^\epsilon_x)^2\phi \mathrm{d}x\mathrm{d}t, \tag{2.6}$$

for all $\phi \in C_0^\infty$, $\phi \geq 0$. The first term on the right hand-side tends to zero as $\epsilon \to 0$. In the second therm we have $U'' > 0$, which means that the the right hand-side of the inequality is non-positive and hence the regularization is entropy dissipative. □

From the Theorem (2.1) we see, that the solution determined by the vanishing-viscosity limit can only converge to the weak solution of (2.1) and the entropy inequality is satisfied for all strictly convex entropy functions. As we mentioned earlier, the entropy inequality will help us to find the unique solution of the problem (2.1-2.2). Let us recall, that the solution may contain centered rarefaction wave or discontinuity, which has to propagate with speed s determined by the Rankine-Hugoniot jump condition. We will refer to this discontinuity as a shock wave.

Theorem 2.2 (Oleinik inequality)**.** *A shock wave solution of (2.1-2.2)*

$$u = \begin{cases} u_l, & x < st, \\ u_r, & x > st, \end{cases} \tag{2.7}$$

satisfy the entropy inequality $U_t + F_x \leq 0$ for all strictly convex entropies if and only if it satisfies the Oleinik entropy inequality

$$\frac{f_r - f_l}{u_r - u_l} \leq \frac{f(u) - f_l}{u - u_l} \tag{2.8}$$

for all u between u_l and u_r, or equivalently

$$\frac{f(u) - f_r}{u - u_r} \leq \frac{f_r - f_l}{u_r - u_l} \tag{2.9}$$

for all u between u_l and u_r.

Proof. We will use the equivalent expression to the entropy inequality (1.38)

$$\begin{aligned} 0 &\geq -s(U_r - U_l) + F_r - F_l \\ &= \int_{u_l}^{u_r} U'(u)(-s + f'(u))\mathrm{d}u \\ &= [U'(u)(-su + f(u))]_{u_l}^{u_r} - \int_{u_l}^{u_r} U''(u)(-s(u-u_l) + f(u) - f_l)\mathrm{d}u. \end{aligned} \tag{2.10}$$

The first expression is equal to zero, since $s = (f_r - f_l)/(u_r - u_l)$. Hence we have

$$-\int_{u_l}^{u_r} U''(u)(u - u_l)\left(\frac{f(u) - f_l}{u - u_l} - \frac{f_r - f_l}{u_r - u_l}\right)\mathrm{d}u \leq 0. \tag{2.11}$$

Since U'' is arbitrary and strictly convex, we get the Oleinik inequality (2.8). Geometrically this condition means, that the graph of f lies below the line connecting points $[u_l, f_l]$ and $[u_r, f_r]$, if $u_r < u_l$ or above the line, if $u_r > u_l$. This is obviously equivalent to (2.9). □

If we let $u \to u_l$ ($u \to u_r$, respectively) in (2.8) (in (2.9), respectively), we get

$$f'_r \leq s \leq f'_l, \tag{2.12}$$

which geometrically means, that characteristics must go into the shock.

Theorem 2.3 (Lax shock inequality)**.** *If the flux function is convex, the Oleinik inequality is equivalent to Lax shock inequality*

$$u_r \leq u_l. \tag{2.13}$$

Proof of this comes straightforward from (2.8).

The Oleinik inequality gives us the condition on uniqueness of solutions we were looking for.

If we return to the problem of two solutions 1.25 and 1.26 of the Riemann problem, we see, that the Oleinik condition excluded the solution with shock and left us with the only one right solution - rarefaction wave.

We will refer to the solutions which satisfy this condition as the classical solutions and the shock wave is said to be the classical shock or compressive shock.

For convex flux, the only admissible solution with shock wave is the one satisfying the Lax condition. If $u_r \geq u_l$, than the solution has to contain the rarefaction wave. In the case of concave flux function the situation is completely analogous.

2.2 Classical entropy solution

Theorem 2.4 (Classical solution to Riemann problem with convex flux)**.** *Suppose that flux function f is convex. Than the solution of the Riemann problem (2.1-2.2) is given by*

- *a shock, connecting u_l and u_r with shock speed s, if $u_r \leq u_l$*

$$u = \begin{cases} u_l, & x < st, \\ u_r, & x > st, \end{cases} \tag{2.14}$$

or by

- *a rarefaction wave, connecting continuously and monotonically u_l and u_r, if $u_r \geq u_l$*

$$u(x,t) = \begin{cases} u_l, & x/t < f'(u_l), \\ (f')^{-1}(x/t) & f'(u_l) < x/t < f'(u_r), \\ u_r, & x/t > f'(u_r). \end{cases} \tag{2.15}$$

The Oleinik inequality, however, holds also for nonconvex flux functions and we will now derive the classical solution for this more general case. The convexity of flux function makes things easier, since the characteristic speed f' changes monotonically with u and the solution than may contain eaither shock or rarefaction wave. On the other hand, if the flux function is nonconvex, the solution may be more complicated and may contain both, shock and rarefaction wave.

First, let us consider a flux function with one inflection point. Suppose the function is concave-convex with the inflection point in $u = 0$ and consider the Riemann problem for $u_l > 0$. If $u_r > u_l$, the flux function is convex between those stages and so the solution is given by rarefaction, as stated above. Therefore, let us concentrate on the case, when $u_r < u_l$. The Oleinik inequality indicates, that the flux function has to lie below the line connecting u_r and u_l. This is true up to the point, where this line becomes tangent of the graph of f. Denote this point as $\tilde{u}_l$ and in this point holds the relation

$$f'(\tilde{u}_l) = \frac{f(\tilde{u}_l) - f_l}{\tilde{u}_l - u_l}, \tag{2.16}$$

see Figures 2.1 and 2.2. When $u_r < \tilde{u}_l$, the line connecting u_l and u_r crosses the graph of f and the Oleinik inequality is violated (Figure 2.3). Hence, the solution must consist of the shock satisfying the Oleinik inequality, connecting the stages u_l and $\tilde{u}_l$, followed by a rarefaction wave to u_r. See Figure 2.4.

The situation is analogous for $u_l < 0$. For $u_r < u_l$ is the flux function concave and the solution is rarefaction wave. For $u_r > u_l$ the graph of f must lie above the line connecting those two stages. The stage $\tilde{u}_l$ is determined the same way as in previous case and for $u_r > \tilde{u}_l$ is the shock wave again followed by rarefaction from $\tilde{u}_l$ to u_r.

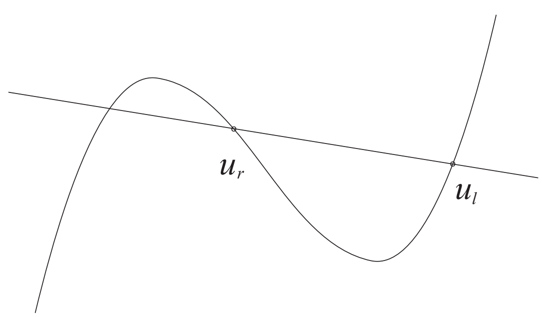


Figure 2.1: Oleinik condition.

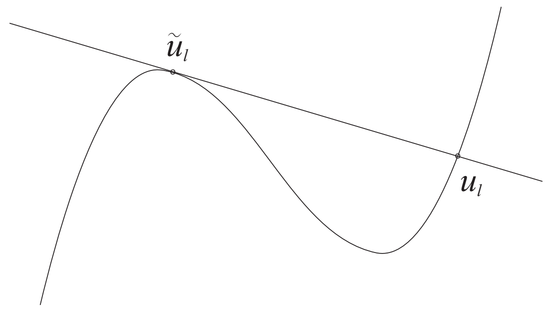


Figure 2.2: The limiting point for the Oleinik condition.

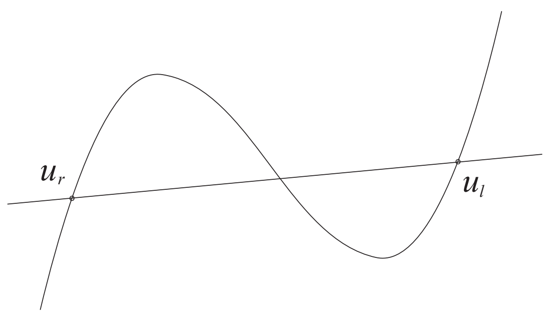


Figure 2.3: Violating the Oleinik condition.

Theorem 2.5 (Classical solution to Riemann problem with concave-convex flux)**.** *Let the flux function f be concave-convex with inflection point at $u = 0$. Then the solution of Riemann problem (2.1-2.2) with $u_l > 0$, satisfying the entropy inequality for all strictly convex entropies U, consists of shock waves and rarefaction waves given as follows:*

- *If $u_r > u_l$, the solution is a rarefaction wave connecting u_l and u_r monotonically and continuously.*
- *If $u_r \in [\tilde{u}_l, u_l)$, where $\tilde{u}_l$ is given by the relation* **??***, the solution is a shock wave connecting u_l and u_r.*
- *If $u_r < \tilde{u}_l$, the solution is composed of the shock wave connecting u_l and $\tilde{u}_l$ and a rarefaction connecting $\tilde{u}_l$ and u_r.*

The solution with $u_l < 0$ is analogous:

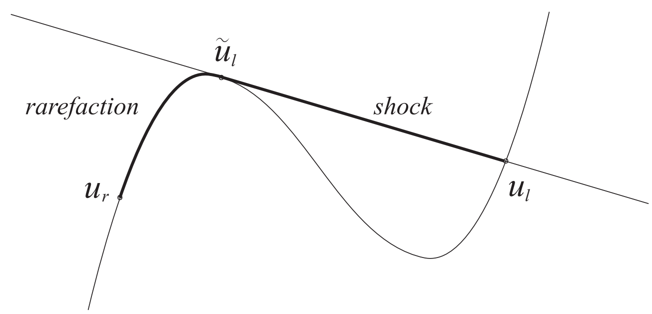


Figure 2.4: Oleinik condition for nonconvex flux.

- *If $u_r < u_l$, the solution is a rarefaction wave connecting u_l and u_r monotonically and continuously.*
- *If $u_r \in (u_l, \tilde{u}_l]$, where $\tilde{u}_l$ is given by the relation (2.16), the solution is a shock wave connecting u_l and u_r.*
- *If $u_r > \tilde{u}_l$, the solution is composed of the shock wave connecting u_l and $\tilde{u}_l$ and a rarefaction connecting $\tilde{u}_l$ and u_r.*

If we stick with the notation of fixed u_l, than the solution for convex-concave flux is given in Theorem 2.6. The difference here is only that the tangent to the graph f is passing through the stage u_r instead of u_l. See Figure 2.5.

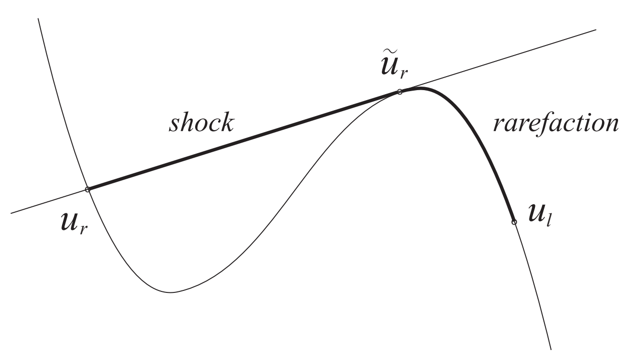


Figure 2.5: Oleinik condition for nonconvex flux.

Theorem 2.6 (Classical solution to Riemann problem with convex-concave flux). *Let the flux function f be convex-concave with inflection point at $u = 0$. Then the solution of Riemann problem (2.1-2.2) with $u_l > 0$, satisfying the entropy inequality for all strictly convex entropies U, consists of shock waves and rarefaction waves given as follows:*

- *If $u_r > u_l$, the solution is a shock wave connecting u_l and u_r.*
- *If $u_r \in [0, u_l)$, the solution is a rarefaction wave connecting u_l and u_r monotonically and continuously.*
- *If $u_r \in [\tilde{u}_l^-, 0)$, where $\tilde{u}_l^-$ is given by*

$$f'(u_l) = \frac{f(\tilde{u}_l^-) - f_l}{\tilde{u}_l^- - u_l}, \tag{2.17}$$

  the solution is composed of a rarefaction wave connecting u_l and $\tilde{u}_r$ and a shock wave connecting $\tilde{u}_r$ and u_r. The stage $\tilde{u}_r$ is given by

$$f'(\tilde{u}_r) = \frac{f(\tilde{u}_r) - f_r}{\tilde{u}_r - u_r}. \tag{2.18}$$

- *If $u_r < \tilde{u}_l^-$, the solution is a shock wave connecting u_l and u_r.*

For convex-concave flux with $u_l < 0$ would be the solution again analogous.

This basic idea of construction of the concave hull for $u_l > u_r$ or the convex hull for $u_l < u_r$ can be generalized for any nonconvex flux function with finitely many inflection points. For example for $u_l > u_r$ would be the interval $[u_r, u_l]$ decomposed by the hull on several intervals, in which the flux function f either coincides with the hull, which corresponds to rarefaction wave, or is strictly below the hull, which is on this interval straight line, and this corresponds to shock wave. In the Figure 2.6 is the situation shown for $u_l < u_r$, so here we are looking for the convex hull.

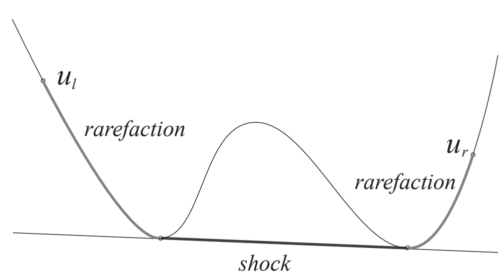


Figure 2.6: Oleinik condition for nonconvex flux.

Example 2.7 (Cubic flux)**.** Consider the Riemann problem

$$u_t + (u^3)_x = 0 \tag{2.19}$$

$$u_0 = \begin{cases} u_l > 0, & x < 0, \\ u_r, & x > 0. \end{cases} \tag{2.20}$$

We will distinguish between three cases, depending on the value of u_r compared to the

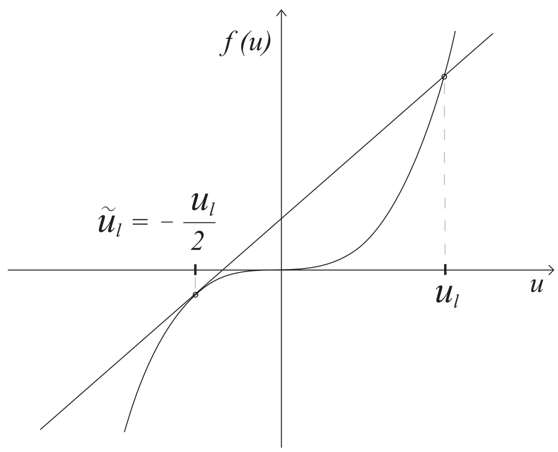


Figure 2.7: Cubic flux function.

fixed value u_l. For $u_r > u_l$ we are looking for the convex hull, which in this case coincides with the flux function. For $u_r < u_l$ the flux function may change its convexity on interval $[u_r, u_l]$ and the situation is more interesting. In this case we are looking for the concave hull and we need to find the point $\tilde{u}_l$, where the shock speed is equal to the speed of right-hand wave. Using the relation (2.16) we have

$$3\tilde{u}_l^2 = \frac{\tilde{u}_l^3 - u_l^3}{\tilde{u}_l - u_l}, \tag{2.21}$$

which gives us the value $\tilde{u}_l = -u_l/2$. The solution of (2.19-2.20) is then given as follows:

- If $u_r > u_l$, the solution is a rarefaction wave

$$u(x,t) = \begin{cases} u_l, & x/t \le 3u_l^2, \\ \sqrt{\frac{x}{3t}} & 3u_l^2 < x/t < 3u_r^2, \\ u_r, & x/t \ge 3u_r^2. \end{cases} \tag{2.22}$$

- If $u_r \in [-u_l/2, u_l)$, the solution is classical shock wave

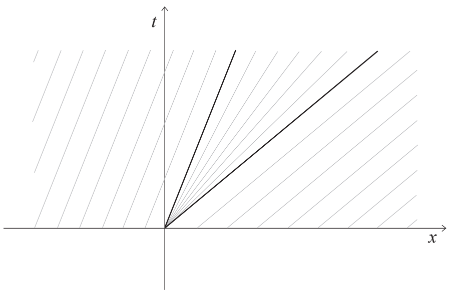
$$u = \begin{cases} u_l, & x < st, \\ u_r, & x > st, \end{cases} \tag{2.23}$$

with the shock speed $s = u_r^2 + u_r u_l + u_l^2$ given by the Rankine-Hugoniot condition.

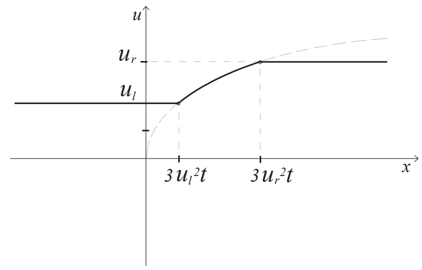
- If $u_r < -u_l/2$, the solution consists of the shock wave from u_l to $-u_l/2$ with shock speed $s = 3/4\,u_l^2$, immediately followed by a rarefaction wave to u_r

$$u(x,t) = \begin{cases} u_l, & x/t \leq 3/4\,u_l^2, \\ -\sqrt{\frac{x}{3t}} & 3/4\,u_l^2 < x/t < 3u_r^2, \\ u_r, & x/t \geq 3u_r^2. \end{cases} \tag{2.24}$$

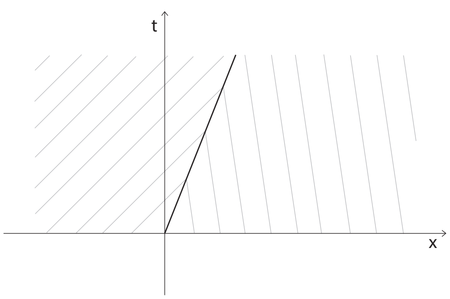
The characteristics and solutions for all cases are shown in Figures 2.8a - 2.8f.



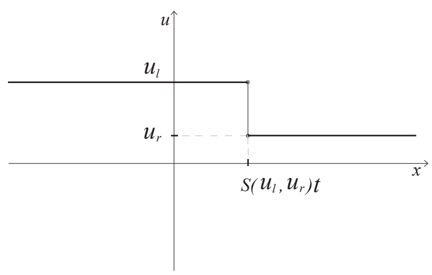
(a) Characteristics for $u_r > u_l$.



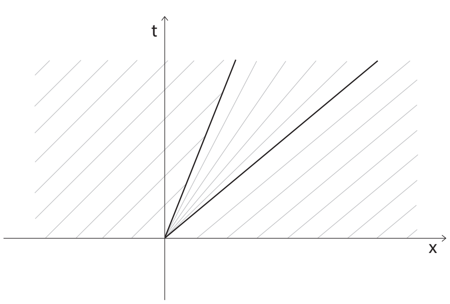
(b) Solution for $u_r > u_l$.



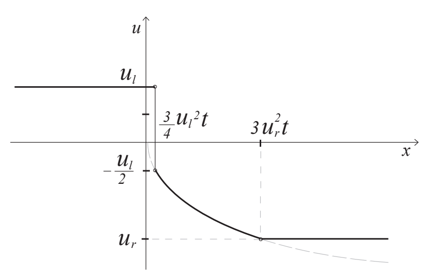
(c) Characteristics for $u_r \in [-u_l/2, u_l)$.



(d) Solution for $u_r \in [-u_l/2, u_l)$.



(e) Characteristics for $u_r < -u_l/2$.



(f) Solution for $u_r < -u_l/2$.

Chapter 3

Nonclassical solution

In this chapter we will sill pay the attention to solving the Riemann problem (2.1-2.2), however we will not impose the entropy inequality for all convex entropies, as we did in previous chapter, but we suppose, that the inequality holds for one given entropy U. We will also assume a flux function to be concave-convex with a single inflection point at $u = 0$ and to be non-degenerate at this point, i.e.

$$uf''(u) > 0 \quad u \neq 0, f'''(0) \neq 0. \tag{3.1}$$

We stick with the notation given in previous chapter, u_l and u_r are related to the Riemann initial data, definitions of stages $\tilde{u}_l$, $\tilde{u}_l^-$ also remains. For consistency of notation it would be, however, useful to define those terms in more general way as functions of u,

$$f'(\tilde{u}(u)) = \frac{f(\tilde{u}(u)) - f(u)}{\tilde{u}(u) - u}, \quad u \in \mathbb{R} \tag{3.2}$$

and the inverse function of $\tilde{u}(u)$, $\tilde{u}^-(u)$

$$f'(u) = \frac{f(\tilde{u}^-(u)) - f(u)}{\tilde{u}^-(u) - u}. \tag{3.3}$$

Hence, obviously, $\tilde{u}_l = \tilde{u}(u_l)$ and $\tilde{u}_l^- = \tilde{u}^-(u_l)$. Assume that the solution contains a shock wave. The stages of this shock will be denoted as u_- and u_+ for $x < st$ and $x > st$, respectively, where $s(u_-, u_+)$ is the speed of the shock. Moreover we will introduce $\tilde{u}_- = \tilde{u}(u_-)$ and $\tilde{u}_-^- = \tilde{u}^-(u_-)$ for the stage u_- and $\tilde{u}_+$ and $\tilde{u}_+^-$ for the stage u_+, that are defined the same way.

3.1 Entropy dissipation

Recall, that the entropy condition for given entropy U holds across the shock (u_-, u_+), if

$$-s(U_- - U_+) + F_- - F_+ \leq 0. \tag{3.4}$$

We will denote the left-hand side of (3.4) as entropy dissipation $D(u_-, u_+)$, which than has to be non-positive. In the following text we will consider fixed $u_- > 0$ and concave-convex flux function and we will present some properties of this function, through which we will come to the concept of nonclassical solutions.

Theorem 3.1 (Entropy dissipation)**.** *Let $u_- > 0$ be any fixed value of left-hand state of Riemann problem (2.1-2.2) with given strictly convex entropy U and entropy flux F. The entropy dissipation*

$$D(u_-, u_+) = -s(u_-, u_+)(U_+ - u_-) + F_+ - F_- \tag{3.5}$$

then has following properties:

- *Sign. We have*

$$\begin{aligned} D(u_-, u_+) &> 0 \quad u_+ \in (-\infty, \tilde{u}_-^-] \cup (u_-, +\infty), \\ D(u_-, u_+) &< 0 \quad u_+ \in [\tilde{u}_-, u_-). \end{aligned} \tag{3.6}$$

- *Monotony.*

$$\begin{aligned} \partial_{u_+} D(u_-, u_+) &< 0 \quad u_+ \in (-\infty, \tilde{u}_-], \\ \partial_{u_+} D(u_-, u_+) &> 0 \quad u_+ \in (\tilde{u}_-, u_-) \cup (u_-, +\infty). \end{aligned} \tag{3.7}$$

- *Zero entropy dissipation. For $u_- > 0$ there exists a value $\tilde{u}_-^0 \in (\tilde{u}_-^-, \tilde{u}_-)$ satisfying $D(u_-, \tilde{u}_-^0) = 0$. Moreover, $D(u_-, u_-) = 0$.*

Proof. The sign of entropy dissipation will be obvious from the formula

$$D(u_-, u_+) = -\int_{u_-}^{u_+} U''(u - u_-) \left(\frac{f - f_l}{u - u_-} - \frac{f_r - f_l}{u_+ - u_-} \right) \mathrm{d}u, \tag{3.8}$$

which comes from the definition of $D(u_-, u_+)$ by integrating.

- If $u_+ > u_-$, the graph of f lies below the line connecting stages u_- and u_+, so the last expression inside the integral (3.8) is negative and $D(u_-, u_+) > 0$.
- If $u_+ \leq \tilde{u}_-^-$, the graph of f lies above the line connecting stages u_- and u_+, so we have the last expression inside the integral (3.8) negative, however $u_+ < u_-$, which leads to $D(u_-, u_+) > 0$.
- If $u_+ \in [\tilde{u}_-, u_-)$, the graph of f lies below the line and $u_+ < u_-$, so $D(u_-, u_+) < 0$. This concludes the proof of the first property of the entropy dissipation.

The property (3.7) comes directly from differentiating (3.5) with respect to u_+

$$\partial_{u_+} D(u_-, u_+) = (u_- - u_+ - U_r'(u_- - u_+))\partial_{u_+} s, \tag{3.9}$$

where

$$\partial_{u_+} s = \frac{f_r' - s}{u_+ - u_-}. \tag{3.10}$$

Since U is strictly convex, the first expression in (3.9) is strictly positive. The sign of $\partial_{u_+} D(u_-, u_+)$ is than given by the sign of $\partial_{u_+} s$, which changes it when $f_r' = s$. This satisfies the stage $\tilde{u}_-$. Hence the entropy dissipation $D(u_-, u_+)$ is decreasing for $u_+ < \tilde{u}_-$ and increasing for $u_+ > \tilde{u}_-$, with maximum negative entropy dissipation in $\tilde{u}_-$.

Since in $\tilde{u}_-$ is the entropy dissipation negative and in $\tilde{u}_-^-$ is positive, there has to exist a point $\tilde{u}_-^0 \in (\tilde{u}_-^-, \tilde{u}_-)$ with zero entropy dissipation. The identity $D(u_-, u_-) = 0$ is obvious.

The whole situation is shown in figure 3.1. □

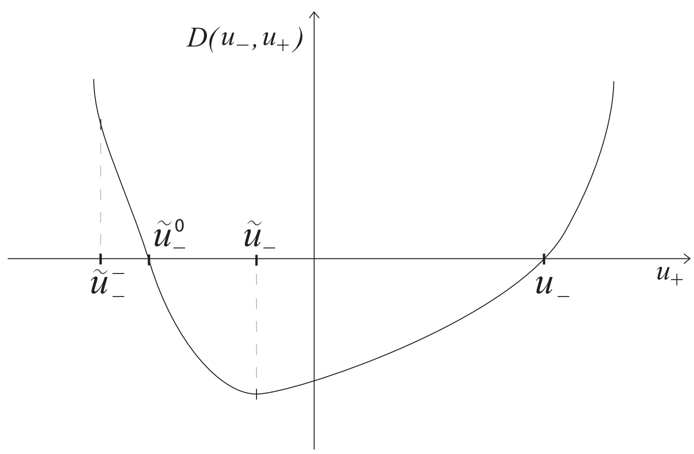


Figure 3.1: Entropy dissipation.

For the case $u_- \leq 0$ a similar result holds, based on the fact, that entropy dissipation D is skew-symmetric, i.e. $D(u_-, u_+) = -D(u_+, u_-)$. Thanks to this we can define a function of zero-entropy dissipation $\tilde{u}^0(u)$, so $\tilde{u}_-^0 = \tilde{u}^0(u_-)$.

Definition 3.2 (Zero-entropy dissipation function)**.** Let $\tilde{u}^0 : \mathbb{R} \to \mathbb{R}$ be the function of zero entropy dissipation, for which

$$D(u, \tilde{u}^0(u)) = 0, \tag{3.11}$$

where

$$\tilde{u}^0(u) \in (\tilde{u}^-(u), \tilde{u}(u)) \quad \text{if} \quad u > 0, \tag{3.12}$$

$$\tilde{u}^0(u) \in (\tilde{u}(u), \tilde{u}^-(u)) \quad \text{if} \quad u < 0. \tag{3.13}$$

$$\tag{3.14}$$

Lemma 3.3. *The zero-entropy dissipation function $\tilde{u}^0(u)$ is monotone decreasing and*

$$\tilde{u}^0(\tilde{u}^0(u)) = u, \quad u \in \mathbb{R}. \tag{3.15}$$

Proof. From definition of $\tilde{u}^0(u)$ we have

$$D(u, \tilde{u}^0(u)) = 0, \quad u \neq \tilde{u}^0(u), D(\tilde{u}^0(u), \tilde{u}^0(\tilde{u}^0(u))) = 0, \quad \tilde{u}^0(u) \neq \tilde{u}^0(\tilde{u}^0(u)). \tag{3.16}$$

Using the skew-symetry of D on the first equation we get

$$-D(\tilde{u}^0(u), u) = D(\tilde{u}^0(u), \tilde{u}^0(\tilde{u}^0(u))) = 0 \tag{3.17}$$

and hence $\tilde{u}^0(\tilde{u}^0(u)) = u$.

Using the skew-symmetry property of $D(u_-, u_+)$ again we have

$$\partial_{u_-} D(u, \tilde{u}^0(u)) = -\partial_{u_+} D(\tilde{u}^0(u), u). \tag{3.18}$$

Differentiating the identity 3.11 and using 3.18 we obtain

$$\frac{\mathrm{d}\tilde{u}^0}{\mathrm{d}u}(u) = -\frac{\partial_{u_-} D(u, \tilde{u}^0(u))}{\partial_{u_+} D(u, \tilde{u}^0(u))} = \frac{\partial_{u_+} D(\tilde{u}^0(u), u)}{\partial_{u_+} D(u, \tilde{u}^0(u))}. \tag{3.19}$$

From properties of $D(u_-, u_+)$ we have

$$\partial_{u_+} D(u, \tilde{u}^0(u)) < 0, \quad u > 0, \tag{3.20}$$

$$\partial_{u_+} D(u, \tilde{u}^0(u)) > 0, \quad u < 0. \tag{3.21}$$

If we take $u > 0$, we have $\tilde{u}^0(u) < 0$ and hence $\partial_{u_+} D(\tilde{u}^0(u), \tilde{u}^0(\tilde{u}^0(u))) > 0$. Using $\tilde{u}^0(\tilde{u}^0(u)) = u$ we obtain $\partial_{u_+} D(\tilde{u}^0(u), u) > 0$, which finally gives us $\mathrm{d}\tilde{u}^0/\mathrm{d}u < 0$ for $u > 0$. For negative u we obtain the same result.

Regularity at $u = 0$ can be proven by using the implicit function theorem and the assumption $f'''(0) \neq 0$.

□

From the theorem 3.1 follows, that for the single entropy inequality is a shock admissible, if $u_+ \in [\tilde{u}^0_-, u_-]$, for fixed $u_- > 0$. Using similar procedure can be shown, that for fixed $u_- < 0$ is the interval of admissible right-hand state $u_+ \in [u_-, \tilde{u}^0_-]$.

The interval $[\tilde{u}^0_-, u_-]$ also contains $[\tilde{u}_-, u_-]$, on which is the Oleinik entropy condition satisfied and the shock is called classical or Lax shock. On the other hand for $u_+ \in [\tilde{u}^0_-, \tilde{u}_-]$ the shock violate the Oleinik entropy condition, but satisfies the single entropy condition and this kind of shocks is called nonclassical.

At the end of this section let us remark, that since the nonclassical shock violates the Oleinik inequality, and hence the Lax inequality, characteristics do not come into the shock on both sides, as they did in the case of classical shock (Figure 3.2), but they pass through the shock, since $f'(u_{l,r}) \geq s$, as shown in the Figure (3.3). These nonclassical shocks are also called slow undercompressive.

For convex-concave function would be the approach entirely analogous by fixing the right-hand state u_+, however the nonclassical shock would be in this case fast undercompressive, $f'(u_{l,r}) \leq s$, see Figure 3.4. Therefore we will further consider only concave-convex flux.

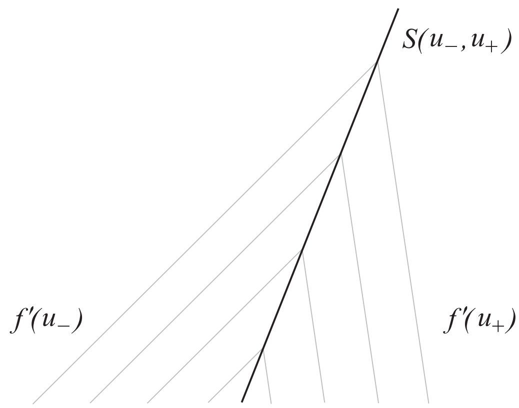


Figure 3.2: Classic shock.

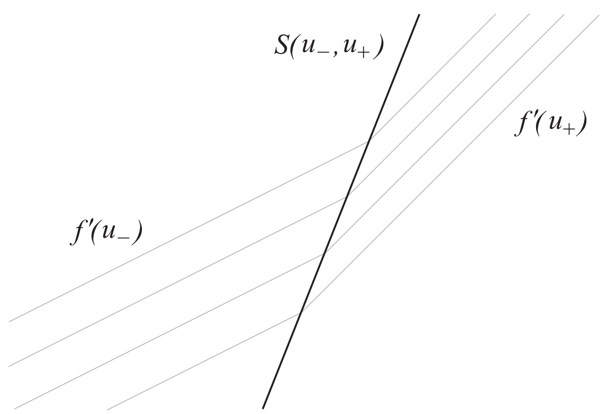


Figure 3.3: Slow undercompressive shock.

3.2 Combination of wave fans

In previous section we have shown, that nonclassical shocks are also admissible in solutions of (2.1-2.2). The question is, how the whole solution looks like.

For a wave to be physically realizable, is necessary that its speed si monotone increasing function of self-similar variable x/t. In other words, if we have combination of two wave fans, the speed of the left-hand wave must be less than or equal to the speed of the right-hand one.

For further argumentation ti will be useful to define the function $p(a, b)$ by relation

$$\frac{f(p(a,b)) - f(a)}{p(a,b) - a} = \frac{f(b) - f(a)}{b - a}. \tag{3.22}$$

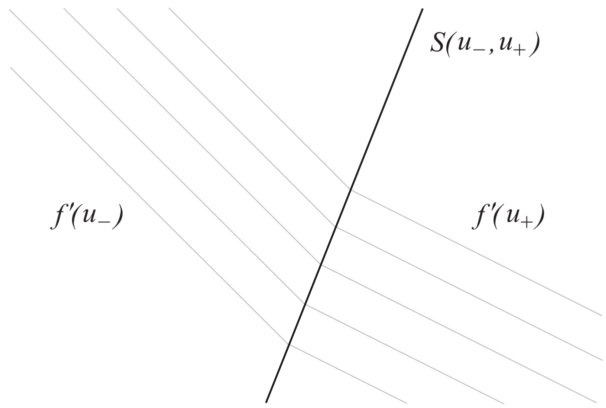


Figure 3.4: Fast undercompressive shock.

This simply means that the point $[p(a,b), f(p(a,b))]$ is in aligned with points $[a, f(a)]$ and $[b, f(b)]$. see Figure 3.5.

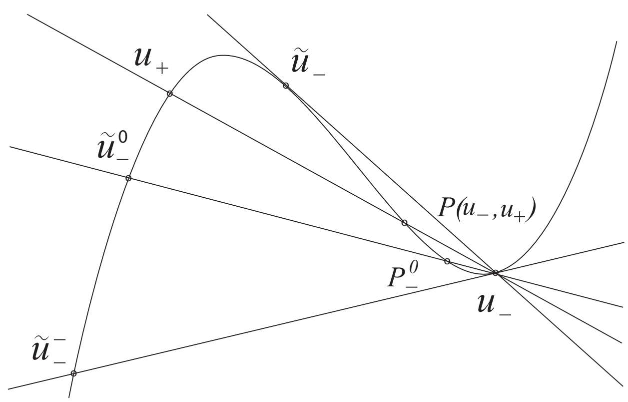


Figure 3.5: Important stages.

First, a rarefaction cannot be followed by any shock, because the shock wave would be always slower. Thus, only a rarefaction wave can be continuously added after another rarefaction.

Now, suppose we have a classical shock connecting $u_- > 0$ to $u_+ \in [\tilde{u}_-, u_-)$. We cannot add another classical shock nor a rarefaction, because it would be slower, except the case $u_+ = \tilde{u}_-$, in which a rarefaction can follow the classical shock, since its speed is equal to the speed of the shock. Consider next a nonclassical shock connecting u_+ and some u_1.

For this nonclassical shock to be admissible one needs

$$u_1 \leq \tilde{u}^0(u_+) \tag{3.23}$$

and it must be faster than the classical one, thus $s(u_+, u_1) > s(u_-, u_+)$ and therefore $u_1 > u_-$ must hold. From Lemma 3.3 we know, that the function $\tilde{u}^0$ is decreasing, so since $u_+ > \tilde{u}^0_-$, we have

$$\tilde{u}^0(\tilde{u}^0(u_-)) \geq \tilde{u}^0(u_+). \tag{3.24}$$

Using the identity $\tilde{u}^0(\tilde{u}^0(u)) = u$ we get $u_- \geq u_1$, which is a contradiction. Therefore the nonclassical shock also cannot follow after the classical one.

Finally, consider a nonclassical shock connecting $u_- > 0$ to $u_+ \in [\tilde{u}^0_-, \tilde{u}_-)$. It is possible to connect any state $u_1 \in [u_+, p(u_-, u_+))$ from u_+ by a classical shock, since its speed is greater. If we would like to add another nonclassical shock after the first one, we would have to take $u_1 \in (\tilde{u}_+, \tilde{u}^0_+]$, but from properties of the function $\tilde{u}^0(u)$ we have $\tilde{u}^0_+ \leq u_-$ and so any second nonclassical shock would be always slower than the first one. However it is possible to add a rarefaction wave to $u_1 \leq u_+$, which travel faster. From another point of view, we can say, that stages $u_1 in [\tilde{u}^0_-, p(\tilde{u}^0_-, u_-)]$ are attainable by combination of nonclassical shock and classical one and furthermore stages $u_1 in [\tilde{u}^0_-, \tilde{u}_-)$ are are attainable by nonclassical shock followed by rarefaction.

If we now denote $u_- = u_l$ and $u_+ = u_m$, we can summarize the whole argumentation given above in the view of u_r as a solution to our Riemann problem.

Theorem 3.4 (Nonclassical solutions to Riemann problem for concave-convex flux)**.** *Let the flux function f be concave-convex with inflection point at $u = 0$ (see 3.1). Then there are admissible solutions of Riemann problem (2.1-2.2) with $u_l > 0$, satisfying the single entropy condition for given strictly convex entropy, consists of shock waves and rarefaction waves given as follows:*

- *If $u_r > u_l$, the solution is a rarefaction wave connecting u_l and u_r monotonically and continuously.*
- *If $u_r \in [p(\tilde{u}^0_l, u_l), u_l)$, the solution is a classical shock wave connecting u_l and u_r.*
- *If $u_r \in [\tilde{u}^0_l, p(\tilde{u}^0_l, u_l))$, there is infinitely many solutions depending on value $u_m \in [\tilde{u}^0_l, \tilde{u}_l]$: It consists of a nonclassical shock connecting u_l and the intermediate state u_m followed by*
  - *a) a classical shock from u_m to u_r if $u_m < p(u_l, u_r)$ or*
  - *b) a rarefaction connecting u_m and u_r if $u_m \geq u_r$.*
- *If $u_r \leq \tilde{u}^0_l$, there is infinitely many solutions consisting of a nonclassical shock connecting u_l and the intermediate state $u_m \in [\tilde{u}^0_l, \tilde{u}_l]$ followed by a rarefaction from u_m to u_r.*

Figures 3.6-3.8 illustrate the situation with nonclassical shock from the Theorem 3.4. As we can see, the nonclassical shock caused nonuniqueness of the solution, therefore we need to impose some additional condition to solution in order to make it unique. This problem will be discussed in the next chapter.

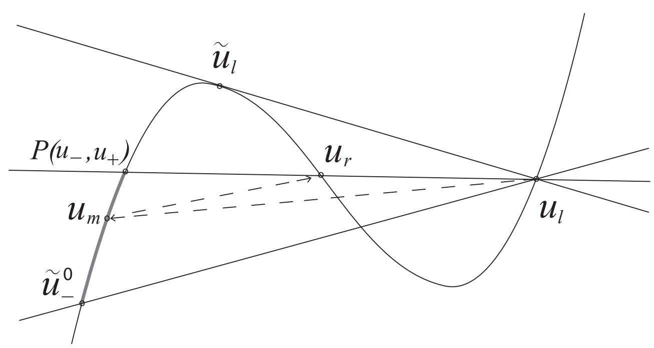


Figure 3.6: The case of nonclassical shock followed by the classical one.

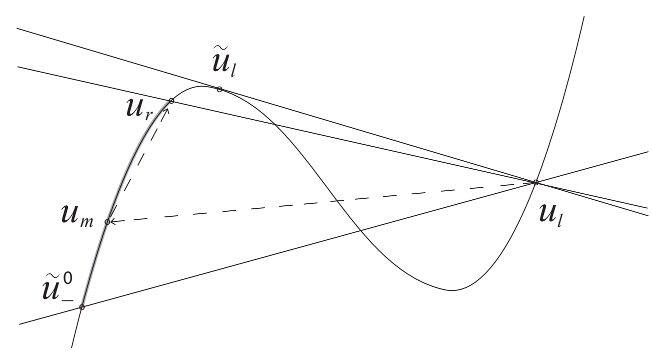


Figure 3.7: The case of nonclassical shock followed by the classical one.

3.3 Diffusive-dispersive traveling waves

Consider the diffusive-dispersive model

$$u_t + f_x(u) = \epsilon u_{xx} + \delta u_{xxx}, \quad u = u^{\epsilon,\delta}(x,t), \tag{3.25}$$

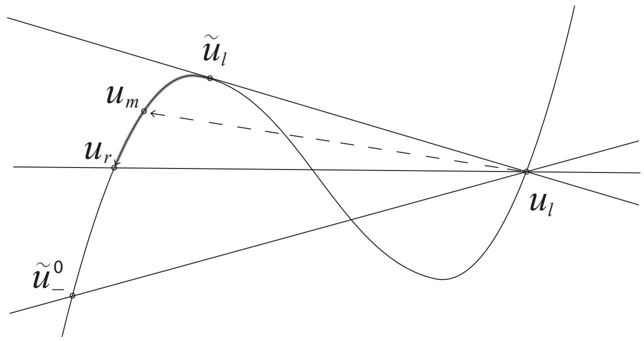


Figure 3.8: The case of nonclassical shock followed by the rarefaction.

where the flux function $f(u)$ is assumed to be concave-convex in the same sense like in previous sections.

In order to obtain the original conservation law

$$u_t + f_x(u) = 0, \quad u = u(x,t), \tag{3.26}$$

we will be interested in the limit case $\epsilon \to 0$ in 3.25 with the ratio $\mu = \epsilon/\sqrt{|\delta|}$ kept constant.

Now we will search for traveling wave solutions of 3.25. Let us recall, that the traveling wave is a solution depending upon the rescaled variable

$$y := \frac{x - st}{\sqrt{|\delta|}} = \mu \frac{x - st}{\epsilon}, \tag{3.27}$$

where s is considered to be a constant speed. The trajectory $y \mapsto u(y)$ than will correspond to a traveling wave connecting fixed u_- and some state u_+. We denote $\eta = \text{sgn}(\delta)$.

If we express derivatives u_t, u_x, u_{xx} and u_{xxx} in terms of y, i.e.

$$u_t = -s\frac{\mu}{\epsilon}u_y, \qquad u_x = \frac{\mu}{\epsilon}u_y \tag{3.28}$$

$$u_{xx} = \frac{\mu^2}{\epsilon^2}u_{yy}, \qquad u_{xxx} = \frac{\mu^3}{\epsilon^3}u_{yyy}, \tag{3.29}$$

we obtain the ordinary differential equation

$$-su_y + f_y = \mu u_{yy} + \eta u_{yyy}, \tag{3.30}$$

which is together with boundary conditions

$$\lim_{y\to+\infty} u(y) = u_+, \quad \lim_{y\to-\infty} u(y) = u_-, \tag{3.31}$$

$$\lim_{y\to\pm\infty} u_y(y) = \lim_{y\to\pm\infty} u_{yy}(y) = 0, \tag{3.32}$$

satisfied by the traveling wave $y \mapsto u(y)$.

Integrating once the equation 1.8 over the interval $(-\infty, y]$ and using the boundary condition we obtain

$$\mu u_y + \eta u_{yy} = -s(u - u_-) + f - f_-. \tag{3.33}$$

Denoting $v = u_y$ we can reformulate the equation 3.33 as a system of two differential equations

$$\frac{\mathrm{d}}{\mathrm{d}y}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\eta\mu v + \eta(f - su) - \eta(f_- - su_-) \end{pmatrix}. \tag{3.34}$$

For definiteness we will consider the left-hand state $u_- > 0$ fixed and the speed s as a parametr. Since we assume the flux function to be concave-convex, we will pay the attention to the most interesting case $s \in (f'(\tilde{u}_-), f'(u_-))$, in which exist three equilibrium points of the system 3.34. In these equilibrium points the identity $v = 0$ holds and we will denote them as $(u_0, 0)$, $(u_1, 0)$ and $(u_2, 0)$, while

$$u_2 < u_1 < u_0 = u_- \tag{3.35}$$

and values u_1 and u_2 satisfy

$$f(u_1) + su_1 = f_- - su_-, \quad \text{and} \quad f(u_2) + su_2 = f_- - su_-. \tag{3.36}$$

This geometrically means, that all three points lie on the line passing through the point (u_-, f_-) with the slope s.

This gives us a connection to the theory of classical and nonclassical shocks. If we take the trajectory $y \mapsto u(y)$ leaving from u_- at $-\infty$, we want to know where it will reach $+\infty$, whether in u_1 or in u_2. It is obvious, that the point u_1 is associated with the classical shock and the point u_2 with nonclassical shock, namely we have classical trajectory connecting u_0 with

$$u_1 \in [\tilde{u}(u_0), u_0], \quad s \in [f'(u_-(u_0)), f'(u_0)] \tag{3.37}$$

and nonclassical trajectory connecting u_0 with

$$u_2 \in [\tilde{u}^0(u_0), \tilde{u}(u_0)), \quad s \in (f'(u_-(u_0)), s(u_0, \tilde{u}^0(u_0))]. \tag{3.38}$$

The eigenvalues of the Jacobian matrix of the system 3.34 at any point $(u, 0)$ are

$$\lambda_1 = \frac{-\eta\mu - \sqrt{\mu^2 + 4\eta(f' - s)}}{2}, \tag{3.39}$$

$$\lambda_2 = \frac{-\eta\mu + \sqrt{\mu^2 + 4\eta(f' - s)}}{2}. \tag{3.40}$$

First, we suppose $\delta < 0$. At the point u_2 we have $f'(u_2) - s > 0$ and so λ_1 and λ_2 are both positive. The situation is the same at the point u_0, which means that both these points are unstable. However, the u_1 point is a saddle, so we can conclude, that, in the case of negative δ, only classical trajectories exist.

If $\delta > 0$, both u_2 and u_0 are saddle points, but the point u_1 is stable, namely it is node, if $\mu^2 + 4(f' - s) > 0$ and spiral, if $\mu^2 + 4(f' - s) < 0$. The nonclassical trajectory than exists only if there is a saddle-saddle connection, which arise only when a special relation between u_0, u_2 and μ holds.

We will now state here without proof an important result presented by Bedjaoui and LeFloch in [11].

Theorem 3.5 (Kinetic function and shock set for general flux)**.** *Given a concave-convex flux function (see 3.1), consider the diffusive-dispersive model 3.25 with the ratio* $\mu = \epsilon/\sqrt{delta}$ *fixed,* $\delta > 0$*. Then, there exists a locally Lipschitz continuous and decreasing kinetic function* $\tilde{u}^\mu : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\tilde{u}(u) \leq \tilde{u}^\mu(u) < \tilde{u}^0(u), \quad u < 0, \tilde{u}^0(u) < \tilde{u}^\mu(u) \leq \tilde{u}(u), \quad u > 0, \tag{3.41}$$

and such that the shock set generated by the model 3.25 is

$$S(u_-) = \{\{\tilde{u}^\mu(u_-)\} \cup (p(u_-, \tilde{u}^\mu(u_-)), u_-]\}, \quad u_- > 0. \tag{3.42}$$

Moreover, there exists a function

$$A : \mathbb{R} \to [0, +\infty), \tag{3.43}$$

called the threshold diffusion-dispersion ratio, which is smooth away from $u = 0$*, Lipschitz continuous at* $u = 0$*, increasing in* $u > 0$*, decreasing in* $u < 0$ *and such that*

$$\tilde{u}^\mu(u) = \tilde{u}(u) \quad \textit{when} \quad \mu \geq A(u). \tag{3.44}$$

Additionaly we have

$$\tilde{u}^\mu(u) \to \tilde{u}^0(u) \quad \textit{as} \quad \mu \to 0, \quad \forall u \in \mathbb{R}. \tag{3.45}$$

The kinetic function from the theorem 3.5 plays very important role, since it completely characterizes nonclassical shocks associated with the diffusive-dispersive model. For a concrete $u_n > 0$ it gives us a value $\tilde{u}^\mu(u_-) \in (\tilde{u}^0_-, \tilde{u}_-]$, which is the unique stage u_m in the nonclassical solution and hence thanks to this we can solve the Riemann problem uniquely.

The shock set $S(u_-)$ given in the theorem 3.5 is a set of right-hand stages u_+, that can be connected by traveling wave from u_-.

The threshold diffusion-dispersion ratio is also important, since it gives us qualitative properties of the nonclassical shocks. The statement 3.44 tells us, that if the ratio μ is chosen to be sufficiently large, the shock leaving from u_- will be always classical. The shock will be also always classical if u_- is chosen to be sufficiently small. On the other hand, by the statement 3.45 we see, that the shock will be always nonclassical, if the ratio μ is sufficiently small.

We will demonstrate these ideas and statements on the example with concrete flux function, namely on the cubic flow.

Example 3.6 (Cubic flow)**.** Consider the diffusive-dispersive model with cubic flux

$$u_t + (u^3)_x = \epsilon u_{xx} + \delta u_{xxx}, \quad u = u^{\epsilon,\delta}(x,t) \tag{3.46}$$

with the ratio $\mu = \epsilon/\sqrt{\delta}$ kept constant and $\delta > 0$. We search for traveling waves solutions depending on the rescaled variable

$$y := \frac{x - st}{\sqrt{\delta}} = \mu\frac{x - st}{\epsilon}. \tag{3.47}$$

The traveling wave $y \mapsto u(y)$ than has to satisfy

$$-su_y + (u^3)_y = \mu u_{yy} + \eta u_{yyy}, \tag{3.48}$$

with boundary conditions

$$\lim_{y\to+\infty} u(y) = u_+, \quad \lim_{y\to-\infty} u(y) = u_-, \tag{3.49}$$

$$\lim_{y\to\pm\infty} u_y(y) = \lim_{y\to\pm\infty} u_{yy}(y) = 0. \tag{3.50}$$

Integrating 3.48 we obtain

$$\mu u_y + \eta u_{yy} = -s(u - u_-) + u^3 - u_-^3. \tag{3.51}$$

Also, taking the limit $y \to +\infty$ we have

$$s = \frac{u_+^3 - u_-^3}{u_+ - u_-} = u_-^2 + u_- u_+ + u_+^2. \tag{3.52}$$

Recall, that we assume $u_- > 0$ fixed and we take s as a parametr. Since we have $\tilde{u}_- = -u_-/2$, the range of speeds

$$s \in (\frac{3}{4}u_-^2, 3u_-^2) \tag{3.53}$$

gives us exactly three points in which the line passing through (u_-, u_-^3) with slope s intersects the graph $f(u) = u^3$. We rewrite the equation 3.51 as a system

$$\frac{\mathrm{d}}{\mathrm{d}y}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\mu v + (u^3 - su) - (u_-^3 - su_-) \end{pmatrix}. \tag{3.54}$$

This system has got three equilibrium points $(u_0, 0)$, $(u_1, 0)$ and $(u_2, 0)$, where $u_0 = u_-$ and u_1 and u_2 are roots of

$$s = u^2 + u_0 u + u_0^2. \tag{3.55}$$

The eigenvalues of Jacobian of the system 3.54 are

$$\lambda_1 = \frac{-\mu + \sqrt{\mu^2 + 4(3u^2 - s)}}{2}, \tag{3.56}$$

$$\lambda_2 = \frac{-\mu - \sqrt{\mu^2 + 4(3u^2 - s)}}{2}, \tag{3.57}$$

Since we have $3u^2 - s > 0$ at points u_0 and u_2, these points are saddles. At the point u_1 we have $3u^2 - s < 0$ and this point is stable. Now we want to derive a relation between u_0, u_2 and μ for which a nonclassical trajectory would exist. Based on the fact, that v must vanish at points u_2 and u_0 and that the expression $u^3 - su$, which appears in the system 3.54, is cubic, we assume, that v is a parabola

$$v(y) = a(u - u_2)(u - u_0), \tag{3.58}$$

where a is a constant we want to determine. Expressing v_y from the system 3.54 we obtain

$$v_y = v(-\mu + \frac{1}{a}(u + u_0 + u_2)), \tag{3.59}$$

where we used 3.58 and 3.55 for the point u_2. On the other hand, by differentiating 3.58 directly, we have

$$v_y = av(2u - u_0 - u_2). \tag{3.60}$$

Comparing those two expressions for v_y, we get

$$\frac{1}{a} = 2a \quad \text{and} \quad -a(u_0 + u_2) = -mu + \frac{1}{a}(u_0 + u_2). \tag{3.61}$$

Hence, $a = 1/\sqrt{2}$ and we get the explicit relation

$$u_2 = -u_0 + \frac{\sqrt{2}}{3}\mu. \tag{3.62}$$

Since the relation 3.55 must hold for both u_1 and u_2, than by comparing those two equations we obtain

$$u_0 + u_1 + u_2 = 0 \tag{3.63}$$

and so $u_1 = -u_0 - u_2$. Since equilibrium points are ordered $u_2 < u_1$, we obtain immedientaly

$$u_0 > \frac{2\sqrt{2}}{3}\mu. \tag{3.64}$$

We have shown, that for some fixed μ and any left-hand state $u_0 > 2\tilde{\mu}$, where $\tilde{\mu} = \frac{\sqrt{2}}{3}\mu$, there exists a saddle-saddle connection from u_0 to $u_2 = -u_0 + \tilde{\mu}$. The kinetic function from theorem 3.5 for $u_- > 0$ takes the form

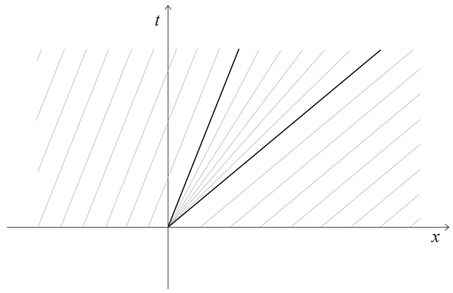
$$\tilde{u}^{\mu}(u_-) = \begin{cases} -u_0 + \tilde{\mu}, & u_- > 2\tilde{\mu} \\ -u_-/2, & u_- < 2\tilde{\mu}. \end{cases} \tag{3.65}$$

Together with theory given in previous sections, we can express the nonclassical solution to the conservation law with cubic flow for given diffusion-dispersion ratio $\tilde{\mu}$ and $u_l > 2\tilde{\mu}$ as follows:

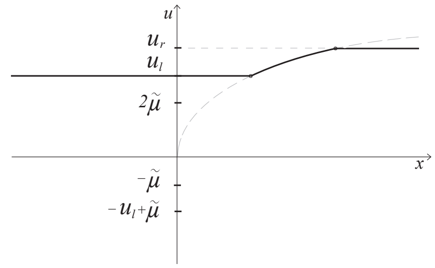
- $u_r > u_l$ The solution si a rarefaction wave from u_l to u_r.

- $u_r \in [-\tilde{\mu}, u_l)$ The solution is a classical shock wave connecting u_l and u_r.
- $u_r \in (-u_l + \tilde{\mu}, -\tilde{\mu})$ The solution consists of a slow nonclassical shock connecting u_l to $u_m = -u_l + \tilde{\mu}$ and fast classical shock from u_m to u_r.
- $u_r < -u_l + \tilde{\mu}$ The solution consists of a slow nonclassical shock connecting u_l to $u_m = -u_l + \tilde{\mu}$ and a rarefaction (not attached to the shock) from u_m to u_r.

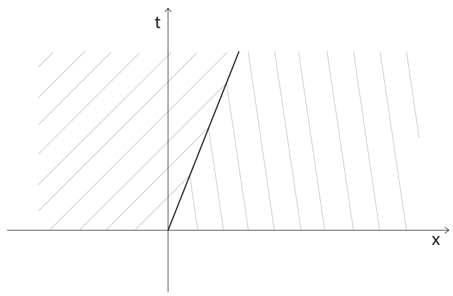
All of those cases are shown in Figures 3.9a-3.9h.



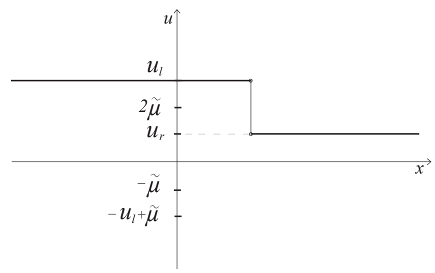
(a) Characteristics for $u_r > u_l$.



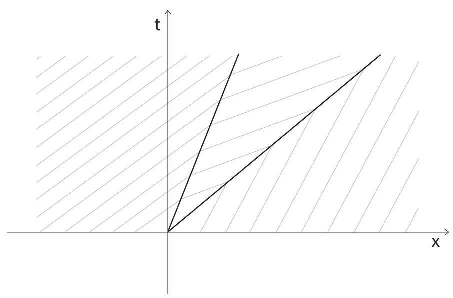
(b) Solution for $u_r > u_l$.



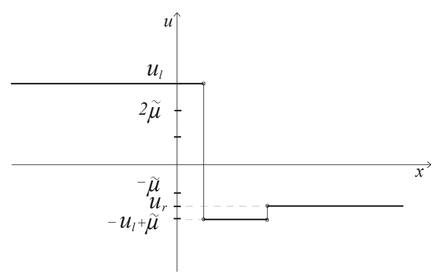
(c) Characteristics for $u_r \in [-\tilde{\mu}, u_l)$.



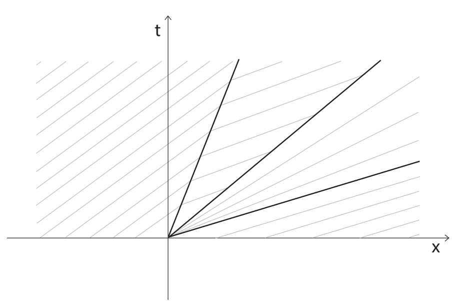
(d) Solution for $u_r \in [-\tilde{\mu}, u_l)$.



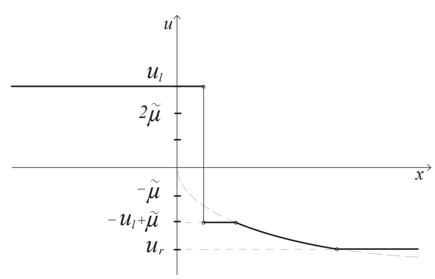
(e) Characteristics for $u_r \in (-u_l + \tilde{\mu}, -\tilde{\mu})$.



(f) Solution for $u_r \in (-u_l + \tilde{\mu}, -\tilde{\mu})$.



(g) Characteristics for $u_r < -u_l + \tilde{\mu}$.



(h) Solution for $u_r < -u_l + \tilde{\mu}$.

Chapter 4

Numerical schemes

Having demonstrated the structure of nonclassical solutions of the Riemann problem for nonconvex fluxes, we now test some numerical schemes in order to find some, that will detect a nonclassical behavior. For the beginning we may say, that schemes for scalar equations, that satisfy the TVD condition are excluded, since the nonclassical shock increase total variation and those schemes then converge to classical solutions.

4.1 Finite difference schemes

Consider the diffusive-dispersive model with cubic flux

$$u_t + (u^3)_x = \epsilon u_{xx} + \alpha\epsilon^2 u_{xxx}, \quad u = u^\epsilon(x,t). \tag{4.1}$$

Its limiting solution with $\epsilon \to 0$ is the solution of the original conservation law $u_t + (u^3)_x = 0$. The parametr $\alpha \in \mathbb{R}$ here is a measuring the ratio of the diffusion to the dispersion. Comparing this to the ratio given in previous section we have $\alpha = 1/\mu^2$.

Denoting $u_j(t)$ the approximation of $u(x_j,t)$, where $x_j = jh$ describes the mash of the length h. If we use high-order accurate, centered finite differences for diffusion and dispersion terms and the discretization of the flux term based on a numerical flux, we obtain continuous in time, discrete in space difference scheme

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} + \frac{g_{j+1/2} - g_{j-1/2}}{h} = \frac{\beta}{2h}(u_{j+1} - 2u_j + u_{j-1}) + \frac{\gamma}{6h}(u_{j+2} - 2u_{j+1} + 2u_{j-1} - u_{j-2}), \tag{4.2}$$

where $g_{j\pm1/2}$ is a discrete flux and $\beta > 0$ and $\gamma \in \mathbb{R}$ are fixed parameters.

We will present a few difference schemes based on different approximation of the flux term.

Scheme I. First order discretization of the flux:

$$g_{j+1/2} = u_j^3. \tag{4.3}$$

The equivalent equation for this scheme is

$$u_t + (u^3)_x = h(\frac{u^3}{2} + \beta\frac{u}{2})_{xx} + h^2(\gamma\frac{u}{3} - \frac{u^3}{6})_{xxx} + O(h^3). \tag{4.4}$$

We will pay the attention to the dispersive term. The linear term $\gamma u/3$ and the cubic term $u^3/6$ are here with opposite sign, which leads to competition between them. The linear term is responsible for producing nonclassical shock, while the cubic one prevents them. Depending on the strength of the shock, one of them wins. For sufficiently strong shocks it would be the cubic one and so the classical behavior. We have done some numerical experiments on this scheme by changing various parameters, however we were only able to detect classical behavior.

Scheme II. Second order discretization of the flux:

$$g_{j+1/2} = \frac{1}{4}\frac{u_{j+1}^4 - u_j^4}{u_{j+1} - u_j}. \tag{4.5}$$

The equivalent equation for this scheme is

$$u_t + (u^3)_x = \frac{h\beta}{2}u_{xx} + \frac{h^2}{2}(\frac{2\gamma}{3}u_{xx} - u^2u_{xx} - uu_x^2)_x + O(h^3). \tag{4.6}$$

The situation here is similar to the first case, there also appears to be a competition between linear and nonlinear terms. This scheme is sensitive on choice of the parametr γ. With different values both classical and nonclassical behavior appears.

Scheme III. Fourth-order flux:

$$g_{j+1/2} = \frac{1}{12}(-u_{j+2}^3 + 7u_{j+1}^3 + 7u_j^3 - u_{j-1}^3). \tag{4.7}$$

The equivalent equation for this scheme is

$$u_t + (u^3)_x = \frac{h\beta}{2}u_{xx} + \frac{h^2\gamma}{3}u_{xxx} + O(h^3), \tag{4.8}$$

which coincides exactly with the original model 4.1 with

$$\beta = 2\frac{\epsilon}{h}, \quad \gamma = 3\alpha\left(\frac{\epsilon}{h}\right)^2. \tag{4.9}$$

Thanks to this analogy the scheme mimic the solution of 4.1 very well. Recall, that the nonclassical shock appears, only if

$$u_l > \frac{2\sqrt{2}}{3\sqrt{\alpha}}. \tag{4.10}$$

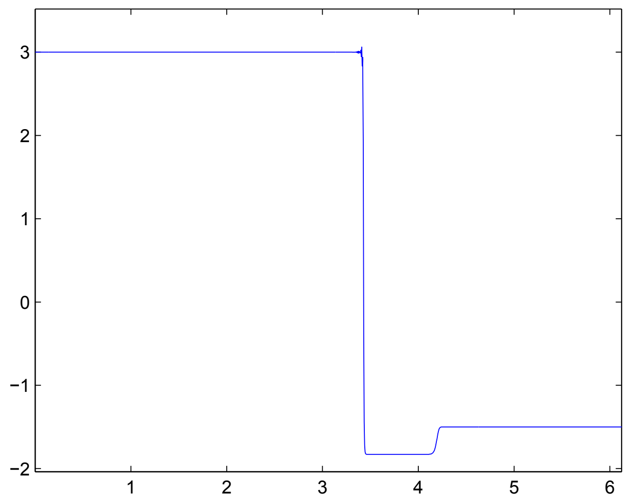


Figure 4.1: The case of nonclassical shock followed by the classical one.

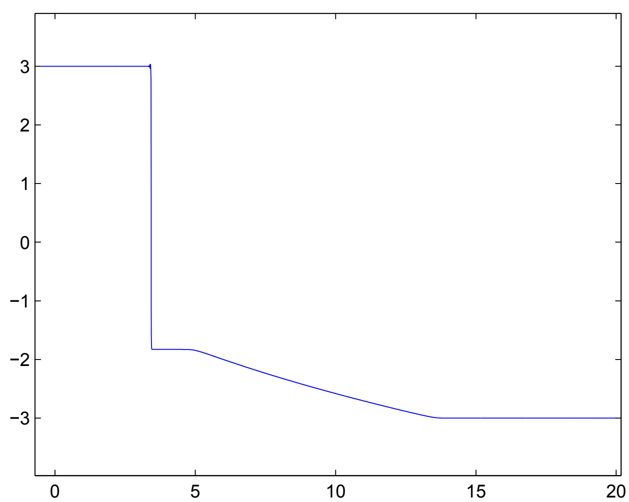


Figure 4.2: The case of nonclassical shock followed by the rarefaction.

Hence, for the shock corresponding to the traveling wave solution we have

$$u_m = \begin{cases} -\frac{u_l}{2}, & 0 \leq u_l \leq \frac{2\sqrt{2}}{3\sqrt{\alpha}} \\ -u_l + \frac{\sqrt{2}}{3\sqrt{\alpha}}, & u_l > \frac{2\sqrt{2}}{3\sqrt{\alpha}}. \end{cases} \tag{4.11}$$

In Figures 4.1 and 4.2 we plot the numerical solution computed by the third scheme with $h = 1/200$, $\epsilon = 1/50$, $\alpha = 2/9$ and we used the fourth-order Runge-Kutta method for time stepping.

Chapter 5

Conclusion

Let us conclude this paper with some practical remarks. As we have shown, classical solutions are independent of regularization mechanisms and can be characterized by entropy inequality. On the other hand, nonclassical solutions are very sensitive to the choice of regularization. The entropy inequality does not characterizes them completly and we need a special relation to select the unique solution.

Because of the sensitivity to reguralizations are nonclassical solutions fundamental in nonlinear elasodynamics and phase transition dynamics with capillarity effects. There are also many other physical models, where nonclassical behavior may occur, for example Thin liquid film model or Magnetohydrodynamic model. It would be interesting to compare some numerical solutions with real experiments.

Since we introduced here the theory of nonclassical solutions only for scalar conservation laws with one inflection point, it would be nice to extend it to more general functions and systems of equations.

Bibliography

[1] P.D. LAX: Hyperbolic systems of conservation laws and the mathematical theory of shock waves, SIAM, 1973.

[2] M. FEISTATUER: Mathematical Methods in Fluid Dynamics, Longman Scientific & Technical, 1993.

[3] R.J. LEVEQUE: Finite volume methods for hyperbolic problems, Cambridge University Press., 2002.

[4] S. DIEHL : On scalar conservation laws with point source and discontinuous flux function, SIAM journal on mathematical analysis, 1995, 1425-1451.

[5] D.S. BALE, et al: A wave propagation method for conservation laws and balance laws with spatially varying flux functions, SIAM Journal on Scientific Computing, 2003, 955-978.

[6] B. HAYES - M. SHEARER : Undercompressive shocks and Riemann problems for scalar conservation laws with nonconvex fluxes, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, Vol 129, 1999, 733-754.

[7] H. HOLDEN - N.H. RISEBRO : Front tracking for hyperbolic conservation laws, Applied Mathematical Sciences 152, 2011, DOI 10.1007/978-3-642-23911-32.

[8] P.G. LEFLOCH: Kinetic relations for undercompressive shock waves, Contemporary mathematics, vol. 526, 2010.

[9] B. HAYES - P.G. LEFLOCH: Nonclassical shock waves in scalar conservation laws, Archive for Rational Mechanics and Analysis, vol. 139, 1997.

[10] B. HAYES - P.G. LEFLOCH: Nonclassical shocks and kinetic relations: Finite difference schemes, SIAM J. Numer. Anal., 35(6), 1998, 2169–2194.

[11] N. BEDJAOUI - P.G. LEFLOCH: Diffusive–Dispersive traveling waves and kinetic relations: Part I: Nonconvex hyperbolic conservation laws, Journal of Differential Equations, vol. 178, 2002.

[12] P.G. LEFLOCH: Hyperbolic systems of conservation laws, The theory of classical and nonclassical shock waves, Lectures in Mathematics – ETH Zurich, Birkhauser Verlag, Basel, 2002.

[13] M. FOSSATI - L. QUARTAPELLE: Exact Riemann solvers for nonconvex hyperbolic problems, Politecnico di Milano, 2011.

[14] YOUNGSOO HA - YONG-JUNG KIM: The fundamental solution of a conservation law without convexity, Quarterly of applied mathematics, 2013.

[15] J. M. BALL - G. G. CHEN: Entropy and convexity for nonlinear partial differential equations,Philos Trans A Math Phys Eng Sci., 2013, 371(2005): 20120340.