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Flexural analysis of deep beam subjected to parabolic load using refined shear deformation theory

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Abstract

A trigonometric shear deformation theory for flexure of thick or deep beams, taking into account transverse shear deformation effects, is developed. The number of variables in the present theory is same as that in the first order shear deformation theory. The sinusoidal function is used in displacement field in terms of thickness coordinate to represent the shear deformation effects. The noteworthy feature of this theory is that the transverse shear stresses can be obtained directly from the use of constitutive relations with excellent accuracy, satisfying the shear stress free conditions on the top and bottom surfaces of the beam. Hence, the theory obviates the need of shear correction factor. Governing differential equations and boundary conditions are obtained by using the principle of virtual work. The thick isotropic beams are considered for the numerical studies to demonstrate the efficiency of the theory. It has been shown that the theory is capable of predicting the local effect of stress concentration due to fixity of support. The fixed isotropic beams subjected to parabolic loads are examined using the present theory. Results obtained are discussed critically with those of other theories.
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Keywords: thick beam, trigonometric shear deformation, principle of virtual work, equilibrium equations, displacement, stress

1. Introduction

It is well-known that elementary theory of bending of beam based on Euler-Bernoulli hypothesis disregards the effects of the shear deformation and stress concentration. The theory is suitable for slender beams and is not suitable for thick or deep beams since it is based on the assumption that the sections normal to neutral axis before bending remain so during bending and after bending, implying that the transverse shear strain is zero. Since theory neglects the transverse shear deformation, it underestimates deflections in case of thick beams where shear deformation effects are significant.

Bresse [5], Rayleigh [16] and Timoshenko [20] were the pioneer investigators to include refined effects such as rotatory inertia and shear deformation in the beam theory. Timoshenko showed that the effect of transverse shear is much greater than that of rotatory inertia on the response of transverse vibration of prismatic bars. This theory is now widely referred to as Timoshenko beam theory or first order shear deformation theory (FSDT) in the literature. In this theory transverse shear strain distribution is assumed to be constant through the beam thickness and thus requires shear correction factor to appropriately represent the strain energy of deformation. Cowper [6] has given refined expression for the shear correction factor for different

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cross-sections of beam. The accuracy of Timoshenko beam theory for transverse vibrations of simply supported beam in respect of the fundamental frequency is verified by Cowper [7] with a plane stress exact elasticity solution. To remove the discrepancies in classical and first order shear deformation theories, higher order or refined shear deformation theories were developed and are available in the open literature for static and vibration analysis of beam.

Levinson [15], Bickford [4], Rehfield and Murty [18], Krishna Murty [14], Baluch et al. [2], Bhimaraddi and Chandrashekhara [3] presented parabolic shear deformation theories assuming a higher variation of axial displacement in terms of thickness coordinate. These theories satisfy shear stress free boundary conditions on top and bottom surfaces of beam and thus obviate the need of shear correction factor. Irretier [12] studied the refined dynamical effects in linear, homogenous beam according to theories, which exceed the limits of the Euler-Bernoulli beam theory. These effects are rotary inertia, shear deformation, axial pre-stress, twist and coupling between bending and torsion.

Hilderbrand and Reissner [11] have given the distribution of stress in built-in beam of narrow rectangular cross section using Airy's stress function and the principle of least work. Timoshenko and Goodier [21] presented the elasticity solutions for simply supported and cantilever beams using Airy's stress polynomial functions and using stress functions in the form of a Fourier series.

Kant and Gupta [13], Heyliger and Reddy [10] presented finite element models based on higher order shear deformation uniform rectangular beams. However, these displacement based finite element models are not free from phenomenon of shear locking (Averill and Reddy [1]; Reddy [17]).

There is another class of refined theories, which includes trigonometric functions to represent the shear deformation effects through the thickness. Vlasov and Leont'ev [22], Stein [19] developed refined shear deformation theories for thick beams including sinusoidal function in terms of thickness coordinate in displacement field. However, with these theories shear stress free boundary conditions are not satisfied at top and bottom surfaces of the beam. A study of literature by Ghugal and Shimpi [8] indicates that the research work dealing with flexural analysis of thick beams using refined trigonometric and hyperbolic shear deformation theories is very scarce and is still in infancy.

In this paper development of theory and its application to thick fixed beams is presented.

2. Development of theory

The beam under consideration as shown in Fig. 1 occupies in $0-x-y-z$ Cartesian coordinate system the region:

$$0 \le x \le L, \qquad 0 \le y \le b, \qquad -\frac{h}{2} \le z \le \frac{h}{2},$$

where x, y, z are Cartesian coordinates, L and b are the length and width of beam in the x and y directions respectively, and h is the thickness of the beam in the z-direction. The beam is made up of homogeneous, linearly elastic isotropic material.

2.1. The displacement field

The displacement field of the present beam theory is of the form:

$$\begin{aligned} u(x,z) &= -z\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{h}{\pi}\sin\frac{\pi z}{h}\phi(x), \\ w(x,z) &= w(x), \end{aligned} \tag{1}$$

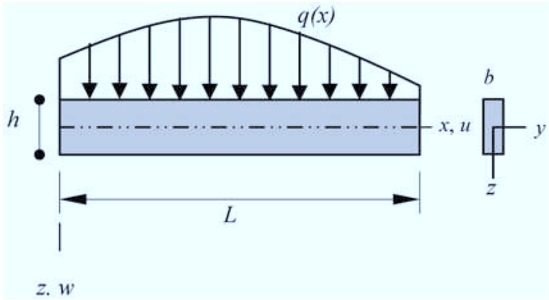

Fig. 1. Beam under bending in x–z plane

where u is the axial displacement in x direction and w is the transverse displacement in z direction of the beam. The sinusoidal function is assigned according to the shear stress distribution through the thickness of the beam. The function ϕ represents rotation of the beam at neutral axis, which is an unknown function to be determined. The normal and shear strains obtained within the framework of linear theory of elasticity using displacement field given by Eq. (1) are as follows:

$$\text{Normal strain: } \varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{h}{\pi}\sin\frac{\pi z}{h}\frac{\mathrm{d}\phi}{\mathrm{d}x}, \tag{2}$$

$$\text{Shear strain: } \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\mathrm{d}w}{\mathrm{d}x} = \cos\frac{\pi z}{h}\phi. \tag{3}$$

The stress-strain relationships used are as follows:

$$\sigma_x = E\varepsilon_x, \qquad \tau_{zx} = G\gamma_{zx}. \tag{4}$$

2.2. *Governing equations and boundary conditions*

Using the expressions for strains and stresses (2) through (4) and using the principle of virtual work, variationally consistent governing differential equations and boundary conditions for the beam under consideration can be obtained. The principle of virtual work when applied to the beam leads to:

$$b\int_{x=0}^{x=L}\int_{z=-h/2}^{z=+h/2}(\sigma_x\delta\varepsilon_x + \tau_{zx}\delta\gamma_{zx})\mathrm{d}x\,\mathrm{d}z - \int_{x=0}^{x=L} q(x)\delta w\,\mathrm{d}x = 0, \tag{5}$$

where the symbol δ denotes the variational operator. Employing Green's theorem in Eq. (4) successively, we obtain the coupled Euler-Lagrange equations which are the governing differential equations and associated boundary conditions of the beam. The governing differential equations obtained are as follows:

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - \frac{24}{\pi^3}EI\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3} = q(x), \tag{6}$$

$$\frac{24}{\pi^3}EI\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} - \frac{6}{\pi^2}EI\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \frac{GA}{2}\phi = 0. \tag{7}$$

The associated consistent natural boundary conditions obtained are of following form:

At the ends $x = 0$ and $x = L$

$$V_x = EI\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} - \frac{24}{\pi^3}EI\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = 0 \quad \text{or } w \text{ is prescribed,} \tag{8}$$

$$M_x = EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \frac{24}{\pi^3}EI\frac{\mathrm{d}\phi}{\mathrm{d}x} = 0 \quad \text{or } \frac{\mathrm{d}w}{\mathrm{d}x} \text{ is prescribed,} \tag{9}$$

$$M_a = EI\frac{24}{\pi^3}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \frac{6}{\pi^2}EI\frac{\mathrm{d}\phi}{\mathrm{d}x} = 0 \quad \text{or } \phi \text{ is prescribed.} \tag{10}$$

Thus the boundary value problem of the beam bending is given by the above variationally consistent governing differential equations and boundary conditions.

2.3. The general solution of governing equilibrium equations of the beam

The general solution for transverse displacement $w(x)$ and warping function $\phi(x)$ is obtained using Eqs. (6) and (7) using method of solution of linear differential equations with constant coefficients. Integrating and rearranging the first governing Eq. (6), we obtain the following equation

$$\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} = \frac{24}{\pi^3}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \frac{Q(x)}{EI}, \tag{11}$$

where $Q(x)$ is the generalized shear force for beam and it is given by $Q(x) = \int_0^x q\,\mathrm{d}x + C_1$.

Now the second governing Eq. (7) is rearranged in the following form:

$$\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} = \frac{\pi}{4}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - \beta\phi. \tag{12}$$

A single equation in terms of ϕ is now obtained using Eqs. (11) and (12) as

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - \lambda^2\phi = \frac{Q(x)}{\alpha EI}, \tag{13}$$

where constants α, β and λ in Eqs. (12) and (13) are as follows

$$\alpha = \left(\frac{\pi}{4} - \frac{24}{\pi^3}\right), \qquad \beta = \left(\frac{\pi^3}{48}\frac{GA}{EI}\right) \quad \text{and} \quad \lambda^2 = \frac{\beta}{\alpha}.$$

The general solution of Eq. (13) is as follows:

$$\phi(x) = C_2\cosh\lambda x + C_3\sinh\lambda x - \frac{Q(x)}{\beta EI}. \tag{14}$$

The equation of transverse displacement $w(x)$ is obtained by substituting the expression of $\phi(x)$ in Eq. (12) and then integrating it thrice with respect to x. The general solution for $w(x)$ is obtained as follows:

$$EIw(x) = \iiiint q\,\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x + \frac{C_1 x^3}{6} + \left(\frac{\pi}{4}\lambda^2 - \beta\right)\frac{EI}{\lambda^3}(C_2\sinh\lambda x + C_3\cosh\lambda x) + C_4\frac{x^2}{2} + C_5 x + C_6, \tag{15}$$

where C_1, C_2, C_3, C_4, C_5 and C_6 are arbitrary constants and can be obtained by imposing boundary conditions of beam.

3. Illustrative example

In order to prove the efficacy of the present theory, the following numerical example is considered. The material properties for beam used are: $E = 210$ GPa, $\mu = 0.3$ and $\rho = 7\,800$ kg/m^3, where E is the Young's modulus, ρ is the density, and μ is the Poisson's ratio of beam material.

A fixed-fixed beam has its origin at left hand side support and is fixed at $x = 0$ and L. The beam is subjected to parabolic load $q(x) = q_0 \left(\frac{x}{L}\right)^2$ on surface $z = -h/2$ acting in the downward z direction with maximum intensity of load q_0 as shown in Fig. 2. The boundary conditions associated with this beam at fixed ends are: $\frac{\mathrm{d}w}{\mathrm{d}x} = \phi = w = 0$ at $x = 0$ and L.

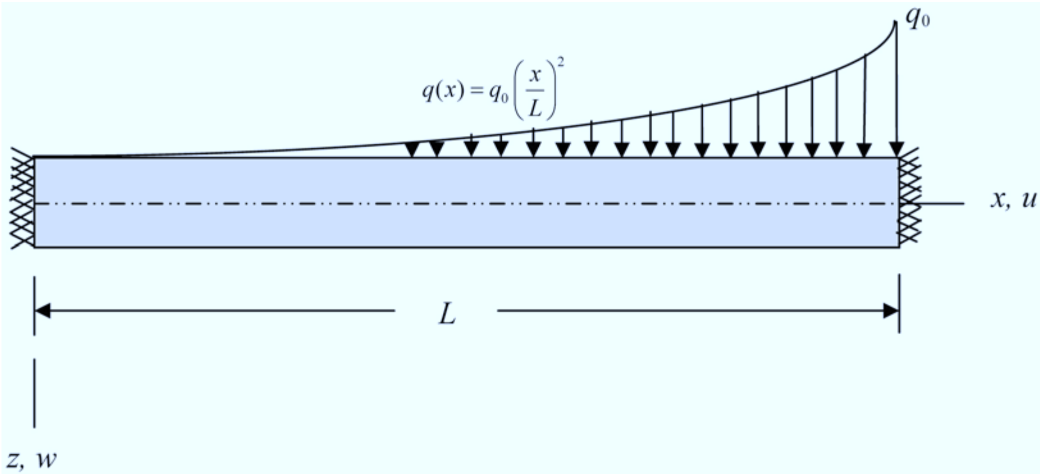

Fig. 2. Fixed beam with parabolic load

General expressions obtained for $w(x)$ and $\phi(x)$ are as follows:

$$w(x) = \frac{q_0 L^4}{120EI}\left[\frac{1}{3}\frac{x^6}{L^6} + \frac{x^2}{L^2} - \frac{4}{3}\frac{x^3}{L^3} - \frac{12}{\pi^2}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{5}{6}\frac{x^4}{L^4} - \frac{5}{3}\frac{x^2}{L^2}\right) - \right. \\ \left. \frac{4}{5}\frac{E}{G}\frac{h^2}{L^2}\left(-\frac{x}{L} + \frac{1}{2}\frac{x^2}{L^2} + \frac{\sinh\lambda x - \cosh\lambda x + 1}{\lambda L}\right)\right], \tag{16}$$

$$\phi(x) = \frac{1}{15}\frac{q_0 L}{\beta EI}\left(1 + 5\frac{x^3}{L^3} + \sinh\lambda x - \cosh\lambda x\right). \tag{17}$$

The expression for axial displacement u is obtained by substituting Eqs. (16) and (17) into the first equation in (1) and it is as follows:

$$u = \frac{q_0 h}{Eb}\left[-\frac{1}{10}\frac{z}{h}\frac{L^3}{h^3}\left(2\frac{x^5}{L^5} + 2\frac{x}{L} - 4\frac{x^2}{L^2} - \frac{40}{\pi^2}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{x^3}{L^3} - \frac{x}{L}\right) - \right.\right. \\ \left. \frac{4}{5}\frac{E}{G}\frac{h^2}{L^2}\left(-1 + \frac{x}{L} + \cosh\lambda x - \sinh\lambda x\right)\right) - \\ \left. \frac{16}{5\pi^4}\sin\frac{\pi z}{h}\frac{E}{G}\frac{L}{h}\left(-1 + 5\frac{x^3}{L^3} + \cosh\lambda x - \sinh\lambda x\right)\right]. \tag{18}$$

The expression for axial stress is obtained using Eqs. (2), (4), (16) and (17) as follows:

$$\sigma_x = \frac{q_0}{b}\left\{-\frac{1}{10}\frac{z}{h}\frac{L^2}{h^2}\left[10\frac{x^4}{L^4}+2-4\frac{x}{L}-\frac{120}{\pi^2}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{x^2}{L^2}-\frac{1}{3}\right)-\right.\right.$$
$$\left.\frac{4}{5}\frac{E}{G}\frac{h^2}{L^2}\big(1+\lambda L(\sinh\lambda x-\cosh\lambda x)\big)\right]- \tag{19}$$
$$\left.\frac{16}{5\pi^4}\sin\frac{\pi z}{h}\frac{E}{G}\left(15\frac{x^2}{L^2}+\lambda L(\sinh\lambda x-\cosh\lambda x)\right)\right\}.$$

The expressions for transverse shear stress is obtained using constitutive relation (4) and using Eq. (17) as follows:

$$\tau_{zx}^{CR} = \frac{16}{5\pi^3}\frac{q_0}{b}\frac{L}{h}\cos\frac{\pi z}{h}\left(1-5\frac{x^3}{L^3}+\sinh\lambda x-\cosh\lambda x\right). \tag{20}$$

Expression for transverse shear stress τ_{zx}^{EE} obtained from equilibrium equation

The alternate approach to determine the transverse shear stress is the use of equilibrium equations. The first stress equilibrium equation of two dimensional theory of elasticity is as follows:

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{zx}}{\partial z}=0. \tag{21}$$

Substituting expression for σ_x into Eq. (21) and integrating it with respect to the thickness coordinate $z$and imposing the boundary condition $\tau_{zx}=0$ at the bounding surfaces $z=\pm h/2$ of the beam one can obtain the final expression of transverse shear stress, which is follows:

$$\tau_{zx}^{EE} = \frac{q_0 L}{80bh}\left(4\frac{z^2}{h^2}-1\right)\left[40\frac{x^3}{L^3}-4-\frac{240}{\pi^2}\frac{x}{L}-\frac{4}{5}\frac{E}{G}\frac{h^2}{L^2}\lambda^2L^2(\cosh\lambda x-\sinh\lambda x)\right]- \tag{22}$$
$$\frac{16}{5\pi^5}\cos\frac{\pi z}{h}\frac{E}{G}\frac{q_0 h}{bL}\left(30\frac{x}{L}+\lambda^2L^2(\cosh\lambda x-\sinh\lambda x)\right).$$

Results are obtained using expressions (16) through (22) for displacements and stresses. The numerical results are presented in Table 1 and graphically presented in Figs. 3 – 11.

Table 1. Non-dimensional axial displacement ($\bar{u}$) at ($x=0.75L$, $z=h/2$), transverse deflection ($\bar{w}$) at ($x=0.75L$, $z=0.0$), axial stress ($\bar{\sigma}_x$) at ($x=0$, $z=h/2$), maximum transverse shear stresses $\bar{\tau}_{zx}^{CR}$ and $\bar{\tau}_{zx}^{EE}$ ($x=0.01L$, $z=0.0$) of the beam for slenderness ratio (S) 4 and 10

Source	S	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
Present		0.2932	0.2513	3.2273	0.1969	−0.4421
Ghugal and Sharma [9]		0.2955	0.2511	3.5277	0.2325	−0.4554
Krishna Murthy [14]	4	0.2979	0.2514	3.2702	0.2053	−0.2834
Timoshenko [20]		−0.8812	0.1107	1.6000	0.0482	0.3999
Bernoulli-Euler		−0.8812	0.0593	1.6000	—	0.3999
Present		−10.8331	0.0902	13.4339	0.8278	−0.0873
Ghugal and Sharma [9]		−10.8275	0.0902	14.1891	0.8851	0.4251
Krishna Murthy [14]	10	−10.8215	0.0902	13.5422	0.8347	0.4197
Timoshenko [20]		−13.7695	0.0675	10.0000	0.7538	0.9999
Bernoulli-Euler		−13.7695	0.0593	10.0000	—	0.9999

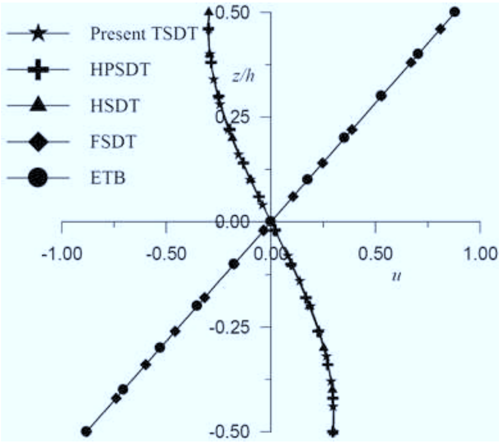

Fig. 3. Variation of axial displacement ($\bar{u}$) through the thickness of fixed-fixed beam at ($x = 0.75L$, z) for slenderness ratio 4

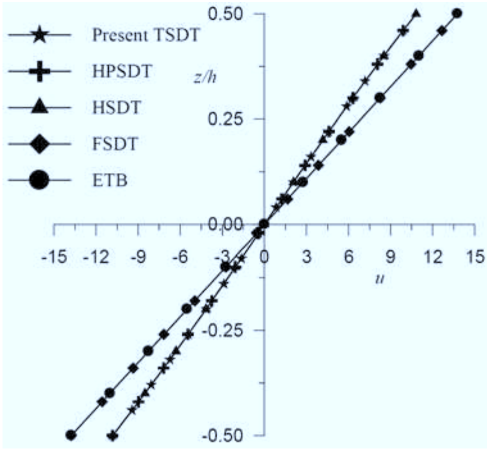

Fig. 4. Variation of axial displacement ($\bar{u}$) through the thickness of fixed-fixed beam at ($x = 0.75L$, z) for slenderness ratio 10

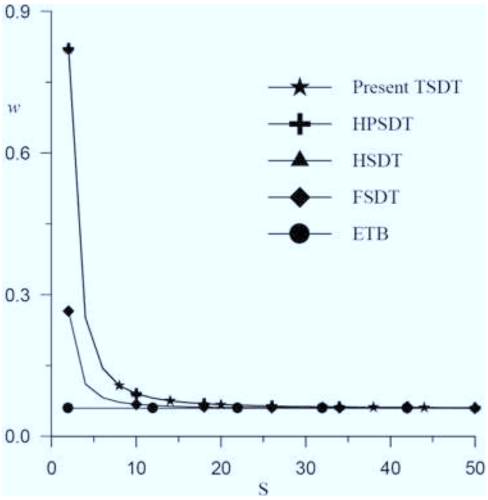

Fig. 5. Variation of maximum transverse displacement ($\bar{w}$) of fixed-fixed beam at ($x = 0.75L$, $z = 0$) with slenderness ratio S

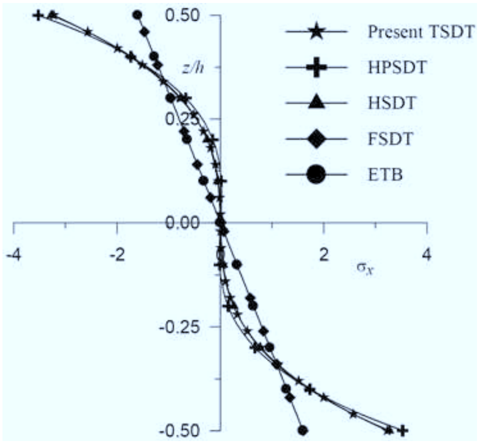

Fig. 6. Variation of axial stress ($\bar{\sigma}_x$) through the thickness of fixed-fixed beam at ($x = 0$, z) for slenderness ratio 4

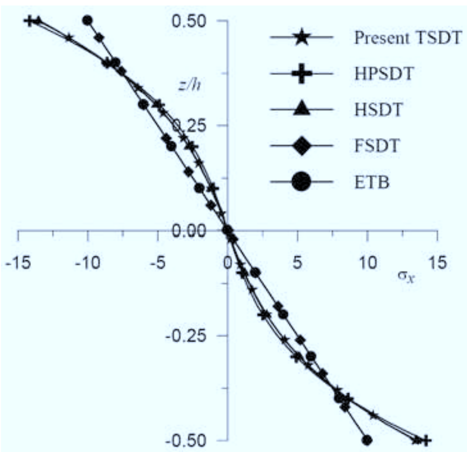

Fig. 7. Variation of axial stress ($\bar{\sigma}_x$) through the thickness of fixed-fixed beam at ($x = 0$, z) for slenderness ratio 10

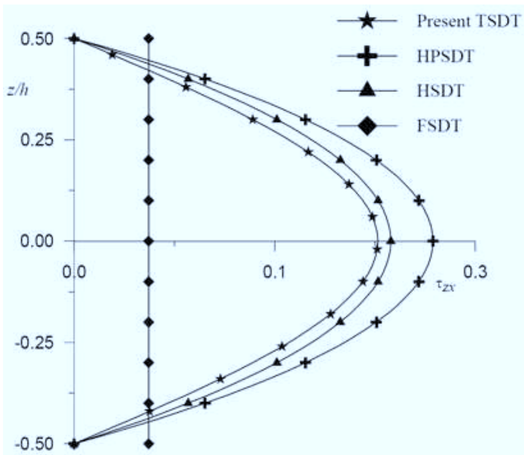

Fig. 8. Variation of transverse shear stress ($\bar{\tau}_{zx}$) through the thickness of fixed-fixed beam at ($x = 0.01L$, z) obtained using constitutive relation for slenderness ratio 4

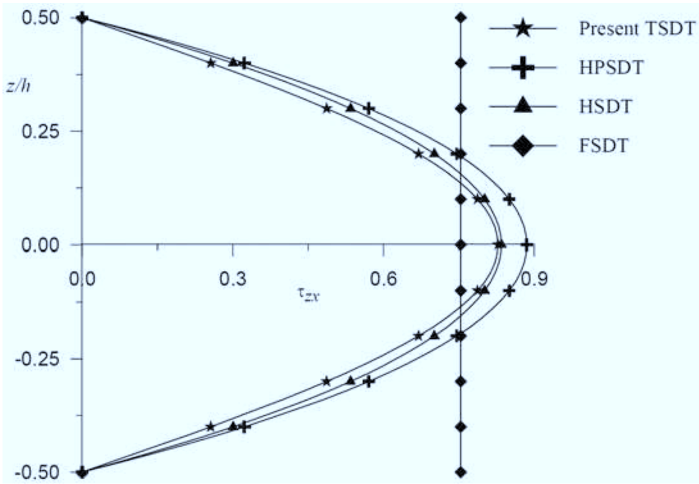

Fig. 9. Variation of transverse shear stress ($\bar{\tau}_{zx}$) through the thickness of fixed-fixed beam at ($x = 0.01L$, z) obtained using constitutive relation for slenderness ratio 4

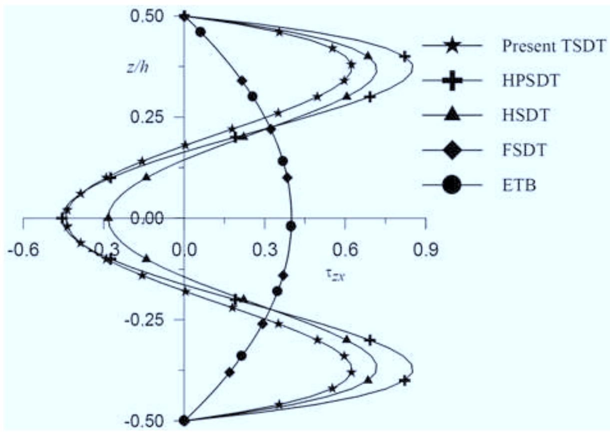

Fig. 10. Variation of transverse shear stress ($\bar{\tau}_{zx}$) through the thickness of fixed-fixed beam at ($x = 0.01L$, z) obtained using equilibrium equation for slenderness ratio 4

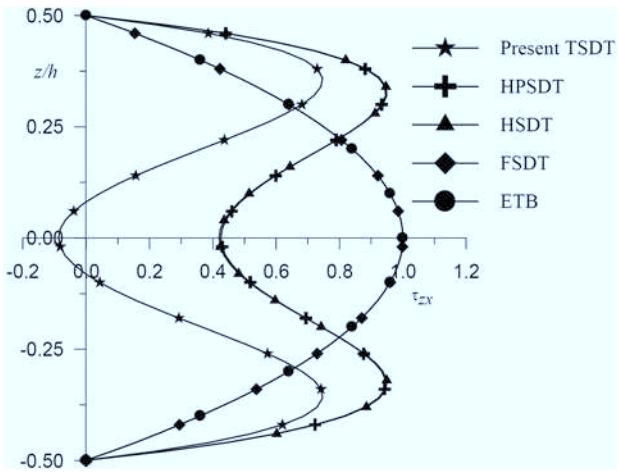

Fig. 11. Variation of transverse shear stress ($\bar{\tau}_{zx}$) through the thickness of fixed-fixed beam at ($x = 0.01L$, z) obtained using equilibrium equation for slenderness ratio 10

4. Results

The results for inplane displacement, transverse displacement, axial and transverse stresses are presented in the following non dimensional form for the purpose of presenting the results in this paper:

$$\bar{u} = \frac{Ebu}{q_0 h}, \qquad \bar{w} = \frac{10Ebh^3 w}{q_0 L^4}, \qquad \bar{\sigma}_x = \frac{b\sigma_x}{q_0}, \qquad \bar{\tau}_{zx} = \frac{b\tau_{zx}}{q_0}, \qquad S = \frac{L}{h}.$$

The numerical results for displacements and stresses are obtained using FORTRAN programs developed based on the non-dimensional expressions for these quantities.

5. Discussion and conclusion

The variationally consistent theoretical formulation of the theory with general solution technique of governing differential equations is presented. The general solutions for beam with parabolic load is obtained in case of thick fixed beams. The displacements and stresses obtained by present theory are in excellent agreement with those of other equivalent refined and higher order theories. The present theory yields the realistic variation of axial displacement and stresses through the thickness of beam. The theory is shown to be capable of predicting the effects of stress concentration on the axial and transverse stresses in the vicinity of the built-in end of the beam which is the region of heavy stress concentration. Thus the validity of the present theory is established.

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Nomenclature

A	Cross sectional area of beam $= bh$
b	Width of beam in y-direction
E, G, μ	Elastic constants of the beam material
h	Thickness of beam
I	Moment of inertia of cross-section of beam
L	Span of the beam
q_0	Intensity of parabolic transverse load
S	Slenderness ratio of the beam $= L/h$
w	Transverse displacement in z-direction
$\bar{w}$	Non-dimensional transverse displacement
$\bar{u}$	Non-dimensional axial displacement
x, y, z	Rectangular Cartesian coordinates
$\bar{\sigma}_x$	Non-dimensional axial stress in x-direction
$\bar{\tau}_{zx}^{CR}$	Non-dimensional transverse shear stress via constitutive relation
$\bar{\tau}_{zx}^{EE}$	Non-dimensional transverse shear stress via equilibrium equation
$\phi(x)$	Unknown function associated with the shear slope

List of abbreviations

CR	Constitutive Relations
EE	Equilibrium Equations
TSDT	Trigonometric Shear Deformation Theory
HPSDT	Hyperbolic Shear Deformation Theory
HSDT	Third Order Shear Deformation Theory
FSDT	First Order Shear Deformation Theory
ETB	Elementary Theory of Beam