University of West Bohemia in Pilsen
Faculty of Applied Sciences
Department of Mathematics

MASTER THESIS

Non-smooth analysis in engineering and economics

Author: **Tomáš Roubal**
Supervisor: **Ing. Radek Cibulka, Ph.D.**

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Poděkování

Nejvíce bych chtěl poděkovat svému vedoucímu Ing. Radku Cibulkovi, Ph.D. za vedení, rady a trpělivost při zpracování celé práce.

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Čestné prohlášení

Prohlašuji, že jsem diplomovou práci vypracoval samostatně a výhradně s použitím odborné literatury a pramenů, jejichž úplný seznam je její součástí.

V Plzni dne ....

Podpis

Abstrakt

Cílem této diplomové práce je představit diferenciálně variační nerovnice a shrnout základní teorii o existenci jejich řešení včetně souvisejících numerických metod. Teoretické výsledky jsou aplikovány na problémy z mechaniky, elektrických obvodů a ekonomie.

Za použití teorie zobecněných rovnic, obyčejných diferenciálních rovnic a diferenciálních inkluzí byla provedena rešerše vět zaručujících existenci řešení různých tříd diferenciálně variačních nerovnic. Dále bylo představeno několik numerických metod pro řešení diferenciálně variačních nerovnic a zobecněných rovnic založených na diskretizaci příslušných úloh.

Existenční věty a numerické metody byly aplikovány na několik vybraných modelů, které popisují reálné úlohy z oblasti kontaktní mechaniky, elektrických obvodů s diodami a modelování ekonomické rovnováhy. U všech příkladů byla provedena numerická simulace a u některých z nich byly odvozeny vztahy pro analytické řešení, což umožnilo posoudit přesnost numericky získaných výsledků.

Klíčová slova: mnohoznačné zobrazení, diferenciálně variační nerovnice, zobecněná rovnice, numerické metody, kontaktní mechanika, elektrické obvody, ekonomická rovnováha

Abstract

The aim of this thesis is to present differential variational inequalities and to summarize basic theory of the existence of solutions together with numerical methods for solving them. These tools are applied to problems from mechanics, electrical circuits and economics.

Using theory of generalized equations, ordinary differential equations, and differential inclusions, we present several theorems guaranteeing the existence of solutions of particular classes of differential variational inequalities. Furthermore, we discuss basic numerical methods for solving differential variational inequalities as well as generalized equations which are based on appropriate discretization schemes.

Theoretical results are applied on selected models, which describe real-world problems arising in contact mechanics, electrical circuits with diodes and economic equilibrium. In all the examples presented, numerical simulations were performed and, for some of them, formulas for the exact solution were obtained, which allowed us to evaluate the precision of the numerically obtained results.

Keywords: set-valued mapping, differential variational inequality, generalized equation, numerical methods, contacts mechanics, electrical circuits, economic equilibrium

Contents

List of symbols

$X \times Y$	Cartesian product of sets X and Y
$x \in X$	x is an element of the set X
$\equiv$	identically equal
$a = b$	a equals b
$a := b$	let a be defined by b
$\mathbb{N}$	positive integers
$\mathbb{R}$	real numbers
$\mathbb{R}_+$	non-negative real numbers
$\mathbb{R}^n$	Euclidean space of $\mathbf{x} = (x_1, ..., x_n)^T$ having n real coordinates
$\mathbb{R}^n_+$	set of $\mathbf{x} \in \mathbb{R}^n$ having non-negative coordinates
$a < b$	$b \in \mathbb{R}$ is greater than $a \in \mathbb{R}$
$a \leq b$	$b \in \mathbb{R}$ is greater than or equal to $a \in \mathbb{R}$
$[a, b]$	closed interval in $\mathbb{R}$ with $a < b$
(a, b)	open interval in $\mathbb{R}$ with $a < b$
$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i$	scalar product of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$
$\lVert \mathbf{x} \rVert = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$	Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$
$\mathbf{x} \perp \mathbf{y}$	$\mathbf{x} \in \mathbb{R}^n$ is perpendicular to $\mathbf{y} \in \mathbb{R}^n$, i.e., $\langle \mathbf{x}, \mathbf{y} \rangle = 0$
$\mathbf{x} \preceq \mathbf{y}$	$x_i \leq y_i$ for each $i \in \{1, .., n\}$
$\mathbf{x} \prec \mathbf{y}$	$x_i < y_i$ for each $i \in \{1, .., n\}$
$\mathbb{B}(\mathbf{x}, r)$	open ball centered at $\mathbf{x} \in \mathbb{R}^n$ with a radius $r > 0$
$\mathbb{B}[\mathbf{x}, r]$	closed ball centered at $\mathbf{x} \in \mathbb{R}^n$ with a radius $r > 0$
$f : X \to Y$	single-valued mapping f from the set X to the set Y
$F : X \rightrightarrows Y$	set-valued mapping F from the set X to the set Y
$\lfloor \cdot \rfloor$	floor function

Chapter 1

Introduction

"This is the problem under our level and beyond our skills."

Unknown student

In this chapter we try to ilustrate our further consideration on easy examples from electronics. Motivated by this, we introduce basic notions from set-valued and variational analysis. More complex eletrical circuits as well as mechanical and economic models can be found in Chapter 4.

1.1 Static problems

In this section, we are going to present two static problems occurring in electrical circuits. We show how these problems can be described by using set-valued functions.

We are going to study the circuit in Figure 1.1, which contains one voltage source and two different components A and B. We denote V_A a voltage across the component A, V_B a voltage across the component B. First, suppose constant source $E > 0$ and the corresponding current i. By Kirchhoff's voltage law, the sum of voltages across all components in a circuit is equal to zero,

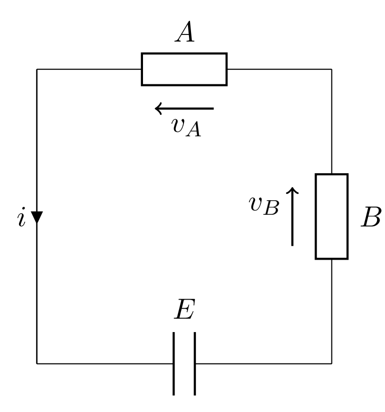


Figure 1.1: Series circuit

that is,

$$V_A + V_B - E = 0. \tag{1.1}$$

Suppose that a component A is a resistor with current-voltage characteristic, which describes the dependence of the voltage on the current, given by $V_A(i) = Ri$, where $R > 0$ is the resistance. Further assume a component B to be an ideal diode with characteristic given by

$$V_B(i) = F(i) := \begin{cases} 0, & \text{for} \quad i > 0, \\ (-\infty, 0], & \text{for} \quad i = 0, \\ \emptyset, & \text{for} \quad i < 0. \end{cases}$$

Therefore (1.1) reads as

$$0 \in Ri - E + F(i). \tag{1.2}$$

The mapping F is a set-valued function. In general we will use the following notation.

Definition 1.1.1 (set-valued mapping) *A set-valued mapping* $\mathbf{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *associates with any* $\mathbf{x} \in \mathbb{R}^m$ *a subset of* $\mathbb{R}^n$*, denoted by* $\mathbf{F}(\mathbf{x})$ *and called the* value *of* $\mathbf{F}$ *at* $\mathbf{x}$*. For such a map, the set*

(i) $\operatorname{dom} \mathbf{F} := \{\mathbf{x} \in \mathbb{R}^m : \ \mathbf{F}(\mathbf{x}) \neq \emptyset\}$ *is the* domain *of* $\mathbf{F}$*,*

(ii) $\operatorname{rge} \mathbf{F} := \{\mathbf{y} \in \mathbb{R}^n : \ \mathbf{y} \in \mathbf{F}(\mathbf{x})$ *for some* $\mathbf{x} \in \mathbb{R}^m\}$ *is the* range *of* $\mathbf{F}$*,*

(iii) $\operatorname{gph} \mathbf{F} := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^n : \mathbf{y} \in \mathbf{F}(\mathbf{x})\}$ *is the* graph *of* $\mathbf{F}$*.*

The problem to find $i \in \mathbb{R}$ such that (1.2) holds is called the *generalized equation*, denoted GE. In higher dimensions, given $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^m$ and $\mathbf{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, we consider the problem of finding a solution $\mathbf{u} \in \mathbb{R}^m$ to the inclusion

$$\mathbf{0}_{\mathbb{R}^m} \in \mathbf{g}(\mathbf{u}) + \mathbf{F}(\mathbf{u}).$$

There are various ways how to write (1.2). The first one uses the notion of the normal cone.

Definition 1.1.2 (normal cone) *Let* $K \subset \mathbb{R}^m$ *be a closed convex set. The* normal cone *to* K *at* $\mathbf{u} \in \mathbb{R}^m$ *is the set*

$$N_K(\mathbf{u}) := \begin{cases} \{\mathbf{p} \in \mathbb{R}^m : \langle \mathbf{p}, \mathbf{v} - \mathbf{u} \rangle \leq 0 \text{ for each } \mathbf{v} \in K\}, & \text{if } \mathbf{u} \in K, \\ \emptyset, & \text{if } \mathbf{u} \notin K. \end{cases}$$

Hence the mapping F appearing in (1.2) satisfies $F(i) = N_{[0,+\infty)}(i)$ for each $i \in \mathbb{R}$. It is easy to see that (1.2) is equivalent to

$$\langle Ri - E, v - i \rangle \geq 0 \quad \text{whenever} \quad v \in [0, +\infty).$$

This problem is called the *variational inequality* in literature, denoted VI. If the set-valued function has the form $\mathbf{F} := N_K$, where K is a convex closed subset of $\mathbb{R}^m$ then GE reads as

$$\mathbf{0}_{\mathbb{R}^m} \in \mathbf{g}(\mathbf{u}) + N_K(\mathbf{u}),$$

so by the definition of a normal cone

$$0 \leq \langle \mathbf{g}(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle \quad \text{whenever} \quad \mathbf{v} \in K.$$

The set of its solutions will be denoted by $\mathrm{SOL}(K, \mathbf{g})$.

The solution of (1.2) has the form

$$i := \begin{cases} 0, & \text{for } E < 0, \\ \dfrac{1}{R}E, & \text{for } E \geq 0, \end{cases}$$

where $E < 0$ means the reversed polarity of the voltage source in the circuit. Let us replace the ideal diode with Zener diode with the current-voltage characteristic given by

$$V_B(i) = F(i) = \mathrm{Sgn}_{V_2,V_1}(i) := \begin{cases} V_1, & \text{for} \quad i > 0, \\ [V_2, V_1], & \text{for} \quad i = 0, \\ V_2, & \text{for} \quad i < 0, \end{cases} \tag{1.3}$$

where $V_1 > 0 > V_2$. In this case the solution of the generalized equation (1.2) has the form

$$i := \begin{cases} \dfrac{1}{R}(E - V_2), & \text{for } E < V_2, \\ 0, & \text{for } E \in [V_2, V_1], \\ \dfrac{1}{R}(E - V_1), & \text{for } E > V_1. \end{cases}$$

Second, suppose that we have a time dependent voltage source $E(t)$ with $t \geq 0$. Then the current also depends on time therefore we are looking for a function $i : \mathbb{R} \to \mathbb{R}$ such that

$$0 \in Ri(t) - E(t) + F(i(t)) \text{ for each } t \in [0, \infty).$$

This is a special case of the *parametric generalized equation* which is the problem for given $\mathbf{g} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m$, $\mathbf{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, and $\Omega \subset \mathbb{R}^d$, find a function $\mathbf{u} : \mathbb{R}^d \to \mathbb{R}^m$ such that

$$\mathbf{0}_{\mathbb{R}^m} \in \mathbf{g}(\mathbf{y}, \mathbf{u}(\mathbf{y})) + \mathbf{F}(\mathbf{u}(\mathbf{y})) \text{ for each } \mathbf{y} \in \Omega.$$

We have to deal with *solution mapping*

$$SOL : \mathbb{R}^d \ni \mathbf{y} \longmapsto \{\mathbf{u} \in \mathbb{R}^m : \mathbf{0}_{\mathbb{R}^m} \in \mathbf{g}(\mathbf{y}, \mathbf{u}) + \mathbf{F}(\mathbf{u})\},$$

which is set-valued in general.

1.2 Dynamic problems

In this section, we will continue studying the circuit in Figure 1.1. Unlike the previous section, a voltage across its components will depend on changes of a current.

Let the component A be an inductor with the relationship between current and voltage given by $V_A(i(t)) = L\frac{d\,i(t)}{dt}$, where $L >$ is a given inductance. Further, let the component B be a resistor with the current-voltage characteristic given by $V_B(i(t)) = Ri(t)$, where $R > 0$ is a resistance. So (1.1) is in the form

$$0 = L\frac{d\,i(t)}{dt} + Ri(t) - E.$$

The previous problem is called the *ordinary differential equation* (see [3]), denoted ODE. These equations are well-known mathematical tool, therefore there exist lots of ways how to (numerically) solve or analyze them.

If we replace the resistor by Zener diode with current-voltage characteristic given by (1.3), then (1.1) has the form

$$0 \in L\frac{d\,i(t)}{dt} + \mathrm{Sgn}_{V_2,V_1}(i(t)) - E.$$

This problem is called the *differential inclusion*, denoted DI. In the general case, for a given set-valued mapping $\mathbf{F} : \mathbb{R}^{n+1} \rightrightarrows \mathbb{R}^n$, it is the problem to find an absolutely continuous[1] function $\mathbf{x} : [a, b] \to \mathbb{R}^n$ such that for almost all $t \in [a, b]$ one has

$$\dot{\mathbf{x}}(t) \in \mathbf{F}(t, \mathbf{x}(t)).$$

[1] See Definition A.1.1.

These problems are difficult to solve because the function Sgn is set-valued at zero. Therefore standard numerical solvers for differential equations cannot be applied. There exist several ways to numerically solve differential inclusions.

For example in [15] the author dealt with difference methods for differential inclusions. The author created numerical methods, which work with a fixed selection of the right-hand side at each step. But this resulted in, that for each type of selection there is a different solution, but the inclusion may have only one solution. Therefore the set-valued right-hand side maybe problematic. Differential inclusions are studied in detail in [4].

We show how to get rid of the set-valued function by creating a new variable. We have

$$\frac{d\,i(t)}{dt} \in -\frac{1}{L}\,\mathrm{Sgn}_{V_2,V_1}(i(t)) + \frac{E}{L}.$$

First, we can write $i = i^+ - i^-$. Further, $-\,\mathrm{Sgn}_{V_2,V_1}(i(t)) = v(t) - V_1$, where v at the time t satisfies

$$0 \le v(t) \perp i^+(t) \ge 0 \quad \text{and} \quad 0 \le V_1 - V_2 - v(t) \perp i^-(t) \ge 0.$$

We obtain

$$\begin{aligned} \frac{d\,i(t)}{dt} &= \frac{1}{L}(v(t) - V_1 + E), \\ 0 &\le v(t) \perp i^+(t) \ge 0, \\ 0 &\le V_1 - V_2 - v(t) \perp i^-(t) \ge 0. \end{aligned}$$

The previous problem is called the *differential variational inequality*, denoted DVI, which is the problem to find an absolutely continuous[2] function $\mathbf{x} : [a,b] \to \mathbb{R}^n$ and an integrable[3] function $\mathbf{u} : [a,b] \to \mathbb{R}^m$ such that for almost all $t \in [a,b]$ one has:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)), \tag{1.4}$$

$$0 \le \langle \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)), \mathbf{v} - \mathbf{u}(t) \rangle \quad \text{whenever} \quad \mathbf{v} \in K, \tag{1.5}$$

$$\mathbf{u}(t) \in K, \tag{1.6}$$

where $\mathbf{f} : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ and $\mathbf{g} : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^m$ are given continuous vector functions, $a < b$ and $K \subset \mathbb{R}^m$ is a non-empty closed convex set. If $\mathbf{u}(t)$ at time t satisfies (1.5)-(1.6), then

$$\mathbf{u}(t) \in \mathrm{SOL}(K, \mathbf{g}(t, \mathbf{x}(t), \cdot)) := \{\mathbf{u} \in \mathbb{R}^m : \mathbf{0}_{\mathbb{R}^m} \in \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}) + N_K(\mathbf{u})\}.$$

[2] In particular applications an absolute continuity of $\mathbf{x}(\dot{})$ can be "too much".

[3] See [13].

Properties of a solution mapping $(t, \mathbf{x}) \rightrightarrows \mathrm{SOL}(K, \mathbf{g}(t, \mathbf{x}, \cdot))$ are important for the existence of a solution. The DVI contains a derivative of $\mathbf{x}(\cdot)$, therefore $\mathbf{x}$ is called a *differential variable*. On the other hand the DVI does not contain the derivative of $\mathbf{u}(\cdot)$, therefore $\mathbf{u}$ is called an *algebraic variable*. Similarly as in the case of ODEs, initial, boundary or another types of conditions are usually added to this problem. We are going to consider DVIs with initial conditions (ICs) only.

This problem was formulated in [6] for the first time. In Chapter 2 we present theorems ensuring the existence of a solution to (1.4)–(1.6). Numerical methods will be discussed in Chapter 3. DVIs provide a powerful modeling paradigm for many applied problems in which dynamics, inequalities, and discontinuities are present. For example contact dynamics, wherein the dry friction occurs or electrical circuits containing diodes. Examples of such models are presented in Chapter 4. The great advantage of DVIs is that ODEs and specific DIs are special types of DVIs.

Chapter 2

Theory of differential variational inequalities

This chapter discusses different types of VIs and DVIs. Then several existence theorems of solutions of DVIs are presented.

2.1 Particular types of VIs and DVIs

In this section, we will present special types of VIs and DVIs. As we will see later, for some special types, it is easier to obtain sufficient conditions for the existence of a solution.

2.1.1 Variational inequalities

First, by Lemma A.2.2, the VI can be written as a non-smooth equation in the form

$$\mathbf{p}_K(\mathbf{u} - \mathbf{g}(\mathbf{u})) = \mathbf{u}.$$

This form of the VI loses some good features of the function $\mathbf{g}$ such as smoothness.

If $K = \mathbb{R}^m$, the VI reduces to an equation $\mathbf{g}(\mathbf{u}) = \mathbf{0}_{\mathbb{R}^m}$. Indeed, for arbitrary $\mathbf{h} \in \mathbb{R}^m$ a vector $\mathbf{v} := \mathbf{u} \pm \mathbf{h}$ lies in K. Therefore

$$\langle \mathbf{g}(\mathbf{u}), \mathbf{h} \rangle = 0 \quad \text{for each} \quad \mathbf{h} \in \mathbb{R}^m,$$

hence $\mathbf{g}(\mathbf{u}) = \mathbf{0}_{\mathbb{R}^m}$.

VIs appear naturaly in conditional minimization of a convex differentiable function over a convex set K, where the vector $\mathbf{g}(\mathbf{u})$ is a gradient of the objective function at $\mathbf{u}$. Solutions of VI correspond to minima.

When the set K is a polyhedron and $\mathbf{g}(\mathbf{u}) := \mathbf{A}\mathbf{u} + \mathbf{b}$, with $\mathbf{A} \in \mathbb{R}^{m\times m}$ and $\mathbf{b} \in \mathbb{R}^m$, then VI is called the *affine variational inequality*, denoted AVI.

If the set K is a cone[1], then the VI is the problem to find $\mathbf{u} \in \mathbb{R}^m$ such that

$$K^* \ni \mathbf{g}(\mathbf{u}) \perp \mathbf{u} \in K,$$

where K^* is a dual cone[2] to K. The previous problem is called the *complementary problem*, denoted CP.

Let us show, that CP is equivalent to VI, when K is a cone. Implication from CP to VI is trivial. For all $\mathbf{v} \in K$ it holds

$$\begin{aligned} \langle \mathbf{v}, \mathbf{g}(\mathbf{u})\rangle \geq 0 \quad &= \langle \mathbf{u}, \mathbf{g}(\mathbf{u})\rangle, \\ \langle \mathbf{v} - \mathbf{u}, \mathbf{g}(\mathbf{u})\rangle \geq 0. & \end{aligned}$$

The implication is proved.

We will prove implication from VI to CP. For all $\mathbf{v} \in K$ it holds

$$\begin{aligned} \langle \mathbf{v} - \mathbf{u}, \mathbf{g}(\mathbf{u})\rangle &\geq 0, \\ \langle \mathbf{v}, \mathbf{g}(\mathbf{u})\rangle &\geq \langle \mathbf{u}, \mathbf{g}(\mathbf{u})\rangle. \end{aligned}$$

We show that

$$\langle \mathbf{u}, \mathbf{g}(\mathbf{u})\rangle = 0. \tag{2.1}$$

Since $\mathbf{u}$ lies in the cone K, so do $\mathbf{v} := \mathbf{0}_{\mathbb{R}^m}$ and $\mathbf{v} := 2\mathbf{u}$. Therefore

$$\begin{aligned} \mathbf{v} = \mathbf{0}_{\mathbb{R}^m} \quad &: \quad 0 \geq \langle \mathbf{u}, \mathbf{g}(\mathbf{u})\rangle, \\ \mathbf{v} = 2\mathbf{u} \quad &: \quad 0 \leq \langle \mathbf{u}, \mathbf{g}(\mathbf{u})\rangle. \end{aligned}$$

The equivalence is proved.

Moreover, if $K = \mathbb{R}^m_+$, then $K^* = \mathbb{R}^m_+$ and the CP is the problem to find $\mathbf{u} \in \mathbb{R}^m$ such that

$$0 \preceq \mathbf{g}(\mathbf{u}) \perp \mathbf{u} \succeq 0,$$

which is called the *nonlinear complementarity problem*, denoted NCP. A NPC with a function $\mathbf{g}(\mathbf{u}) := \mathbf{A}\mathbf{u} + \mathbf{b}$, $\mathbf{A} \in \mathbb{R}^{m\times m}$ and $\mathbf{b} \in \mathbb{R}^m$, i.e. the problem to find $\mathbf{u} \in \mathbb{R}^m$, such that

$$0 \preceq \mathbf{A}\mathbf{u} + \mathbf{b} \perp \mathbf{u} \succeq 0,$$

[1] See Definition A.1.4

[2] See Definition A.1.5

is called the *linear complementarity problem*, denoted LCP. A parametric version of a LCP has the form

$$0 \preceq \mathbf{Au} + \mathbf{By} + \mathbf{b} \perp \mathbf{u} \succeq 0,$$

where $\mathbf{A} \in \mathbb{R}^{m\times m}$, $\mathbf{B} \in \mathbb{R}^{m\times d}$ and $\mathbf{b} \in \mathbb{R}^{m}$.

Complementarity problems and variational inequalities are studied in detail in [2].

2.1.2 Differential variational inequalities

Similarly to VIs, DVIs have several particular forms depending on $\mathbf{f}, \mathbf{g}$, and K.

If $K := \mathbb{R}^m$, then a DVI reduces to the *differential algebraic equation*, denoted DAE, i.e. the problem to find functions $\mathbf{x} : [a, b] \to \mathbb{R}^n$ and $\mathbf{u} : [a, b] \to \mathbb{R}^m$, such that

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)) \quad \text{and} \quad \mathbf{0}_{\mathbb{R}^m} = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)) \quad \text{for almost all} \quad t \in [a, b].$$

DAEs are studied in detail in [7].

Let K be a cone in $\mathbb{R}^m$ with a dual cone K^*, then DVI is reduced to the *differential complementarity problem*, denoted DCP, i.e. the problem to find functions $\mathbf{x} : [a, b] \to \mathbb{R}^n$ and $\mathbf{u} : [a, b] \to \mathbb{R}^m$, such that

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)), \\ K &\ni \mathbf{u}(t) \perp \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)) \in K^* \quad \text{for almost all } t \in [a, b]. \end{aligned}$$

Moreover, if $K = \mathbb{R}^m_+$, then the previous system is called the *differential nonlinear complementarity problem*, denoted DNCP, and if

$$\begin{aligned} \mathbf{f}(t, \mathbf{x}, \mathbf{u}) &:= \mathbf{Ax} + \mathbf{Bu} + \mathbf{p}, \\ \mathbf{g}(t, \mathbf{x}, \mathbf{u}) &:= \mathbf{Cx} + \mathbf{Du} + \mathbf{q}, \end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{m\times n}, \mathbf{D} \in \mathbb{R}^{m\times m}, \mathbf{p} \in \mathbb{R}^n$ and $\mathbf{q} \in \mathbb{R}^m$ are given, then NDCP reduces to a *differential linear complementarity problem*, denoted DLCP.

Another problem mentioned in [12] is the *differential mixed variational inequality*, denoted DMVI, which is the problem to find an absolutely continuous function $\mathbf{x} : [a, b] \to \mathbb{R}^n$ and an integrable function $\mathbf{u} : [a, b] \to \mathbb{R}^m$ such that for almost all $t \in [a, b]$ one has:

(2.2)

$$\begin{aligned} \dot{\mathbf{x}}(t) = &\quad \mathbf{f}(t, \mathbf{x}(t)) + \mathbf{B}(t, \mathbf{x}(t))\mathbf{u}(t), \\ 0 \leq &\quad \langle \mathbf{h}(t, \mathbf{x}(t)) + \mathbf{g}(\mathbf{u}(t)), \mathbf{v} - \mathbf{u}(t) \rangle + \varphi(\mathbf{u}) - \varphi(\mathbf{v}) \quad \text{whenever} \quad \mathbf{v} \in K, \\ \mathbf{u}(t) \in &\quad K, \end{aligned}$$

where $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \mathbf{h} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m, \mathbf{B} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ and $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^m$ are given, $\varphi : \mathbb{R}^m \to (-\infty, \infty]$ is lower semicontinuous convex function and K is a non-empty closed convex set.

2.2 Reduction of DVIs to ODEs

First approach to obtain the existence of a solution of a DVI is to reduce it to an ODE. We present theorems, which ensure that there is the only one function $\mathbf{u}$, such that $\mathbf{u}(t, \mathbf{x}) \in \mathrm{SOL}(K, \mathbf{g}(t, \mathbf{x}, \cdot))$ for each $t \in [a, b]$ and also that $\mathbf{u}$ is continuous in t and (locally) Lipschitz continuous in $\mathbf{x}$. Then a composition of functions $\mathbf{f}$ and $\mathbf{u}$ satisfies assumptions of the classical ODE theory. More precisely we reduce a DVI either locally or globally to the initial value problem in the form

$$\dot{\mathbf{x}}(t) = \mathbf{h}(t, \mathbf{x}(t)), \tag{2.3}$$

$$\mathbf{x}(a) = \mathbf{x}_a, \tag{2.4}$$

where $\mathbf{h} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{x}_a \in \mathbb{R}^n$.

2.2.1 Global reduction

In this section, we are going to focus on sufficient conditions for the existence of a solution of DVIs, such that $\mathbf{u} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous on $[a, b] \times \Omega$ and the function $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$ is continuously differentable on (a, b) and satisfies the initial condition $\mathbf{x}(a) = \mathbf{x}_a \in \Omega$. Here and further Ω is a non-empty closed subset of $\mathbb{R}^n$.

We begin with unique existence of a solution of the parametric VI (see for example [2]).

Theorem 2.2.1 *Consider a parametric VI in the form*

$$\mathbf{0}_{\mathbb{R}^m} \in \mathbf{g}(\mathbf{y}, \mathbf{u}) + N_K(\mathbf{u}), \tag{2.5}$$

where $\mathbf{g} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m$*,* K *is a non-empty closed convex subset of* $\mathbb{R}^m$ *and* Ω *is a non-empty closed subset of* $\mathbb{R}^d$*. Suppose that*

(i) $\mathbf{g}$ *is continuous on* $\Omega \times K$*,*

(ii) there is $L > 0$ *such that, for each* $\mathbf{u} \in K$*, the mapping* $\mathbf{g}(\cdot, \mathbf{u})$ *is Lipschitz continuous on* Ω *with the constant* L*,*

(iii) there is $\mu > 0$ *such that*

$$\langle \mathbf{g}(\mathbf{y},\mathbf{u}) - \mathbf{g}(\mathbf{y},\mathbf{w}), \mathbf{u} - \mathbf{w} \rangle \geq \mu \|\mathbf{u} - \mathbf{w}\|^2 \text{ whenever } \mathbf{u}, \mathbf{w} \in K \text{ and } \mathbf{y} \in \Omega.$$

Then the solution mapping of (2.5) *is single-valued for all* $\mathbf{y} \in \Omega$ *and Lipschitz continuous on* Ω *with the constant* L/μ*.*

Proof. See [5, Theorem 2.2.1, p. 23]. ■

The above statement will be combined with the following well-known result from ODEs.

Theorem 2.2.2 (global unique existence) *Consider the problem* (2.3) *and suppose that* $\mathbf{h}$ *is continuous and let* $(a, \mathbf{x}_a) \in \mathbb{R} \times \mathbb{R}^n$ *be given. Then the following holds:*

1. *There is* $\varepsilon > 0$ *and a solution of* (2.3) *on an open interval* $(a-\varepsilon, a+\varepsilon)$ *satisfying* $\mathbf{x}(a) = \mathbf{x}_a$*.*

2. *If in addition we assume that* $\mathbf{h}$ *is linearly bounded, i.e. there exists number* α *such that*

   $$\|\mathbf{h}(t,\mathbf{x})\| \leq \alpha(\|\mathbf{x}\| + 1) \text{ whenever } (t,\mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n,$$

   then there is a solution of (2.3) *on* $(-\infty, \infty)$ *such that* $\mathbf{x}(a) = \mathbf{x}_a$*.*

3. *Moreover if* $\mathbf{h}$ *is globally Lipschitz in the second variable uniformly with respect to the first one, i.e. there is* $L > 0$ *such that*

   $$\|\mathbf{h}(t,\mathbf{x}) - \mathbf{h}(t,\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\| \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ and } t \in \mathbb{R},$$

   then there is a unique solution of (2.3) *on* $(-\infty, \infty)$ *such that* $\mathbf{x}(a) = \mathbf{x}_a$*.*

Proof. See [14, Theorem 1.1, p. 178]. ■

Now we show, how to apply the previous theorems to the autonomous DVI in the form

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \\
0 &\leq \langle \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)), \mathbf{v} - \mathbf{u}(t) \rangle \quad \text{whenever} \quad \mathbf{v} \in K, \\
\mathbf{u}(t) &\in K.
\end{aligned}$$

where $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are given mappings and K is a non-empty closed convex subset of $\mathbb{R}^m$.

Suppose that $\mathbf{g}$ satisfies conditions (i)-(iii) in Theorem 2.2.1. Then there is a (unique) Lipschitz continuous function $\mathbf{u} : \Omega \ni \mathbf{x} \to \mathbf{u}(\mathbf{x}) \in \mathrm{SOL}(K, \mathbf{g}(\mathbf{x}, \cdot))$ having Lipschitz constant $\frac{L}{\mu}$ on Ω. Moreover assume that $\mathbf{f}$ is Lipschitz continuous on $\Omega \times K$, that is, there exist numbers $L_{\mathbf{x}}, L_{\mathbf{u}} > 0$, such that

$$\|\mathbf{f}(\mathbf{x}_1, \mathbf{u}_1) - \mathbf{f}(\mathbf{x}_2, \mathbf{u}_2)\| \leq L_{\mathbf{x}}\|\mathbf{x}_1 - \mathbf{x}_2\| + L_{\mathbf{u}}\|\mathbf{u}_1 - \mathbf{u}_2\|,$$

for each $(\mathbf{x}_1, \mathbf{u}_1), (\mathbf{x}_2, \mathbf{u}_2) \in \Omega \times K$. We show that the composition $\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}))$, $\mathbf{x} \in \Omega$, is Lipschitz continuous. For any $\mathbf{x}_1, \mathbf{x}_2 \in \Omega$, we have

$$\begin{aligned}\|\mathbf{f}(\mathbf{x}_1, \mathbf{u}(\mathbf{x}_1)) - \mathbf{f}(\mathbf{x}_2, \mathbf{u}(\mathbf{x}_2))\| \leq & \quad L_{\mathbf{x}}\|\mathbf{x}_1 - \mathbf{x}_2\| + L_{\mathbf{u}}\|\mathbf{u}(\mathbf{x}_1) - \mathbf{u}(\mathbf{x}_2)\| \leq \\ \leq & \quad (L_{\mathbf{x}} + \frac{L_{\mathbf{u}}L}{\mu})\|\mathbf{x}_1 - \mathbf{x}_2\|.\end{aligned}$$

We arrive at

$$\dot{\mathbf{x}} = \mathbf{h}(\mathbf{x}).$$

By Theorem 2.2.2, this ODE has a unique solution on $[a, +\infty]$ with $\mathbf{x}(a) = \mathbf{x}_a \in \Omega$. We showed, that there is a unique C^1 function $\mathbf{x}(\cdot)$ and a unique Lipschitz continuous function $\mathbf{u}(\cdot)$ solving the DVI.

Now consider non-empty sets $\Omega \subset \mathbb{R}^n, K \subset \mathbb{R}^m, [a, b] \subset \mathbb{R}$ and a function $\mathbf{g} : [a, b] \times \Omega \times K \to \mathbb{R}^n$. Impose the following assumptions on $\mathbf{g}$:

(A) $\mathbf{g}(t, \mathbf{x}, \cdot)$ is continuous, uniformly *P-function* on the set K with a modulus that is independent of $(t, \mathbf{x})$, i.e. there is a constant $\kappa > 0$ such that

$$\max_{1 \leq i \leq N} (\mathbf{u}_i - \mathbf{u}'_i)^T(\mathbf{g}_i(t, \mathbf{x}, \mathbf{u}) - \mathbf{g}_i(t, \mathbf{x}, \mathbf{u}')) \geq \kappa\|\mathbf{u} - \mathbf{u}'\|^2 \tag{2.6}$$

for all $(t, \mathbf{x}) \in [a, b] \times \Omega$ and $\mathbf{u} := (\mathbf{u}_i)_{i=1}^N, \mathbf{u}' := (\mathbf{u}'_i)_{i=1}^N$ in $K := \Pi_{i=1}^N K^i$, where K^i is a closed convex subset of $\mathbb{R}^{m_i}$,

(B) $\mathbf{g}(\cdot, \cdot, \mathbf{u})$ is Lipschitz continuous with a constant independent of $\mathbf{u}$, i.e. there is $L > 0$ such that

$$\|\mathbf{g}(t_1, \mathbf{x}, \mathbf{u}) - \mathbf{g}(t_2, \mathbf{y}, \mathbf{u})\| \leq L(|t_1 - t_2| + \|\mathbf{x} - \mathbf{y}\|), \tag{2.7}$$

for each $(t_1, \mathbf{x}) \in [a, b] \times \Omega$, $(t_2, \mathbf{y}) \in [a, b] \times \Omega$ and $\mathbf{u} \in K$.

The following theorem was presented in [6, Theorem 5.1].

Theorem 2.2.3 *Let $K := \Pi_{i=1}^{N} K^i$ where each K^i is a closed convex subset of $\mathbb{R}^{m_i}$, with $m_1+m_2+...+m_N := m$. Assume $\mathbf{g}$ satisfies (A) and (B). Then there is a Lipschitz continuous function $\mathbf{u} : [a,b] \times \Omega \to K$ such that for each pair $(t, \mathbf{x}) \in [a,b] \times \Omega$, $\mathbf{u}(t,\mathbf{x})$ is the unique solution of the VI $(K, \mathbf{g}(t,\mathbf{x},\cdot))$.*

The previous theorem can be applied to non-autonomous DVI in the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)), \\ 0 &\leq \langle \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)), \mathbf{v} - \mathbf{u}(t) \rangle \quad \text{whenever} \quad \mathbf{v} \in K, \\ \mathbf{u}(t) &\in K. \end{aligned}$$

Similarly as in the previous case. By Theorem 2.2.3 the DVI can be reduced to the ODE in the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t, \mathbf{x}(t))) := \mathbf{h}(t, \mathbf{x}(t)),$$

where $\mathbf{u}(t,\mathbf{x}) \in \mathrm{SOL}(K, \mathbf{g}(t,\mathbf{x},\cdot))$. Then we use Theorem 2.2.2 with appropriate properties of $\mathbf{f}$ and an initial condition.

The following proposition shows necessary and sufficient condition for existence of an unique C^1 trajectory $\mathbf{x}(t)$ on $[a, +\infty]$ solving an initial value problem for the semi-affine autonomous DVI in the form

$$\begin{aligned} \dot{\mathbf{x}}(t) =& \quad \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(a) = \mathbf{x}_a \in \mathbb{R}^n, \\ 0 \leq& \quad \langle \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t) \rangle \quad \text{whenever} \quad \mathbf{v} \in K, \\ \mathbf{u}(t) \in& \quad K, \end{aligned} \tag{2.8}$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n, \mathbf{B} \in \mathbb{R}^{n \times m}, \mathbf{C} \in \mathbb{R}^{m \times n}$ and $\mathbf{D} \in \mathbb{R}^{m \times m}$ are given and K is a closed convex subset of $\mathbb{R}^m$.

The following proposition was presented in [6].

Proposition 2.2.1 *Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz continuous and let $K \subset \mathbb{R}^m$ be polyhedral. A necessary and sufficient condition for (2.8) to have a unique C^1 solution trajectory $\mathbf{x}(t)$ on $[a, +\infty)$ for all $\mathbf{x}_a \in \mathbb{R}^n$ is that $\mathbf{B}SOL(K, \mathbf{C}\mathbf{x} + \mathbf{D})$ is a singleton for all $\mathbf{x} \in \mathbb{R}^n$.*

Proof. See [6, Proposition 5.1, p. 367]. ■

Now, we can formulate the following corollary.

Corollary 2.2.1 *Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz continuous and let $K \subset \mathbb{R}^m$ be polyhedral. Suppose symmetric part of $\mathbf{D} \in \mathbb{R}^{m \times m}$ has positive eigenvalues, then (2.8) has unique C^1 solution trajectory $\mathbf{x}(t)$ on $[a, +\infty)$ for arbitrary $\mathbf{x}_a \in \mathbb{R}^n$.*

Proof.

1. Clearly the vector function $\mathbf{Cx} + \mathbf{Du}$ is Lipschitz continuous on $\mathbb{R}^n \times \mathbb{R}^m$.

2. Let $\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2$ for $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^m : \mathbf{u}_1 \neq \mathbf{u}_2$. Denote by $\widetilde{\mathbf{D}}$ a symmetric part of $\mathbf{D}$ and $\lambda_{\min}$ its smallest eigenvalue. By using properties of Rayleigh quotient and Lemmas A.2.3 and A.2.4, it holds

$$\frac{\langle \mathbf{Du}, \mathbf{u} \rangle}{\|\mathbf{u}\|^2} = \frac{\langle \widetilde{\mathbf{D}}\mathbf{u}, \mathbf{u} \rangle}{\|\mathbf{u}\|^2} \geq \lambda_{\min} > 0,$$

therefore

$$\langle \mathbf{Du}_1 - \mathbf{Du}_2, \mathbf{u}_1 - \mathbf{u}_2 \rangle \geq \lambda_{\min} \|\mathbf{u}_1 - \mathbf{u}_2\|^2,$$

for each $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^m$.

Assumptions of Theorem 2.2.1 are satisfied with $\Omega := \mathbb{R}^n, \mathbf{g}(\mathbf{x}, \mathbf{u}) := \mathbf{Cx} + \mathbf{Du}$. Therefore for arbitrary $\mathbf{B} \in \mathbb{R}^{m \times m}$ the set $\mathbf{B}\mathrm{SOL}(K, \mathbf{Cx} + \mathbf{D})$ is singleton for each $\mathbf{x} \in \mathbb{R}^n$. By Proposition 2.2.1 the initial value problem (2.8) has unique C^1 solution trajectory $\mathbf{x}(t)$ on $[a, +\infty]$ for arbitrary $\mathbf{x}(a) = \mathbf{x}_a \in \mathbb{R}^n$. ■
Consider an initial value problem for DVI in the form

$$\begin{aligned}
&\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}(\mathbf{x}(t))\mathbf{u}(t), \quad \mathbf{x}(a) = \mathbf{x}_a \in \mathbb{R}^n, \\
&0 \leq \langle \mathbf{l}(\mathbf{x}(t)) + \mathbf{g}(\mathbf{u}(t)), \mathbf{v} - \mathbf{u}(t) \rangle \quad \text{whenever} \quad \mathbf{v} \in K, \\
&\mathbf{u}(t) \in K,
\end{aligned} \tag{2.9}$$

where K is a non-empty closed convex subset of $\mathbb{R}^m, \mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n, \mathbf{B} : \mathbb{R}^n \to \mathbb{R}^{n \times m}, \mathbf{l} : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^m$ are given.

The following theorem gives conditions for an uniqueness of a solution $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ of the DVI on the interval $[a, b]$ and was presented in [11].

Theorem 2.2.4 *Suppose that*

1. $K \subset \mathbb{R}^m$ *is a closed and convex set,*

2. $\mathbf{f}, \mathbf{B}$ *and* $\nabla \mathbf{l}$ *are locally Lipschitz continuous on* $\mathbb{R}^n$*, where* $\nabla \mathbf{l}(\mathbf{x})$ *is the Jacobi matrix of* $\mathbf{l}$ *at* $\mathbf{x}$*,*

3. $\mathbf{g}$ *is monotone on* $\mathbb{R}^m$*,*

4. $\nabla \mathbf{l}(\mathbf{x})\mathbf{B}(\mathbf{x})$ *is symmetric positive definite matrix for all* $\mathbf{x} \in \mathbb{R}^n$*,*

5. *all solutions of the problem* (2.9) *have bounded* $\mathbf{u}(\cdot)$ *on the interval* $[a, b]$.

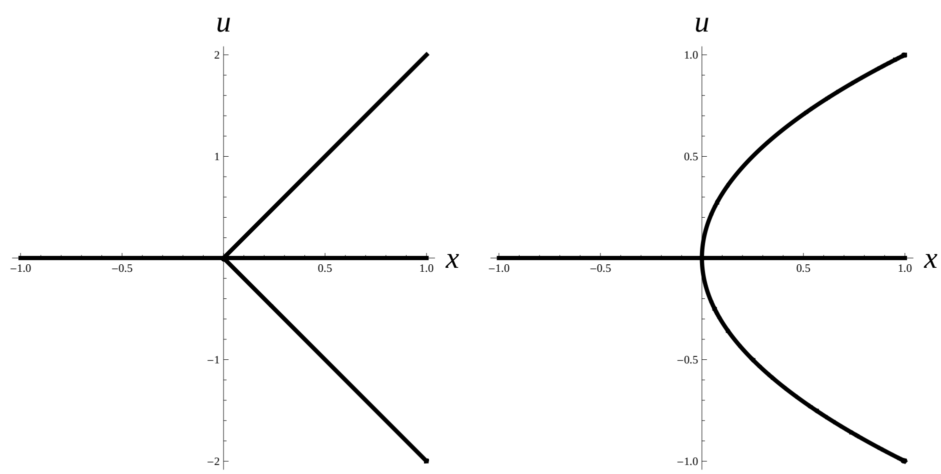
Then the solution $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ *to* (2.9) *is unique on* $[a, b]$.

Proof. See [11, Theorem 4.3, p. 816] ■
In general, there may exist more then one $\mathbf{u}(\cdot)$ solving the algebraic constraint in a DVI and not all of them are Lipschitz continuous on $[a, b]$. Let us illustrate this on a simple DAE.

Example 2.2.1 *Consider the DAE in the form*

$$\begin{aligned} \dot{x}(t) &= -x(t) - u(t), \\ 0 &= |u(t) + x(t)| - |u(t) - x(t)| - u(t). \end{aligned}$$

The graph of the solution mapping is in Figure 2.1a. It is easy to see,



(a) (b)

Figure 2.1: Solution mappings.

that this mapping has three global selections and all of them are Lipschitz continuous on $\mathbb{R}$. *More precisely*

$$\mathbb{R} \ni x \longmapsto SOL(\mathbb{R}, |\cdot + x| - |\cdot - x| - \cdot) = \{2x^+, 0, -2x^+\}.$$

So the DAE consists of three different differential equations:

1. $\dot{x}(t) = -x(t) - 2x^+(t)$,

2. $\dot{x}(t) = -x(t)$,

3. $\dot{x}(t) = -x(t) + 2x^+(t)$.

Further, consider the DAE in the form

$$\begin{aligned} \dot{x}(t) &= -x(t) - u(t), \\ 0 &= u(t)^3 - u(t)x(t). \end{aligned}$$

The solution mapping of the algebraic equation is in Figure 2.1b. Obviously, the mapping has three continuous selections. More precisely

$$\mathbb{R} \ni x \longmapsto SOL(\mathbb{R}, (\cdot)^3 - (\cdot)x) = \{(\mathrm{sgn}\,(x)\sqrt{|x|})^+, 0, -(\mathrm{sgn}\,(x)\sqrt{|x|})^+\}.$$

So this mapping has only one global selection, which is Lipschitz continuous on $\mathbb{R}$. This motivates our investigation of local selections instead of global ones in the next subsection.

2.2.2 Local reduction

Unlike the previous section, we are going to focus on the local solvability of an initial value problem for DVIs. We present sufficient conditions, such that for $\mathbf{u}_a \in \mathrm{SOL}(K, \mathbf{g}(a, \mathbf{x}_a, \cdot))$, where $(a, \mathbf{x}_a) \in \mathbb{R} \times \mathbb{R}^n$ is a given point, there are $\alpha > 0$, a neighborhood U of $\mathbf{x}_a$ and the unique (Lipschitz) continuous function $\mathbf{u} : [a, a+\alpha] \times U \to \mathbb{R}^m$ satisfying $\mathbf{u}(a) = \mathbf{u}_a$ and

$$\mathbf{u}(t, \mathbf{x}) \in \mathrm{SOL}(K, \mathbf{g}(t, \mathbf{x}, \cdot)) \quad \text{for each } (t, \mathbf{x}) \in [a, a+\alpha] \times U.$$

Therefore a DVI is equivalent to an ODE with a (Lipschitz) continuous right-hand side on the set $[a, a+\alpha] \times U$. Then by ODE theory, there is a (unique) smooth solution $\mathbf{x} : [a, a+\alpha] \to \mathbb{R}^n$ satisfying (2.3)–(2.4) with $\mathbf{h} = \mathbf{f} \circ \mathbf{u}$.

Recall some important properties of set-valued mappings. The mapping $\Phi : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ with $\bar{\mathbf{y}} \in \Phi(\bar{\mathbf{u}})$ is *strongly metrically regular* at $\bar{\mathbf{u}}$ for $\bar{\mathbf{y}}$, if the mapping $\mathbf{S} := \Phi^{-1}$ has a Lipschitz continuous single-valued localization around $\bar{\mathbf{y}}$ for $\bar{\mathbf{u}}$, i.e. for some neighbourhoods U of $\bar{\mathbf{u}}$ and V of $\bar{\mathbf{y}}$ we have

$$\mathbf{S}(\mathbf{y}) \cap U = \{\mathbf{s}(\mathbf{y})\} \quad \text{if} \quad \mathbf{y} \in V,$$

where $\mathbf{s} : V \to U$ is a Lipschitz function. The set-valued mapping Φ is called *metrically regular* at $\bar{\mathbf{u}}$ for $\bar{\mathbf{y}}$ if there is a constant $\kappa > 0$ along with neighborhoods V of $\bar{\mathbf{y}}$ and U of $\bar{\mathbf{u}}$ such that

$$d\left(\mathbf{u}, \Phi^{-1}(\mathbf{y})\right) \le \kappa\, d(\mathbf{y}, \Phi(\mathbf{u})) \quad \text{whenever} \quad (\mathbf{u}, \mathbf{y}) \in U \times V.$$

Consider the parametric generalized equation in the form

$$\mathbf{0}_{\mathbb{R}^m} \in \mathbf{g}(\mathbf{p}, \mathbf{u}) + \mathbf{F}(\mathbf{u}), \tag{2.10}$$

where $\mathbf{g} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m$ and $\mathbf{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ are given.

Now we present some results from GE theory (see for example [12]).

Theorem 2.2.5 *Let* $\mathbf{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ *be a given set-valued mapping,* $\mathbf{g} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m$ *be a continuously differentiable function and* $\mathbf{S}$ *be a solution mapping of* (2.10) *with* $\bar{\mathbf{u}} \in \mathbf{S}(\bar{\mathbf{p}})$*. Suppose that* $\mathbf{g}(\bar{\mathbf{p}}, \bar{\mathbf{u}}) + \nabla_{\mathbf{u}} \mathbf{g}(\bar{\mathbf{p}}, \bar{\mathbf{u}})(\cdot - \bar{\mathbf{u}}) + \mathbf{F}$ *is strongly metrically regular at* $\bar{\mathbf{u}}$ *for* $\mathbf{0}_{\mathbb{R}^m}$*. Then* $\mathbf{S}$ *has a localization around* $\bar{\mathbf{p}}$ *for* $\bar{\mathbf{u}}$*, which is Lipschitz continuous.*

Proof. See [5]. ■

Let $\Omega \subset \mathbb{R}^d$ be a non-empty set, which contains a point $\bar{\mathbf{x}}$, the *regular normal cone* to Ω at $\bar{\mathbf{x}}$ is the set

$$\widehat{N}_\Omega(\bar{\mathbf{x}}) := \left\{ \xi \in \mathbb{R}^d : \limsup_{\Omega \ni \mathbf{x} \to \bar{\mathbf{x}}} \frac{\langle \xi, \mathbf{x} - \bar{\mathbf{x}} \rangle}{\|\mathbf{x} - \bar{\mathbf{x}}\|} \leq 0 \right\}.$$

The *limiting normal cone* $N_\Omega(\bar{\mathbf{x}})$ to Ω at $\bar{\mathbf{x}}$ contains all $\xi \in \mathbb{R}^d$ for which there are sequences $(\mathbf{x}_k)_{k \in \mathbb{N}}$ in Ω and $(\xi_k)_{k \in \mathbb{N}}$ in $\mathbb{R}^d$ converging to $\bar{\mathbf{x}}$ and ξ, respectively, such that $\xi^k \in \widehat{N}_\Omega(\mathbf{x}^k)$ for each $k \in \mathbb{N}$. Note that for a convex set Ω we get the normal cone introduced in Definition 1.1.2.

The *Bouligand paratingent cone* $\widetilde{T}_\Omega(\bar{\mathbf{x}})$ to Ω at $\bar{\mathbf{x}}$ contains those $\mathbf{v} \in \mathbb{R}^d$ for which there are sequences $(t_k)_{k \in \mathbb{N}}$ in $(0, +\infty)$, $(\mathbf{v}_k)_{k \in \mathbb{N}}$ in $\mathbb{R}^d$, and $(\mathbf{x}_k)_{k \in \mathbb{N}}$ in Ω converging to 0, $\mathbf{v}$, and $\bar{\mathbf{x}}$, respectively, such that $\mathbf{x}_k + t_k \mathbf{v}_k \in \Omega$ whenever $k \in \mathbb{N}$.

The following theorem gives us the criterion for verification of the strong metric regularity (for example see [16]) in Theorem (2.2.5).

Theorem 2.2.6 *Consider the mapping* $\Phi := \mathbf{g} + \mathbf{F}$ *where* $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^m$ *is continuously differentiable on* $\mathbb{R}^m$*,* $\mathbf{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ *has closed graph and there are* $F_i : \mathbb{R} \rightrightarrows \mathbb{R}$*,* $i \in \{1, \ldots, m\}$*, such that* $\mathbf{F}(\mathbf{u}) = \prod_{i=1}^{m} F_i(u_i)$ *whenever* $\mathbf{u} = (u_1, \ldots, u_m)^T \in \mathbb{R}^m$*. Let* $\bar{\mathbf{y}}, \bar{\mathbf{u}} \in \mathbb{R}^m$ *and* $\bar{\mathbf{v}} := \bar{\mathbf{y}} - \mathbf{g}(\bar{\mathbf{u}})$*. Then*

1. *Φ is metrically regular at $\bar{\mathbf{u}}$ for $\bar{\mathbf{y}}$ if and only if*

$$\big(\nabla \mathbf{g}(\bar{\mathbf{u}})^T \xi, \xi\big) \in -N_{\mathrm{gph}\,\mathbf{F}}\big((\bar{\mathbf{u}}, \bar{\mathbf{v}})\big) \implies \xi = \mathbf{0}_{\mathbb{R}^m},$$

2. *Φ is strongly metrically regular at $\bar{\mathbf{u}}$ for $\bar{\mathbf{y}}$ if and only if*

(a) for each neighborhood U of $\bar{\mathbf{u}}$ there is a neighborhood V of $\bar{\mathbf{y}}$ such that $\Phi^{-1}(\mathbf{y}) \cap U \neq \emptyset$ whenever $\mathbf{y} \in V$,

(b)
$$(\mathbf{b}, -\nabla \mathbf{g}(\bar{\mathbf{u}})\mathbf{b}) \in \widetilde{T}_{\operatorname{gph} \mathbf{F}}((\bar{\mathbf{u}}, \bar{\mathbf{v}})) \implies \mathbf{b} = \mathbf{0}_{\mathbb{R}^m}.$$

Note that instead of verifying the condition 2.(a) it is sufficient to verify the metric regularity of Φ at $\bar{\mathbf{u}}$ for $\bar{\mathbf{y}}$.

Recall that for a given set-valued mapping $\mathbf{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ and a point $(\bar{\mathbf{p}}, \bar{\mathbf{u}}) \in \operatorname{gph} \mathbf{S}$, a *selection for* $\mathbf{S}$ *around* $\bar{\mathbf{p}}$ *for* $\bar{\mathbf{u}}$ is any single-valued mapping $\mathbf{s}$ defined on a neighbourhood V of $\bar{\mathbf{p}}$ such that

$$\mathbf{s}(\bar{\mathbf{p}}) = \bar{\mathbf{u}} \quad \text{and} \quad \mathbf{s}(\mathbf{p}) \in \mathbf{S}(\mathbf{p}) \quad \text{for each} \quad \mathbf{p} \in V.$$

And $\mathbf{S}$ is *locally monotone* at $(\bar{\mathbf{p}}, \bar{\mathbf{u}}) \in \operatorname{gph} \mathbf{S}$ if there is a neighborhood W of $(\bar{\mathbf{p}}, \bar{\mathbf{u}})$ such that

$$\langle \hat{\mathbf{p}} - \tilde{\mathbf{p}}, \hat{\mathbf{u}} - \tilde{\mathbf{u}} \rangle \geq 0 \quad \text{whenever} \quad (\hat{\mathbf{p}}, \hat{\mathbf{u}}), (\tilde{\mathbf{p}}, \tilde{\mathbf{u}}) \in \operatorname{gph} \mathbf{S} \cap W. \tag{2.11}$$

The following statement guarantees that if we have a selection of a set-valued mapping in hand, then it is a localization of the mapping and conversely. In addition, the theorem holds even if we replace the continuity by the Lipschitz continuity.

Lemma 2.2.1 *A set-valued mapping $\mathbf{S} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, which is locally monotone at $(\bar{\mathbf{p}}, \bar{\mathbf{u}}) \in \operatorname{gph} \mathbf{S}$, has a single-valued continuous localization around $\bar{\mathbf{p}}$ for $\bar{\mathbf{u}}$ if and only if it has a continuous selection around $\bar{\mathbf{p}}$ for $\bar{\mathbf{u}}$.*

Proof. We shall imitate the proof of [5, Theorem 2.4.1]. Find W such that (2.11) holds. By assumptions there are $r_1, r_2 > 0$ and a continuous function $\mathbf{s} : \mathbb{R}^m \to \mathbb{R}^m$, such that

$$\mathbb{B}(\bar{\mathbf{p}}, r_1) \times \mathbb{B}(\bar{\mathbf{u}}, r_2) \subseteq W \quad \text{and} \quad \mathbf{s}(\mathbb{B}(\bar{\mathbf{p}}, r_1)) \subseteq \mathbb{B}(\bar{\mathbf{u}}, r_2).$$

Fix any $\mathbf{p} \in \mathbb{B}(\bar{\mathbf{p}}, r_1)$. The continuity of $\mathbf{s}$ at $\mathbf{p}$ means that

$$\forall \varepsilon > 0, \exists \delta > 0, \forall \mathbf{p} \in \mathbb{R}^m : \|\mathbf{p}' - \mathbf{p}\| < \delta \Rightarrow \|\mathbf{s}(\mathbf{p}') - \mathbf{s}(\mathbf{p})\| < \varepsilon. \tag{2.12}$$

Clearly we have $\mathbf{s}(\mathbf{p}) \in \mathbb{B}(\bar{\mathbf{u}}, r_2)$. Therefore, the point $\mathbf{s}(\mathbf{p})$ lies in $\mathbf{S}(\mathbf{p}) \cap \mathbb{B}(\bar{\mathbf{u}}, r_2)$. It suffices to show that the latter set is singleton. Suppose that this is not the case. Find $\mathbf{u} \in \mathbb{R}^m$ such that

$$\mathbf{u} \in \mathbf{S}(\mathbf{p}) \cap \mathbb{B}(\bar{\mathbf{u}}, r_2) \quad \text{with} \quad \mathbf{u} \neq \mathbf{s}(\mathbf{p}).$$

Let $b := \|\mathbf{u} - \mathbf{s}(\mathbf{p})\|$ and $\mathbf{c} := (\mathbf{u} - \mathbf{s}(\mathbf{p}))/b$, which means that

$$(2.13) \qquad b > 0, \quad \|\mathbf{c}\| = 1, \quad \text{and} \quad \langle \mathbf{u}, \mathbf{c}\rangle = b + \langle \mathbf{s}(\mathbf{p}), \mathbf{c}\rangle.$$

Fix $\varepsilon > 0$ such that $\varepsilon < b$. There is a positive δ such that the implication in (2.12) holds. Find $\tau > 0$ that $\mathbf{p} + \tau\mathbf{c} \in \mathbb{B}(\bar{\mathbf{p}}, r_1) \cap \mathbb{B}(\mathbf{p}, \delta)$. Since $\|\mathbf{c}\| = 1$, the Cauchy-Schwartz inequality and the continuity of $\mathbf{s}$ imply that

$$(2.14) \qquad \langle \mathbf{s}(\mathbf{p} + \tau\mathbf{c}) - \mathbf{s}(\mathbf{p}), \mathbf{c}\rangle \;\le\; \|\mathbf{s}(\mathbf{p} + \tau\mathbf{c}) - \mathbf{s}(\mathbf{p})\| \, \|\mathbf{c}\| \;<\; \varepsilon.$$

Since $(\mathbf{p} + \tau\mathbf{c}, \mathbf{s}(\mathbf{p} + \tau\mathbf{c}))$ and $(\mathbf{p}, \mathbf{u})$ are in $\operatorname{gph} \mathbf{S} \cap W$, (2.11) reveals that

$$(2.15) \qquad 0 \;\le\; \langle \mathbf{s}(\mathbf{p} + \tau\mathbf{c}) - \mathbf{u}, \mathbf{p} + \tau\mathbf{c} - \mathbf{p}\rangle \;=\; \tau\langle \mathbf{s}(\mathbf{p} + \tau\mathbf{c}) - \mathbf{u}, \mathbf{c}\rangle.$$

Now, we may estimate

$$b + \langle \mathbf{s}(\mathbf{p}), \mathbf{c}\rangle \overset{(2.13)}{=} \langle \mathbf{u}, \mathbf{c}\rangle \overset{(2.15)}{\le} \langle \mathbf{s}(\mathbf{p} + \tau\mathbf{c}), \mathbf{c}\rangle \overset{(2.14)}{<} \langle \mathbf{s}(\mathbf{p}), \mathbf{c}\rangle + \varepsilon < \langle \mathbf{s}(\mathbf{p}), \mathbf{c}\rangle + b.$$

We arrived at a contradiction, therefore $\mathbf{S}(\mathbf{p}) \cap \mathbb{B}(\bar{\mathbf{u}}, r_2) = \{\mathbf{s}(\mathbf{p})\}$ for each $\mathbf{p} \in \mathbb{B}(\bar{\mathbf{p}}, r_1)$. The opposite direction is trivial. ■

If we want to apply the previous result to a DVI, it is necessary to have a VI in the form

$$(2.16) \qquad \mathbf{0}_{\mathbb{R}^m} \in \mathbf{g}(\mathbf{p}, \mathbf{u}) + N_K(\mathbf{u}),$$

where $\mathbf{g} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ and $K \subset \mathbb{R}^m$ is a non-empty closed convex set. By the setting $\mathbf{p} = (t, \mathbf{x})$ in the non-autonomous case of the VI or $\mathbf{p} = \mathbf{x}$ in the autonomous case and $m = n$ we get (1.5). Similarly, let $\bar{\mathbf{p}} = (a, \mathbf{x}_a)$ or $\bar{\mathbf{p}} = \mathbf{x}_a$. Suppose that the solution mapping $\mathbf{S}$ of (2.16) satisfies (2.11) around $(\bar{\mathbf{p}}, \mathbf{u}_a) \in \operatorname{gph} \mathbf{S}$ and that there is a continuous selection for $\mathbf{u}(\mathbf{p})$ of $\mathbf{S}$ around $\bar{\mathbf{p}}$ for $\mathbf{u}_a$. Then we can reduce a DVI to an ODE with a continuous right-hand side around $(\bar{\mathbf{p}}, \mathbf{u}_a)$.

The following theorem guarantees local existence of a solution of an initial value problem for an ODE, which has a continuous right-hand side (see for example [3]).

Theorem 2.2.7 (Cauchy-Peano Theorem) *Consider the initial value problem* (2.3)–(2.4). *Assume* $\mathbf{h}(\cdot, \cdot) \in C(\Omega, \mathbb{R}^n)$, *where*

$$\Omega := \{(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n : |t - a| \le \alpha, \|\mathbf{x} - \mathbf{x}_a\| \le \beta\}$$

for some $\alpha > 0$ *and* $\beta > 0$. *Then the problem* (2.3)-(2.4) *has a solution on* $[a - \gamma, a + \gamma]$ *with* $\|\mathbf{x}(t) - \mathbf{x}_a\| \le b$, *for* $t \in [a - \gamma, a + \gamma]$, *where*

$$\gamma := \min\left\{\alpha, \frac{\beta}{M}\right\}$$

and

$$M := \max_{(t,\mathbf{x})\in\Omega} \|\mathbf{h}(t,\mathbf{x})\|.$$

Proof. See [3, Theorem 8.13, p. 358]. ■

We can apply Theorem 2.2.5 to a parametric VI, which is a part of a DVI. This theorem guarantees, that a VI has a locally unique solution, which is locally Lipschitz continuous. If an ODE that appears in a DVI, has a (locally) Lipschitz continuous right-hand side, then the composition of the right-hand side and the solution of the VI is a locally Lipschitz continuous function. Therefore we can reduce the DVI into an ODE with a locally Lipschitz continuous right-hand side. The following well-know theorem guarantees, that an initial value problem for such an ODE has a locally unique solution.

Theorem 2.2.8 (Picard-Lindelöf Theorem) *Consider the initial value problem* (2.3)–(2.4). *Let the assumptions of Theorem 2.2.7 be satisfied and suppose that for some* $\kappa > 0$ *we have*

$$\|\mathbf{h}(t,\mathbf{x}) - \mathbf{h}(t,\mathbf{y})\| \leq \kappa\|\mathbf{x} - \mathbf{y}\| \text{ whenver } \mathbf{x},\mathbf{y} \in \mathbb{B}[\mathbf{x}_a, \beta] \text{ and } t \in [a, a+\alpha].$$

Then there is $\gamma \in (0, \alpha]$ *such that the initial value problem* (2.3)–(2.4) *has a unique solution* $\mathbf{x}(\cdot)$ *on* $[a, a+\gamma]$. *Furthermore,*

$$\mathbf{x}(t) \in \mathbb{B}[\mathbf{x}_a, \beta] \quad \text{whenever} \quad t \in [a, a+\gamma].$$

Proof. See [3, Theorem 8.27, p. 350]. ■

Finally, we present the existence theorem for DMVIs, which was stated in [12].

Theorem 2.2.9 *Let* $K \subset \mathbb{R}^m$ *be a non-empty compact convex set. Let* $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \mathbf{h} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ *and* $\mathbf{B} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ *be Lipschitz continuous and* $\mathbf{B}$ *be a bounded matrix-valued function on* $[a, b] \times \mathbb{R}^n$, *i.e.*

$$\sup_{(t,\mathbf{x})\in[a,b]\times\mathbb{R}^n} \|\mathbf{B}(t,\mathbf{x})\| < +\infty,$$

Let $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^m$ *be a monotone, continuous vector function and* $\varphi : \mathbb{R}^m \to (-\infty, +\infty]$ *be a proper lower semi-continuous convex function. Then an initial value problem for* (2.2) *with* $\mathbf{x}(a) = \mathbf{x}_a \in \mathbb{R}^n$ *has a solution on* $[a, b]$.

Proof. See [12]. ■

If $\varphi(\mathbf{u}) \equiv 0$, a DMVI (2.2) reduces to a DVI and Theorem 2.2.9 can be applied.

2.3 Reformulation of DVIs as DIs

Second approach to obtain existence of a solution of a DVI with IC is to formulate it as an initial value problem for DIs. Given an open subset $D \subset \mathbb{R}^{n+1}$ containing a point $(a, \mathbf{x}_a)$ we are looking for the solution of the DI on $[a, a+\alpha]$, for some $\alpha > 0$, with

$$\mathbf{F}(t, \mathbf{x}) := \{\mathbf{f}(t, \mathbf{x}, \mathbf{u}) : \mathbf{u} \in \mathrm{SOL}(K, \mathbf{g}(t, \mathbf{x}, \cdot))\}$$

such that $\mathbf{x}(a) = \mathbf{x}_a$. It reads as

$$\begin{cases} \dot{\mathbf{x}}(t) \in \mathbf{F}(t, \mathbf{x}(t)) \text{ for almost all } t \in (a, a+\alpha), \\ \mathbf{x}(a) = \mathbf{x}_a. \end{cases} \tag{2.17}$$

If the set-valued function $\mathbf{F}$ has appropriate properties, then we can apply the following existence theorem by A. F. Filippov from [4].

Theorem 2.3.1 (Local existence) *Let Ω be an open convex subset of $\mathbb{R}^{n+1}$ which contains a point $(a, \mathbf{x}_a) \in \mathbb{R}^{n+1}$. Suppose that $\mathbf{F} : \mathbb{R}^{n+1} \rightrightarrows \mathbb{R}^n$ satisfies for each $(t, \mathbf{x}) \in \Omega$ the following conditions:*

(i) the set $\mathbf{F}(t, \mathbf{x})$ is non-empty, bounded, closed and convex,

(ii) $\mathbf{F}$ is Pompeiu-Hausdorff upper/outer semi-continuous *at $(t, \mathbf{x})$, i.e. for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\|(s, \mathbf{y}) - (t, \mathbf{x})\| < \delta \quad \textit{implies that} \quad \mathbf{F}(s, \mathbf{y}) \subset \mathbf{F}(t, \mathbf{x}) + \mathbb{B}(\mathbf{0}_{\mathbb{R}^n}, \varepsilon).$$

Then the differential inclusion (2.17) *has a solution.*

If, in addition, $\mathbf{F}$ satisfies one-sided Lipschitz condition *in Ω, i.e. there is a non-negative Lebesgue integrable[3] function l such that*

$$\langle \mathbf{u} - \mathbf{v}, \mathbf{x} - \mathbf{y} \rangle \;\leq\; l(t)\|\mathbf{x} - \mathbf{y}\|^2 \tag{2.18}$$

whenever $(t, \mathbf{x})$, $(t, \mathbf{y}) \in \Omega$, $\mathbf{u} \in \mathbf{F}(t, \mathbf{x})$, $\mathbf{v} \in \mathbf{F}(t, \mathbf{y})$, then the solution is unique.

Proof. See [5, Theorem 3.1.1, p. 42]. ■

Let us mention also global version.

Theorem 2.3.2 (Global existence) *Given $(a, \mathbf{x}_a) \in \mathbb{R} \times \mathbb{R}^n$ and $b > a$, let $\Omega := [a, b] \times \mathbb{R}^n$. Suppose that $\mathbf{F} : \Omega \rightrightarrows \mathbb{R}^n$ satisfies the following conditions*

[3] see [13]

(i) $\mathbf{F}(t,\mathbf{x})$ *is non-empty, closed and convex for each* $(t,\mathbf{x}) \in \Omega$*,*

(ii) $\mathbf{F}$ *is Pompeiu-Hausdorff upper/outer semi-continuous at each* $(t,\mathbf{x}) \in \Omega$*,*

(iii) $\mathbf{F}$ *is a linearly bounded on* Ω*, i.e. there exists* $\alpha > 0$ *such that*

$$\|\mathbf{z}\| \le \alpha(\|\mathbf{x}\| + 1) \quad \textit{whenever} \quad \mathbf{z} \in \mathbf{F}(t,\mathbf{x}) \ \textit{and}\ (t,\mathbf{x}) \in \Omega.$$

Then the differential inclusion (2.17) *has a solution on* $[a,b]$*.*

If $\mathbf{F}$ *satisfies the one-sided Lipschitz condition* (2.18) *in* Ω*, then the solution is unique.*

Now, consider an IVP for an autonomous DI in the form

$$\begin{cases} \dot{\mathbf{x}}(t) \in -\mathbf{F}(\mathbf{x}(t)), \\ \mathbf{x}(a) = \mathbf{x}_a. \end{cases}, \tag{2.19}$$

where $\mathbf{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $\mathbf{x}_a \in \mathbb{R}^n$.

Theorem 2.3.3 (Unique existence) *Suppose that* $\mathbf{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *is a maximal monotone set-valued function, which is a linearly bounded, i.e. there exists* $\alpha > 0$ *such that*

$$\|\mathbf{z}\| \le \alpha(\|\mathbf{x}\| + 1) \quad \textit{whenever} \quad \mathbf{z} \in \mathbf{F}(\mathbf{x}) \ \textit{and}\ \mathbf{x} \in \mathbb{R}^n.$$

Then the differential inclusion (2.19) *has a unique solution on* $[a,+\infty)$ *for arbitrary* $\mathbf{x}_a \in \mathbb{R}^n$*.*

Proof. See [8]. ■

Consider a DVI in the form

$$\begin{aligned} &\dot{\mathbf{x}}(t) = \mathbf{f}(t,\mathbf{x}(t)) + \mathbf{B}(t,\mathbf{x}(t))\mathbf{u}(t), \\ &0 \le \langle \mathbf{h}(t,\mathbf{x}(t)) + \mathbf{g}(\mathbf{u}(t)), \mathbf{v} - \mathbf{u}(t)\rangle \quad \text{whenever} \quad \mathbf{v} \in K, \\ &\mathbf{u}(t) \in K, \end{aligned} \tag{2.20}$$

where K is a non-empty closed convex subset of $\mathbb{R}^m, \mathbf{f} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \mathbf{B} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}, \mathbf{h} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^m$.

Presented existence theorems can be combined with the following proposition.

Proposition 2.3.1 ([6]) *Suppose that*

1. $\mathbf{g}$ *is continuous,*

2. $\mathbf{f}, \mathbf{B}$ *and* $\mathbf{h}$ *are locally Lipschitz continuous on* $[a,b] \times \mathbb{R}^n$*,*

3. $\mathbf{B}$ *is a bounded matrix-valued function on* $[a,b] \times \mathbb{R}^n$*, i.e. for some matrix norm* $\|\cdot\|$ *it holds*

$$\sup_{(t,\mathbf{x}) \in [a,b] \times \mathbb{R}^n} \|\mathbf{B}(t,\mathbf{x})\| < +\infty.$$

Moreover, suppose that there exists a constant $\rho > 0$ *such that*

$$\sup\{\|\mathbf{u}\| : \mathbf{u} \in SOL(K, \mathbf{q} + \mathbf{g})\} \leq \rho(1 + \|\mathbf{q}\|) \tag{2.21}$$

for all $\mathbf{q} \in \mathbf{h}([a,b] \times \mathbb{R}^n)$ *and that an initial value problem for the DI in the form*

$$\begin{cases} \dot{\mathbf{x}} \in \mathbf{F}(t,\mathbf{x}) := \{\mathbf{f}(t,\mathbf{x}) + \mathbf{B}(t,\mathbf{x})\mathbf{u} : \mathbf{u} \in SOL(K, \mathbf{h}(t,\mathbf{x}) + \mathbf{g})\}, \\ \mathbf{x}(a) = \mathbf{x}_a \in \mathbb{R}^n, \end{cases}$$

has a solution on $[a,b]$*. Then* (2.20) *has a solution on* $[a,b]$ *with* $\mathbf{x}(a) = \mathbf{x}_a$*.*

Proof. See [6, Proposition 6.1, p. 375]. ■

Let us mention [6, Proposition 6.2] which says that if $\mathbf{g}$ is monotone and there exists $\bar{\mathbf{u}} \in K$ such that

$$\liminf_{\mathbf{u} \in K, \|\mathbf{u}\| \to +\infty} \frac{(\mathbf{u} - \bar{\mathbf{u}})^T \mathbf{g}(\mathbf{u})}{\|\mathbf{u}\|} > 0,$$

then $\mathrm{SOL}(K, \mathbf{q} + \mathbf{g})$ is non-empty, closed and convex and (2.21) holds for for all $\mathbf{q} \in \mathbf{h}([a,b] \times \mathbb{R}^n)$.

Example 2.3.1 *Consider a DVI in the form*

$$\begin{aligned} &\dot{x}(t) = \sin(t) + 2 - v(t), x(a) = x_a \in \mathbb{R}, \\ &0 \leq x(t) \perp 2 - v(t) \geq 0, \\ &0 \leq x(t) \perp v(t) \geq 0. \end{aligned} \tag{2.22}$$

Clearly, the solution mapping of the VI has the form

$$SOL(\mathbb{R}_+, x) = \begin{cases} \{2\}, & \text{for} \quad x > 0, \\ [0,2], & \text{for} \quad x = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

We reduce the DVI into the DI in the form

$$\dot{x}(t) \in \sin(t) + 2 - SOL(\mathbb{R}_+, x) =: F(t,x).$$

Then

$$F(t,x) = \begin{cases} \{\sin t\}, & \text{for} \quad x > 0, \\ [\sin t, \sin t + 2] & \text{for} \quad x = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Apparently, the right hand side satisfies assumptions (i) and (iii) in Theorem 2.3.2 with $\alpha = 3$ *on* $\mathbb{R} \times \mathbb{R}^n$.

Because $\operatorname{gph} F$ *is a closed set and* $F(t,x)$ *is a bounded set for each* $(t,x) \in \mathbb{R} \times \mathbb{R}_+$, *then by Lemma A.2.1 the right-hand side* F *is Pompeiu-Hausdorff upper/outer semi-continuous at each* $(t,x) \in \mathbb{R} \times \mathbb{R}_+$, *therefore (ii) is satisfied too. By Theorem 2.3.2 an IVP for the DI has a solution. By setting* $K = \mathbb{R}^2_+, \mathbf{u} = (2-v, v)^T, f(t,x) = \sin t, \mathbf{h}(t,x) = (x,x)^T, \mathbf{g}(u) = (0,0)^T$ *and the element* $b_{1,1}$ *of the matrix* $\mathbf{B}(t,x) \in \mathbb{R}^{2\times 2}$ *is equal to one and other elements are zeros, then we get* (2.20). *It is easy to see, that* $\mathbf{g}$ *is continuous, that functions* $f, \mathbf{B}, \mathbf{h}$ *are Lipschitz continuous,* $\mathbf{B}(t,x)$ *is the bounded matrix-valued function for any* $(t,x) \in \mathbb{R} \times \mathbb{R}$ *and*

$$\sup\{|u| : u \in SOL(\mathbb{R}_+, q)\} \le 2 \le 2(|q| + 1) \quad \text{whenever} \quad q \in \mathbb{R}.$$

Therefore by Proposition 2.3.1 the IVP (2.22) *has a solution.*

Chapter 3

Numerical methods

"One should not do math with numbers."

Jakub Janoušek
during ODEs class

In this chapter we will present numerical methods for solving GEs and DVIs. If we will looking for a numerical solution on the fixed interval $[a, b]$ with $a < b$, we are consider the uniform grid with $N + 1$ points such that

$$a = t_0 < t_1 < t_2 < ...t_{N-1} < t_N = b,$$

where

$$t_i = a + hi \quad \text{with} \quad h := \frac{b-a}{N}, \quad i = 0, 1, 2, ..., N.$$

3.1 Euler-Newton path-following method for GEs

In this section, we will present the Euler-Newton path-following method from [18] and its modification. We apply these methods on a parametric GE in the form

$$\mathbf{p}(t) \in \mathbf{g}(\mathbf{u}(t)) + \mathbf{F}(\mathbf{u}(t)) \quad \text{for} \quad t \in [a, b], \tag{3.1}$$

where $\mathbf{p} : \mathbb{R} \to \mathbb{R}^m$ is Lipschitz continuous function, $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^m$ is differentiable with locally Lipschitz continuous derivative and $\mathbf{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ has closed graph. Denote $\bar{\mathbf{u}}(\cdot)$ an exact solution of the previous problem.

We begin with the original method. In addition, it is requested that $\mathbf{p}(\cdot) \in C^1([a, b], \mathbb{R}^m)$ and $\mathbf{g}(\cdot) \in C^2(\mathbb{R}^m, \mathbb{R}^m)$. The predictor and corrector steps of the method consist of solving two generalized equations with linear single-valued part:

$$(3.2)\begin{cases} \mathbf{g}(\mathbf{u}_i) - \mathbf{p}(t_i) - h\mathbf{p}'(t_i) + \nabla\mathbf{g}(\mathbf{u}_i)(\mathbf{v}_{i+1} - \mathbf{u}_i) + \mathbf{F}(\mathbf{v}_{i+1}) \ni \mathbf{0}_{\mathbb{R}^m}, \\ \mathbf{g}(\mathbf{v}_{i+1}) - \mathbf{p}(t_{i+1}) + \nabla\mathbf{g}(\mathbf{v}_{i+1})(\mathbf{u}_{i+1} - \mathbf{v}_{i+1}) + \mathbf{F}(\mathbf{u}_{i+1}) \ni \mathbf{0}_{\mathbb{R}^m}, \end{cases}$$

where $\mathbf{u}_0$ is an exact solution of (3.1) at time a. In the general case, we cannot expect the exact solution $\bar{\mathbf{u}}(\cdot)$ to be smooth but it is only Lipschitz continuous. Therefore a piecewise linear interpolation will have error of order $O(h)$ in the uniform norm over the interval $[a,b]$. On the other hand, the following theorem guarantees that the grid error has order $O(h^4)$.

Theorem 3.1.1 ([18]) *Let $\bar{\mathbf{u}}(\cdot)$ be a Lipschitz continuous solution of the problem (3.1) with a continuously differentiable $\mathbf{p} : [a,b] \to \mathbb{R}^m$, a twice continuously differentiable $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^m$, and $\mathbf{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ having closed graph. Suppose that for each $t \in [a,b]$ the mapping*

$$\mathbb{R}^m \ni \mathbf{v} \mapsto \mathbf{H}_t(\mathbf{v}) := \mathbf{g}(\bar{\mathbf{u}}(t)) - \mathbf{p}(t) + \nabla\mathbf{g}(\bar{\mathbf{u}}(t))(\mathbf{v} - \bar{\mathbf{u}}(t)) + \mathbf{F}(\mathbf{v}) \subset \mathbb{R}^m$$

is strongly metrically regular at $\bar{\mathbf{u}}(t)$ for $\mathbf{0}_{\mathbb{R}^m}$. Let $\mathbf{u}_0 = \bar{\mathbf{u}}(a)$. Then there exist positive constants c and β and $N_0 \in \mathbb{N}$ such that for any natural $N \geq N_0$ the iteration (3.2) generates unique $(\mathbf{u}_i)_{i=1}^N$ starting from $\mathbf{u}_0$ and such that $\mathbf{u}_i \in \mathbb{B}(\bar{\mathbf{u}}(t_i), \beta)$ for $i = 0, 1, ..., N$. Moreover, we have

$$(3.3) \qquad \max_{0\leq i\leq N} \|\mathbf{u}_i - \bar{\mathbf{u}}(t_i)\| \leq ch^4.$$

We are going present our modification of (3.2). In general case, the exact solution at time a of (3.1) is difficult to get, especially if there is not the only one. Hence it is better idea to use a point near the exact solution instead, such that for given $\Delta > 0$, $\mathbf{u}_0 \in \mathbb{B}[\bar{\mathbf{u}}(a), \Delta h^4]$. This approximate solution can be obtained by using another numerical method, for example see Section 3.2.

Further, in the first equation in (3.2) we replace derivative of $\mathbf{p}(\cdot)$ by a backward finite-difference quotient. Thus we get rid of the assumptions that the function $\mathbf{p}(\cdot)$ is continuously differentiable. Moreover we admit that the first equation can be solved inaccurately. Therefore we can add to the right-hand side elements of a sequence which are close to zero, such that $\mathbf{e}_i \in \mathbb{B}[\mathbf{0}_{\mathbb{R}^m}, \Delta h^2]$. After these changes, we get

$$(3.4)\begin{cases} \mathbf{g}(\mathbf{u}_i) - \mathbf{p}(t_{i+1}) + \nabla\mathbf{g}(\mathbf{u}_i)(\mathbf{v}_{i+1} - \mathbf{u}_i) + \mathbf{F}(\mathbf{v}_{i+1}) \ni \mathbf{e}_i, \\ \mathbf{g}(\mathbf{v}_{i+1}) - \mathbf{p}(t_{i+1}) + \nabla\mathbf{g}(\mathbf{v}_{i+1})(\mathbf{u}_{i+1} - \mathbf{v}_{i+1}) + \mathbf{F}(\mathbf{u}_{i+1}) \ni \mathbf{0}_{\mathbb{R}^m}. \end{cases}$$

It turns out that (3.2) and (3.4) have same order of the grid error.

Theorem 3.1.2 *Let $\bar{\mathbf{u}}(\cdot)$ be a Lipschitz continuous solution of the problem* (3.1) *with a Lipschitz continuous $\mathbf{p} : [a,b] \to \mathbb{R}^m$, a differentiable $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^m$ such that $\nabla \mathbf{g}(\cdot)$ is locally Lipchitz continuous at each point of $\mathbb{R}^m$, and $\mathbf{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ having closed graph. Suppose that for each $t \in [a,b]$ the mapping*

$$\mathbb{R}^m \ni \mathbf{v} \mapsto \mathbf{H}_t(\mathbf{v}) := \mathbf{g}(\bar{\mathbf{u}}(t)) - \mathbf{p}(t) + \nabla \mathbf{g}(\bar{\mathbf{u}}(t))(\mathbf{v} - \bar{\mathbf{u}}(t)) + \mathbf{F}(\mathbf{v}) \subset \mathbb{R}^m$$

is strongly metrically regular at $\bar{\mathbf{u}}(t)$ for $\mathbf{0}_{\mathbb{R}^m}$. Then for any $\Delta > 0$ there are $N_0 \in \mathbb{N}$, $\alpha > 0$, and $c > 0$ such that for each $N > N_0$ and each $\mathbf{u}_0 \in \mathbb{B}[\bar{\mathbf{u}}(a), \Delta h^4]$, the iteration (3.4)*, with the initial point $\mathbf{u}_0$, generates unique $(\mathbf{u}_i)_{i=1}^N$ verifying* (3.3) *such that $\mathbf{u}_i \in \mathbb{B}[\bar{\mathbf{u}}(t_i), \alpha]$ for each $i \in \{1, \dots, N\}$.*

The great advantage of (3.4) is its usability if the function $\mathbf{p}(\cdot)$ is not exactly known but $\mathbf{p}(t_i)$ are discrete values of some measurement.

3.2 Solving GEs by minimalization

In this section, we present one idea how to solve GEs by using minimalization. Let $g : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function and $h : \mathbb{R}^m \to \mathbb{R}$ be a locally Lipschitz function. It is known that if $\mathbf{u}$ is a minimum of $g + h$ then

$$\mathbf{0}_{\mathbb{R}^m} \in \partial_C(g(\mathbf{u}) + h(\mathbf{u})),$$

and by Proposition A.2.2 we have the GE in the form

$$\mathbf{0}_{\mathbb{R}^m} \in \nabla g(\mathbf{u}) + \partial_C h(\mathbf{u}).$$

So if we want to solve the GE in the previous form, we can reformulate it to finding minima of convex function $g + h$. To this problem can be applied some numerical methods for non-smooth minimization (see for example [22]).

Example 3.2.1 *Consider GE in the form*

$$\mathbf{0}_{\mathbb{R}^m} \in \mathbf{A}\mathbf{u} + \mathbf{b} + \mathbf{F}(\mathbf{u}),$$

where $\mathbf{A} \in \mathbb{R}^{m \times m}$ is symmetric positive definite matrix, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{F} = \Pi_{i=1}^m \operatorname{Sgn}_{V_1^i, V_2^i}(u_i)$, where $V_1^i < V_2^i$ for each $i \in \{1, 2, ..., m\}$. This GE can be reformulated as finding minima of the function

$$\frac{1}{2}\mathbf{u}^T \mathbf{A}\mathbf{u} + \mathbf{u}^T \mathbf{b} + \sum_{i=1}^m f_i(u_i)$$

where

$$f_i(u_i) = \begin{cases} V_1^i u_i & \textit{for} \quad u_i < 0, \\ V_2^i u_i & \textit{for} \quad u_i \geq 0. \end{cases}$$

3.3 Time-stepping schemes for DVIs

In this section, we present time-stepping methods for solving initial value problems for the DVI on the interval $[a, b]$ with $\mathbf{x}(a) = \widetilde{\mathbf{x}}_a \in \mathbb{R}^n$. We replace the time derivative $\dot{\mathbf{x}}$ by a backward finite-difference quotient such that

$$\dot{\mathbf{x}}(t_{i+1}) \approx \frac{\mathbf{x}(t_{i+1}) - \mathbf{x}(t_i)}{h} \quad \text{for} \quad i = 0, 1, 2, ..., N.$$

In addition we denote $\mathbf{u}_i = \mathbf{u}(t_i)$ and $\mathbf{x}_i = \mathbf{x}(t_i)$ with $\mathbf{x}_0 = \widetilde{\mathbf{x}}_a$. We want to compute two finite sets of vectors

$$\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N\} \subset \mathbb{R}^n \quad \text{and} \quad \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_N\} \subset \mathbb{R}^m.$$

For a number $\theta \in [0, 1]$, we distinguish an explicit ($\theta = 1$), an implicit ($\theta = 0$), or a semi-implicit ($\theta \in (0, 1)$) discretization of an ODE.

In [6], authors introduced two Moreau's time-stepping schemes for a general DVI.

The first variant has the form

$$\begin{aligned} \mathbf{x}_{i+1} &= \mathbf{x}_i + h\mathbf{f}(t_{i+1}, \theta\mathbf{x}_i + (1-\theta)\mathbf{x}_{i+1}, \mathbf{u}_{i+1}), \\ \mathbf{u}_{i+1} &\in \text{SOL}(K, \mathbf{g}(t_{i+1}, \mathbf{x}_{i+1}, \cdot)), \end{aligned} \tag{3.5}$$

in which the VI is satisfied exactly by each iterate $(\mathbf{x}_{i+1}, \mathbf{u}_{i+1})$ at time t_{i+1}.

The second variant has the form

$$\begin{aligned} \mathbf{x}_{i+1} &= \mathbf{x}_i + h\mathbf{f}(t_{i+1}, \theta\mathbf{x}_i + (1-\theta)\mathbf{x}_{i+1}, \mathbf{u}_{i+1}), \\ \mathbf{u}_{i+1} &\in \text{SOL}(K, \mathbf{g}(t_{i+1}, \mathbf{x}_i, \cdot)), \end{aligned}$$

which allows to solve the VI first and then plug it into the first equation.

Clearly, the scheme (3.5) can be written as VI $(\mathbb{R}^n \times K, \mathbf{H}_{i+1})$, where

$$\mathbf{H}_{i+1}(\mathbf{x}, \mathbf{u}) = \begin{pmatrix} \mathbf{x} - \mathbf{x}_i - h\mathbf{f}(t_{i+1}, \theta\mathbf{x}_i + (1-\theta)\mathbf{x}, \mathbf{u}) \\ \mathbf{g}(t_{i+1}, \mathbf{x}, \mathbf{u}) \end{pmatrix} \quad \text{for} \quad (\mathbf{x}, \mathbf{u}) \in \mathbb{R}^n \times K.$$

The scheme (3.5) for the DLCP reads as

$$\begin{aligned} \frac{\mathbf{x}_{i+1} - \mathbf{x}_i}{h} &= \mathbf{A}((1-\theta)\mathbf{x}_{i+1} + \theta\mathbf{x}_i) + \mathbf{B}\mathbf{u}_{i+1} + \mathbf{p}, \\ \mathbf{0}_{\mathbb{R}^m} &\preceq \mathbf{y}_{i+1} \perp \mathbf{u}_{i+1} \succeq \mathbf{0}_{\mathbb{R}^m}, \\ \mathbf{y}_{i+1} &= \mathbf{C}\mathbf{x}_{i+1} + \mathbf{D}\mathbf{u}_{i+1} + \mathbf{q}, \end{aligned} \tag{3.6}$$

For h small enough, the matrix $\mathbf{I}_n - h(1-\theta)\mathbf{A}$ is non-singular and denote

$$\mathbf{W} := (\mathbf{I}_n - h(1-\theta)\mathbf{A})^{-1},$$

one sees that the next step $\mathbf{u}_{i+1}$ solves

$$\mathbf{0}_{\mathbb{R}^m} \preceq \mathbf{u}_{i+1} \perp \underbrace{\mathbf{CW}\Big(\mathbf{I}_n + h\theta\mathbf{A}\Big)\mathbf{x}_i + h\mathbf{CWp} + \mathbf{q}}_{\mathbf{d}_i} + \underbrace{(h\mathbf{CWB} + \mathbf{D})}_{\mathbf{M}}\mathbf{u}_{i+1} \succeq \mathbf{0}_{\mathbb{R}^m}.$$

Having $\mathbf{u}_{i+1}$ in hand, we compute the new state $\mathbf{x}_{i+1}$ by

$$\mathbf{x}_{i+1} = \mathbf{W}\big((\mathbf{I}_n + h\theta\mathbf{A})\mathbf{x}_i + h(\mathbf{B}\mathbf{u}_{i+1} + \mathbf{p})\big).$$

The above discussion reveals that, at each step, one has for given $\mathbf{d} \in \mathbb{R}^m$ find a solution $\mathbf{u} \in \mathbb{R}^m$ of

$$\mathbf{0}_{\mathbb{R}^m} \preceq \mathbf{u} \perp \mathbf{d} + \mathbf{M}\mathbf{u} \succeq \mathbf{0}_{\mathbb{R}^m}, \tag{3.7}$$

where the matrix $\mathbf{M} \in \mathbb{R}^{m\times m}$ depends on a particular choice of the scheme. If $\mathbf{M}$ is a P-matrix then (3.7) always has a unique solution. Further, we focus on the numerical solving of (2.20). This problem has important applications in mechanics, for example in case of unilateral constraints and Coulomb friction. The corresponding discretization scheme introduced in [6] has the form

$$\begin{aligned} &\mathbf{x}_{i+1} = \mathbf{x}_i + h\mathbf{f}(t_{i+1}, \theta\mathbf{x}_i + (1-\theta)\mathbf{x}_{i+1}) + \mathbf{B}(t_i, \mathbf{x}_i)\mathbf{u}_{i+1}, \\ &\mathbf{u}_{i+1} \in \mathrm{SOL}(K, \mathbf{h}(t_{i+1}, \mathbf{x}_{i+1}) + \mathbf{g}). \end{aligned} \tag{3.8}$$

Now we present sufficient conditions for the convergence.

Let $\Psi : \mathbb{R}^l \to \mathbb{R}^l$ be a Lipschitz continuous function, $\mathbf{E} \in \mathbb{R}^{l\times m}$ be a matrix, $\ker\mathbf{E}$ be the null space of $\mathbf{E}$ with dimension k and $(\ker\mathbf{E})^\perp$ be its orthogonal complement with dimension $m-k$. Further K_1 and K_2 be the orthogonal projection of the set K onto $\ker\mathbf{E}$ and $(\ker\mathbf{E})^\perp$, respectively. Let columns of a matrix $\mathbf{W} \in \mathbb{R}^{m\times(m-k)}$ be elements of orthonormal basis of $(\ker\mathbf{E})^\perp$ and $\Upsilon := (\mathbf{EW})^T \circ \Psi \circ \mathbf{EW}$. In addition, we assume:

(A) $\mathbf{f}, \mathbf{B}$, and $\mathbf{g}$ are Lipschitz continuous functions on $[a,b]\times\mathbb{R}^n$,

(B) $\mathbf{B}$ is bounded on $[a,b]\times\mathbb{R}^n$, that is

$$\sup_{(t,\mathbf{x})\in[a,b]\times\mathbb{R}^n} \|\mathbf{B}(t,\mathbf{x})\| < +\infty,$$

(C) there is $\eta > 0$ such that

$$(\mathbf{u}_1 - \mathbf{u}_2)^T(\mathbf{h}(t_1, \mathbf{r} + \mathbf{B}(t_2, \mathbf{x})\mathbf{u}_1) - \mathbf{h}(t_1, \mathbf{r} + \mathbf{B}(t_2, \mathbf{x})\mathbf{u}_2)) \geq \eta\|\mathbf{u}_1 - \mathbf{u}_2\|^2$$

for all $\mathbf{x}, \mathbf{r} \in \mathbb{R}^n, \mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^m$ and $t_1, t_2 \in [a, b]$,

(D) $\mathbf{g} := \mathbf{E}^T \circ \Psi \circ \mathbf{E}$,

(E) $K_1 \oplus K_2 = K$,

(F) there is $\eta' > 0$ such that

$$(\Upsilon(\lambda) - \Upsilon(\lambda'))^T(\lambda - \lambda') \geq \eta'\|\lambda - \lambda'\|^2 \quad \text{for each} \quad \lambda, \lambda' \in \mathbb{R}^{m-k}.$$

The following theorem guarantees convergence of the scheme under previous conditions.

Theorem 3.3.1 *Let K be a closed convex cone in $\mathbb{R}^m$ and $\theta \in [0, 1]$ be a given scalar. Let (A)–(F) be satisfied. Moreover assume that $\Psi(\mathbf{0}_{\mathbb{R}^l}) = \mathbf{0}_{\mathbb{R}^l}$. A positive scalar $\bar{h} > 0$ exist such that for all $\mathbf{x}_a \in \mathbb{R}^n$ for which $SOL(K, \mathbf{h}(a, \mathbf{x}_a) + \mathbf{g}) \neq \emptyset$ and for all $h \in (0, \bar{h}]$, a unique pair $(\mathbf{x}_{i+1}, \mathbf{u}_{i+1})$ exist satisfying (3.8) for all $i = 0, 1, ..., N$.*

Proof. See [6, Theorem 8.1, p. 405] ■

Chapter 4

Models

> "Are you familiar with old robot saying, "Does not compute"?"
>
> *Bender Bending Rodríguez*
> *series Futurama*

In this chapter, we present some problems that appear in mechanics, in electrical circuit theory and in economics. We use DVIs and GEs to describe them.

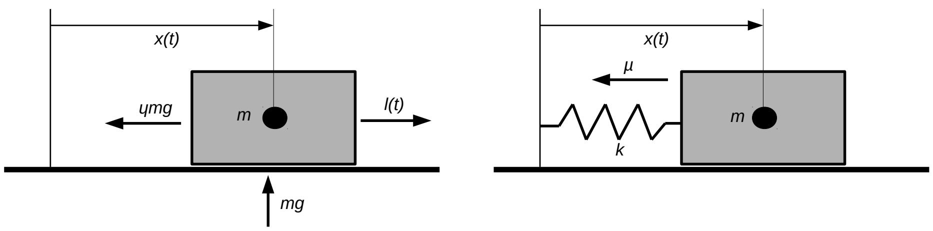
We begin with contact dynamics involving Coulomb friction, an impact of a rigid body and an impact of a non-rigid body. Further, we focus on the behavior of electrical circuits containing non-smooth elements such a diodes. At the end, we present one economic model describing a market equilibrium of pure exchange economy which depends on time.

We illustrate how to verify the existence of a solution by utilizing theory from the previous chapters. In all examples we provide numerical simulations. We also derive formulas for the exact solution to some of the above mentioned problems with initial value conditions.

4.1 Mechanical models

We present a DVI formulation of some problems arising in contact mechanics. A contact of a body with a rigid or a non-rigid surface involving Coulomb friction, an impact or an interpenetration of the surface is considered.

We denote by $m > 0$ a mass of a moving body, by g a gravitational constant, by $\mu \geq 0$ a coefficient of Coulomb friction, and by $k \geq 0$ a stiffness of a spring. For $k = 0$ there is no spring and for $\mu = 0$ there is no Coulomb friction.



(a) Example 4.1.1.

(b) Example 4.1.2.

Figure 4.1: Bodies with Coloumb friction.

Example 4.1.1 (Coulomb friction between body and surface) *Consider a rigid body having a contact with a solid surface in Figure 4.1a. The external force, given by a function* $l : \mathbb{R} \to \mathbb{R}$*, drags the body across the surface. Let* $x(t)$ *be a horizontal position of the body at time* t *and we denote* $v(t) := \dot{x}(t)$*. Then Newton's second law of motion gives us the differential inclusion in the form*

$$m\dot{v}(t) \in l(t) - \mu mg \operatorname{Sgn}_{-1,1}(v(t)). \tag{4.1}$$

Further, we can write $\operatorname{Sgn}_{-1,1}(v) = 1 - w$*, where* w *satisfies*

$$\begin{aligned} 0 &\leq w \perp v + v^- \geq 0, \\ 0 &\leq v^- \perp 2 - w \geq 0, \end{aligned}$$

then the DVI formulation of this problem has the form

$$\begin{aligned} m\dot{v}(t) &= l(t) - \mu mg(1 - w(t)), \\ 0 &\leq w(t) \perp v(t) + v^-(t) \geq 0, \\ 0 &\leq v^-(t) \perp 2 - w(t) \geq 0. \end{aligned}$$

Now, we focus on the existence of a solution. If $l(t) \equiv 0$*, then it holds*

$$\partial_C(\mu mg|\cdot|)(c) = \mu mg \operatorname{Sgn}_{-1,1}(v) := F(v),$$

hence F *is both maximal monotone*[1] *and linearly bounded, because*

$$\sup\{|z| : z \in \mu mg \operatorname{Sgn}_{-1,1}(v)\} \leq \mu mg \leq \mu mg(|v| + 1)$$

[1] See Definition A.1.3 and Proposition A.2.1.

for each $v \in \mathbb{R}$. *Therefore* (4.1) *has a unique solution on* $[0,+\infty)$ *by Theorem 2.19 for an arbitrary* $v(0) = v_0 \in \mathbb{R}$.

By setting $K = \mathbb{R}^2_+, x = v, \mathbf{u} = (w, v^-), f(t,x) = -\mu mg, B(t,x) = \mu mg, \mathbf{h}(t,x) = (v + v^-, 2)$ *and* $\mathbf{g}(u) = (0, -w)$ *we get* (2.20). *It is easy to see, that* $\mathbf{g}$ *is continuous,* $f, B, \mathbf{h}$ *are Lipschitz continuous and* B *is a bounded.*

The solution mapping of the VI has the form

$$\mathrm{SOL}(\mathbb{R}^2_+, (v + v^-, 2 - \cdot)) = \mathrm{Sgn}_{-1,1}(v),$$

and hence it holds

$$\sup\{|z| : z \in \mathrm{Sgn}_{-1,1}(q)\} \leq 1 \leq |q| + 1$$

for each $q \in \mathbb{R}$. *Therefore by Proposition 2.3.1 the DVI has a solution with* $v(0) = v_0$.

An exact solution of an IVP for this problem has the following forms:

1. *If* $f(t) \equiv 0$, *then*

$$v(t) = \begin{cases} -\operatorname{sgn}(v_0)\mu g t + v_0 & \text{for } t \in \left[0, \dfrac{|v_0|}{g\mu}\right], \\ 0 & \text{for } t > \dfrac{|v_0|}{g\mu}, \end{cases}$$

$$w(t) = \begin{cases} 1 - \operatorname{sgn}(v_0) & \text{for } t \in \left[0, \dfrac{|v_0|}{g\mu}\right], \\ 1 & \text{for } t > \dfrac{|v_0|}{g\mu}. \end{cases}$$

2. *If* $f(t) = \sin(t)$ *and* $\mu g m \geq 1$, *then*

$$v(t) = \begin{cases} -\operatorname{sgn}(v_0)\mu g t + v_0 - \dfrac{\cos(t) - 1}{m} & \text{for } t \in [0, t_0], \\ 0 & \text{for } t > t_0, \end{cases}$$

$$w(t) = \begin{cases} 1 - \operatorname{sgn}(v_0) & \text{for } t \in [0, t_0], \\ 1 & \text{for } t > t_0, \end{cases}$$

where

$$t_0 = \min\{t \geq 0 : -\operatorname{sgn}(v_0) g\mu t + v_0 - \frac{\cos t - 1}{m} = 0\}.$$

The exact solution was obtained using considerations about its physical meaning. The body moves until it stops and then stays in place if the external force on the body is smaller than or equal to the force caused by the Coulomb friction.

For a numerical implementation we use the scheme (3.8) *with a discretization step $h > 0$, therefore we have*

$$\begin{aligned} mv_{i+1} &= v_i + h\big(l_{i+1} - \mu m g(1 - w_{i+1})\big), \\ 0 &\leq w_{i+1} \perp v_{i+1} + (v_{i+1})^- \geq 0, \\ 0 &\leq (v_{i+1})^- \perp 2 - w_{i+1} \geq 0, \end{aligned}$$

where we denote $l(t_{i+1}) = l_{i+1}$. Now, we are able to get w_{i+1} independently of v_{i+1}, that is

$$w_{i+1} = \begin{cases} 2 & \text{for } v_i \leq -hl_{i+1} - ghm\mu, \\ 0 & \text{for } v_i > ghm\mu - hl_{i+1}, \\ 1 - \dfrac{hl_{i+1} + v_i}{ghm\mu} & \text{for } ghm\mu - hl_{i+1} \geq v_i > -hl_{i+1} - ghm\mu. \end{cases}$$

Graphs of solutions and the absolute errors of the numerical solutions are in Figure 4.2. Note, that the first components of solutions are non-smooth and the second ones are discontinuous.

Example 4.1.2 (Harmonic oscillator with Coulomb friction) *Consider a rigid body in the gravitation field having a contact with a solid surface in Figure 4.1b. In addition, the Coulomb friction arises between the body and the surface. The body and the zero point are connected by a solid spring. Denote $x(t)$ a horizontal position of the body at time t. Then the second Newton's law gives us*

$$m\ddot{x}(t) \in -kx(t) - \mu\,\mathrm{Sgn}_{-1,1}(\dot{x}(t)) := L(x(t), \dot{x}(t)). \tag{4.2}$$

Now we focus on the existence of a solution of the previous DI. For each $r > 0$, it holds

$$L(\mathbb{B}((x_1, x_2), r)) \subset [-k(x_1 + r) - \mu, -k(x_1 - r) + \mu],$$

for each $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$, hence L is locally bounded. Clearly, $\mathrm{gph}\, L$ *is a closed set, therefore the function L is Pompeiu-Hausdorff upper/outer semicontinuous by Theorem A.2.1. Apparently, $L(x_1, x_2)$ is a non-empty convex set for each $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$ and*

$$\sup\{|y|, y \in [-k(x_1 + r) - \mu, -k(x_1 - r) + \mu]\} \leq (kr + \mu)(\|(x_1, x_2)\| + 1),$$

for each $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$*. Then the system*

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ m\ddot{x}_2(t) &\in -kx_1(t) - \mu \operatorname{Sgn}_{-1,1}(x_2(t)), \end{aligned}$$

where $x_1(t) := x(t)$ *and* $x_2(t) := \dot{x}(t)$*, has a solution on* $[0, +\infty)$ *with any initial condition by Theorem 2.3.2.*

We introduce a new variable u*, such that*

$$\operatorname{Sgn}_{-1,1}(x_2) = 1 - u,$$

and u *satisfies*

$$\begin{aligned} 0 &\leq u \perp x_2^+ \geq 0, \\ 0 &\leq x_2^- \perp 2 - u \geq 0. \end{aligned}$$

Hence, the problem can be reformulated as the DVI in the form

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ m\dot{x}_2(t) &= -kx_1(t) - \mu(1 - u(t)), \\ 0 &\leq u(t) \perp x_2^+(t) \geq 0, \\ 0 &\leq x_2^-(t) \perp 2 - u(t) \geq 0. \end{aligned}$$

The exact solution of the initial value problem for this DVI has the following form. First, we define a function $\lfloor \cdot \rceil$ *for* $y \in \mathbb{R}$ *by*

$$\lfloor y \rceil := \begin{cases} \lfloor y \rfloor & \text{for} \quad y - \lfloor y \rfloor \in [0, 0.5], \\ \lfloor y \rfloor + 1 & \text{for} \quad y - \lfloor y \rfloor \in (0.5, 1), \\ 0 & \text{otherwise} \end{cases}$$

where $\lfloor \cdot \rfloor$ *is a floor function.*

For simplicity we put $\lambda := \dfrac{\mu}{m}$ *and* $\beta := \dfrac{k}{m}$*. If* $|x_1(0)| > \lambda\beta^{-1}$ *and* $x_2(0) = 0$*, we define the sequence*

$$t_n = \frac{n\pi}{\sqrt{\beta}}, \quad n \in \mathbb{N}$$

and the sequence of functions

$$f_n(t) := \frac{(-1)^n \lambda \operatorname{sgn}(x_1(0)) + \cos(\sqrt{\beta} t)(x_1(0)\beta - (\lambda + 2\lambda n) \operatorname{sgn}(x_1(0))}{\beta}$$

for $n \in \mathbb{N} \cup \{0\}$ *and the natural number*

$$\widetilde{n} := \left\lceil \frac{\beta |x_1(0)|}{2\lambda} \right\rceil - 1.$$

Then the solution at time t *is given by*

$$x_1(t) := \begin{cases} f_0(t) & \text{for} \quad t \in [0, t_1], \\ f_1(t) & \text{for} \quad t \in (t_1, t_2], \\ & \vdots \\ f_{\widetilde{n}}(t) & \text{for} \quad t \in (t_{\widetilde{n}-1}, t_{\widetilde{n}}], \\ f_{\widetilde{n}}(t_{\widetilde{n}}) & \text{for} \quad t > t_{\widetilde{n}}, \end{cases}$$

$$\begin{aligned} x_2(t) &:= \dot{x}_1(t), \\ u(t) &:= \frac{m\dot{x}_2(t) + \beta x_1(t)}{\lambda} + 1. \end{aligned}$$

Note that the derivative is considered only where it exists.

If $x_1(0) \in [-\lambda\beta^{-1}, \lambda\beta^{-1}]$ *and* $x_2(0) = 0$*, then the solution at time* t *has the form*

$$\begin{aligned} x_1(t) &:= x_1(0) && \text{for} \quad t \in [0, +\infty), \\ x_2(t) &:= 0 && \text{for} \quad t \in [0, +\infty), \\ u(t) &:= \frac{\beta x_1(0)}{\lambda} + 1 && \text{for} \quad t \in [0, +\infty). \end{aligned}$$

If $|x_1(0)| > \lambda\beta^{-1}$ *or* $x_2(0) \neq 0$ *(or both the variants), we define the number*

$$t_h := \frac{1}{\sqrt{\beta}} \arccos\left(\frac{x_1\beta + \lambda \operatorname{sgn}(x_2(0))}{\sqrt{\beta\big((x_2(0))^2 + (x_1(0))^2\big) + \lambda \operatorname{sgn}(x_2(0))\big(2x_1\beta + \lambda \operatorname{sgn}(x_2(0))\big)}} \right),$$

the function

$$g(t) := \frac{-\lambda \operatorname{sgn}(x_2(0)) + \cos(\sqrt{\beta}t)\big(x_1(0)\beta + \lambda \operatorname{sgn}(x_2(0))\big) + x_2(0)\sqrt{\beta}\sin(\sqrt{\beta}t)}{\beta},$$

the sequence of functions

$$g_n(t) := \frac{(-1)^n \lambda \operatorname{sgn}(g(t_h)) + \cos(\sqrt{\beta}(t - t_h))(x_1(0)\beta - (\lambda + 2\lambda n) \operatorname{sgn}(g(t_h))}{\beta}.$$

for $n \in \mathbb{N} \cup \{0\}$ *and the natural number*

$$\bar{n} := \left\lfloor \frac{\beta |g(t_h)|}{2\lambda} \right\rceil - 1.$$

Then the solution is given by

$$\begin{aligned}
x_1(t) &:= \begin{cases} g(t) & \text{for} \quad t \in [0, t_h], \\ g_0(t) & \text{for} \quad t \in [t_h, t_1 + t_h], \\ g_1(t) & \text{for} \quad t \in (t_1 + t_h, t_2 + t_h], \\ & \vdots \\ g_{\bar{n}}(t) & \text{for} \quad t \in (t_{\bar{n}-1} + t_h, t_{\bar{n}} + t_h], \\ g_{\bar{n}}(t_{\bar{n}}) & \text{for} \quad t > t_{\bar{n}} + t_h, \end{cases} \\
x_2(t) &:= \dot{x}_1(t), \\
u(t) &:= \frac{\dot{x}_2(t) + \beta x_1(t)}{\lambda} + 1.
\end{aligned}$$

Note that we consider derivative only where it exists.

For numerical implementation, we use the explicit scheme (3.5)*. Therefore we have*

$$\begin{aligned}
x_{1,i+1} &= x_{1,i} + h x_{2,i}, \\
x_{2,i+1} &= x_{2,i} - h(\beta x_{1,i} + \lambda(1 - u_{i+1})), \\
0 &\leq u_{i+1} \perp (x_{2,i+1})^+ \geq 0, \\
0 &\leq (x_{2,i+1})^- \perp 2 - u_{i+1} \geq 0.
\end{aligned}$$

We are able to get u_{i+1} *independently of* $x_{2,i+1}$ *in such a way that*

$$u_{i+1} = \begin{cases} 2 & \text{for} \quad x_{2,i} \leq -h\beta x_{1,i} - hm\lambda, \\ 0 & \text{for} \quad x_{2,i} > hm\lambda - h\beta x_{1,i}, \\ 1 - \dfrac{h\beta x_{1,i} + x_{2,i}}{hm\lambda} & \text{for} \quad hm\lambda - h\beta x_{1,i} \geq x_{2,i} > -h\beta x_{1,i} - hm\lambda. \end{cases}$$

Graphs of solution and the absolute error of the numerical solution are in Figure 4.3. Note, that the first two components of the solution are continuous and the third one is discontinuous.

Example 4.1.3 (Body with an impact to a non-rigid surface) *Consider a rigid body in the plane. The body is in the gravitational field. Let* $x(t)$ *be horizontal position of the center of mass at time* t *and* $y(t)$ *be a vertical* *position of the center of mass at time t. The body may fall to the surface, which is given by a function* $r : \mathbb{R} \to \mathbb{R}$*. Contact between the body and the surface is described by the* normal compliance*, which represents the contact by a stiff spring applying no force when there is no interpenetration. But when there is an interpenetration, the force in the spring is proportional to the depth of this interpenetration. The contact force of the surface at time t is denoted* $n(t)$ *and* $k > 0$ *is a stiffness of the surface.*

The DVI formulation of this problem has the form

$$\begin{aligned} m\ddot{x}(t) &= n(t)\sin(\alpha), \\ m\ddot{y}(t) &= -mg + n(t)\cos(\alpha), \\ 0 &\leq y(t) - r(x(t)) + \frac{n(t)}{k} \perp n(t) \geq 0, \\ \alpha &= -\arctan(\dot{r}(x(t))), \end{aligned} \tag{4.3}$$

We show, that if $r(\cdot)$ *is Lipschitz continuous on* $\mathbb{R}$ *with a constant* $L > 0$*, then the solution mapping of* (4.3) *is single-valued and Lipschitz continuous.*

Clearly, the function $y - r(x) + \frac{n}{k}$ *is continuous on* $\mathbb{R}^2 \times \mathbb{R}$*. For any* $(x_1, y_1) \in \mathbb{R}^2$ *and* $(x_2, y_2) \in \mathbb{R}^2$*, we have*

$$\begin{aligned} \|y_1 - r(x_1) + \frac{n}{k} - y_2 + r(x_2) - \frac{n}{k}\| &\leq \|y_1 - y_2\| + \|r(x_2) - r(x_1)\| \leq \\ &\leq \|y_1 - y_2\| + L\|x_2 - x_1\|, \end{aligned}$$

for each $n \in \mathbb{R}$*. Further, for any* $n_1, n_2 \in \mathbb{R}$ *we have*

$$\|y - r(x) + \frac{n_1}{k} - y + r(x) - \frac{n_2}{k}\| \leq \frac{1}{k}\|n_1 - n_2\|,$$

for each $x, y \in \mathbb{R}$*. Therefore by Theorem 2.2.1, with* $\mathbf{y} = (x, y), u = n, g(\mathbf{y}, u) = y - r(x) + \frac{n}{k}$ *and* $K = \mathbb{R}_+$*, the solution mapping of* (4.3) *is a single-valued and Lipschitz continuous. We can compute the corresponding solution as follows:*

If $y(t) - r(x(t)) < 0$ *then* $n(t) = k(r(x(t)) - y(t))$ *and if* $y(t) - r(x(t)) \geq 0$ *then* $n(t) = 0$*. Therefore the solution mapping is given by*

$$\mathrm{SOL}(\mathbb{R}_+, y(t) - r(x(t)) + (\cdot)/k) = \{k(r(x(t)) - y(t))^+\}.$$

Hence the DVI can be reduced to the system of ODEs in the form

$$\begin{aligned} m\ddot{x}(t) &= k(r(x(t)) - y(t))^+ \sin(\alpha), \\ m\ddot{y}(t) &= -mg + k(r(x(t)) - y(t))^+ \cos(\alpha), \\ \alpha &= -\arctan(\dot{r}(x(t))). \end{aligned}$$

The solvability of an IVP for this system is highly dependent on the form of the function $r(\cdot)$. If $r(\cdot)$ is continuously differentiable then the right-hand side is continuous, therefore the system has a solution by Theorem 2.2.7. In addition, if $\dot{r}(\cdot)$ is Lipschitz continuous, then the solution is unique by Theorem 2.2.8.

Graphs of the components of the solution are in Figure 4.4 and were obtained by using a function ODE45 *in MATLAB. Note that all components of the solution are smooth functions.*

Example 4.1.4 (Body with impact to a surface) *Consider the one-dimensional version of the problem in Example 4.1.3. We denote $x_1(t)$ a vertical position of the body at time t with a corresponding velocity $x_2(t)$ and r a vertical position of the surface. The contact between the body and the surface, with a stiffness $k > 0$, is described by normal compliance[2] for $k \in (0, \infty)$. The contact force of the surface at time t is denoted $n(t)$. The DVI formulation of the problem has the form*

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ m\dot{x}_2(t) &= -gm + n(t), \\ 0 &\leq x_1(t) - r + \frac{n(t)}{k} \perp n(t) \geq 0. \end{aligned}$$

Similarly as in Example 4.1.3 we can reduce this DVI to the system of ODEs in the form

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ m\dot{x}_2(t) &= -gm + k(r - x_1(t))^+. \end{aligned} \tag{4.4}$$

Graphs of components of numerical solution are in Figure 4.5. The solution was obtained by using the function ODE45 *in MATLAB.*

For $k \to +\infty$ the surface becomes solid and the model loses the normal compliance. Therefore, an impact occurs during contact between the body and the surface. This causes that the function of a velocity $x_2(\cdot)$ is discontinuous at the time of the impact and the contact force $n(\cdot)$ contains a Dirac impulse. Therefore we have

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ m\dot{x}_2(t) &= -gm + n(t), \\ 0 &\leq x_1(t) - r \perp n(t) \geq 0. \end{aligned} \tag{4.5}$$

[2] See Example 4.1.3.

It is easy to see that the solution mapping of the VI at x_1 has the form

$$\mathrm{SOL}(\mathbb{R}_+, x_1 - r) = \begin{cases} [0, +\infty) & \textit{for} \quad x_1 = r, \\ \{0\} & \textit{for} \quad x_1 > r, \\ \emptyset & \quad\textit{otherwise.} \end{cases}$$

Clearly, the condition (2.21) *cannot be satisfied, hence presented theory fails. On the other hand, we are able to find an exact solution, which is not absolutely continuous and not even continuous, in the form*

$$\begin{aligned} x_1(t) \quad &:= \quad \begin{cases} -\dfrac{gt^2}{2} + x_1(0) + tx_2(0) & \textit{for } t < t_0, \\ r & \textit{otherwise,} \end{cases} \\ x_2(t) \quad &:= \quad \begin{cases} -gt + x_2(0) & \textit{for } t < t_0, \\ 0 & \textit{otherwise,} \end{cases} \end{aligned}$$

where

$$t_0 := \min\{t \geq 0 : -g/2t^2 + x_1(0) + tx_2(0) = r\},$$

and by using the weak derivative, we have

$$n(t) := -gm\theta(t_0 - t) - gm\delta(t - t_0) + mg,$$

where $\theta(\cdot)$ is the Heaviside function and $\delta(\cdot)$ is the Dirac impulse.

For numerical implementation, we use the implicit scheme (3.5) *for this problem and we have*

$$\begin{aligned} \frac{x_{1,k+1} - x_{1,k}}{h} &= \quad x_{2,k+1}, \\ \frac{x_{2,k+1} - x_{2,k}}{h} &= \quad -g + \frac{n_{k+1}}{m}, \\ 0 \leq & \quad x_{1,k+1} - r \perp n_{k+1} \geq 0. \end{aligned}$$

Now we compute $x_{2,k+1}$ from the second equation and substitute it into the first one. Further, we express $x_{1,k+1}$ from the first equation, thus we have

$$\begin{aligned} x_{2,k+1} \quad &= \quad -h(g - \frac{n_{k+1}}{m}) + x_{2,k}, \\ x_{1,k+1} \quad &= \quad h\big(h(\frac{n_{k+1}}{m} - g) + x_{2,k}\big) + x_{1,k}, \\ 0 \quad &\leq \quad x_{1,k+1} - r \perp n_{k+1} \geq 0. \end{aligned}$$

Now we are able to get n_{k+1}. It is easy to see, that if $n_{k+1} = 0$ then $x_{1,k} - h(gh - x_{2,k}) - r \geq 0$ and if $n_{k+1} > 0$, then $x_{1,k} - h(gh - x_{2,k}) - r < 0$ and $x_{1,k+1} - r = 0$, therefore

$$n_{k+1} = \begin{cases} 0 & \textit{for} \quad x_{1,k} - h(gh - x_{2,k}) - r \geq 0, \\ \dfrac{m\big(r - x_{1,k} + h(gh - x_{2,k})\big)}{h^2} & \textit{for} \quad x_{1,k} - h(gh - x_{2,k}) - r < 0. \end{cases}$$

The solution and the absolute error of the numerical solution is in Figure 4.6. Note that the first component of the solution is non-smooth but continuous while the second one is discontinuous.

4.2 Models of electrical circuits

In Chapter 1 we presented a simple electrical circuit. Now we continue in investigating of more complicated circuits. The voltages and currents in these circuits are related to each other by DVIs or GEs.

Example 4.2.1 (Simple series circuit) *Consider the circuit in Figure 4.7a involving a non-linear resistor with current-voltage characteristic given by $g(i) := \operatorname{argsinh}(i)$, a source $E > 0$, an input-signal source u with the corresponding instantaneous current i, and a practical diode with current–voltage characteristic given by $F(i) := \operatorname{Sgn}_{V_1,V_2}(i)$, where $V_1 < 0 < V_2$ are given constants and $i \in \mathbb{R}$. Let $p := u - E$. By Kirchhoff's voltage law, for a fixed time interval $[a, b]$ the current i in the circuit is solution to the GE in the form*

$$p(t) \in \underbrace{\operatorname{argsinh}(i(t))}_{V_R} + \underbrace{\operatorname{Sgn}_{V_1,V_2}(i(t))}_{V_D} := \Phi(i(t)) \quad \textit{for} \quad t \in [a, b],$$

Now we use Theorem 2.2.6 to verify the strong metric regularity of the mapping Φ at any point $\bar{i} \in \operatorname{dom}\Phi$ for any $p \in \operatorname{rge}\Phi$ and use the note under this theorem. It is known, that the derivative of $\operatorname{argsinh}(i)$ is $\dfrac{1}{\sqrt{1+i^2}}$. Let denote $\bar{v} := p - \operatorname{argsinh}(\bar{i})$. The Bouligand paratingent cone $\widetilde{T}_{\operatorname{gph}\, \operatorname{Sgn}_{V_1,V_2}}((\bar{i}, \bar{v}))$ is in Figure 4.8 and the limiting normal cone $N_{\operatorname{gph}\, \operatorname{Sgn}_{V_1,V_2}}((\bar{i}, \bar{v}))$ is in Figure 4.9. It is easy to see, that

$$\left(\xi, -\frac{1}{\sqrt{1+(\bar{i})^2}}\xi\right) \in \widetilde{T}_{\operatorname{gph}\, \operatorname{Sgn}_{V_1,V_2}}((\bar{i}, \bar{v}))$$

and

$$\left(\xi, \frac{1}{\sqrt{1+(\bar{i})^2}}\xi\right) \in -N_{\text{gph Sgn}_{V_1,V_2}}((\bar{i},\bar{v}))$$

hold if only if $\xi = 0$. Hence the assumptions of Theorems 3.1.1 and 3.1.2 are satisfied and we can apply the scheme (3.4) *with Lipschitz continuous $p(\cdot)$ on $[a,b]$. The exact solution of the generalized equation has the form*

$$i(p) := \begin{cases} 0, & \text{if } p \in [V_1, V_2], \\ \sinh(p - V_2), & \text{if } p > V_2, \\ \sinh(p - V_1), & \text{if } p < V_1. \end{cases}$$

The graph of the solution and errors of the numerical solution are in Figure 4.10. Note that the solution is non-smooth and Lipschitz continuous.

Example 4.2.2 ([23]) *Consider the circuit in Figure 4.7b involving load resistances $R_B > 0$ and $R_L > 0$, two input-signal sources u_1 and u_2, and a P-N-P transistor (see Figure 4.11) having three terminals labeled emitter, base and collector. Its behavior can be described by the Ebers-Moll model [20, p. 409] involving two diodes placed back to back and two dependent current-controlled sources $\alpha_I I'$ and $\alpha_N I$ shunting the diodes. Here $\alpha_N \in [0,1)$ is known as the current gain in normal operation and $\alpha_I \in (0,1]$ is known as the inverted common-base gain current. Therefore $i_E = I - \alpha_I I'$ and $i_C = I' - \alpha_N I$. This means that*

$$\begin{pmatrix} i_E \\ i_C \end{pmatrix} = \begin{pmatrix} 1 & -\alpha_I \\ -\alpha_N & 1 \end{pmatrix} \begin{pmatrix} I \\ I' \end{pmatrix}.$$

Kirchhoff's laws also reveal that $i_B = -(i_E + i_C)$, so $R_B(-i_C - i_E) + u_1 - V_E = 0$ and $0 = V_C + u_2 + R_L i_C - V_E = V_C + u_2 + R_L i_C + R_B(i_C + i_E) - u_1$. Given $V_{E1} < 0 < V_{E2}$, $V_{C1} < 0 < V_{C2}$, $\alpha > 0$, and $\beta > 0$, assume that the characteristics of the diodes involved in the Ebers-Moll model are defined by

$$G_1(x) := \begin{cases} [V_{E1}, V_{E2}], & x = 0, \\ V_{E1} + \alpha \operatorname{argsinh}(x), & x < 0, \\ V_{E2} + \alpha \operatorname{argsinh}(x), & x > 0, \end{cases}$$

$$G_2(x) := \begin{cases} [V_{C1}, V_{C2}], & x = 0, \\ V_{C1} + \beta \operatorname{argsinh}(x), & x < 0, \\ V_{C2} + \beta \operatorname{argsinh}(x), & x > 0. \end{cases}$$

Then $G_1(I) = \alpha \operatorname{argsinh}(I) + F_1(I)$ *and* $G_2(I') = \beta \operatorname{argsinh}(I') + F_2(I')$, *where*

$$F_1(x) := \begin{cases} [V_{E1}, V_{E2}], & x = 0, \\ V_{E1}, & x < 0, \\ V_{E2}, & x > 0, \end{cases} \quad \text{and} \quad F_2(x) := \begin{cases} [V_{C1}, V_{C2}], & x = 0, \\ V_{C1}, & x < 0, \\ V_{C2}, & x > 0. \end{cases}$$

Therefore we have

$$\underbrace{\begin{pmatrix} u_1 \\ u_1 - u_2 \end{pmatrix}}_{=\mathbf{p}} \in \underbrace{\begin{pmatrix} R_B(1-\alpha_N) & R_B(1-\alpha_I) \\ R_B - \alpha_N(R_B+R_L) & R_B + R_L - \alpha_L R_B \end{pmatrix}\begin{pmatrix} I \\ I' \end{pmatrix} + \begin{pmatrix} \alpha \operatorname{argsinh}(I) \\ \beta \operatorname{argsinh}(I') \end{pmatrix}}_{=\mathbf{g}(\mathbf{u})} + \underbrace{\begin{pmatrix} F_1(I) \\ F_2(I') \end{pmatrix}}_{=\mathbf{F}(\mathbf{u})},$$

where $\mathbf{u} = (I, I')$. *In Appendix B Example 4.3, we verify the strong metric regularity in the similar system of GEs. In the same way we can do it for this problem.*

Further we use the scheme (3.4) *to obtain graphs of the numerical solution depicted in Figure 4.12.*

Example 4.2.3 (Electrical circuit with Zener diode [9]) *Consider a circuit in Figure 4.13 involving a resistor with a resistance* $R > 0$, *a coil with an inductance* $L > 0$, *a capacitor with a capacity* $E > 0$ *and a Zener diode with a knee voltage* $V_z > 0$ *and a breakdown voltage* 0, *therefore the characteristic of the Zener diode is given by*

$$F(x) := \operatorname{Sgn}_{0,V_z}(-x) \quad x \in \mathbb{R}.$$

We denote $x_1(t)$ *a current across the coil at time* t *and* $x_2(t)$ *a charge across a capacitor at the time* t. *These quantities satisfy*

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &\in -\frac{R}{L}x_2(t) - \frac{1}{LE}x_1(t) + \frac{1}{L}F(x_2(t)). \end{aligned}$$

The DVI formulation of this problem has the form

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -\frac{R}{L}x_2(t) - \frac{1}{LC}x_1(t) + \frac{V_z}{L}\lambda(t), \\ 0 &\leq \lambda(t) \perp x_2^+(t) \geq 0, \\ 0 &\leq x_2^-(t) \perp 1 - \lambda(t) \geq 0. \end{aligned}$$

One can verify the conditions guaranteeing the existence of a solution in the similar way as in Example 4.1.2. The numerical solution, computed by the implicit scheme (3.5)*, does not match the properties of the model, hence we use the explicit scheme for the numerical simulation. Therefore we have*

$$\begin{aligned} x_{1,i+1} &= x_{1,i} + hx_{2,i}, \\ x_{2,i+1} &= x_{2,i} - h\left(\frac{1}{LE}x_{1,i} + \frac{R}{L}x_{2,i} - \frac{V_z}{L}u_{i+1}\right), \\ 0 &\leq u_{i+1} \perp (x_{2,i+1})^+ \geq 0, \\ 0 &\leq (x_{2,i+1})^- \perp 1 - u_{i+1} \geq 0. \end{aligned}$$

Now, we are able to get u_{i+1} independently of $x_{2,i+1}$ in such a way

$$u_{i+1} = \begin{cases} 0 & \text{for} \quad x_{1,i} < \dfrac{E(L-hR)x_{2,i}}{h}, \\ 1 & \text{for} \quad x_{1,i} > E\left(V_z + \dfrac{Lx_{2,i}}{h} - Rx_{2,i}\right), \\ \dfrac{hx_{1,i} - ELx_{1,i} + EhRx_{2,i}}{EhV_z} & \text{otherwise.} \end{cases}$$

Graphs of components of the solution are in Figure 4.14. Note that its first component is smooth, the second one is non-smooth but continuous while the third one is even discontinuous..

Example 4.2.4 (**Electrical oscillator with 4 diodes bridge full-wave rectifier**) *Consider the circuit in Figure 4.15 involving the four-diodes bridge full-wave rectifier, a resistor with a resistance $R > 0$, a capacitor with the capacitance $C_0 > 0$ and an inductor with the inductance $L > 0$. Further, we denote v_C a voltage across the capacitor, i_C a current across the capacitor, i_L a current across the inductor and $i_{DF1}, i_{DF2}, i_{DR1}, i_{DR2}$ currents across the diodes and $v_{DF1}, v_{DF2}, v_{DR1}, v_{DR2}$ voltages across the diodes. Then the Kirchhoff's laws can be written as*

$$\begin{aligned} v_L &= v_C, \\ v_{DF1} - v_{DR1} &= v_L, \\ v_{DF2} + v_R + v_{DR1} &= 0, \\ i_C + i_L + i_{DF1} - i_{DR2} &= 0, \\ i_{DF1} + i_{DR1} &= i_R, \\ i_{DF2} + i_{DR2} &= i_R. \end{aligned}$$

Then the problem is described by the DLCP in the form

$$\begin{array}{rcl}\dot{\mathbf{x}}(t) & = & \mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t),\\ \mathbf{0}_{\mathbb{R}^4} & \preceq & \mathbf{C}\mathbf{x}(t)+\mathbf{D}\mathbf{u}(t)\perp\mathbf{u}(t)\succeq\mathbf{0}_{\mathbb{R}^4},\end{array}$$

where

$$\mathbf{x}=\begin{pmatrix} v_C\\ i_L\end{pmatrix},\mathbf{A}=\begin{pmatrix} 0 & -\frac{1}{C_0}\\ \frac{1}{L} & 0\end{pmatrix},\mathbf{B}=\begin{pmatrix} 0 & 0 & -\frac{1}{C_0} & \frac{1}{C_0}\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\mathbf{u}=\begin{pmatrix} -v_{DR1}\\ -v_{DF2}\\ i_{DF1}\\ i_{DR2}\end{pmatrix},\mathbf{C}=\begin{pmatrix} 0 & 0\\ 0 & 0\\ -1 & 0\\ 1 & 0\end{pmatrix},\mathbf{D}=\begin{pmatrix} \frac{1}{R} & \frac{1}{R} & -1 & 0\\ \frac{1}{R} & \frac{1}{R} & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}.$$

For the numerical implementation of this problem we use the scheme (3.6) *in the semi-implicit form. The LCP, at every step, was solved by LCP/MCP solver*[3] *in MATLAB. Graphs of the solution components are in Figure 4.16.*

4.3 The model of economic equilibrium

In this section we present one economic problem, which concerns the finding the economic equilibrium of the pure exchange economy which evolves in time.

Example 4.3.1 ([18]) *Consider a model of the economic equilibrium for exchange n different types of goods in a single time period. There are k agents, each of which starts with a vector $\mathbf{x}_i^0 \in \mathbb{R}^n$ of goods and trades them for another goods vectors $\mathbf{x}_i \in \mathbb{R}^n$. Suppose that a vector $\mathbf{l} \in \mathbb{R}^n_+$ is a vector of prices of the goods in the market. The agent $i \in \{1, ..., k\}$ has an initial amount of money $m_i^0 \in \mathbb{R}_+$ and ends up, after trading, with an amount of money $m_i \in \mathbb{R}_+$. Each agent wants to maximize his utility function $u_i(m_i, \mathbf{x}_i)$ over the set $\mathbb{R}_+ \times U_i$ subject to the budget constraint*

$$m_i - m_i^0 + \langle \mathbf{l}, \mathbf{x}_i - \mathbf{x}_i^0\rangle \leq 0, \tag{4.6}$$

[3] The function is available on `http://www.mathworks.com/matlabcentral/fileexchange/20952-lcp---mcp-solver--newton-based-`.

where the sets $U_i \subset \mathbb{R}^n$ are non-empty, closed and convex and the functions u_i are continuously differentiable, concave and non-decreasing over $\mathbb{R}_+ \times U_i$. In addition, there are constrains on money and goods in the form

$$\sum_{i=1}^{k}[m_i - m_i^0] \leq 0 \quad \textit{and} \quad \sum_{i=1}^{k}[\mathbf{x}_i - \mathbf{x}_i^0] \preceq \mathbf{0}_{\mathbb{R}^n}. \tag{4.7}$$

If the agents have choices $(\bar{m}_i, \bar{\mathbf{x}}_i) \in \mathbb{R}_+ \times \mathbb{R}^n$ available to them for which

(a) $\mathbf{x}_i \leq \mathbf{x}_i^0$ *but* $\bar{m}_i < m_i^0$,

(b) $\sum_{i=1}^{k} \bar{\mathbf{x}}_i \prec \sum_{i=1}^{k} \mathbf{x}_i$,

and for every good there is at least one agent i such that the utility u_i always increases on $\mathbb{R}_+ \times U_i$ when that good component increases. Then an equilibrium always exists, and moreover, it satisfies a first-order optimality condition for each agent involving the Lagrange functions

$$L_i(\mathbf{l}, m_i, \mathbf{x}_i, \lambda_i) = -u(m_i, \mathbf{x}_i) + \lambda_i(m_i - m_i^0 + \langle \mathbf{l}, \mathbf{x}_i - \mathbf{x}_i^0 \rangle)$$

with a Lagrange multiplier $\lambda_i \geq 0, i = 1, ..., k$, associated with the budget constraint (4.6). In addition with (4.7) we get the variational inequality in the form

$$-\mathbf{f}(\mathbf{l}, \mathbf{m}, \mathbf{x}, \lambda, \mathbf{m}^0, \mathbf{x}^0) \in N_C(\mathbf{l}, \mathbf{m}, \mathbf{x}, \lambda)$$

with

$$\mathbf{f}(\mathbf{l}, \mathbf{m}, \mathbf{x}, \lambda, \mathbf{m}^0, \mathbf{x}^0) = \begin{pmatrix} \sum_{i=1}^{k}[\mathbf{x}_i^0 - \mathbf{x}_i] \\ \cdots \\ \lambda_i + \nabla_{m_i} u_i(m_i, \mathbf{x}_i) \\ \cdots \\ \lambda_i \mathbf{l} + \nabla_{\mathbf{x}_i} u_i(m_i, \mathbf{x}_i) \\ \cdots \\ m_i^0 - m_i + \langle \mathbf{l}, \mathbf{x}_i^0 - x_i \rangle \\ \cdots \end{pmatrix}$$

where $C = \mathbb{R}_+^n \times \mathbb{R}_+^k \times U_1 \times ... \times U_k \times \mathbb{R}_+^k$, $\mathbf{l} \in \mathbb{R}_+^n, \mathbf{m} = (m_1, m_2, ..., m_k) \in \mathbb{R}^k, \mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_k) \in U_1 \times ... \times U_k, \lambda = (\lambda_1, ..., \lambda_k) \in \mathbb{R}_+^k, \mathbf{m}^0 = (m_1^0, ..., m_k^0) \in \mathbb{R}_+^k$ and $\mathbf{x}^0 = (\mathbf{x}_1^0, .., \mathbf{x}_k^0) \in U_1 \times ... \times U_k$. The initial endowments are represented by the vectors $\mathbf{x}^0$ and $\mathbf{m}^0$.

Further, we consider a parametric version of the previous problem, such that a market has time dependent initial endowments $(\mathbf{x}^0(t), \mathbf{m}^0(t)), t \in [a, b]$. For each $t \in [a, b]$ the endowments $(\mathbf{x}^0(t), \mathbf{m}^0(t))$ are traded to obtain *an equilibrium vector* $(\mathbf{l}(t), \mathbf{m}(t), \mathbf{x}(t))$ *with an associated Lagrange multiplier* $\lambda(t)$. *For given functions* $(\mathbf{x}^0(\cdot), \mathbf{m}^0(\cdot))$, *consider*

$$-\mathbf{f}(\mathbf{l}(t), \mathbf{m}(t), \mathbf{x}(t), \lambda(t), \mathbf{m}^0(t), \mathbf{x}^0(t)) \in N_C(\mathbf{l}(t), \mathbf{m}(t), \mathbf{x}(t), \lambda(t))$$

for $t \in [a, b]$.

Now we focus on the model with two agents, which have utility functions

$$u_i(m_i, x_i) = \alpha_i \log(m_i) + \beta_i \log(x_i) \quad i = 1, 2,$$

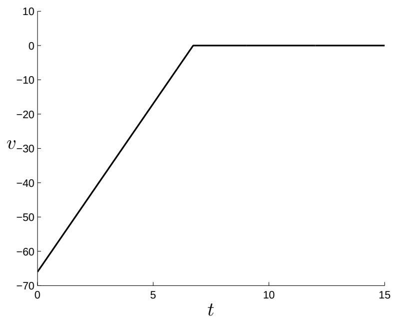
and a single good subject to the constraints

$$x_i \in U_i = [\zeta_i, \eta_i] \quad i = 1, 2,$$

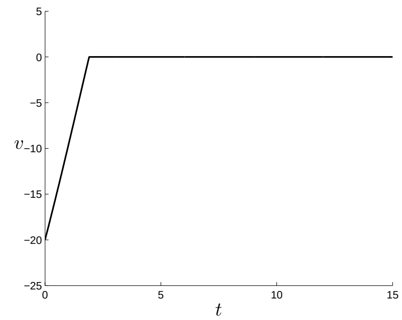
for $0 < \zeta_i < \eta_i$. *Then the variational inequality has the form*

$$\underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ m_1^0 \\ m_2^0 \end{pmatrix}}_{=\mathbf{p}} + \underbrace{\begin{pmatrix} \sum_{i=1}^2 [x_i^0 - x_i] \\ \lambda_1 - \dfrac{\alpha_2}{m_2} \\ \lambda_2 - \dfrac{\alpha_1}{m_1} \\ \lambda_1 l - \dfrac{\beta_1}{x_1} \\ \lambda_2 l - \dfrac{\beta_2}{x_2} \\ -m_1 + l[x_1^0 - x_1] \\ -m_2 + l[x_2^0 - x_2] \end{pmatrix}}_{=-\mathbf{g}(\mathbf{u})} \in \underbrace{\begin{pmatrix} N_{\mathbb{R}_+}(l) \\ N_{\mathbb{R}_+}(m_1) \\ N_{\mathbb{R}_+}(m_2) \\ N_{U_1}(x_1) \\ N_{U_2}(x_2) \\ N_{\mathbb{R}_+}(\lambda_1) \\ N_{\mathbb{R}_+}(\lambda_2) \end{pmatrix}}_{=\mathbf{F}(\mathbf{u})},$$

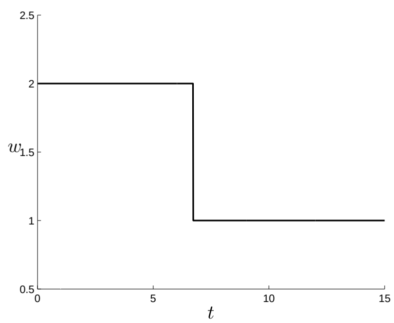
where $\mathbf{u} = (l, m_1, m_2, x_1, x_2, \lambda_1, \lambda_2)^T$. *For a numerical simulation we implemented the scheme* (3.2), *using the LCP/MCP solver at each step, in MATLAB. Note that the assumptions guaranteeing that the numerical solution has the grid error* $O(h^4)$ *are satisfied (see [21]). Graphs of the components of the solution are in Figure 4.17.*



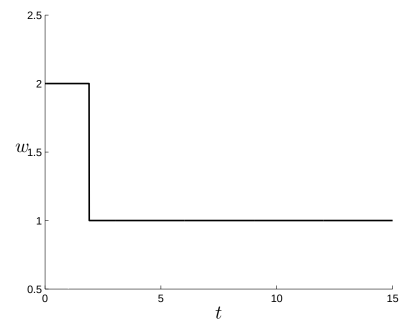
(a) The first component with $f(t) \equiv 0$ and $v_0 = -66$.



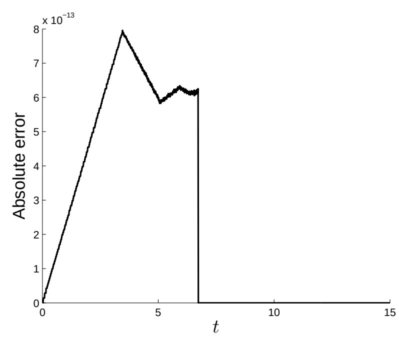
(b) The first component with $f(t) = \sin(t)$ and $v_0 = -20$.



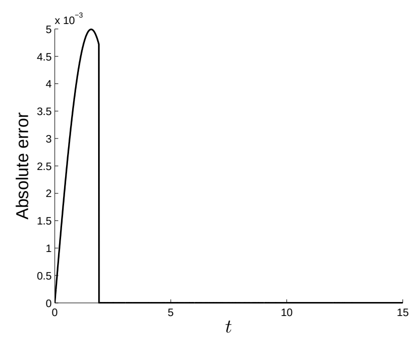
(c) The second component with $f(t) \equiv 0$ and $v_0 = -66$.



(d) The second component with $f(t) = \sin(t)$ and $v_0 = -20$.

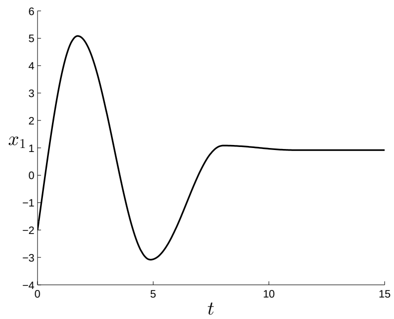


(e) The absolute error of the first component of the numerical solution with $f(t) \equiv 0$ and $v_0 = -66$.

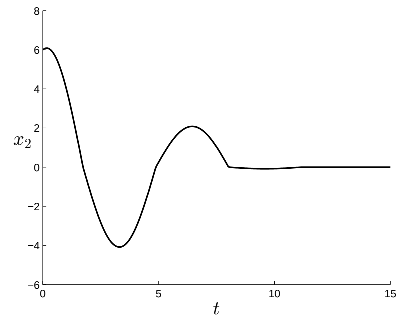


(f) The absolute error of the first component of the numerical solution with $f(t) = \sin(t)$ and $v_0 = -20$.

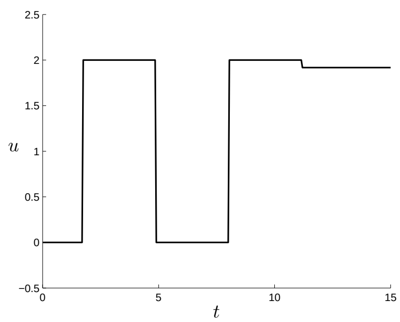
Figure 4.2: Example 4.1.1 with $\mu = 1, m = 1, g = 9.81$ and $h = 0.01$.



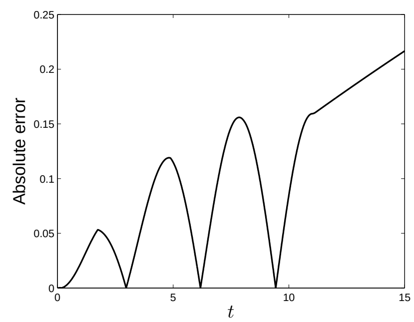
(a) The first component.



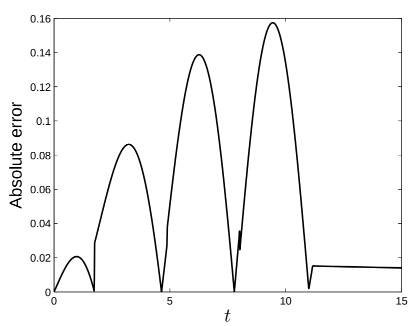
(b) The second component.



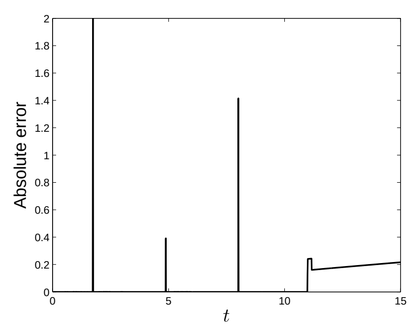
(c) The third component.



(d) The absolute error of the numerical solution of the first component.

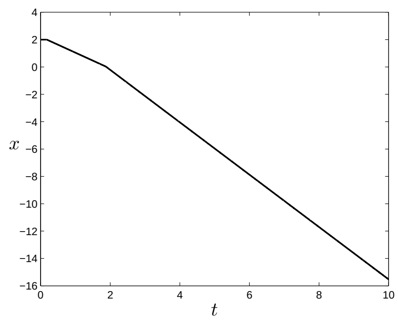


(e) The absolute error of the numerical solution of the second component.

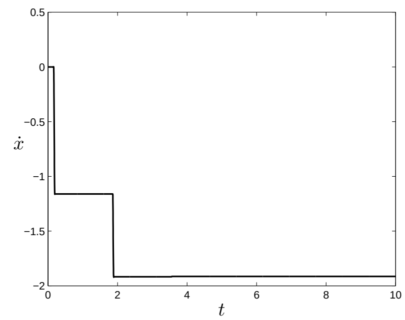


(f) The absolute error of the numerical solution of the third component.

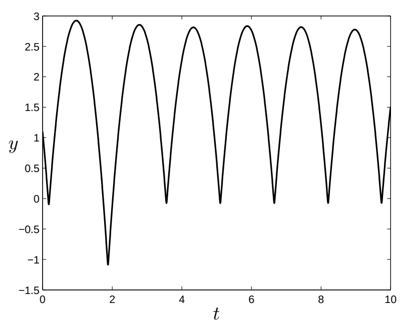
Figure 4.3: The solution for Example 4.1.2 with $\mu = 1, m = 1, k = 1, x(0) = -2, x_2(0) = 6$ and $h = 0.01$.



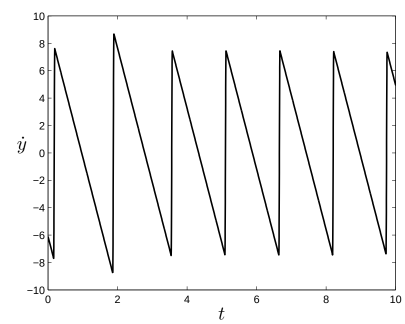
(a) The first component.



(b) The second component.

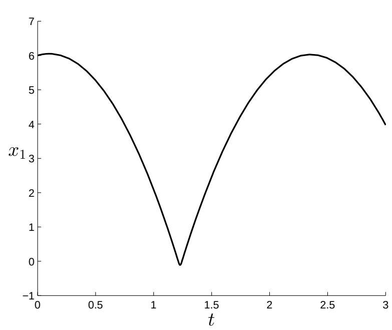


(c) The third component.

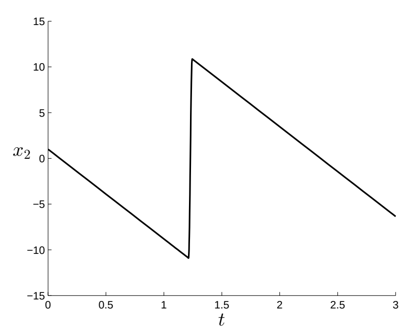


(d) The fourth component.

Figure 4.4: The solution for Example 4.1.3 with $x(0) = 2, \dot{x}(0) = 0, y(0) = 1.1, \dot{y}(0) = -6.1, r(x) = -e^{-x^2}, m = 1, g = 9.81$ and $k = 10000$.

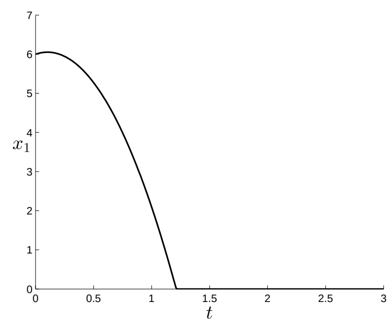


(a) The first component of the solution.

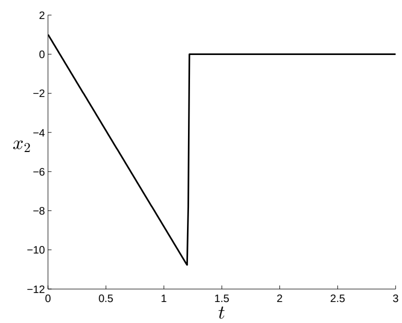


(b) The second component of the solution.

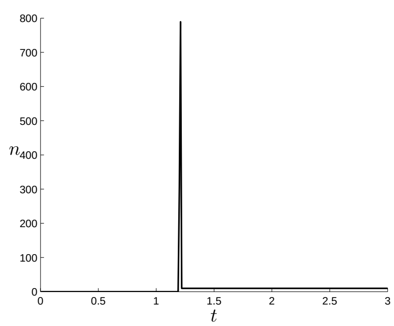
Figure 4.5: The solution to 4.4 with $x_1(0) = 6, x_2(0) = 1, r = 0, m = 1, g = 9.81$ and $k = 10000$.



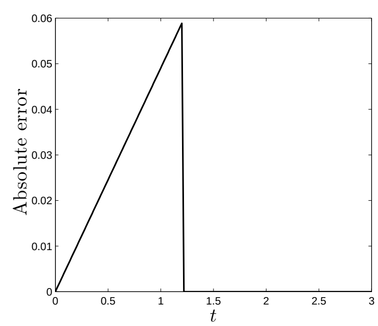
(a) The first component.



(b) The second component.

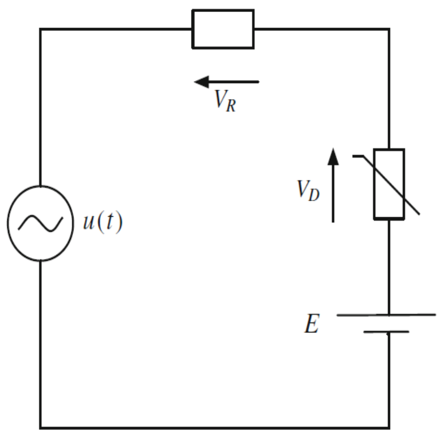


(c) The third component.

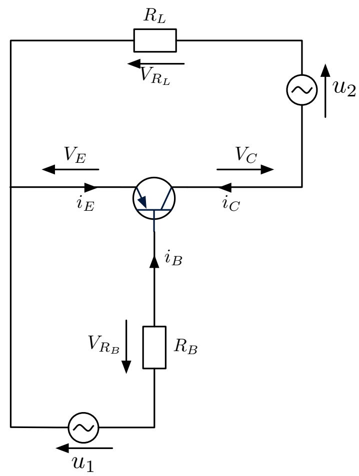


(d) The absolute error of the first component.

Figure 4.6: The solution and the absolute error for Example 4.5 with $x_1(0) = 6, x_2(0) = 1, r = 0, m = 1, g = 9.81$ and $h = 0.01$.

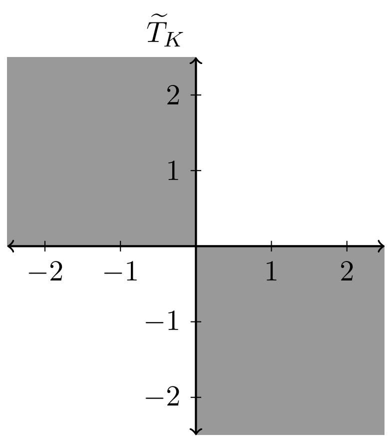


(a) Example 4.2.1.

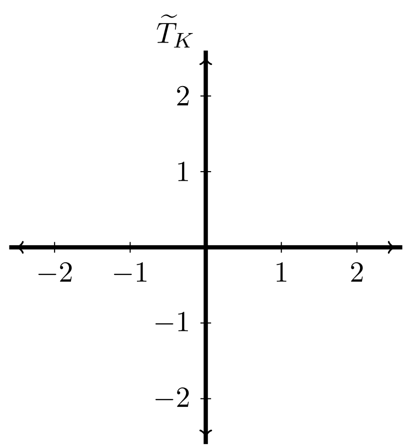


(b) Example 4.2.2.

Figure 4.7: The circuits considered.

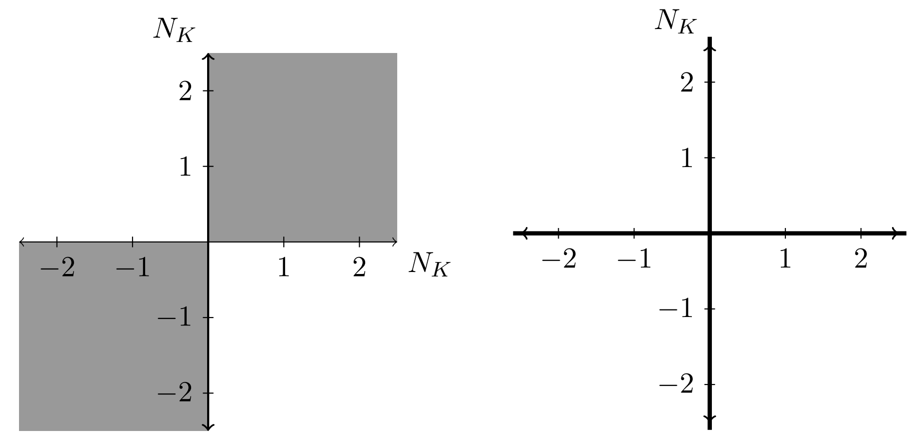


(a) Union of cones at points $(0, V_2)$ and $(0, V_1)$.



(b) Union of cones at points $(p, V_2), (-p, V_1)$ and $(0, w)$ for $p > 0$ and $w \in (V_1, V_2)$.

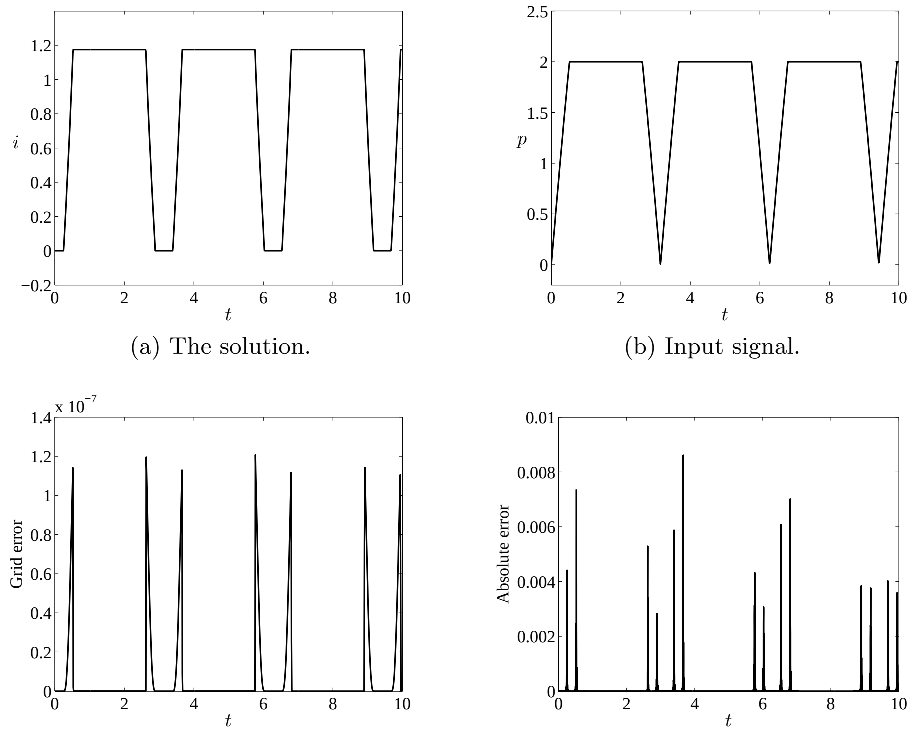
Figure 4.8: Bouligand paratingent cones for Example 4.2.1 with $K = \operatorname{gph} \operatorname{Sgn}_{V_1,V_2}$.



(a) Union of cones at points $(0, V_2)$ and $(0, V_1)$.

(b) Union of cones at points $(p, V_2), (-p, V_1)$ and $(0, w)$ for $p > 0$ and $w \in (V_1, V_2)$.

Figure 4.9: Normal cones for Example 4.2.1 with $K = \text{gph Sgn}_{V_1,V_2}$.



(a) The solution. (b) Input signal.

(c) The grid error. (d) The absolute error.

Figure 4.10: The solution, the input signal and errors for Example 4.2.1 with $u(t) = 4\min\{|\sin(t), 0|\}, V_1 = -1, V_2 = 1, [a, b] = [0, 10], E = 0, e_i = 0$ for each $i \in \mathbb{N}$ and $h = 0.01$.

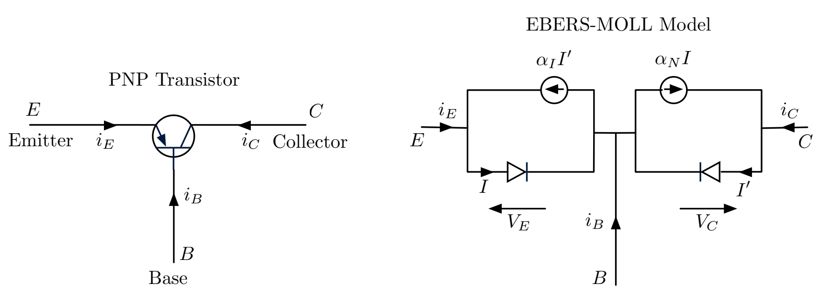
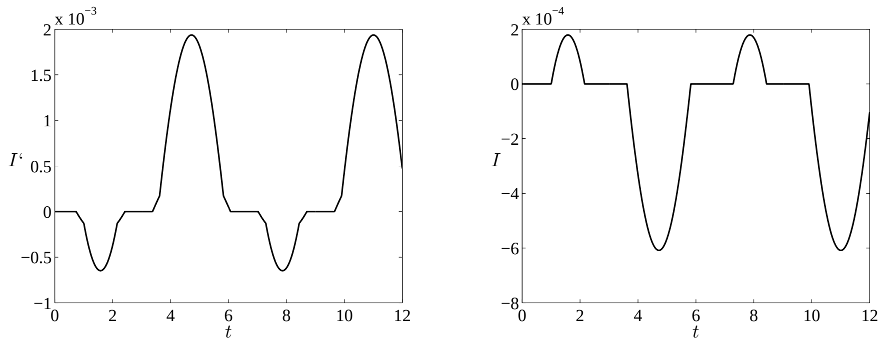


Figure 4.11: The P-N-P transistor and its Ebers-Moll model.



(a) The first component. (b) The second component.

Figure 4.12: The solution for Example 4.2.2 with $V_{E1} = -2$, $V_{E2} = 2$, $V_{C1} = -4$, $V_{C2} = 4$, $\alpha = 2/\pi$, $\beta = 2$, $u_1(t) = \sin(t)$, $u_2(t) = 10\sin(t)$, $R_L = 3000$, $R_B = 30000$, $\alpha_I = 0.7$, $\alpha_N = 0.1, \mathbf{e}_i = \mathbf{0}_{\mathbb{R}^2}$ for each $i \in \mathbb{N}$ and $h = 0.01$.

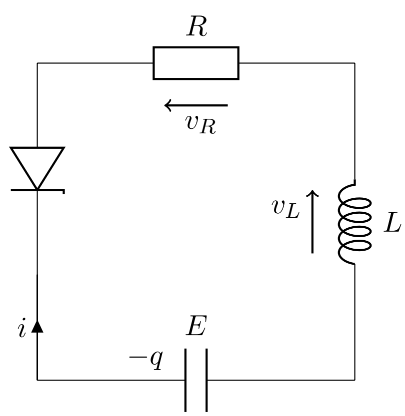
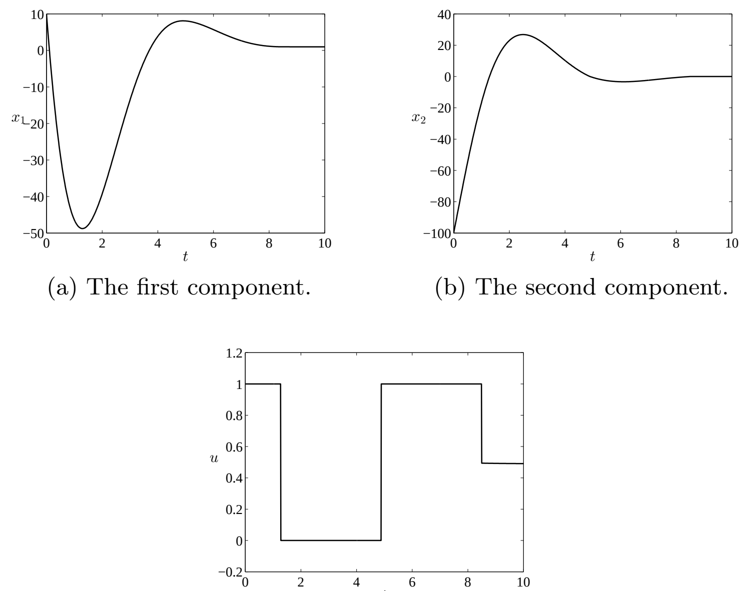


Figure 4.13: Circuit considered in Example 4.2.3



(a) The first component.

(b) The second component.

(c) The third component.

Figure 4.14: The numerical solution for Example 4.2.3 with $L = 1, E = 1, R = 1, V_z = 2, x_1(0) = 10, x_2(0) = -100$ and $h = 0.01$.

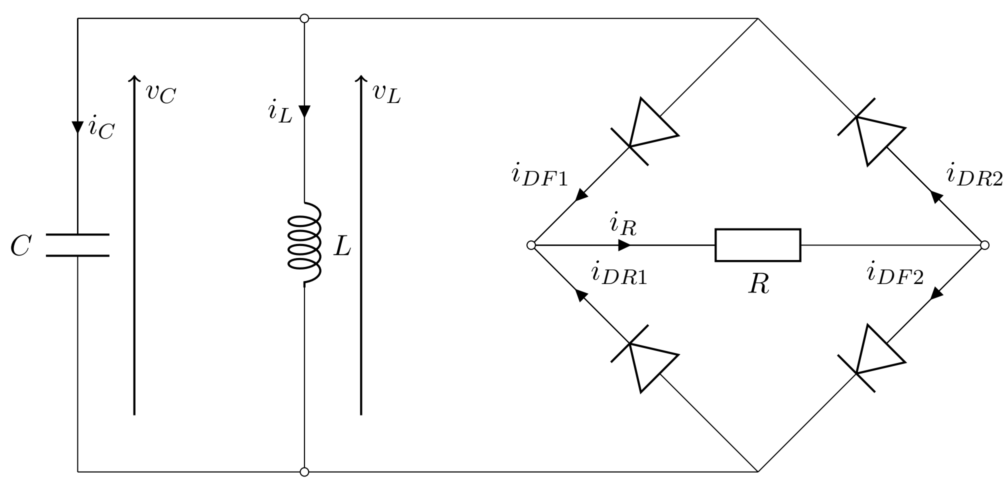
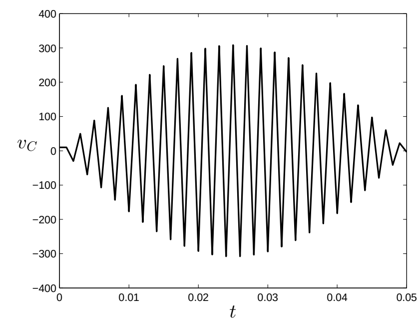
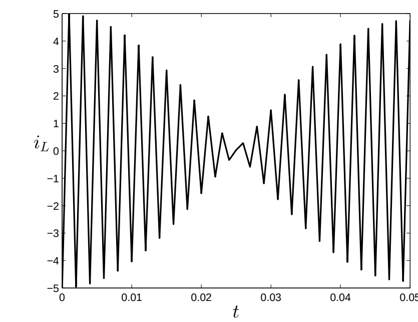


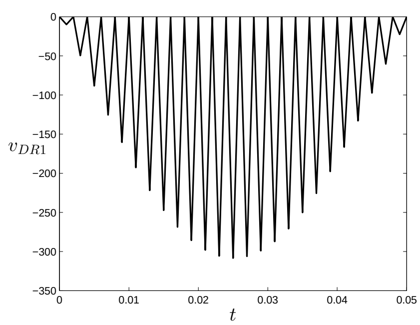
Figure 4.15: Electrical oscillator with 4 diodes bridge full-wave rectifier considered in Example 4.2.4.



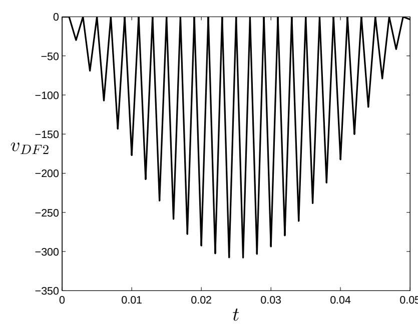
(a) The fisrt component.



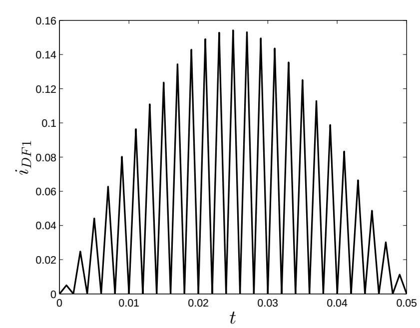
(b) The second component.



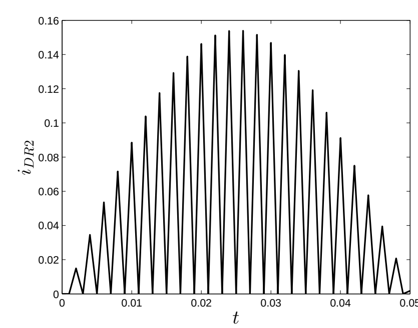
(c) The third component.



(d) The fourth component.

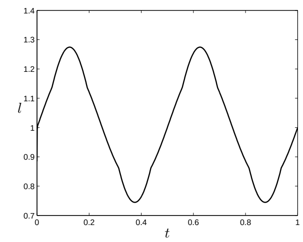


(e) The fifth component.

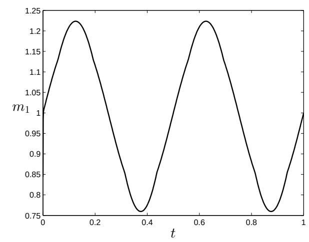


(f) The sixth component.

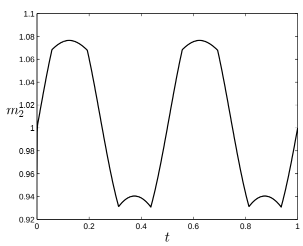
Figure 4.16: The numerical solution for Example 4.2.4 with $C_0 = 10^{-10}, R = 2000, L = 0.001, v_C(0) = 10, i_L(0) = -5, \theta = 0.5$ and $h = 0.001$.



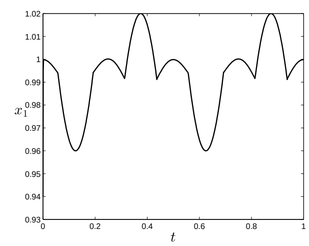
(a) The price.



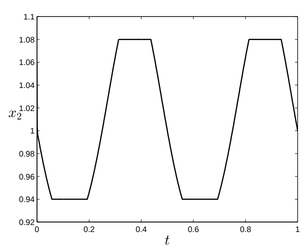
(b) The amount of money of the first agent



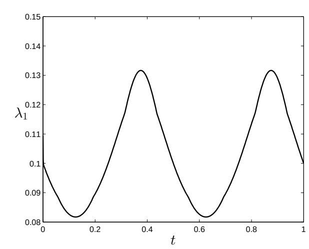
(c) The amount of money of the second agent.



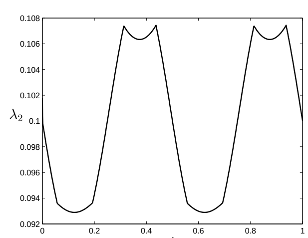
(d) The good of the first agent.



(e) The good of the second agent.



(f) Lagrange multiplier in time.



(g) Lagrange multiplier in time.

Figure 4.17: The numerical solution for Example 4.3.1 with $m_1^0(t) = 1 + \sin(4\pi t), m_1^0(t) = 1, x_1^0(t) = 1 - 0.1\sin(4\pi t), x_2^0(t) = 1, U_2 = U_1 = [0.94, 1.08], [a, b] = [0, 1]$ and the step $h = 10^{-12}$.

Chapter 5

Conclusion

In this thesis we set ourselves the task to present existence theorems and numerical methods for differential variational inequalities.

In Chapter 1, we motivated our consideration by basic problems, which appear in electrical circuits containing non-smooth elements such as diodes and that can be described by DVIs, ODEs, DIs or GEs.

Chapter 2 is divided into three parts. In the first and the second section we presented sufficient conditions for global reduction and local reduction of DVIs to ODEs, respectively. In the third section, we focused on reformulating of a DVI as a DI in such a way that if the DI has a solution then so does the DVI.

In Chapter 3, we presented numerical methods for solving DVIs and GEs. Numerical schemes for DVIs are based on finite differences and schemes for GEs are based on an Euler–Newton continuation method.

In Chapter 4, we applied numerical methods and the above mentioned existence theory on real-world problems, which are described by DVIs or GEs. For some problems we obtained formulas for the exact solution.

In future work, we plan to focus on reduction of DVIs to ODEs by using selections of the solution mapping of the corresponding VI and on one-parametric bifurcations in generalized equations similar to bifurcations in the regular equations.

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Appendix A

Definitions and statements

A.1 Definitions

Definition A.1.1 (absolutely continuous function) *We say that a function* $\mathbf{f} : [a, b] \to \mathbb{R}^n$ *is* absolutely continuous *on* $[a, b]$ *if for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that for any collection of nonoverlapping subintervals* $\{[a_j, b_j]\}$, $j \in \mathbb{N}$, *of* $[a, b]$, *we have*

$$\sum_j (b_j - a_j) < \delta \quad \Longrightarrow \quad \sum_j \|\mathbf{f}(b_j) - \mathbf{f}(a_j)\| < \varepsilon.$$

Definition A.1.2 (monotone mapping) *A function* $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ *is called* monotone*, if*

$$\langle \mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq 0, \text{ whenever } \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

A set-valued function $\mathbf{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *is called* monotone*, if*

$$\langle \mathbf{u} - \mathbf{v}, \mathbf{x} - \mathbf{y} \rangle \geq 0 \text{ whenever } \quad (\mathbf{x}, \mathbf{u}), (\mathbf{y}, \mathbf{v}) \in \operatorname{gph} \mathbf{F}.$$

Definition A.1.3 (maximal monotone mapping) *A monotone set-valued function* $\mathbf{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *is called maximal monotone, if there exists no other monotone set-valued function whose graph strictly contains the graph of* $\mathbf{F}$.

Definition A.1.4 (cone) *A set* $C \subset \mathbb{R}^n$ *is called the* cone*, if*

$$\forall \mathbf{x} \in C, \forall \lambda \geq 0, \lambda \mathbf{x} \in C.$$

Definition A.1.5 (dual cone) *Let* $C \subset \mathbb{R}^n$ *be non-empty.* The dual cone *to* C *is the set*

$$C^* := \{\mathbf{p} \in \mathbb{R}^n : \langle \mathbf{p}, \mathbf{x} \rangle \geq 0, \forall \mathbf{x} \in C\}.$$

Definition A.1.6 (Clarke subdifferential) *Let* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}$ *be locally Lipschitz continuous. The* Clarke subdifferential *of* $\mathbf{f}$ *at* $\mathbf{x}$ *is defined by*

$$\partial_C \mathbf{f}(\mathbf{x}) := \overline{\mathrm{co}}\{\lim_{i\to+\infty} \nabla\mathbf{f}(\mathbf{x}_i) : \mathbf{x}_i \to \mathbf{x}, \nabla\mathbf{f}(\mathbf{x}_i) \text{ exist}\} \subset \mathbb{R}^n,$$

*where "*co*" denotes the convex hull.*

Definition A.1.7 (P-matrix) *A matrix* $\mathbf{A} \in \mathbb{R}^{m\times m}$ *is called a* P-matrix*, if all principal minors of* $\mathbf{A}$ *are positive.*

Definition A.1.8 (symmetric and antisymmetric matrix) *Let* $\mathbf{A} \in \mathbb{R}^{n\times n}$*. We say that the matrix is* symmetric *if*

$$\mathbf{A} = \mathbf{A}^T.$$

We say that the matrix is antisymmetric *if*

$$\mathbf{A} = -\mathbf{A}^T.$$

Definition A.1.9 *For* $\alpha < \beta$ *define*

$$Sgn_{\alpha,\beta}(x) := N^{-1}_{[\alpha,\beta]}(x) = \begin{cases} \alpha, & \text{if } x < 0, \\ \beta, & \text{if } x > 0, \\ [\alpha,\beta], & \text{if } x = 0. \end{cases}$$

Define

$$x := x^+ - x^-,$$

where $x^+ := \max\{x, 0\}$ *and* $x^- := \max\{-x, 0\}$*.*

Definition A.1.10 *For a subset* K *of* $\mathbb{R}^m$ *and a point* $\mathbf{u} \in \mathbb{R}^m$ *the* distance from $\mathbf{u}$ to C *and the* projection of $\mathbf{u}$ on K *are defined by*

$$d(\mathbf{u}, K) = \inf\big\{\|\mathbf{v}-\mathbf{u}\| : \mathbf{v} \in K\big\} \quad \text{and} \quad P_K(\mathbf{u}) = \big\{\mathbf{v} \in K : \|\mathbf{v}-\mathbf{u}\| = d(\mathbf{u}, K)\big\},$$

respectively.

A.2 Statements

Theorem A.2.1 (Rayleigh quotient) *If* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *is a symmetric matrix with eigenvalues* $\lambda_1 \geq \lambda_2 \geq .... \geq \lambda_n$*, then*

$$\frac{\langle \mathbf{Au}, \mathbf{u}\rangle}{\|\mathbf{u}\|^2} \in [\lambda_n, \lambda_1] \quad \text{whenever} \quad \mathbf{u} \in \mathbb{R}^n \setminus \{\mathbf{0}_{\mathbb{R}^n}\}.$$

The previous fraction is called the Rayleigh quotient*.*

Lemma A.2.1 *Any locally bounded set-valued mapping* $\mathbf{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$*, i.e. for each* $\mathbf{x} \in \operatorname{dom} \mathbf{F}$ *there is* $r > 0$ *such that* $\mathbf{F}(\mathbb{B}(\mathbf{x}, r))$ *is bounded, having a closed graph is Pompeiu-Hausdorff upper/outer semi-continuous.*

Proof. See [5]. ■

Lemma A.2.2 *Let* K *be a non-empty closed convex subset of* $\mathbb{R}^m$ *and* $\mathbf{u} \in \mathbb{R}^m$*. Then*

(*i*) $P_K(\mathbf{u})$ *contains the only point,* $\mathbf{p}_{_K}(\mathbf{u})$ *say. Moreover,*

$$\langle \mathbf{z} - \mathbf{p}_{_K}(\mathbf{u}), \mathbf{u} - \mathbf{p}_{_K}(\mathbf{u}) \rangle \leq 0 \quad \textit{whenever} \quad \mathbf{z} \in K;$$

(*ii*) $N_K(\mathbf{u})$ *is a non-empty closed convex cone. If, in addition,* $\mathbf{u}$ *is an interior point of* K*, then* $N_K(\mathbf{u}) = \{\mathbf{0}\}$*;*

(*iii*) $\mathbf{p} \in N_K(\mathbf{u})$ *if and only if* $\mathbf{p}_{_K}(\mathbf{u} + \mathbf{p}) = \mathbf{u}$.

Proof. See [5]. ■

Proposition A.2.1 ([9]) *Let* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}$ *be a convex function. Then*

1. $\mathbf{f}$ *is locally Lipschitz;*
2. $\partial_C \mathbf{f}$ *is maximal monotone and bounded on bounded sets;*
3. $\partial_C \mathbf{f}$ *is Pompeiu-Hausdorff upper/outer semi-continuous with non-empty, convex and compact values.*

Lemma A.2.3 *Each matrix* $\mathbf{A} \in \mathbb{R}^{n \times n}$ *can be split into the sum of a symmetric and an antisymmetric matrix.*

Proof. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$, then

$$\mathbf{A} = \frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{A}^T + \frac{1}{2}\mathbf{A} - \frac{1}{2}\mathbf{A}^T.$$

Then

$$\left(\frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{A}^T\right)^T = \frac{1}{2}\mathbf{A}^T + \frac{1}{2}\mathbf{A},$$

therefore $\frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{A}^T$ is a symmetric matrix and

$$\left(\frac{1}{2}\mathbf{A} - \frac{1}{2}\mathbf{A}^T\right)^T = \frac{1}{2}\mathbf{A}^T - \frac{1}{2}\mathbf{A} = -\left(\frac{1}{2}\mathbf{A} - \frac{1}{2}\mathbf{A}^T\right),$$

so $\frac{1}{2}\mathbf{A} - \frac{1}{2}\mathbf{A}^T$ is an antisymmetric matrix. ■

Lemma A.2.4 *Let* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *be an antisymmetric matrix, then*

$$\langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle = \mathbf{x}^T\mathbf{A}\mathbf{x} = 0,$$

for any $\mathbf{x} \in \mathbb{R}^n$.

Proof. Let $\mathbf{x} \in \mathbb{R}^n$, then

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = (\mathbf{x}^T\mathbf{A}\mathbf{x})^T = \mathbf{x}^T\mathbf{A}^T\mathbf{x} = -\mathbf{x}^T\mathbf{A}\mathbf{x},$$

therefore

$$\langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle = 0.$$

■

Proposition A.2.2 ([5]) *Suppose that a function* $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}$ *is locally Lipschitz continuous. Then*

(i) if $\mathbf{h}$ *is continuously differentiable at* $\mathbf{u} \in \mathbb{R}^n$*, then* $\partial_C\mathbf{h}(\mathbf{u}) = \{\nabla\mathbf{h}(\mathbf{u})\}$*;*

(ii) if $\mathbf{h} = \mathbf{h}_1 + \mathbf{h}_2$ *for a continuously differentiable* $\mathbf{h}_1$ *and a locally Lipschitz continuous* $\mathbf{h}_2$*, then*

$$\partial_C\mathbf{h}(\mathbf{u}) = \nabla\mathbf{h}_1(\mathbf{u}) + \partial_C\mathbf{h}_2(\mathbf{u}) \quad \textit{for each} \quad \mathbf{u} \in \mathbb{R}^n.$$

Proof. See [5]. ■

Appendix B

Regularity Properties of Generalized Equations Arising in Electronics.

Finally, we attach a revised version of a note which was submitted to the Journal of Set-valued and Variational Analysis and currently undergoes second round of reviewing process.

Regularity Properties of Generalized Equations Arising in Electronics.

R. Cibulka[1] and **T. Roubal** [2]

Abstract. We study strong metric (sub-)regularity of a special non-monotone generalized equation with either smooth or locally Lipschitz single-valued part. The existence of a Lipschitz continuous response to a Lipschitz continuous input signal is proved. An inexact Euler-Newton continuation method for tracking a solution trajectory is introduced and demonstrated to have an accuracy of order $O(h^4)$. The theoretical results are applied in the study of non-regular electrical circuits involving devices like diodes and transistors.

Key Words. generalized equation, Lipschitz selection, strong metric regularity, strong metric sub-regularity, Clarke generalized Jacobian, non-smooth analysis, path-following, non-regular electrical circuits.

2010 AMS Subject Classification. 49J53, 49J52, 49K40, 90C31.

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1 Introduction

Given matrices $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ with $m \leq n$, a vector $p \in \mathbb{R}^n$, a single-valued mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, and a set-valued $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, we consider the problem of finding a solution $z \in \mathbb{R}^n$ to the generalized equation

$$p \in f(z) + BF(Cz). \tag{1}$$

In [3], the authors considered the special case of the above inclusion with the linear single-valued part f, with $B = C^T$, and F being the Clarke subdifferential [6] of the super-potential j defined by

$$j(x) := j_1(x_1) + j_2(x_2) + \cdots + j_m(x_m) \quad \text{whenever} \quad x = (x_1, \ldots, x_m)^T \in \mathbb{R}^m, \tag{2}$$

with $j_i : \mathbb{R} \to \mathbb{R}$ being a locally Lipchitz continuous function for each index $i \in \{1, \ldots, m\}$. They investigated two important stability properties called Aubin/Lipschitz-like property and the isolated calmness of the solution mapping corresponding to (1). A generalization to the present setting with a smooth single-valued part can be found in [1], where also the calmness of the solution mapping is considered. In the second section, we derive conditions guaranteeing the strong metric regularity

[1]NTIS - New Technologies for the Information Society and Department of Mathematics, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 22, 306 14 Pilsen, Czech Republic, cibi@kma.zcu.cz

[2]Faculty of Applied Sciences, University of West Bohemia, Univerzitní 22, 306 14 Pilsen, Czech Republic, roubalt@students.zcu.cz

of the mapping $\Phi = f + BF(C\cdot)$ when f is either smooth or non-smooth but locally Lipschitz continuous. The latter statement is based on a recent result from [10] by A. F. Izmailov. We prove similar result on strong metric sub-regularity and apply it to the mapping Φ.

In the third section, we study the case when the parameter p in (1) varies as a given Lipschitz function over a fixed time interval. Based on a generalization of [8, Theorem 2.4.] we prove the existence of a Lipchitz continuous response.

The last section is devoted to numerical simulations and applications in electronics. A note [2] is devoted to the simulation issues which are performed by using Xcos (a component of Scilab). In order to use this software package, the set-valued part in (1) is approximated by a single-valued one. In [8], the authors considered an Euler-Newton continuation method for tracking solution trajectories of parametric variational inequalities. We derive similar method for tracking solution trajectories of parametric generalized equations under slightly weaker assumptions on differentiability of the corresponding single-valued part. Finally, implementing this method (in Matlab) we provide a simulation of the behavior of some basic non-regular circuits, i.e. the circuits where various types of diodes are present. Unlike [2], we work directly with a set-valued model here. Based on analytic expressions for a solution, we compare the efficiency of several numerical methods.

The notation is fairly standard. In $\mathbb{R}^d$, the norm, the scalar product, the closed and the open ball with the center $x \in \mathbb{R}^d$ and the radius $r \geq 0$, are denoted by $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, $\mathbb{B}[x,r]$, and $\mathbb{B}(x,r)$, respectively. We set $\mathbb{B} = \mathbb{B}[x,1]$. Fix a non-empty subset Ω of $\mathbb{R}^d$ containing a point $\bar{x}$ for a while; the *Fréchet/regular normal cone* to Ω at $\bar{x}$ is the set

$$\widehat{N}(\bar{x};\Omega) := \left\{ \xi \in \mathbb{R}^d : \limsup_{\Omega \ni x \to \bar{x}} \frac{\langle \xi, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq 0 \right\};$$

the *general/limiting normal cone* $N(\bar{x};\Omega)$ to Ω at $\bar{x}$ contains all $\xi \in \mathbb{R}^d$ for which there are sequences $(x^k)_{k\in\mathbb{N}}$ in Ω and $(\xi^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^d$ converging to $\bar{x}$ and ξ, respectively, such that $\xi^k \in \widehat{N}(x^k;\Omega)$ for each $k \in \mathbb{N}$; the *Bouligand-Severi tangent cone* $T(\bar{x};\Omega)$ to Ω at $\bar{x}$ contains those $v \in \mathbb{R}^d$ for which there are sequences $(t^k)_{k\in\mathbb{N}}$ in $(0,\infty)$ and $(v^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^d$ converging to 0 and v, respectively, such that $\bar{x} + t^k v^k \in \Omega$ whenever $k \in \mathbb{N}$; and finally the *Bouligand paratingent cone* $\widetilde{T}(\bar{x};\Omega)$ to Ω at $\bar{x}$ contains those $v \in \mathbb{R}^d$ for which there are sequences $(t^k)_{k\in\mathbb{N}}$ in $(0,\infty)$, $(v^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^d$, and $(x^k)_{k\in\mathbb{N}}$ in Ω converging to 0, v, and $\bar{x}$, respectively, such that $x^k + t^k v^k \in \Omega$ whenever $k \in \mathbb{N}$. The *distance* from a point $x \in \mathbb{R}^d$ to Ω is denoted by $d(x,\Omega)$ with convention that $d(x,\emptyset) = \infty$. Throughout $s : \mathbb{R}^d \to \mathbb{R}^l$ means that s is single-valued while $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^l$ denotes a general mapping which may be set-valued. For such a mapping S, the domain, the graph, and the range are denoted by $\operatorname{dom} S$, $\operatorname{gph} S$ and $\operatorname{rge} S$. Fix a point $(\bar{x},\bar{y}) \in \operatorname{gph} S$. Then the *selection for S around $\bar{x}$ for $\bar{y}$* is any single-valued mapping s defined on a neighborhood U of $\bar{x}$ such that

$$s(\bar{x}) = \bar{y} \quad \text{and} \quad s(x) \in S(x) \quad \text{for each } x \in U;$$

the *(graphical) localization of S around $\bar{x}$ for $\bar{y}$* is any set-valued mapping $\widetilde{S}$ such that for some neighborhoods V of $\bar{y}$ and U of $\bar{x}$ we have $\operatorname{gph} \widetilde{S} = \operatorname{gph} S \cap (U \times V)$ and $\operatorname{dom} \widetilde{S} \supset U$. Recall that $\Phi : \mathbb{R}^l \rightrightarrows \mathbb{R}^d$ is *strongly metrically regular* at $\bar{y}$ for $\bar{x}$ provided that $S := \Phi^{-1}$ has a Lipschitz continuous single-valued localization around $\bar{x}$ for $\bar{y}$; the mapping Φ is called *metrically regular at*

$\bar{y}$ *for* $\bar{x}$ if there is a constant $\kappa > 0$ along with neighborhoods V of $\bar{y}$ and U of $\bar{x}$ such that

$$(3) \qquad d\left(y, \Phi^{-1}(x)\right) \leq \kappa\, d(x, \Phi(y)) \quad \text{whenever} \quad (y, x) \in V \times U;$$

and Φ is *strongly metrically sub-regular at* $\bar{y}$ *for* $\bar{x}$ provided that there is $\kappa > 0$ along with a neighborhood V of $\bar{y}$ such that

$$(4) \qquad \|y - \bar{y}\| \leq \kappa\, d(\bar{x}, \Phi(y)) \quad \text{for each} \quad y \in V.$$

The infimum over all $\kappa > 0$ such that (3) holds for some neighborhoods U of $\bar{x}$ and V of $\bar{y}$ is the *regularity modulus* of Φ at $\bar{y}$ for $\bar{x}$ denoted by $\operatorname{reg}(\Phi; \bar{y}|\bar{x})$. Similarly, the infimum over all $\kappa > 0$ such that (4) holds for some neighborhood V of $\bar{y}$ is the *subregularity modulus* of Φ at $\bar{y}$ for $\bar{x}$ denoted by $\operatorname{subreg}(\Phi; \bar{y}|\bar{x})$. We will need the *paratingent/strict graphical derivative* of Φ at $(\bar{y}, \bar{x})$ which is the mapping $\widetilde{D}\Phi(\bar{y}, \bar{x}) : \mathbb{R}^l \rightrightarrows \mathbb{R}^d$ defined by

$$\widetilde{D}\Phi(\bar{y}, \bar{x})(u) := \{v \in \mathbb{R}^d : \ (u, v) \in \widetilde{T}((\bar{y}, \bar{x}); \operatorname{gph}\Phi)\}, \quad u \in \mathbb{R}^l.$$

Consider a locally Lipschitz continuous $h : \mathbb{R}^l \to \mathbb{R}^d$, i.e. for any $\bar{u} \in \mathbb{R}^l$ there is a neighborhood U of $\bar{u}$ along with a constant $L_{\bar{u}} > 0$ such that $\|h(\hat{u}) - h(\tilde{u})\| \leq L_{\bar{u}} \|\hat{u} - \tilde{u}\|$ whenever $\hat{u}, \tilde{u} \in U$. The infimum over all $L_{\bar{u}} > 0$ such that the previous inequality holds for some neighborhood U of $\bar{u}$ is the *Lipschitz modulus* of h at $\bar{u}$ and is denoted by $\operatorname{lip}(h; \bar{u})$. The *Bouligand's limiting Jacobian of* h *at* $\bar{u}$ is the (non-empty compact) set $\partial_B h(\bar{u})$ consisting of all matrices $A \in \mathbb{R}^{d \times l}$ for which there is a sequence $(u_n)_{n \in \mathbb{N}}$ such that h is differentiable at each u_n and $\nabla h(u_n) \to A$ as $n \to \infty$. The *Clarke's generalized Jacobian of* h *at* $\bar{u}$, denoted by $\partial h(\bar{u})$, is the convex hull of $\partial_B h(\bar{u})$. Finally, a function $h : \mathbb{R}^l \to \mathbb{R}^d$ is *calm* at $\bar{u}$ relative to $U \subset \operatorname{dom} h$ with the constant $\mu > 0$ provided that

$$\|h(u) - h(\bar{u})\| \leq \mu \|u - \bar{u}\| \quad \text{for each} \quad u \in U.$$

Standing assumptions. Denote by Φ the set-valued mapping from $\mathbb{R}^n$ into itself defined by $\Phi(z) := f(z) + BF(Cz)$ whenever $z \in \mathbb{R}^n$. Let us define the mappings $Q : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $F_C : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ by $Q(z) := BF(Cz)$, $z \in \mathbb{R}^n$, and $F_C(u) := F(u)$ if $u = Cz$ for some $z \in \mathbb{R}^n$ and $F_C(u) := \emptyset$ otherwise. We also suppose that we have in hand a point $(\bar{z}, \bar{p}) \in \operatorname{gph}\Phi$. Finally, put $\bar{v} := (B^T B)^{-1} B^T (\bar{p} - f(\bar{z}))$.

We will also refer to several combinations of the additional assumptions in the main results. Namely,

($A1$) B is injective;

($A2$) f is continuously differentiable on $\mathbb{R}^n$;

($\widetilde{A2}$) f is locally Lipschitz continuous on $\mathbb{R}^n$;

($A3$) F has closed graph;

($A4$) C is surjective; and

($A5$) there are $F_i : \mathbb{R} \rightrightarrows \mathbb{R}$, $i \in \{1, \ldots, m\}$ such that $F(x) = \prod_{i=1}^{m} F_i(x_i)$ whenever $x = (x_1, \ldots, x_m)^T \in \mathbb{R}^m$.

2 Regularity properties of Φ at $\bar{z}$ for $\bar{p}$

First, we are going to compute the strict graphical derivative of Φ at the reference point. Let us start with the following geometric lemma which is an analogue of Lemma 4.1 in [1], where the classical Bouligand-Severi tangent cone was considered.

Lemma 2.1. *Let $E \in \mathbb{R}^{k\times d}$ be any matrix, let $G \in \mathbb{R}^{l\times d}$ be injective, and let Γ be a subset of* $\operatorname{rge} E$. *Put $\Xi := E^{-1}(\Gamma)$ and $\Lambda := G(\Xi)$. For $\bar{x} \in \Lambda$ denote by $\bar{y}$ the (unique) point in Ξ with $G\bar{y} = \bar{x}$. Then*

$$\widetilde{T}(\bar{x};\Lambda) = \{u \in \mathbb{R}^l : \exists w \in \mathbb{R}^d \text{ such that } u = Gw \text{ and } Ew \in \widetilde{T}(E\bar{y};\Gamma)\}.$$

Proof. We `claim` that $\widetilde{T}(\bar{y};\Xi) = \{w \in \mathbb{R}^d : \ Ew \in \widetilde{T}(E\bar{y},\Gamma)\}$. First, take any $w \in \widetilde{T}(\bar{y};\Xi)$. Find $(t^n)_{n\in\mathbb{N}}$ in $(0,\infty)$, $(y^n)_{n\in\mathbb{N}}$ in Ξ, and $(w^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ converging to 0, $\bar{y}$ and w, respectively, such that $y^n + t^n w^n \in \Xi$ whenever $n \in \mathbb{N}$. Then we have that $Ey^n + t^n Ew^n = E(y^n + t^n w^n) \in \Gamma$ for each $n \in \mathbb{N}$. Hence $Ew \in \widetilde{T}(E\bar{y};\Gamma)$. On the other hand, let $w \in \mathbb{R}^d$ be such that $Ew \in \widetilde{T}(E\bar{y},\Gamma)$. Pick $(t^n)_{n\in\mathbb{N}}$ in $(0,\infty)$, and $(u^n)_{n\in\mathbb{N}}$ in Γ, and $(v^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^k$ converging to 0, $E\bar{y}$ and Ew, respectively, such that $u^n + t^n v^n \in \Gamma$ whenever $n \in \mathbb{N}$. As $\Gamma \subset \operatorname{rge} E$, where the latter set is a closed subspace of $\mathbb{R}^k$, one infers that $v^n \in \operatorname{rge} E$ for each $n \in \mathbb{N}$. Therefore, by Banach open mapping theorem there are sequences $(y^n)_{n\in\mathbb{N}}$ converging to $\bar{y}$ and $(w^n)_{n\in\mathbb{N}}$ converging to w, both in $\mathbb{R}^d$, such that

$$Ey^n = u^n \quad \text{and} \quad Ew^n = v^n \quad \text{for each} \quad n \in \mathbb{N}.$$

Thus, for an arbitrary index n, we have $Ey^n \in \Gamma$ and $E(y^n + t^n w^n) \in \Gamma$, hence both y^n and $y^n + t^n w^n$ are in $E^{-1}(\Gamma) = \Xi$. So $w \in \widetilde{T}(\bar{y};\Xi)$. The claim is proved.
Now, we show that $\widetilde{T}(\bar{x};\Lambda) = \{Gw : \ w \in \widetilde{T}(\bar{y},\Xi)\}$. To prove that $G\big(\widetilde{T}(\bar{y};\Xi)\big) \subset \widetilde{T}(\bar{x};\Lambda)$, pick any $w \in G\big(\widetilde{T}(\bar{y};\Xi)\big)$. Find $v \in \widetilde{T}(\bar{y};\Xi)$ with $Gv = w$. Thus there is $(t^n)_{n\in\mathbb{N}}$ in $(0,\infty)$ converging to 0, $(y^n)_{n\in\mathbb{N}}$ in Ξ converging to $\bar{y}$ and $(v^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ converging to v such that $y^n + t^n v^n \in \Xi$ whenever $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, put $u^n = Gy^n$ and $w^n = Gv^n$. Clearly, $(u^n)_{n\in\mathbb{N}}$ converges to $G\bar{y} = \bar{x}$ and $(w^n)_{n\in\mathbb{N}}$ converges to w. Moreover,

$$u^n + t^n w^n = G(y^n + t^n v^n) \in G(\Xi) = \Lambda \quad \text{whenever} \quad n \in \mathbb{N}.$$

So $w \in \widetilde{T}(\bar{x};\Lambda)$. To see the opposite inclusion, pick any $w \in \widetilde{T}(\bar{x};\Lambda)$. Find $(t^n)_{n\in\mathbb{N}}$ in $(0,\infty)$ converging to 0, $(x^n)_{n\in\mathbb{N}}$ in Λ converging to $\bar{x}$, and $(w^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^l$ converging to w such that

$$x^n + t^n w^n \in G(\Xi) \quad \text{for each} \quad n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, find $y^n \in \Xi$ such that $x^n = Gy^n$. Then $(y^n)_{n\in\mathbb{N}}$ is bounded. Indeed, if this is not the case, find a cluster point $\bar{h}$ of $(y^n/\|y^n\|)_{n\in\mathbb{N}}$. Let N be an infinite subset of $\mathbb{N}$ such that $\lim_{N\ni n\to\infty} y^n/\|y^n\| = \bar{h}$. Then

$$0 = \lim_{N\ni n\to\infty} \frac{x^n}{\|y^n\|} = \lim_{N\ni n\to\infty} G\left(\frac{y^n}{\|y^n\|}\right) = G\bar{h}.$$

This contradicts the injectivity of G because $\|\bar{h}\| = 1$. Therefore there is an infinite subset N of $\mathbb{N}$ such that $(y^n)_{n\in N}$ converges to $\tilde{y} \in \mathbb{R}^d$, say. Then

$$G\bar{y} = \bar{x} = \lim_{N\ni n\to\infty} x^n = \lim_{N\ni n\to\infty} Gy^n = G\tilde{y}.$$

Employing, the injectivity once more, we get $\bar{y} = \tilde{y}$. For each $n \in N$, find v^n in Ξ such that $w^n = G\big((v^n - y^n)/t^n\big)$, and put $u^n = (v^n - y^n)/t^n$. Similar argument as in the case of $(y^n)_{n\in\mathbb{N}}$ shows that $(u^n)_{n\in N}$ is bounded. Therefore there is an infinite subset N' of N such that $(u^n)_{n\in N'}$ converges to some $u \in \mathbb{R}^d$. For each $n \in N'$, we have $y^n + t^n u^n = v^n \in \Xi$, therefore $u \in \widetilde{T}(\bar{y}; \Xi)$. Moreover, $w = G(u)$. The proof is finished. This and the claim yield the assertion. □

Proposition 2.1. *Under the assumptions (A1) – (A2), for any $b \in \mathbb{R}^n$ one has*

$$\widetilde{D}\Phi(\bar{z},\bar{p})(b) = \nabla f(\bar{z})b + B\,\widetilde{D}F_C(C\bar{z},\bar{v})(Cb).$$

Proof. As in [7, Proposition 4A.2], it is elementary to show that

$$\widetilde{D}\Phi(\bar{z},\bar{p})(b) = \nabla f(\bar{z})b + \widetilde{D}Q(\bar{z},\bar{p}-f(\bar{z}))(b) \quad \text{for each} \quad b \in \mathbb{R}^n.$$

Further, observe that

$$\operatorname{gph} Q = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^{2n} : \exists \begin{pmatrix} b \\ c \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^m : \begin{pmatrix} u \\ v \end{pmatrix} = G \begin{pmatrix} b \\ c \end{pmatrix} \quad \text{and} \quad E \begin{pmatrix} b \\ c \end{pmatrix} \in \operatorname{gph} F_C \right\},$$

with

$$G := \begin{pmatrix} I_n & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad E := \begin{pmatrix} C & 0 \\ 0 & I_m \end{pmatrix}.$$

As B is injective, so is G. Lemma 2.1 (with $k := 2m$, $l := 2n$, $d := n+m$, $\Gamma := \operatorname{gph} F_C$, $\bar{x} := (\bar{z},\bar{p}-f(\bar{z}))^T$, and $\bar{y} := (\bar{z},\bar{v})^T$) reveals that

$$\widetilde{T}\big((\bar{z},\bar{p}-f(\bar{z})); \operatorname{gph} Q\big) = \left\{ \begin{pmatrix} b \\ Bc \end{pmatrix} : \begin{pmatrix} Cb \\ c \end{pmatrix} \in \widetilde{T}\big((C\bar{z},\bar{v}); \operatorname{gph} F_C\big) \right\}.$$

This means that $\widetilde{D}Q(\bar{z},\bar{p}-f(\bar{z}))(b) = B\,\widetilde{D}F_C(C\bar{z},\bar{v})(Cb)$. The assertion is proved. □

The following statement is a slight modification of the condition by B. Kummer (e.g, see [7, Theorem 4D.1]) guaranteeing the strong metric regularity of a set-valued mapping.

Proposition 2.2. *Consider a set-valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and a point $(\bar{x},\bar{y}) \in \operatorname{gph} H$. Then H is strongly metrically regular at $\bar{x}$ for $\bar{y}$ if and only if it verifies the following three conditions:*

(a) for each neighborhood U of $\bar{x}$ there is a neighborhood V of $\bar{y}$ such that $H^{-1}(y) \cap U \neq \emptyset$ whenever $y \in V$;

(b) the set $\operatorname{gph} H \cap \big(\mathbb{B}[\bar{x},r] \times \mathbb{B}[\bar{y},r]\big)$ is closed for some $r > 0$;

(c) $0 \in \widetilde{D}H(\bar{x}|\bar{y})(u) \quad \Longrightarrow \quad u = 0.$

Proof. Suppose that H is strongly metrically regular at $\bar{x}$ for $\bar{y}$. Then (c) holds by [7, Theorem 4D.1]. Observe also that H has necessarily locally closed graph at the reference point. Finally, (a) is satisfied since H is open at $(\bar{x}, \bar{y})$, i.e. for any neighborhood U of $\bar{x}$ the set $V := H(U)$ is a neighborhood of $\bar{y}$. The converse implication is proved in [7, Theorem 4D.1]. □

Trivial examples show that the premise (a) cannot be omitted. Indeed, define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x$, $x \geq 0$. Then both (b) and (c) are valid, (a) fails and h is not strongly metrically regular at 0 for 0.

Theorem 2.1. *Assume that (A1) – (A3) hold true. Then Φ is strongly metrically regular at $\bar{z}$ for $\bar{p}$ if and only if*

(a) for each neighborhood U of $\bar{z}$ there is a neighborhood V of $\bar{p}$ such that $\Phi^{-1}(p) \cap U \neq \emptyset$ whenever $p \in V$;

(b) $0 \in \nabla f(\bar{z})b + B\,\widetilde{D}F_C(C\bar{z}, \bar{v})(Cb) \quad \Longrightarrow \quad b = 0.$

Moreover, its regularity modulus is given by

$$\operatorname{reg}(\Phi; \bar{z}|\bar{p}) = \sup\Big\{\|b\| : \ \Big(\nabla f(\bar{z})b + B\widetilde{D}F_C(C\bar{z}, \bar{v})(Cb)\Big) \cap \mathbb{B} \neq \emptyset\Big\}.$$

Proof. Observe that Φ has closed graph and combine Proposition 2.2 and Proposition 2.1 to conclude the proof. □

Now, we are in position to formulate the main statement of this section.

Theorem 2.2. *Suppose that the assumptions (A1) – (A4) hold true. Then*

(i) Φ is metrically regular at $\bar{z}$ for $\bar{p}$ if and only if

$$\left.\begin{array}{c} \Big(\big(CC^T\big)^{-1}C\nabla f(\bar{z})^T\xi, B^T\xi\Big) \in -N\big((C\bar{z}, \bar{v}); \operatorname{gph} F\big) \\ \nabla f(\bar{z})^T\xi \in \operatorname{rge} C^T \end{array}\right\} \quad \Longrightarrow \quad \xi = 0;$$

(ii) Φ is strongly metrically sub-regular at $\bar{z}$ for $\bar{p}$ if and only if

$$\left.\begin{array}{c} \Big(Cb, -\big(B^TB\big)^{-1}B^T\nabla f(\bar{z})b\Big) \in T\big((C\bar{z}, \bar{v}); \operatorname{gph} F\big) \\ \nabla f(\bar{z})b \in \operatorname{rge} B \end{array}\right\} \quad \Longrightarrow \quad b = 0;$$

(iii) Φ is strongly metrically regular at $\bar{z}$ for $\bar{p}$ if and only if

(a) for each neighborhood U of $\bar{z}$ there is a neighborhood V of $\bar{p}$ such that $\Phi^{-1}(p) \cap U \neq \emptyset$ whenever $p \in V$;

(b)

$$\left.\begin{array}{c}\Big(Cb, -\big(B^TB\big)^{-1}B^T\nabla f(\bar z)b\Big) \in \widetilde{T}\big((C\bar z,\bar v);\operatorname{gph} F\big)\\ \nabla f(\bar z)b \in \operatorname{rge} B\end{array}\right\} \implies b = 0,$$

Proof. The statement (i) is [1, Corollary 3.1], whereas (ii) is [1, Corollary 4.1]. To see the last one, note that if C is surjective, then $F_C = F$. Moreover, $(A1)$ ensures that $B^TB \in \mathbb{R}^{m\times m}$ is non-singular. It suffices to show that (b) is equivalent to Theorem (2.1) (b).

First, let $b \in \mathbb{R}^n$ be such that $0 \in \nabla f(\bar z)b + B\widetilde{D}F(C\bar z,\bar v)(Cb)$. Find a point $w \in \widetilde{D}F(C\bar z,\bar v)(Cb)$ with $\nabla f(\bar z)b + Bw = 0$. Thus $-\big(B^TB\big)^{-1}B^T\nabla f(\bar z)b$ is in $\widetilde{D}F(C\bar z,\bar v)(Cb)$. Clearly, we have $\nabla f(\bar z)b \in \operatorname{rge} B$ and the definition of the paratingent derivative of F yields the rest.

On the other hand, pick any $b \in \mathbb{R}^n$ with $\Big(Cb, -\big(B^TB\big)^{-1}B^T\nabla f(\bar z)b\Big)$ in $\widetilde{T}\big((C\bar z,\bar v);\operatorname{gph} F\big)$ and $\nabla f(\bar z)b \in \operatorname{rge} B$. The definition of the paratingent derivative says that $w := -\big(B^TB\big)^{-1}B^T\nabla f(\bar z)b \in \widetilde{D}F(C\bar z,\bar v)(Cb)$. Thus we have $B^TBw = -B^T\nabla f(\bar z)b$. So $Bw + \nabla f(\bar z)b \in \ker B^T \cap \operatorname{rge} B = \{0\}$. Therefore $0 \in \nabla f(\bar z)b + B\widetilde{D}F(C\bar z,\bar v)(Cb)$. ◻

Now, we derive a simple sufficient condition guaranteeing the strong metric regularity which will be applied to the problems arising in electronics in Section 4. A matrix $M \in \mathbb{R}^{n\times n}$ is called *P-matrix* provided that all its k-by-k principal minors are positive whenever $k \in \{1,\dots,n\}$. It is well known, that M is a P-matrix if and only if for any non-zero $x \in \mathbb{R}^n$ there is $j \in \{1,\dots,n\}$ such that $x_j(Mx)_j > 0$.

Corollary 2.1. *In addition to (A1)–(A5), assume that $n = m$, that $B = C = I_n$, that $\nabla f(\bar z)$ is a P-matrix, and that for each $i \in \{1,2,\dots,n\}$, the mapping $F_i : \mathbb{R} \rightrightarrows \mathbb{R}$ is maximal monotone. Then Φ is strongly metrically regular at $\bar z$ for $\bar p$.*

Proof. By (A5), we have $\prod_{i=1}^n \operatorname{gph} F_i = \varphi\big(\operatorname{gph} F\big)$, where

$$\varphi(x,y) = \big((x_1,y_1),\dots,(x_n,y_n)\big), \quad x = (x_1,\dots,,x_n), y = (y_1,\dots,,y_n) \in \mathbb{R}^n.$$

Clearly, φ is linear and one-to-one. The definition of the paratingent cone and Lemma 2.1 imply that

$$\prod_{i=1}^n \widetilde{T}\big((\bar z_i,\bar v_i);\operatorname{gph} F_i\big) \supset \widetilde{T}\Big(\varphi(\bar z,\bar v);\prod_{i=1}^n \operatorname{gph} F_i\Big) = \varphi\Big(\widetilde{T}\big((\bar z,\bar v);\operatorname{gph} F\big)\Big).$$

Also, it is well-known that

$$\prod_{i=1}^n N\big((\bar z_i,\bar v_i);\operatorname{gph} F_i\big) = N\Big(\varphi(\bar z,\bar v);\prod_{i=1}^n \operatorname{gph} F_i\Big) = \varphi\Big(N\big((\bar z,\bar v);\operatorname{gph} F\big)\Big).$$

As all F_i's are maximal monotone, we have $N\big((\bar z_i,\bar v_i);\operatorname{gph} F_i\big) \subset \{(a,b) \in \mathbb{R}^2 :\ ab \le 0\}$ and $\widetilde{T}\big((\bar z_i,\bar v_i);\operatorname{gph} F_i\big) \subset \{(a,b) \in \mathbb{R}^2 :\ ab \ge 0\}$ for each $i \in \{1,\dots,n\}$. Fix any non-zero $\eta \in \mathbb{R}^n$. Since $\nabla f(\bar z)$ is a P-matrix, so is $\nabla f(\bar z)^T$. There are k, $l \in \{1,\dots,n\}$ such that $\eta_k(\nabla f(\bar z)\eta)_k > 0$ and

$\eta_l(\nabla f(\bar{z})^T\eta)_l > 0$, which means that $(\eta_k, -(\nabla f(\bar{z})\eta)_k) \notin \widetilde{T}\big((\bar{z}_k, \bar{v}_k); \mathrm{gph} F_k\big)$ and $\big((\nabla f(\bar{z})^T\eta)_l, \eta_l\big) \notin -N((\bar{z}_l, \bar{v}_l); \mathrm{gph} F_l\big)$. The above relations for the normal and parantingent cone and the fact that φ is one-to-one imply that both the conditions in Theorem 2.2 (iii) hold (the first one thanks to the statement (i) of this theorem). □

Remark 2.1. D. Goeleven [9] considered the case when $f(z) := Az$, $z \in \mathbb{R}^n$, with a given P-matrix $A \in \mathbb{R}^{n\times n}$ and F is the Fenchel-Moreau-Rockafellar subdifferential of the super-potential j defined by (2) with $j_i : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, $i \in \{1, \ldots, n\}$, being a proper, lower semi-continuous convex function such that

$$\lambda j_i(x) = j_i(\lambda x) \quad \text{for each} \quad \lambda \geq 0 \quad \text{and} \quad x \in \mathrm{dom}\, j_i.$$

In fact, this assumption guarantees that each j_i is differentiable at each non-zero point of its domain (the existence of which is assumed but never explicitly mentioned in [9]). More precisely, for each $i \in \{1, \ldots, n\}$, the mapping $F_i := \partial j_i$ is "piece-wise constant", in sense that $F_i(x)$ equals either

$$\begin{cases} \{\alpha\}, & x < 0, \\ \{\beta\}, & x > 0, \\ [\alpha, \beta], & x = 0; \end{cases} \quad \text{or} \quad \begin{cases} \{\alpha\}, & x < 0, \\ [\alpha, +\infty), & x = 0; \end{cases} \quad \text{or} \quad \begin{cases} \{\beta\}, & x > 0, \\ (-\infty, \beta], & x = 0, \end{cases}$$

with $\alpha := -j_i(-1) \leq j_i(1) =: \beta$ provided that the corresponding value is finite. In this case, [9, Theorem 2.1] says that Φ^{-1} is single-valued with the whole of $\mathbb{R}^n$ as its domain. By Corollary 2.1, we get a generalization of [9, Proposition 2.1] proving that the solution mapping is not only continuous but locally Lipschitz (and therefore Lipschitz on any compact set).

Izmailov's theorem, see e.g. [5, Theorem 2], provides a sufficient condition for strong metric regularity even when the single-valued part is locally Lipschitz continuous only. Let us present similar result for the strong metric sub-regularity. We start with a quantitative version of the well-known fact that the strong metric sub-regularity is stable with respect to a calm single-valued perturbation.

Lemma 2.2. *Let $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be such that $\bar{y} \in G(\bar{x})$. Suppose that there is $\kappa > 0$ along with a neighborhood U of $\bar{x}$ such that*

$$\|x - \bar{x}\| \leq \kappa d\big(\bar{y}, G(x)\big) \quad \textit{whenever} \quad x \in U.$$

Then for any function $g : \mathbb{R}^n \to \mathbb{R}^m$ which is calm at $\bar{x}$ relative to $U \subset \mathrm{dom}\, g$ with the constant $\mu < 1/\kappa$ one has

$$\|x - \bar{x}\| \leq \frac{\kappa}{1 - \kappa\mu} d\big(\bar{y} + g(\bar{x}), g(x) + G(x)\big) \quad \textit{for each} \quad x \in U.$$

Proof. Fix any $x \in U$. The calmness of g means that $\|g(x) - g(\bar{x})\| \leq \mu\|x - \bar{x}\|$. Therefore

$$\begin{aligned} \|x - \bar{x}\| &\leq \kappa d\big(\bar{y}, G(x)\big) \leq \kappa\|g(x) - g(\bar{x})\| + \kappa d\big(\bar{y} + g(\bar{x}) - g(x), G(x)\big) \\ &\leq \kappa\mu\|x - \bar{x}\| + \kappa d\big(\bar{y} + g(\bar{x}), g(x) + G(x)\big). \end{aligned}$$

Performing a small rearrangement and dividing by $1 - \kappa\mu > 0$, we obtain the desired inequality. □

Theorem 2.3. *Let $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be such that $\bar{y} \in g(\bar{x}) + G(\bar{x})$. Suppose that there exists a compact subset $\mathcal{A}$ of $\mathbb{R}^{m\times n}$ such that*

(i) for each $\varepsilon > 0$ there exists $r > 0$ such that for each $u \in \mathbb{B}(\bar{x}, r)$ one can find $A \in \mathcal{A}$ such that

$$\|g(u) - g(\bar{x}) - A(u - \bar{x})\| \leq \varepsilon \|u - \bar{x}\|;$$

(ii) for every $A \in \mathcal{A}$ the mapping

$$G_A : \mathbb{R}^n \ni x \longmapsto g(\bar{x}) + A(x - \bar{x}) + G(x) \subset \mathbb{R}^m$$

is strongly metrically sub-regular at $\bar{x}$ for $\bar{y}$ and let $m := \sup_{A\in\mathcal{A}} \operatorname{subreg}(G_A; \bar{x}\,|\,\bar{y})$.

Then $g + G$ is strongly metrically sub-regular at $\bar{x}$ for $\bar{y}$; and $\operatorname{subreg}(g + G; \bar{x}\,|\,\bar{y}) \leq m$.

Proof. Without any loss of generality assume that $\bar{y} = 0$. Since $\mathcal{A}$ is compact, we have $m < \infty$. Fix any $\kappa > m$. Find $\varepsilon > 0$ such that $2\varepsilon\kappa < 1$. Let $r > 0$ be as in (i). First, we show that there exists $a \in (0, r]$ such that

$$\|x - \bar{x}\| \leq \frac{\kappa}{1 - \kappa\varepsilon} d\big(0, G_A(x)\big) \quad \text{whenever} \quad x \in \mathbb{B}(\bar{x}, a) \quad \text{and} \quad A \in \mathcal{A}. \tag{5}$$

As $\mathcal{A}$ is compact, there is a finite set $\mathcal{A}_F \subset \mathcal{A}$ such that

$$\mathcal{A} \subset \mathcal{A}_F + \varepsilon\mathbb{B}. \tag{6}$$

Pick any $\tilde{A} \in \mathcal{A}_F$. Then there exists $\alpha_{\tilde{A}} > 0$ such that

$$\|x - \bar{x}\| \leq \kappa d\big(0, G_{\tilde{A}}(x)\big) \quad \text{whenever} \quad x \in \mathbb{B}(\bar{x}, \alpha_{\tilde{A}}).$$

Fix any $A' \in \varepsilon\mathbb{B}$. As $G_{\tilde{A}+A'} = G_{\tilde{A}} + A'(x - \bar{x})$, Lemma 2.2 reveals that

$$\|x - \bar{x}\| \leq \frac{\kappa}{1 - \kappa\varepsilon} d\big(0, G_{\tilde{A}+A'}(x)\big) \quad \text{for any} \quad x \in \mathbb{B}(\bar{x}, \alpha_{\tilde{A}}).$$

Thus for any $\tilde{A} \in \mathcal{A}_F$ there is $\alpha_{\tilde{A}} > 0$ such that for each $A' \in \varepsilon\mathbb{B}$ the above inequality holds. Let $a = \min\big\{r, \min_{\tilde{A}\in\mathcal{A}_F} \alpha_{\tilde{A}}\big\}$. Taking into account (6), we obtain (5).

Fix any $x \in \mathbb{B}(\bar{x}, a)$. Use (i) to find $A \in \mathcal{A}$ such that $\|g(x) - g(\bar{x}) - A(x - \bar{x})\| \leq \varepsilon\|x - \bar{x}\|$. This and (5) implies that

$$\begin{aligned}
\|x - \bar{x}\| &\leq \frac{\kappa}{1 - \kappa\varepsilon} d\big(0, G_A(x)\big) = \frac{\kappa}{1 - \kappa\varepsilon} d\big(-g(\bar{x}) - A(x - \bar{x}), G(x)\big) \\
&\leq \frac{\kappa}{1 - \kappa\varepsilon} \Big(d\big(-g(x), G(x)\big) + \|g(x) - g(\bar{x}) - A(x - \bar{x})\|\Big) \\
&\leq \frac{\kappa}{1 - \kappa\varepsilon} d\big(0, g(x) + G(x)\big) + \frac{\kappa\varepsilon}{1 - \kappa\varepsilon}\|x - \bar{x}\|.
\end{aligned}$$

Since $\kappa\varepsilon/(1 - \kappa\varepsilon) < 1$, we get that

$$\|x - \bar{x}\| \leq \frac{\kappa}{1 - 2\varepsilon\kappa} d\big(0, g(x) + G(x)\big).$$

Thus $g + G$ is strongly metrically sub-regular at $\bar{x}$ for 0. As $\varepsilon > 0$ and $\kappa > m$ were arbitrary, we get the desired estimate on the sub-regularity modulus. ∎

For a locally Lipschitz continuous function g, the assumption (i) holds when $\mathcal{A} := \partial g(\bar{x})$. Another possible choice is Bouligand's limiting Jacobian under an additional assumption on g.

Corollary 2.2. *Let $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be such that $\bar{y} \in g(\bar{x}) + G(\bar{x})$. Suppose that g is locally Lipschitz continuous at $\bar{x}$ and that for every $\varepsilon > 0$ there exists $r > 0$ along with a selection h for $\partial_B g$ such that*

$$\|g(u) - g(\bar{x}) - h(u)(u - \bar{x})\| \leq \varepsilon \|u - \bar{x}\| \quad \text{whenever} \quad u \in \mathbb{B}(\bar{x}, r). \tag{7}$$

Assume that the assumption (ii) in Theorem 2.3 is satisfied with $\mathcal{A} := \partial_B g(\bar{x})$. Then $g + G$ is strongly metrically sub-regular at $\bar{x}$ for $\bar{y}$; and $\operatorname{subreg}(g + G; \bar{x}\,|\,\bar{y}) \leq m$.

Proof. Let $\gamma \in (0, 1)$ be such that $\gamma(m + \gamma) < 1$. Set $\mathcal{A} = \partial_B g(\bar{x}) + \gamma \mathbb{B}$. Then $\mathcal{A}$ is compact. By Lemma 2.2, for any $\tilde{A} \in \partial_B g(\bar{x})$ and any $A' \in \gamma \mathbb{B}$, the mapping $G_{\tilde{A}+A'} = G_{\tilde{A}} + A'(x - \bar{x})$ is strongly metrically sub-regular at $\bar{x}$ for $\bar{y}$ with the modulus at most $(m+\gamma)/(1-(m+\gamma)\gamma)$. Thus for every $A \in \mathcal{A}$ the mapping G_A is strongly metrically sub-regular at $\bar{x}$ for $\bar{y}$; and

$$m' := \sup_{A \in \mathcal{A}} \operatorname{subreg}(G_A; \bar{x}\,|\,\bar{y}) \leq \frac{m + \gamma}{1 - (m + \gamma)\gamma}.$$

Let $\varepsilon \in (0, \gamma)$ be arbitrary. By the outer semi-continuity of $\partial_B g$ and (7), there is $r > 0$ and a selection h for $\partial_B g$ such that, for each $u \in \mathbb{B}(\bar{x}, r)$, one has

$$h(u) \in \partial_B g(u) \subset \partial_B g(\bar{x}) + \varepsilon \mathbb{B} \subset \mathcal{A} \quad \text{and} \quad \|g(u) - g(\bar{x}) - h(u)(u - \bar{x})\| \leq \varepsilon \|u - \bar{x}\|.$$

Thus for each $u \in \mathbb{B}(\bar{x}, r)$ one can find $A \in \mathcal{A}$ such that $\|g(u) - g(\bar{x}) - A(u - \bar{x})\| \leq \varepsilon \|u - \bar{x}\|$. Theorem 2.3 implies that the mapping $g + G$ is strongly metrically sub-regular at $\bar{x}$ for $\bar{y}$; and $\operatorname{subreg}(g + G; \bar{x}\,|\,\bar{y}) \leq m'$. As $\gamma > 0$ can be arbitrarily small the proof is finished. □

Note that (7) is satisfied when g is semi-smooth at the reference point. To conclude this section, let us present strong metric (sub-)regularity criteria for (1) with a non-smooth f.

Theorem 2.4. *Under the assumptions (A1), ($\widetilde{A2}$), (A3), and (A4), for any $A \in \partial f(\bar{z})$, define the mapping*

$$J_A : \mathbb{R}^n \ni z \to f(\bar{z}) + A(z - \bar{z}) + BF(Cz).$$

(i) The mapping Φ is strongly metrically sub-regular at $\bar{z}$ for $\bar{p}$ provided that for each $A \in \partial f(\bar{z})$, one has that

$$\left.\begin{matrix} \left(Cb, -\left(B^T B\right)^{-1} B^T A b\right) \in T\big((C\bar{z}, \bar{v}); \operatorname{gph} F\big) \\ Ab \in \operatorname{rge} B \end{matrix}\right\} \implies b = 0_{\mathbb{R}^n};$$

(ii) The mapping Φ is strongly metrically regular at $\bar{z}$ for $\bar{p}$, provided that for each $A \in \partial f(\bar{z})$, one has that

(a) *for each neighborhood* U *of* $\bar{z}$ *there is a neighborhood* V *of* $\bar{p}$ *such that* $J_A^{-1}(p) \cap U \neq \emptyset$ *whenever* $p \in V$*;*

(b)

$$\left.\begin{array}{c}\Big(Cb, -\big(B^TB\big)^{-1}B^TAb\Big) \in \widetilde{T}\big((C\bar{z},\bar{v}); \operatorname{gph} F\big)\\ Ab \in \operatorname{rge} B\end{array}\right\} \implies b = 0_{\mathbb{R}^n}.$$

Proof. (i) For each $A \in \partial f(\bar{z})$, the mapping J_A is strongly metrically sub-regular at $\bar{z}$ for $\bar{p}$ by Theorem 2.2 (ii) with $\Phi := J_A$. Apply Theorem 2.3 to get the conclusion.

(ii) The conditions (a) and (b) guarantee that, for each $A \in \partial f(\bar{z})$, the mapping J_A is strongly metrically regular at $\bar{z}$ for $\bar{p}$. By [5, Theorem 2], Φ is strongly metrically regular at $\bar{z}$ for $\bar{p}$. ◻

3 Existence of a Lipschitz continuous response

In this section, we study parameters depending on time. More precisely, a function $p : [a, b] \to \mathbb{R}^n$ is given and one wants to find $z : [a, b] \to \mathbb{R}^n$ such that (1) holds at each instant of time, i.e.

$$p(t) \in f(z(t)) + BF(Cz(t)) \quad \text{whenever} \quad t \in [a, b]. \tag{8}$$

Given $g : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, consider the parametric generalized equation:

$$\text{For} \quad y \in \mathbb{R}^d \quad \text{find} \quad z \in \mathbb{R}^n \quad \text{such that} \qquad 0 \in g(y, z) + G(z), \tag{9}$$

along with the corresponding solution mapping $S : \mathbb{R}^d \ni y \longmapsto \{z \in \mathbb{R}^n : 0 \in g(y, z) + G(z)\}$. Solving the problem (8) with a fixed $p(\cdot)$ means to find $z(\cdot)$ such that $z(t) \in S(t)$ for each $t \in [a, b]$ with $d = 1$, $g(t, z) := f(z) - p(t)$, and $G(z) = BF(Cz)$, $(t, z) \in \mathbb{R} \times \mathbb{R}^n$.

The following statement is a slight generalization of a parametric version of [8, Theorem 2.4], where the existence of a Lipschitz localization instead of a Lipschitz selection is considered.

Theorem 3.1. *Given* $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *and* $g : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$*, let* S *be a solution mapping for* (9)*. Suppose that a point* $(\bar{y}, \bar{z}) \in \mathbb{R}^d \times \mathbb{R}^n$ *and positive constants* α*,* β*,* δ*,* κ*,* μ *and* ν *are such that*

(i) *there is a function* $s : \mathbb{B}[0, \delta] \to \mathbb{R}^n$ *which is Lispchitz continuous with the constant* $\kappa < 1/\mu$*,*

$$s(0) = \bar{z} \quad \text{and} \quad s(y) \in G^{-1}(y) \quad \text{for each} \quad y \in \mathbb{B}[0, \delta];$$

(ii) $\|g(\bar{y}, \bar{z})\| \leq \beta$*;*

(iii) $\|g(y, \hat{z}) - g(y, \tilde{z})\| \leq \mu \|\hat{z} - \tilde{z}\| \quad$ *whenever* $\quad y \in \mathbb{B}[\bar{y}, \delta]$ *and* $\hat{z}, \tilde{z} \in \mathbb{B}[\bar{z}, \delta]$*;*

(iv) $\|g(\hat{y}, z) - g(\tilde{y}, z)\| \leq \nu \|\hat{y} - \tilde{y}\| \quad$ *whenever* $\quad z \in \mathbb{B}[\bar{z}, \delta]$ *and* $\hat{y}, \tilde{y} \in \mathbb{B}[\bar{y}, \delta]$*;*

(v) $\beta\kappa'(1 + \nu) \leq \alpha$ *and* $\alpha \leq \delta \min\{1, \kappa\}$*, where* $\kappa' := \kappa/(1 - \mu\kappa)$*.*

Then there is $\sigma : \mathbb{R}^d \to \mathbb{R}^n$ which is Lipschitz continous on $\mathbb{B}[\bar{y}, \beta]$ with the constant $\kappa'\nu$ such that

$$\|\bar{z} - \sigma(\bar{y})\| \leq \kappa' \|g(\bar{y}, \bar{z})\| \quad \textit{and} \quad \sigma(y) \in S(y) \cap \mathbb{B}[\bar{z}, \alpha] \quad \textit{whenever} \quad y \in \mathbb{B}[\bar{y}, \beta].$$

Proof. First, we observe that

$$\|g(y,z)\| \leq \alpha/\kappa \quad \text{for each} \quad (y,z) \in \mathbb{B}[\bar{y}, \beta] \times \mathbb{B}[\bar{z}, \alpha]. \tag{10}$$

Indeed, fix any such (y, z). Note that $\alpha \leq \delta$ and $\beta < \delta$ due to (v). Using (ii)–(v), we get

$$\begin{aligned} \|g(y,z)\| &\leq \|g(y,z) - g(y,\bar{z})\| + \|g(y,\bar{z}) - g(\bar{y},\bar{z})\| + \|g(\bar{y},\bar{z})\| \leq \mu\|z - \bar{z}\| + \nu\|y - \bar{y}\| + \beta \\ &\leq \mu\alpha + \beta(1+\nu) \leq \frac{\alpha\kappa\mu}{\kappa} + \frac{\alpha(1-\kappa\mu)}{\kappa} = \frac{\alpha}{\kappa} \quad (\leq \delta). \end{aligned}$$

Fix any $y \in \mathbb{B}[\bar{y}, \beta]$. Consider the function

$$\varphi_y : \mathbb{B}[\bar{z}, \alpha] \ni z \mapsto \varphi_y(z) = s(-g(y,z)),$$

where $s : \mathbb{B}[0, \delta] \to \mathbb{R}^n$ satisfies (i). By (10), φ_y is well-defined. For any $z \in \mathbb{B}[\bar{z}, \alpha]$, (i) and (10) imply that

$$\|\bar{z} - \varphi_y(z)\| = \|s(0) - s(-g(y,z))\| \leq \kappa\|g(y,z)\| \leq \alpha.$$

Moreover, for any $\hat{z}$, $\tilde{z} \in \mathbb{B}[\bar{z}, \alpha]$, the Lipschitz continuity of s and (iii) reveal that

$$\|\varphi_y(\hat{z}) - \varphi_y(\tilde{z})\| = \|s(-g(y,\hat{z})) - s(-g(y,\tilde{z}))\| \leq \kappa\|g(y,\hat{z}) - g(y,\tilde{z})\| \leq \kappa\mu\|\hat{z} - \tilde{z}\|.$$

As $\kappa\mu < 1$, φ_y is a contraction from $\mathbb{B}[\bar{z}, \alpha]$ into itself, hence it has a unique fixed point in $\mathbb{B}[\bar{z}, \alpha]$.

For each $y \in \mathbb{B}[\bar{y}, \beta]$, denote by $\sigma(y)$ the unique fixed point of φ_y in $\mathbb{B}[\bar{z}, \alpha]$. Moreover, for each $y \in \mathbb{B}[\bar{y}, \beta]$, we have that

$$\sigma(y) = z \iff z = \varphi_y(z) \implies 0 \in g(y,z) + G(z) \implies \sigma(y) \in S(y). \tag{11}$$

To show that σ is Lipschitz continuous on $\mathbb{B}[\bar{y}, \beta]$ with the constant $\kappa\nu/(1 - \kappa\mu)$, fix arbitrary $\hat{y}, \tilde{y} \in \mathbb{B}[\bar{y}, \beta]$. The first equivalence in (11) together with the definitions of $\varphi_{\hat{y}}$ and $\varphi_{\tilde{y}}$ yields that

$$\begin{aligned} \|\sigma(\hat{y}) - \sigma(\tilde{y})\| &= \|s(-g(\hat{y}, \sigma(\hat{y}))) - s(-g(\tilde{y}, \sigma(\tilde{y})))\| \leq \kappa\|g(\hat{y}, \sigma(\hat{y})) - g(\tilde{y}, \sigma(\tilde{y}))\| \\ &\leq \kappa\|g(\hat{y}, \sigma(\hat{y})) - g(\tilde{y}, \sigma(\hat{y}))\| + \kappa\|g(\tilde{y}, \sigma(\hat{y})) - g(\tilde{y}, \sigma(\tilde{y}))\| \\ &\leq \kappa\nu\|\hat{y} - \tilde{y}\| + \kappa\mu\|\sigma(\hat{y}) - \sigma(\tilde{y})\|. \end{aligned}$$

The inequality $\|\bar{z} - \sigma(\bar{y})\| \leq \kappa'\|g(\bar{y}, \bar{z})\|$ is implied by the following chain of estimates

$$\begin{aligned} \|\bar{z} - \sigma(\bar{y})\| &= \|s(0) - s(-g(\bar{y}, \sigma(\bar{y})))\| \leq \kappa\|g(\bar{y}, \sigma(\bar{y}))\| \leq \kappa\|g(\bar{y}, \sigma(\bar{y})) - g(\bar{y}, \bar{z})\| + \kappa\|g(\bar{y}, \bar{z})\| \\ &\leq \kappa\mu\|\sigma(\bar{y}) - \bar{z}\| + \kappa\|g(\bar{y}, \bar{z})\|. \end{aligned}$$

□

We immediately get the following non-parametric version.

Corollary 3.1. *Consider a mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ along with a point $(\bar{z}, \bar{y}) \in \operatorname{gph} G$. Assume that there exist positive constants δ and κ and a function $s : \mathbb{B}[\bar{y}, \delta] \to \mathbb{R}^n$ such that*

$$(12) \qquad s(\bar{y}) = \bar{z} \quad \textit{and} \quad s(y) \in G^{-1}(y) \quad \textit{for each} \quad y \in \mathbb{B}[\bar{y}, \delta],$$

and that s is Lipschitz continuous on $\mathbb{B}[\bar{y}, b]$ with the constant κ. Let $\mu > 0$ be such that $\kappa\mu < 1$. Then for every positive α and β such that

$$(13) \qquad 2\beta\kappa \le \alpha(1 - \mu\kappa) \quad \textit{and} \quad \alpha \le \delta \min\{1, \kappa\},$$

and for every function $g : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$(14) \qquad \|g(\bar{z})\| \le \beta \quad \textit{and} \quad \|g(\hat{z}) - g(\tilde{z})\| \le \mu\|\hat{z} - \tilde{z}\| \qquad \textit{for every} \quad \hat{z}, \tilde{z} \in \mathbb{B}[\bar{z}, \delta],$$

there is $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ which is Lipschitz continous on $\mathbb{B}[\bar{y}, \beta]$ with the constant $\kappa/(1 - \kappa\mu)$ such that

$$\sigma(\bar{y} + g(\bar{z})) = \bar{z} \quad \textit{and} \quad \sigma(y) \in (g + G)^{-1}(y) \cap \mathbb{B}[\bar{z}, \alpha] \quad \textit{whenever} \quad y \in \mathbb{B}[\bar{y}, \beta].$$

Proof. It suffices to apply the previous result with $g(y, z) := g(z) + \bar{y} - y$, $G(\cdot) := G(\cdot) - \bar{y}$, and $s(\cdot) := s(\cdot + \bar{y})$. Then $\nu = 1$ and all the conditions (i)–(v) of Theorem 3.1 are satisfied. Hence we obtain the latter inclusion in the statement. To see the first equality, note that the function φ_y, from the proof of Theorem 3.1, is such that $\varphi_{\bar{y}+g(\bar{z})}(\bar{z}) = s(-g(\bar{z}) + (\bar{y} + g(\bar{z}))) = s(\bar{y}) = \bar{z}$. Thus $\sigma(\bar{y} + g(\bar{z})) = \bar{z}$. ☐

Remark 3.1. If there is a single-valued Lipschitz localization of G^{-1} around $\bar{y}$ for $\bar{z}$ then this also is the selection for G^{-1} around $\bar{y}$ for $\bar{z}$ and all such selections coincide locally. Therefore φ_y is uniquely determined by y and the uniqueness of the fixed point of φ_y in $\mathbb{B}[\bar{z}, \alpha]$ gives that

$$\mathbb{B}[\bar{y}, \beta] \ni y \longmapsto (g + G)^{-1}(y) \cap \mathbb{B}[\bar{z}, \alpha]$$

is single-valued. Also the implications in (11) become equivalences. Therefore one arrives at [8, Theorem 2.4]. More precisely, the assumption involving (12) reads as: *There are positive constants δ and κ such that the mapping*

$$\mathbb{B}[\bar{y}, \delta] \ni y \longmapsto s(y) := G^{-1}(y) \cap \mathbb{B}[\bar{z}, \kappa\delta]$$

is single-valued and Lipschitz continuous with the constant κ.

If the mapping in question is (locally) monotone then the assumption on the existence of a single-valued Lipschitz localization is equivalent to the existence of a Lipschitz selection. Recall that $S : \mathbb{R}^l \rightrightarrows \mathbb{R}^l$ is *locally monotone* at $(\bar{y}, \bar{x}) \in \operatorname{gph} S$ if there is a neighborhood W of $(\bar{y}, \bar{x})$ such that

$$(15) \qquad \langle \hat{y} - \tilde{y}, \hat{x} - \tilde{x} \rangle \ \ge\ 0 \quad \text{whenever} \quad (\hat{y}, \hat{x}), (\tilde{y}, \tilde{x}) \in \operatorname{gph} S \cap W.$$

Lemma 3.1. *A set-valued mapping $S : \mathbb{R}^l \rightrightarrows \mathbb{R}^l$, which is locally monotone at $(\bar{y}, \bar{x}) \in \operatorname{gph} S$, has a single-valued Lipschitz continuous localization around $\bar{y}$ for $\bar{x}$ if and only if it has a Lipschitz continuous selection around $\bar{y}$ for $\bar{x}$.*

Proof. We shall imitate the proof of [7, Theorem 3G.5]. Find W such that (15) holds. Let s be a local selection for S which is both defined and Lipschitz continuous on $\mathbb{B}(\bar{y}, r)$ for some $r > 0$ such that $\mathbb{B}(\bar{y}, r) \times \mathbb{B}(\bar{x}, \kappa r) \subset W$, where $\kappa > 0$ is the corresponding Lipschitz constant. Fix any $y \in \mathbb{B}(\bar{y}, r)$. As $s(\bar{y}) = \bar{x}$, we have $s(y) \in \mathbb{B}(\bar{x}, \kappa r)$. Therefore, the point $s(y)$ lies in $S(y) \cap \mathbb{B}(\bar{x}, \kappa r)$. It suffices to show that the latter set is singleton. Suppose that this is not the case. Find $x \in \mathbb{R}^l$ such that

$$x \in S(y) \cap \mathbb{B}(\bar{x}, \kappa r) \quad \text{with} \quad x \neq s(y).$$

Let $b := \|x - s(y)\|$ and $c := (x - s(y))/b$, which means that

$$b > 0, \quad \|c\| = 1, \quad \text{and} \quad \langle x, c \rangle = b + \langle s(y), c \rangle. \tag{16}$$

Find $\tau > 0$ such that $\kappa\tau < b$ and that $y + \tau c \in \mathbb{B}(\bar{y}, r)$. Since $\|c\| = 1$, the Cauchy-Schwartz inequality and the Lipschitz continuity of s imply that

$$\langle s(y + \tau c) - s(y), c \rangle \;\leq\; \|s(y + \tau c) - s(y)\| \, \|c\| \;\leq\; \kappa\tau. \tag{17}$$

Since $(y + \tau c, s(y + \tau c))$ and (y, x) are in $\operatorname{gph} S \cap W$, (15) reveals that

$$0 \;\leq\; \langle s(y + \tau c) - x, y + \tau c - y \rangle \;=\; \tau \langle s(y + \tau c) - x, c \rangle. \tag{18}$$

Now, we may estimate

$$b + \langle s(y), c \rangle \overset{(16)}{=} \langle x, c \rangle \overset{(18)}{\leq} \langle s(y + \tau c), c \rangle \overset{(17)}{\leq} \langle s(y), c \rangle + \kappa\tau < \langle s(y), c \rangle + b.$$

We arrived at a contradiction, therefore $S(y) \cap \mathbb{B}(\bar{x}, \kappa r) = \{s(y)\}$ for each $y \in \mathbb{B}(\bar{y}, r)$. The opposite direction is trivial. □

Now, we are in position to state conditions guaranteeing that the problem (8) is solvable.

Theorem 3.2. *Under the assumptions (A1)–(A3), consider the problem (8) along with its solution mapping*

$$S : [a, b] \ni t \longmapsto S(t) := \{z \in \mathbb{R}^n : \; p(t) \in f(z) + BF(Cz)\},$$

where $p : [a, b] \to \mathbb{R}^n$ is a given Lipschitz function. Suppose that there exits $r > 0$ such that $\emptyset \neq S(t) \subset r\mathbb{B}$ whenever $t \in [a, b]$, and that for each $(t, z) \in \operatorname{gph} S$ the inverse of the mapping

$$\mathbb{R}^n \ni v \mapsto H_{t,z}(v) := f(z) - p(t) + \nabla f(z)(v - z) + BF(Cv) \subset \mathbb{R}^n$$

has a Lispchitz continous selection around 0 for z. Then S has a Lipschitz selection around t for z for any $(t, z) \in \operatorname{gph} S$.

If, in addition,

$$L := \sup_{(t,z) \in \operatorname{gph} S} \{\operatorname{lip}(s; t) : \; s \text{ is a selection for } S \text{ around } t \text{ for } z\} < \infty, \tag{19}$$

then S has a Lipschitz selection.

Proof. Fix any $(t,z) \in \operatorname{gph} S$. Note that for any $(\tau, v) \in [a,b] \times \mathbb{R}^n$ we have

$$g(\tau, v) + H_{t,z}(v) = f(v) - p(\tau) + BF(Cv),$$

where $g(\tau, v) = f(v) - f(z) - \nabla f(z)(v-z) + p(t) - p(\tau)$. By Theorem 3.1 (with the reference point (t,z), μ arbitrary small, ν being the Lipschitz constant of $p(\cdot)$, and $g(t,z) = 0$), there exists a closed neighborhood $T_{t,z}$ of t in $[a,b]$, a closed neighborhood $U_{t,z}$ of z in $\mathbb{R}^n$, and a Lipschitz continous function $u_{t,z} : T_{t,z} \to \mathbb{R}^n$ such that $u_{t,z}(\tau) \in S(\tau) \cap U_{t,z}$ for each $\tau \in T_{t,z}$ and $u_{t,z}(t) = z$.

Let

$$\bar{t} := \sup\{t \geq a : \text{ the selection } u \text{ for } S \text{ exists and is Lipschitz on } [a,t]\}.$$

By the first part of the proof, we have $\bar{t} > a$. We show that $\bar{t} \geq b$. Suppose that this is not the case. Fix any strictly increasing sequence $(t^k)_{k\in\mathbb{N}}$ converging to $\bar{t}$ such that $z^k := u(t^k)$ is defined for each $k \in \mathbb{N}$. Then $(z^k)_{k\in\mathbb{N}}$ is bounded and $z^k \in S(t^k)$ for each $k \in \mathbb{N}$. Choose an infinite set $N \subset \mathbb{N}$ such that $\bar{z} := \lim_{N \ni k \to \infty} z^k$ exists. Since the graph of S is closed, we have $\bar{z} \in S(\bar{t})$. Set $u(\bar{t}) = \bar{z}$. Further, there is $T_{\bar{t},\bar{z}} = [\bar{t}, \hat{t}]$, for some $\hat{t} > \bar{t}$, along with a Lipschitz function $u_{\bar{t},\bar{z}} : [\bar{t}, \hat{t}] \to \mathbb{R}^n$ such that $u(\tau) := u_{\bar{t},\bar{z}}(\tau) \in S(\tau)$ for each $\tau \in [\bar{t}, \hat{t}]$ and $u_{\bar{t},\bar{z}}(\bar{t}) = \bar{z} \in S(\bar{t})$.

For any $k \in \mathbb{N}$, the function u is Lipschitz continous on $[a, t_k]$, with the constant less or equal than L, and the same is true on $[\bar{t}, \hat{t}]$. Fix any $t \in [a, \bar{t})$ and $\tau \in [\bar{t}, \hat{t}]$. Find $k_0 \in \mathbb{N}$ such that $t^{k_0} > t$. Then, for any $k > k_0$, we have

$$\begin{aligned} \|u(t) - u(\tau)\| &\leq \|u(t) - u(t^k)\| + \|u(t^k) - u(\bar{t})\| + \|u(\bar{t}) - u(\tau)\| \\ &\leq L|t - t^k| + \|z^k - \bar{z}\| + L|\bar{t} - \tau|. \end{aligned}$$

Passing to the limit as $k \to \infty$, we get that $\|u(t) - u(\tau)\| \leq L|t - \bar{t}| + L|\bar{t} - \tau| = L|t - \tau|$. Consequently, u is Lipschitz on $[a, \hat{t}]$, a contradiction. □

Note that the condition (19), which is satisfied when all the points of $\operatorname{gph} S$ are strongly regular in the sense of Robinson, cannot be omitted. Indeed, it suffices to consider $[a,b] := [-1,1]$ and $S(t) := \{0, \sqrt{-t}\}$ if $t < 0$ and $S(t) = 0$ otherwise. To conclude this section, let us comment on the existence of a Lipschitz continuous selection briefly.

Remark 3.2. Let $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^l$ be a mapping with $(\bar{p}, \bar{u}) \in \operatorname{gph} S$. Assume that there is $\kappa > 0$ along with closed convex neighborhoods U of $\bar{u}$ and V of $\bar{p}$ such that $S(p) \cap U$ is closed convex and $S(\tilde{p}) \cap U \subset S(\hat{p}) + \kappa\|\tilde{p} - \hat{p}\|\,\mathbb{B}$ for each $\tilde{p}, \hat{p} \in V$. Note that the last inclusion holds, for some U and V, provided that S^{-1} is metrically regular at $\bar{u}$ for $\bar{p}$. By [7, Theorem 3E.3], there is $\kappa_1 > 0$ together with closed convex neighborhoods U_1 of $\bar{u}$ and V_1 of $\bar{p}$ such that $S(p) \cap U_1$ is closed convex and $S(\tilde{p}) \cap U_1 \subset S(\hat{p}) \cap U_1 + \kappa_1\|\tilde{p} - \hat{p}\|\,\mathbb{B}$ for each $\tilde{p}, \hat{p} \in V_1$. Using Steiner selection, the remark following [4, Theorem 9.4.3] implies that S has a Lipschitz continuous selection around $\bar{p}$ for $\bar{u}$.

4 Numerical simulation and applications in electronics

In the following section, we discuss several basic examples from electronics. For the input-output simulation, we use a modification of an Euler-Newton path-following method from [7, Section 6G] which, for variational inequalities, was introduced in [8]. We will apply this method to a generalized equation (8) with $B = C = I_n$, that is,

$$p(t) \in f(z(t)) + F(z(t)), \quad t \in [a,b], \tag{20}$$

where $p(\cdot)$ is Lipschitz continuous, $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable and its derivative mapping is locally Lipchitz at each point of $\mathbb{R}^n$, and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has a closed graph. For $N > 1$, we consider the uniform grid $t^i := a + ih$, $i \in \{0, 1, \ldots, N\}$, with a step size $h = (b-a)/N$. Given $\Delta > 0$, the corresponding predictor and corrector steps will be

$$\begin{cases} f(z^i) - p(t^{i+1}) + \nabla f(z^i)(v^{i+1} - z^i) + F(v^{i+1}) \ni e^i \quad \text{for a fixed} \quad e^i \in \mathbb{B}[0, \Delta h^2], \\ f(v^{i+1}) - p(t^{i+1}) + \nabla f(v^{i+1})(z^{i+1} - v^{i+1}) + F(z^{i+1}) \ni 0, \end{cases} \tag{21}$$

where z^0 is sufficiently close to the exact solution of (20) at time $t := a$. The algorithm proposed in [7, Section 6G] reads as

$$\begin{cases} f(z^i) - p(t^i) - hp'(t^i) + \nabla f(z^i)(v^{i+1} - z^i) + F(v^{i+1}) \ni 0, \\ f(v^{i+1}) - p(t^{i+1}) + \nabla f(v^{i+1})(z^{i+1} - v^{i+1}) + F(z^{i+1}) \ni 0, \end{cases} \tag{22}$$

and requests that $p(\cdot) \in \mathcal{C}^1([a,b], \mathbb{R}^n)$. Then [7, Theorem 6G.2] contains conditions guaranteeing that, for all sufficiently large N, this method generates the iterates $(z^i)_{i=1}^N$ with the *grid error*

$$\max_{0 \le i \le N} \|z^i - \bar{z}(t^i)\| \le ch^4, \tag{23}$$

where c is a fixed constant and $\bar{z}(\cdot)$ is the (exact) solution to (20). This statement requires f to be twice continuously differentiable, but the original proof relies on the Lipschitz continuity of ∇f only. Let us present a similar convergence result for the method (21).

Theorem 4.1. *Let $\bar{z}(\cdot)$ be a Lipschitz continuous solution of the problem (20) with a Lipschitz continuous $p : [a,b] \to \mathbb{R}^n$, a differentiable $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $\nabla f(\cdot)$ is locally Lipchitz at each point of $\mathbb{R}^n$, and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ having closed graph. Suppose that for each $t \in [a,b]$ the mapping*

$$\mathbb{R}^n \ni v \mapsto H_t(v) := f(\bar{z}(t)) - p(t) + \nabla f(\bar{z}(t))(v - \bar{z}(t)) + F(v) \subset \mathbb{R}^n$$

is strongly metrically regular at $\bar{z}(t)$ for 0. Then for any $\Delta > 0$ there are $N_0 \in \mathbb{N}$, $\alpha > 0$, and $c > 0$ such that for each $N > N_0$ and each $z^0 \in \mathbb{B}[\bar{z}(a), \Delta h^4]$, where $h := (b-a)/N$, the iteration (21), with the initial point z^0, generates unique $(z^i)_{i=1}^N$ verifying (23) such that $z^i \in \mathbb{B}[\bar{z}(t^i), \alpha]$ for each $i \in \{1, \ldots, N\}$.

Proof. We will imitate the proofs of Theorem 6G.1 and Thoerem 6G.2 in [7]. Denote by L_1 and L_2 the Lipschitz constant of $\bar{z}(\cdot)$ and $p(\cdot)$, respectively. Let $r > 0$ be such that $\bar{z}([a,b]) + \mathbb{B} \subset r\mathbb{B}$. As $\nabla f(\cdot)$ is locally Lipschitz, it is Lipschitz on the (compact) set $r\mathbb{B}$ with the constant $L_3 > 0$, say. Let $L := \max\{1, L_1, L_2, L_3\}$. Then, for any $\hat{z}$, $\tilde{z} \in r\mathbb{B}$, we have

$$\begin{aligned} \|f(\hat{z}) - f(\tilde{z}) - \nabla f(\tilde{z})(\hat{z} - \tilde{z})\| &= \left\| \int_0^1 \big(\nabla f(\tilde{z} + t(\hat{z} - \tilde{z})) - \nabla f(\tilde{z})\big)(\hat{z} - \tilde{z}) \, \mathrm{d}t \right\| \\ &\le L \|\hat{z} - \tilde{z}\|^2 \int_0^1 t \, \mathrm{d}t = \frac{L}{2} \|\hat{z} - \tilde{z}\|^2 . \end{aligned} \tag{24}$$

STEP 1. We show that *there are positive δ and κ such that, for each $t \in [a,b]$, the mapping $\mathbb{B}[0,\delta] \ni w \longmapsto H_t^{-1}(w) \cap \mathbb{B}[\bar{z}(t), \kappa\delta]$ is single-valued and Lipschitz continuous with the constant κ.*

To prove this, fix any $t \in [a,b]$. Let $\delta_t > 0$ and $\kappa_t > 0$ be such that the mapping $\mathbb{B}[0,b_t] \ni w \longmapsto H_t^{-1}(w) \cap \mathbb{B}[\bar{z}(t), \delta_t\kappa_t]$ is single-valued and Lipschitz continuous with the constant κ_t. We `claim` that there are positive α_t, β_t, κ_t', and ρ_t such that for each $\tau \in [a,b] \cap (t-\rho_t, t+\rho_t)$ the mapping

$$\mathbb{B}[0,\beta_t] \ni w \longmapsto H_\tau^{-1}(w) \cap \mathbb{B}[\bar{z}(\tau), \alpha_t/2]$$

is single-valued and Lipschitz continuous with the constant κ_t'. Let $\alpha_t := \delta_t \min\{1, \kappa_t\}$. Choose $\rho_t > 0$ such that

$$2\rho_t L^2 < \min\{1, 1/\kappa_t, \alpha_t/(4\kappa_t), \alpha_t\}. \tag{25}$$

Fix any $\tau \in [a,b] \cap (t-\rho_t, t+\rho_t)$. Define the function $g_{t,\tau} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$g_{t,\tau}(v) := f(\bar{z}(t)) - p(t) - f(\bar{z}(\tau)) + p(\tau) + \nabla f(\bar{z}(t))(v - \bar{z}(t)) - \nabla f\big(\bar{z}(\tau)\big)\big(v - \bar{z}(\tau)\big), \quad v \in \mathbb{R}^n.$$

Note that $H_t = H_\tau + g_{t,\tau}$. For any $\hat{v}$, $\tilde{v} \in \mathbb{R}^n$, we have

$$\begin{aligned} \|g_{t,\tau}(\hat{v}) - g_{t,\tau}(\tilde{v})\| &= \big\|\big(\nabla f(\bar{z}(t)) - \nabla f\big(\bar{z}(\tau)\big)\big)(\hat{v} - \tilde{v})\big\| \le L\|\bar{z}(t) - \bar{z}(\tau)\| \, \|\hat{v} - \tilde{v}\| \\ &\le L^2|t-\tau|\|\hat{v} - \tilde{v}\| \le \rho_t L^2 \|\hat{v} - \tilde{v}\|. \end{aligned}$$

Let $\mu_t := \rho_t L^2$. By (25) we have $2\kappa_t\mu_t < 1$. Furthermore, (24) and (25) imply that

$$\begin{aligned} \big\|g_{t,\tau}\big(\bar{z}(t)\big)\big\| &= \big\|f(\bar{z}(t)) - f(\bar{z}(\tau)) - \nabla f\big(\bar{z}(\tau)\big)\big(\bar{z}(t) - \bar{z}(\tau)\big) + p(\tau) - p(t)\big\| \\ &\le L\|\bar{z}(t) - \bar{z}(\tau)\|^2 + \|p(\tau) - p(t)\| \le L^3|t-\tau|^2 + L|t-\tau| \\ &\le L\rho_t(L^2\rho_t + 1) < 2\rho_t L =: \beta_t. \end{aligned} \tag{26}$$

As $\mu_t\kappa_t < 1/2$ and $L \ge 1$, we have $1 - \mu_t\kappa_t > 1/2$ and $\rho_t L \le \rho_t L^2$. Using (25) we get

$$2\kappa_t\beta_t = 4\kappa_t\rho_t L \le 4\kappa_t\rho_t L^2 < \alpha_t/2 < \alpha_t(1 - \kappa_t\mu_t).$$

Let $\kappa_t' := \kappa_t/(1 - \mu_t\kappa_t) > 0$. The strong regularity version of Corollary 3.1 (see Remark 3.1), with $g := -g_{t,\tau}$, $G := H_t$, $\bar{z} := \bar{z}(t)$, and $\bar{y} := 0$, implies that

$$\mathbb{B}[0,\beta_t] \ni w \longmapsto \sigma_{t,\tau}(w) := H_\tau^{-1}(w) \cap \mathbb{B}[\bar{z}(t), \alpha_t]$$

is single-valued and Lipschitz continuous with the constant κ_t'. Note that $\bar{z}(\tau) \in H_\tau^{-1}(0)$. By (25), we have $\|\bar{z}(\tau) - \bar{z}(t)\| \le L\rho_t < \alpha_t/2$, hence $\bar{z}(\tau) \in H_\tau^{-1}(0) \cap \mathbb{B}(\bar{z}(t), \alpha_t/2) = \{\sigma_{t,\tau}(0)\}$. Fix any $w \in \mathbb{B}[0,\beta_t]$. Since $\kappa_t' < 2\kappa_t$ and $2\kappa_t\beta_t < \alpha_t/2$, we have

$$\|\sigma_{t,\tau}(w) - \bar{z}(\tau)\| = \|\sigma_{t,\tau}(w) - \sigma_{t,\tau}(0)\| \le \kappa_t'\|w\| < 2\kappa_t\beta_t < \alpha_t/2.$$

As $\bar{z}(\tau) \in \mathbb{B}(\bar{z}(t), \alpha_t/2)$, we have $\sigma_t\big(\mathbb{B}[0,\beta_t]\big) \subset \mathbb{B}(\bar{z}(\tau), \alpha_t/2) \subset \mathbb{B}(\bar{z}(t), \alpha_t)$. The claim is proved.

From the open covering $\cup_{t\in[a,b]}(t-\rho_t, t+\rho_t)$ of $[a,b]$ choose a finite sub-covering by intervals $(t_i-\rho_{t_i}, t_i+\rho_{t_i})$ with some $t_i \in [a,b]$, $i\in\{1,2,\ldots,m\}$. To finish the proof of STEP 1, let $\kappa := \max\left\{\kappa'_{t_i} :\ i\in\{1,2,\ldots,m\}\right\}$ and find $\delta>0$ such that

$$\delta < \min\Big\{\min\left\{\beta_{t_i} :\ i\in\{1,2,\ldots,m\}\right\}, \min\left\{\alpha_{t_i}/2 :\ i\in\{1,2,\ldots,m\}\right\}/\kappa\Big\}.$$

STEP 2. Pick any $\Delta>0$. Use STEP 1 to find the corresponding $\kappa>0$ and $\delta>0$. Let

$$\alpha := \delta\min\{1,\kappa\} \quad \text{and} \quad K:=\max\{L,\Delta,1/\alpha,\kappa,b-a\} \quad \text{and} \quad c:=36K^{10}. \tag{27}$$

Choose $N_0\in\mathbb{N}$ bigger than c. Fix any $N>N_0$ and let $h:=(b-a)/N$. Then

$$h<(b-a)/N_0 \le K/N_0 < K/c. \tag{28}$$

Pick any $z^0\in\mathbb{B}[\bar z(a),\Delta h^4]$. As $c>K\ge\Delta$ and $K\ge 1/\alpha$, by (28), we have $\|z^0-\bar z(t^0)\| < ch^4 < K^4/c^3 < 1/K \le \alpha$. We proceed by induction. Suppose that z^i verifies $\|z^i-\bar z(t^i)\|\le ch^4$ for some $i\ge 0$. We will show that there is a unique z^{i+1} such that

$$\|z^{i+1}-\bar z(t^{i+1})\| \le ch^4 \quad (<\alpha).$$

Pick any $e^i\in\mathbb{B}[0,\Delta h^2]$. Since $K\ge L\ge 1$, (28) implies that $ch^4<1$. Thus $z^i=\bar z(t^i)+\big(z^i-\bar z(t^i)\big)\in \bar z([a,b])+\mathbb{B}\subset r\mathbb{B}$. Define the function $g:\mathbb{R}^n\to\mathbb{R}^n$ by

$$g(v):=f(z^i)-f\big(\bar z(t^{i+1})\big)+\nabla f(z^i)(v-z^i)-\nabla f\big(\bar z(t^{i+1})\big)\big(v-\bar z(t^{i+1})\big), \quad v\in\mathbb{R}^n.$$

Then v^{i+1} satisfies the first inclusion in (21) if and only if it solves the generalized equation

$$g(v)+H_{t^{i+1}}(v)\ni e^i.$$

As (28) guarantees that $ch^3<K^3/c^2<K$, we get that

$$\|z^i-\bar z(t^{i+1})\| \le \|z^i-\bar z(t^i)\|+\|\bar z(t^i)-\bar z(t^{i+1})\| \le ch^4+Kh = h(ch^3+K) < 2hK. \tag{29}$$

This implies that, for any $\hat v$, $\tilde v\in\mathbb{R}^n$, we have

$$\|g(\hat v)-g(\tilde v)\| \;=\; \big\|\big(\nabla f(z^i)-\nabla f\big(\bar z(t^{i+1})\big)\big)(\hat v-\tilde v)\big\| \le K\|z^i-\bar z(t^{i+1})\|\,\|\hat v-\tilde v\| \le 2hK^2\|\hat v-\tilde v\|.$$

Let $\mu:=2hK^2$. Since $K\ge\kappa$, (28) implies that $2\mu\kappa\le 4hK^3<4K^4/c<1$. Moreover, (24) and (29) imply that

$$\begin{aligned}\big\|g\big(\bar z(t^{i+1})\big)\big\| \;&=\; \big\|f\big(\bar z(t^{i+1})\big)-f(z^i)-\nabla f(z^i)\big(\bar z(t^{i+1})-z^i\big)\big\| \le \frac{L}{2}\|\bar z(t^{i+1})-z^i\|^2\\ &\le\; 2h^2K^3 =: \beta.\end{aligned} \tag{30}$$

Then $\beta=\mu hK$. As $\mu\kappa<1/2$, we have $1-\mu\kappa>1/2$, and the relations (28) and (27) imply that

$$2\kappa\beta \;=\; 2\kappa\mu hK < hK < K^2/c < 1/(2K) \le \alpha/2 < \alpha(1-\kappa\mu).$$

By the strong regularity version of Corollary 3.1, with $G := H_{t^{i+1}}$, $\bar{z} := \bar{z}(t^{i+1})$, and $\bar{y} := 0$, we get that the function

$$\mathbb{B}[0,\beta] \ni y \longmapsto \sigma(y) := (g + H_{t^{i+1}})^{-1}(y) \cap \mathbb{B}[\bar{z}(t^{i+1}), \alpha]$$

is single-valued and Lipschitz continuous with the constant $\kappa/(1-\mu\kappa)$. Note that $\|e^i\| \le \Delta h^2 \le Kh^2 < \beta$. Let $v^{i+1} := \sigma(e^i)$. Thus v^{i+1} is the unique solution to the first inclusion of (21) in $\mathbb{B}[\bar{z}(t^{i+1}), \alpha]$. Note that $\sigma\big(g(\bar{z}(t^{i+1}))\big) = \bar{z}(t^{i+1})$. Since $K \ge \kappa$ and $1 - \kappa\mu > 1/2$, we get

$$\begin{aligned} \big\|v^{i+1} - \bar{z}(t^{i+1})\big\| &= \big\|\sigma(e^i) - \sigma\big(g(\bar{z}(t^{i+1}))\big)\big\| \le \frac{\kappa}{1-\mu\kappa}\left(\|e^i\| + \big\|g\big(\bar{z}(t^{i+1})\big)\big\|\right) \\ &\overset{(30)}{\le} \frac{\kappa(h^2K + 2h^2K^3)}{1-\mu\kappa} \le \frac{3\kappa h^2 K^3}{1-\mu\kappa} < 6\kappa h^2 K^3 \le 6K^4h^2. \end{aligned} \tag{31}$$

As (28) implies that $6K^4h^2 < 6K^6/c^2 < 1$, we have $v^{i+1} \in \bar{z}([a,b]) + \mathbb{B} \subset r\mathbb{B}$. Define the function $\tilde{g} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\tilde{g}(u) := f(v^{i+1}) - f\big(\bar{z}(t^{i+1})\big) + \nabla f(v^{i+1})(u - v^{i+1}) - \nabla f\big(\bar{z}(t^{i+1})\big)\big(u - \bar{z}(t^{i+1})\big), \quad u \in \mathbb{R}^n.$$

Then z^{i+1} satisfies the second inclusion in (21) if and only if it solves the generalized equation

$$\tilde{g}(u) + H_{t^{i+1}}(u) \ni 0.$$

Now, (28) means that $3K^3h < 3K^4/c < 1$. By (31), for any $\hat{u}$, $\tilde{u} \in \mathbb{R}^n$, we have

$$\begin{aligned} \|\tilde{g}(\hat{u}) - \tilde{g}(\tilde{u})\| &= \big\|\big(\nabla f(v^{i+1}) - \nabla f\big(\bar{z}(t^{i+1})\big)\big)(\hat{u} - \tilde{u})\big\| \le K\|v^{i+1} - \bar{z}(t^{i+1})\|\,\|\hat{u} - \tilde{u}\| \\ &\le 6K^5h^2\|\hat{u} - \tilde{u}\| = 3K^3h\mu\|\hat{u} - \tilde{u}\| \le \mu\|\hat{u} - \tilde{u}\|. \end{aligned}$$

Moreover, (28) implies that $9K^6h^2 < 9K^8/c^2 < 1$. Using (24), (31), and (30), we infer that

$$\begin{aligned} \big\|\tilde{g}\big(\bar{z}(t^{i+1})\big)\big\| &= \big\|f\big(\bar{z}(t^{i+1})\big) - f(v^{i+1}) - \nabla f(v^{i+1})\big(\bar{z}(t^{i+1}) - v^{i+1}\big)\big\| \le \frac{L}{2}\|\bar{z}(t^{i+1}) - v^{i+1}\|^2 \\ &\le 18K^9h^4 = 9K^6h^2\beta < \beta. \end{aligned} \tag{32}$$

By the strong regularity version of Corollary 3.1, with $g := \tilde{g}$, $G := H_{t^{i+1}}$, $\bar{z} := \bar{z}(t^{i+1})$, and $\bar{y} := 0$, we get that the function

$$\mathbb{B}[0,\beta] \ni y \longmapsto \tilde{\sigma}(y) := (\tilde{g} + H_{t^{i+1}})^{-1}(y) \cap \mathbb{B}[\bar{z}(t^{i+1}), \alpha]$$

is single-valued and Lipschitz continuous with the constant $\kappa/(1-\mu\kappa)$. Let $z^{i+1} := \tilde{\sigma}(0)$. This means that z^{i+1} is the unique solution to the second inclusion of (21) in $\mathbb{B}[\bar{z}(t^{i+1}), \alpha]$. Note that $\tilde{\sigma}\big(\tilde{g}(\bar{z}(t^{i+1}))\big) = \bar{z}(t^{i+1})$. Since $K \ge \kappa$ and $1 - \kappa\mu > 1/2$, we get that

$$\big\|z^{i+1} - \bar{z}(t^{i+1})\big\| = \big\|\tilde{\sigma}(0) - \tilde{\sigma}\big(\tilde{g}(\bar{z}(t^{i+1}))\big)\big\| \le \frac{\kappa}{1-\mu\kappa}\big\|\tilde{g}\big(\bar{z}(t^{i+1})\big)\big\| \overset{(32)}{\le} \frac{18\kappa K^9h^4}{1-\mu\kappa} < 36K^{10}h^4.$$

By the very definition of c, we see that z^{i+1} is the point we were looking for. □

The possibility to choose any sufficiently small e^i in the first inclusion of (21) shows that, at each step i, we have to solve the generalized equation $f(z^i) - p(t^{i+1}) + \nabla f(z^i)(v - z^i) + F(v) \ni 0$ only with the residual proportional to h^2. Moreover, taking $e^i := p(t^i) - p(t^{i+1}) + hp'(t^i)$, $i \geq 0$, we have $\|e^i\| \leq \Delta h^2$ provided that $p'(\cdot)$ exists and is Lipschitz on $[a, b]$. Hence (22) is a particular case of (21). Similarly, instead of 0 in the latter inclusion of (21) one can take any $\tilde{e}^i \in \mathbb{B}[0, \Delta h^4]$.

In the remaining part, we discuss some elementary examples from electronics.

Example 4.1. Consider the circuit in Fig. 6.1a involving a non-linear resistor with current-voltage (i–v) characteristic given by $f(z) := \operatorname{argsinh}(z)$, $z \in \mathbb{R}$, a source $E > 0$, an input-signal source u with the corresponding instantaneous current i, and a practical diode with i–v characteristic

$$F(z) := \begin{cases} [V_1, V_2], & \text{if} \quad z = 0, \\ V_2, & \text{if} \quad z > 0, \\ V_1, & \text{if} \quad z < 0, \end{cases}$$

where $V_1 < 0 < V_2$ are given constants. Letting $p := u - E$ and $z := i$, Kirchhoff's voltage law reveals that, during a fixed time interval $[a, b]$, the problem (8) reads as

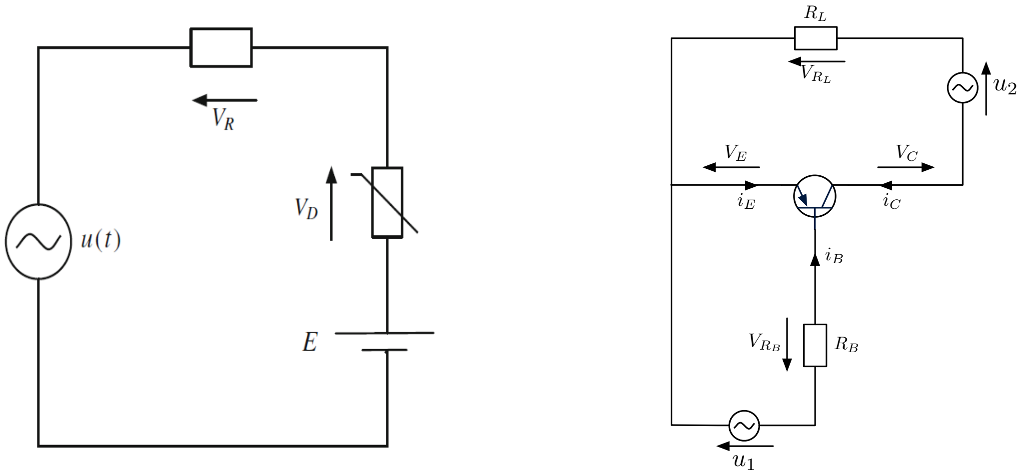
$$p(t) \in \underbrace{\operatorname{argsinh}(z(t))}_{V_R} + \underbrace{F(z(t))}_{V_D} \quad \text{for} \quad t \in [a, b].$$

Corollary 2.1 and Remmark 2.1 imply that Φ^{-1} is a Lipschitz function on any compact interval in $\mathbb{R}$, where $\Phi := f + F$ is the mapping corresponding to the static problem (1). Note that, for each $p \in \mathbb{R}$,

$$\Phi^{-1}(p) := \begin{cases} 0, & \text{if} \quad p \in [V_1, V_2], \\ \sinh(p - V_2), & \text{if} \quad p > V_2, \\ \sinh(p - V_1), & \text{if} \quad p < V_1. \end{cases}$$

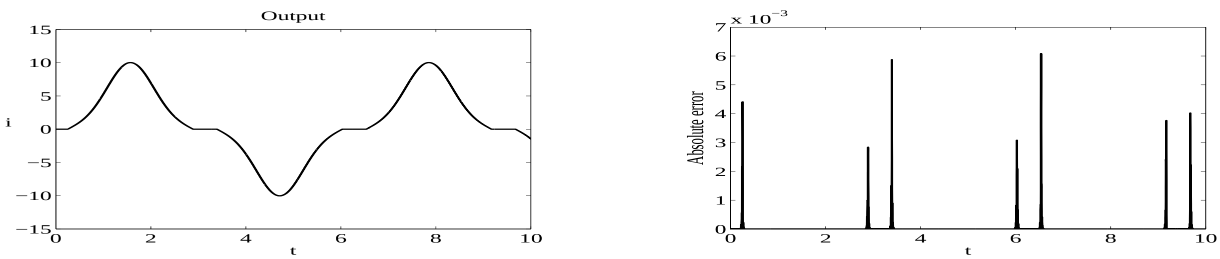
The assumptions of Theorem 4.1 are satisfied. For a particular choice of parameters, the solution of the time-dependent problem along with the absolute error of the Euler-Newton path-following method (21) with $e^i = 0$, $i \geq 0$, and $p(\cdot) \in C^\infty(\mathbb{R})$ can be found in Fig. 6.2a and Fig. 6.2b, respectively. In this setting, the precision of both the methods (21) and (22) is almost the same as can be seen from Table 1. The input-output simulation for a non-smooth $p(\cdot)$ along with the errors can be found in Fig. 6.3.

Example 4.2. Consider the circuit in Fig. 6.1b involving load resistances $R_B > 0$ and $R_L > 0$, two input-signal sources u_1 and u_2, and a P-N-P transistor (see Fig. 6.4) having three terminals labeled emitter, base and collector. Its behavior can be described by the Ebers-Moll model [12, p. 409] involving two diodes placed back to back and two dependent current-controlled sources $\alpha_I I'$ and $\alpha_N I$ shunting the diodes. Here $\alpha_N \in [0, 1)$ is known as the current gain in normal operation and $\alpha_I \in (0, 1]$ is known as the inverted common-base gain current. Therefore $i_E = I - \alpha_I I'$ and



(a) Example 4.1. (b) Example 4.2.

Figure 6.1: The circuits considered.



(a) The output current. (b) The absolute error.

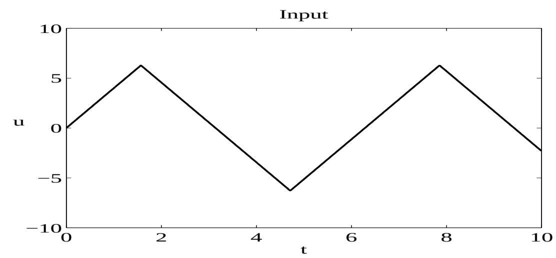
Figure 6.2: Example 4.1 with $[a, b] = [0, 10]$, $V_1 = -1, V_2 = 1, p(t) = 4\sin t, h = 0.01, E = 0$.

$i_C = I' - \alpha_N I$. This means that

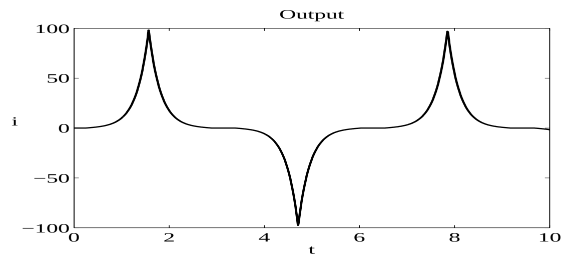
$$\begin{pmatrix} i_E \\ i_C \end{pmatrix} = \begin{pmatrix} 1 & -\alpha_I \\ -\alpha_N & 1 \end{pmatrix} \begin{pmatrix} I \\ I' \end{pmatrix}.$$

Kirchhoff's laws also reveal that $i_B = -(i_E + i_C)$, so $R_B(-i_C - i_E) + u_1 - V_E = 0$ and $0 = V_C + u_2 + R_L i_C - V_E = V_C + u_2 + R_L i_C + R_B(i_C + i_E) - u_1$. Given $V_{E1} < 0 < V_{E2}$, $V_{C1} < 0 < V_{C2}$, $\alpha > 0$, and $\beta > 0$, assume that the characteristics of the diodes involved in Ebers-Moll model are defined by

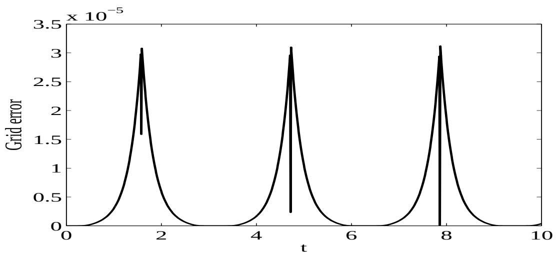
$$G_1(x) := \begin{cases} [V_{E1}, V_{E2}], & x = 0, \\ V_{E1} - \alpha \arctan(x), & x < 0, \\ V_{E2} - \alpha \arctan(x), & x > 0, \end{cases} \quad \text{and} \quad G_2(x) := \begin{cases} [V_{C1}, V_{C2}], & x = 0, \\ V_{C1} - \beta \arctan(x), & x < 0, \\ V_{C2} - \beta \arctan(x), & x > 0. \end{cases}$$



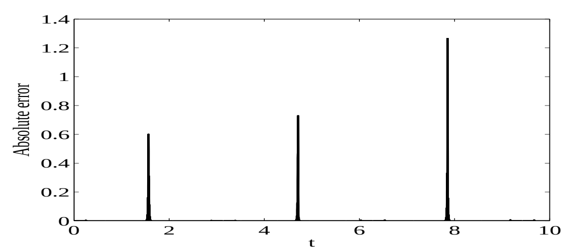
(a) The input signal.



(b) The output current.



(c) The grid error.



(d) The absolute error.

Figure 6.3: Example 4.1 with $[a, b] = [0, 10]$, $V_1 = -1, V_2 = 1, h = 0.01, E = 0$.

h	Method (22)	Method (21) with $e^i = 0$
0.1	0.0020	0.0027
0.01	2.0035e-07	2.2500e-07
0.001	2.0197e-11	2.2057e-11
0.0001	2.4514e-13	2.4869e-13

Table 1: The grid error $\max_{0\le i\le N} \|z^i - \bar{z}(t^i)\|$ for different step sizes in Example 4.1.

Then $V_E \in G_1(I) = -\alpha \arctan(I) + F_1(I)$ and $V_C \in G_2(I') = -\beta \arctan(I) + F_2(I')$, where

$$F_1(x) := \begin{cases} [V_{E1}, V_{E2}], & x = 0, \\ V_{E1}, & x < 0, \\ V_{E2}, & x > 0, \end{cases} \quad \text{and} \quad F_2(x) := \begin{cases} [V_{C1}, V_{C2}], & x = 0, \\ V_{C1}, & x < 0, \\ V_{C2}, & x > 0. \end{cases}$$

To sum up, we obtained that

$$\begin{pmatrix} u_1 \\ u_1 - u_2 \end{pmatrix} \in \begin{pmatrix} R_B & R_B \\ R_B & R_B + R_L \end{pmatrix} \begin{pmatrix} 1 & -\alpha_I \\ -\alpha_N & 1 \end{pmatrix} \begin{pmatrix} I \\ I' \end{pmatrix} - \begin{pmatrix} \alpha \arctan(I) \\ \beta \arctan(I') \end{pmatrix} + \begin{pmatrix} F_1(I) \\ F_2(I') \end{pmatrix}.$$

So we arrived at (1) with $n = m = 2$, $B = C = I_2$, $p := (u_1, u_1 - u_2)^T$, $z := (I, I')^T$. Fix any $\bar{z} \in \mathbb{R}^2$. Then

$$\nabla f(\bar{z}) = \begin{pmatrix} (1-\alpha_N)R_B - \alpha/(1+\bar{z}_1^2) & (1-\alpha_I)R_B \\ (1-\alpha_N)R_B - \alpha_N R_L & (1-\alpha_I)R_B + R_L - \beta/(1+\bar{z}_2^2) \end{pmatrix}.$$

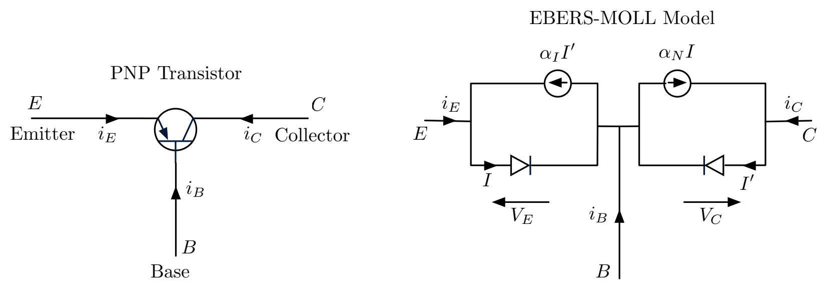
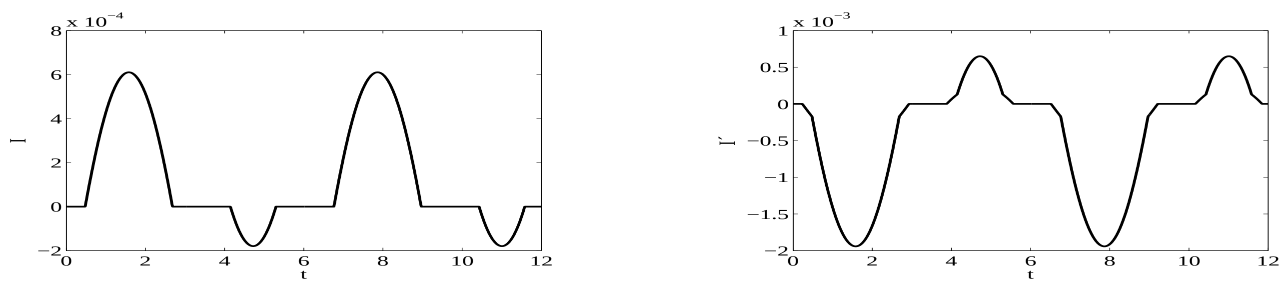


Figure 6.4: The P-N-P transistor and its Ebers-Moll model.

Assume that $2\alpha < (1-\alpha_N)R_B$ and $2\beta < (1-\alpha_I)R_B + R_L$. Then $A := \nabla f(\bar{z})$ a P-matrix, since the principal minors are

$$a_{11} > (1-\alpha_N)R_B/2 > 0, \quad a_{22} > \big((1-\alpha_I)R_B + R_L\big)/2 > 0, \quad \text{and} \quad \det A > (1-\alpha_I\alpha_N)R_L R_B > 0.$$

As both F_1 and F_2 are maximal monotone, Corollary 2.1 says that Φ is strongly regular at any reference point. The solution obtained by Euler-Newton method (21), with $e^i = 0$, $i \geq 0$, is in Fig. 6.5a and 6.5b. The solutions of the corresponding "partially" linearized sub-problems can be found precisely in this case (see Remark 4.1).



(a) The first component I of the solution.

(b) The second component I' of the solution.

Figure 6.5: Example 4.3 with $V_{E1} = -2$, $V_{E2} = 2$, $V_{C1} = -4$, $V_{C2} = 4$, $\alpha = 2/\pi$, $\beta = 2$, $u_1(t) = \sin(t)$, $u_2(t) = 10\sin(t)$, $R_L = 3000$, $R_B = 30000$, $\alpha_I = 0.7$, $\alpha_N = 0.1$, and $h = 0.01$.

Remark 4.1. Consider the generalized equation $p \in Az + F(z)$. Suppose that $A \in \mathbb{R}^{2\times 2}$ is a P-matrix and that $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is defined by $F(z) = (\partial|\cdot|(z_1), \partial|\cdot|(z_2))^T$, $z = (z_1, z_2) \in \mathbb{R}^2$. Pick arbitrary p_1, $p_2 \in \mathbb{R}$. Then

$$\begin{aligned}
g_1 &:= (a_{22} + \partial|\cdot|)^{-1}(p_2),\\
g_2 &:= (a_{22} - \frac{a_{21}a_{12}}{a_{11}} + \partial|\cdot|)^{-1}\left(p_2 - \frac{a_{21}}{a_{11}}p_1 + \frac{a_{21}}{a_{11}}\right), \text{ and}\\
g_3 &:= (a_{22} - \frac{a_{21}a_{12}}{a_{11}} + \partial|\cdot|)^{-1}\left(p_2 - \frac{a_{21}}{a_{11}}p_1 - \frac{a_{21}}{a_{11}}\right)
\end{aligned}$$

are well-defined numbers. Let

$$z_2 := \begin{cases} g_1, & \text{if } \ |p_1 - a_{12}g_1| \le 1, \\ g_2, & \text{if } \ p_1 - a_{12}g_2 \ge 1, \\ g_3, & \text{if } \ p_1 - a_{12}g_3 \le -1, \end{cases}$$

and let $z_1 := (a_{11} + \partial|\cdot|)^{-1}(p_1 - a_{12}z_2)$. It is easy to show that (z_1, z_2) is well-defined and solves our generalized equation.

Example 4.3. Suppose that a continuously differentiable function $\varphi : \mathbb{R} \to (0, \infty)$ and $\alpha > 0$ are such that $\varphi(0) < \varphi(\alpha), \varphi'(0) > 0$ and $\varphi'(\alpha) > 0$. Let us replace the practical diode in Example 4.1 by Sillicon Controlled Rectifier (SCR) with characteristic described by

$$G(z) := \begin{cases} [V_1, \varphi(0)], & \text{if } \ z = 0, \\ az + V_1, & \text{if } \ z < 0, \\ \varphi(z), & \text{if } \ z \in [0, \alpha], \\ a(z - \alpha) + \varphi(\alpha), & \text{if } \ z > \alpha, \end{cases}$$

where $V_1 < \varphi(0)$ and $a > \varphi'(\alpha)$ are given. Suppose that $\varphi'(z) > -R$ for each $z \in (0, \alpha)$. Note that $G = g + F$ with

$$F(z) := \begin{cases} [V_1, \varphi(0)], & \text{if } \ z = 0, \\ V_1, & \text{if } \ z < 0, \\ \varphi(0), & \text{if } \ z > 0, \end{cases}$$

and

$$g(z) := \begin{cases} az, & \text{if } \ z < 0, \\ \varphi(z) - \varphi(0), & \text{if } \ z \in [0, \alpha], \\ a(z - \alpha) + \varphi(\alpha) - \varphi(0), & \text{if } \ z > \alpha. \end{cases}$$

The current i solves $u - E \in Ri + g(i) + F(i)$. By setting $p = u - E$ and $z = i$ we get (1) with $m = n = 1$, that $B = C = 1$, $f(z) = Rz + g(z), z \in \mathbb{R}$. Then f is locally Lipschitz continuous on $\mathbb{R}$ with

$$\partial f(z) = \begin{cases} R + a, & \text{if } \ z < 0, \\ [R + a, R + \varphi'(0)], & \text{if } \ z = 0, \\ R + \varphi'(z), & \text{if } \ z \in (0, \alpha), \\ [R + \varphi'(\alpha), R + a], & \text{if } \ z = \alpha, \\ R + a, & \text{if } \ z > \alpha. \end{cases}$$

For any $z \in \mathbb{R}$, all the elements of $\partial f(z)$ are positive. Given $(\bar{z}, \bar{p}) \in \operatorname{gph} \Phi$, repeating the steps in Example 4.1, we get that the assumptions of Theorem 2.4 (ii) are satisfied. Thus Φ is strongly metrically regular at any reference point.

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