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On shape optimization of conduits with incompressible flow

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Abstract

In this paper we present an optimal flow problem in the context of internal laminar incompressible flows of Newtonian liquid. The flow optimization problem is introduced in variational form of the stationary Navier-Stokes equations. The parametrization of design domain using the free-form deformation (FFD) approach allows an easy mesh adaptation w.r.t. shape changes. Numerical examples are given for two kinds of optimization objectives: enhancing flow uniformity in a control region and minimizing pressure losses.

Keywords: shape optimization, stationary laminar flow, incompressible Navier-Stokes equations, free-form deformation

1. Introduction

Our interest in shape optimization for the problems of fluid mechanics has been motivated by vast applications in automotive, aerospace and other industries. Those applications involve both internal (e.g. efficient cooling, exhaust piping) and external (e.g. wing, blade profiles, vehicle aerodynamics) flows. Shapes of channels or obstacles to flow play an important role in mechanical and (bio)chemical processes such as convected reaction-diffusion (catalysis, drug delivery, …), combustion, mixing, etc.

Our ultimate aim is to develop shape optimization tools for real life problems (3D complex geometries). We focus mainly on internal flows (ducts, channels) from the point of view of various merits of optimization, namely obtaining a desired velocity profile at a “control” part of the channel to improve efficiency of downstream parts, minimizing pressure losses, reducing wear of downstream parts or reducing noise. In this paper we restrict this broad topic to laminar incompressible flows of Newtonian liquid with sufficiently high kinematic viscosity ($\sim 10^{-3}$).

In [5] we have proposed a variational formulation of an optimal flow problem in closed channels. The shape sensitivity formulas as well as computational domain parametrization using the free-form deformation (FFD) approach, cf. [3], [6], were presented and employed to compute numerical examples using a simplified Stokes problem.

Here we briefly recall the formulation, introducing the flow optimization problem in variational form in Section 2. Then we show the first results of the shape optimization procedure with full incompressible Navier-Stokes equations for stationary laminar flow in Section 3. Finally we present, as an outlook, a promising approach to stabilization of a Navier-Stokes problem solution in Section 4, according to [2], as a way of tackling low viscosity (air) flows in the context of finite elements.

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2. Problem setting

Let us recall the variational formulation of an optimal flow problem, as introduced in [5].

2.1. Variational formulation of an optimal flow problem

The problem is defined in an open bounded domain $\Omega \subset \mathbb{R}^3$ with two (possibly overlapping) subdomains defined as

$$\overline{\Omega} = \overline{\Omega_D \cup \Omega_C} \quad \text{with} \quad \Gamma_C = \partial\Omega_D \cap \partial\Omega_C \;, \tag{1}$$

where Ω_C is the *control domain* and Ω_D is the *design domain*, see Fig. 1. The shape of Ω_D is modified exclusively through the *design boundary*, $\Gamma_D \subset \partial\Omega_D \setminus \Gamma_{\text{in-out}}$ where $\Gamma_{\text{in-out}} \subset \partial\Omega$ is the union of the "inlet-outlet" boundary of the channel; in general $\Gamma_{\text{in-out}}$ consists of two disjoint parts, $\Gamma_{\text{in-out}} = \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$.

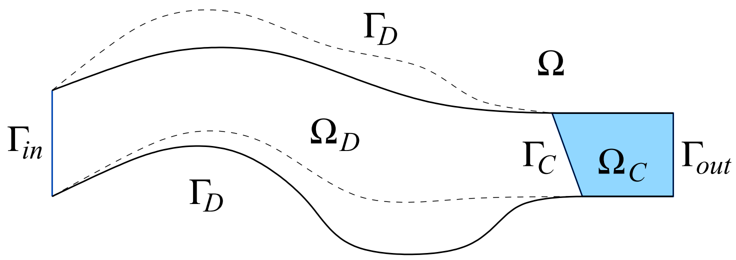


Fig. 1. The decomposition of domain Ω, control domain Ω_C at the outlet sector of the channel.

We seek a steady state of an incompressible flow in Ω by solving the following problem: find a velocity, $\boldsymbol{u}$, and pressure, p, fields in Ω such that (ν is the kinematic viscosity)

$$\begin{aligned} -\nu\nabla^2\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = 0 \quad & \text{in } \Omega \;, \\ \nabla\cdot\boldsymbol{u} = 0 \quad & \text{in } \Omega \;, \end{aligned} \tag{2}$$

with the boundary conditions

$$\begin{aligned} \boldsymbol{u} = 0 \quad \text{on } \partial\Omega \setminus \Gamma_{\text{in-out}} \;, \quad \boldsymbol{u} = \bar{\boldsymbol{u}} \quad \text{on } \Gamma_{\text{in}} \;, \\ -p\boldsymbol{n} + \nu\frac{\partial\boldsymbol{u}}{\partial n} = -\bar{p}\boldsymbol{n} \quad \text{on } \Gamma_{\text{out}} \;, \end{aligned} \tag{3}$$

where $\boldsymbol{n}$ is the unit outward-normal vector on Γ_{out}, $\frac{\partial}{\partial n} = \boldsymbol{n}\cdot\nabla$ and $\bar{\boldsymbol{u}}$ is a given inlet velocity profile. Note that by $(3)_2$ we prescribe the stress in the form of pressure $\bar{p}$, so that we enforce the condition of $\frac{\partial\boldsymbol{u}}{\partial n} = 0$, i.e. the flow is uniform in the normal direction w.r.t. Γ_{out}.

Now we introduce the following functional forms ($i = 1, 2$ or $i = 1, 2, 3$, summation convention is employed):

$$\begin{aligned} a_\Omega(\boldsymbol{u}, \boldsymbol{v}) &:= \nu\int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v} = \nu\int_\Omega \frac{\partial u_i}{\partial x_k}\frac{\partial v_i}{\partial x_k} \;, \\ c_\Omega(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) &:= \int_\Omega (\boldsymbol{w}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{v} = \int_\Omega w_k\frac{\partial u_i}{\partial x_k}v_i \;, \\ b_\Omega(\boldsymbol{u}, p) &:= \int_\Omega p\,\nabla\cdot\boldsymbol{u} \;, \quad g_{\Gamma_{\text{out}}}(\boldsymbol{v}) := -\int_{\Gamma_{\text{out}}} \bar{p}\,\boldsymbol{v}\cdot\boldsymbol{n}\,dS \;, \end{aligned} \tag{4}$$

and the space of admissible velocities

$$\boldsymbol{V}_0 = \{\boldsymbol{v} \in \mathbf{H}^1(\Omega) \mid \boldsymbol{v} = 0 \text{ on } \partial\Omega \setminus \Gamma_{\rm out}\} \;, \tag{5}$$

where $\mathbf{H}^1(\Omega) = [H^1(\Omega)]^3$. Using the forms (4) we obtain the following weak problem: find $\boldsymbol{u} \in \boldsymbol{V}_0(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} a_\Omega(\boldsymbol{u}, \boldsymbol{v}) + c_\Omega(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - b_\Omega(\boldsymbol{v}, p) = g_{\Gamma_{\rm out}}(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}_0 \;, \\ b_\Omega(\boldsymbol{u}, q) = 0 \quad \forall q \in L^2(\Omega) \;. \end{aligned} \tag{6}$$

2.2. Shape optimization problem

Our objective is to minimize the objective function $\Psi(\boldsymbol{u}, p)$ w.r.t. some criterion (see below) by means of varying Γ_D:

$$\begin{aligned} &\min_{\Gamma_D} \Psi(\boldsymbol{u}, p) \;, \\ \text{subject to: } &(\boldsymbol{u}, p) \text{ satisfy (6)} \;, \\ &\Gamma_D \text{ in } \mathcal{U}_{ad}(\Omega_0) \;. \end{aligned} \tag{7}$$

Above $(7)_2$ imposes the admissibility of the velocity and pressure fields, whereas $(7)_3$ restricts shape variation of Γ_D w.r.t. some "initial" shape inherited from the reference domain Ω_0 which defines the associated *set of admissible shapes*, $\mathcal{U}_{ad}(\Omega_0)$, given by the parametrization of Ω_D shape, see [5].

In the examples below we use the following objective functions (possibly in combination):

1. Uniform flow in control region:

$$\Psi_1(\boldsymbol{u}) = \frac{\nu}{2} \int_{\Omega_C} |\nabla \boldsymbol{u}|^2 = \frac{1}{2} a_{\Omega_C}(\boldsymbol{u}, \boldsymbol{u}) \;. \tag{8}$$

Here we wish to enhance flow uniformity by reducing the gradients of flow velocities in Ω_C. The objective function does not depend on the pressure p. Moreover, if $\Gamma_D \subset \partial\Omega_D \setminus (\Gamma_{\rm in-out} \cup \partial\Omega_C)$, the control domain Ω_C does not depend on design modifications, which simplifies the sensitivity formulae.

2. Inlet-outlet pressure difference:

$$\Psi_2(p) = (\int_{\Gamma_{\rm in}} p) - \bar{p} \;. \tag{9}$$

In this case the pressure loss is minimized. Recall that $\bar{p}$ is a given outlet pressure.

2.3. Numerical solution

The weak problem (6) is discretized by an inf-sup stable finite element discretization (fulfilling the Babuška-Brezzi condition), namely by P1B/P1 elements (piecewise-linear velocities enriched by a bubble function and piecewise-linear pressures). The resulting system on nonlinear algebraic equations can be solved by either the Newton iteration or Oseen iteration, see also Section 4.1. All computations were performed by our software which can be found at http://ui505p06-mbs.ntc.zcu.cz/sfe, cf. [7].

3. Numerical examples

In the examples presented below we use $\nu = 1.25 \cdot 10^{-3}$. A consistent unit set $\{$m, s, kg$\}$ is used. Concerning the boundary conditions, the velocity component in the tube direction is set to 1 on the inlet part of the boundary. On the walls we assume no-slip condition $\boldsymbol{u} = 0$. On the outlet we specify $\bar{p} = 0$. The boundary of the control domain Ω_C does not depend on design changes: $\Gamma_D \cap \partial\Omega_C = \emptyset$. The results are summarized in figures which show the domain shape and the fluid flow within, as well as control boxes that govern the FFD parametrization of the domain and hence the domain shape.

3.1. Simple elbow shape

In Figs. 2 3 we can see results of shape optimization of a simple elbow tube with the following parameters: diameter 6 cm, inner elbow diameter: 14 cm. Two choices of objective function were considered: a) $\Psi(\boldsymbol{u}) = \Psi_1(\boldsymbol{u})$ and b) $\Psi(\boldsymbol{u}, p) = 0.9\Psi_1(\boldsymbol{u}) + 0.1\Psi_2(p)$. The control domain Ω_C of $\Psi_1(\boldsymbol{u})$ was situated next to the outlet of the tube. We can see that in case of a) a flow uniformity was improved by straightening the tube segment preceding Ω_C, the inlet-outlet pressure difference increased, though, as the tube was squeezed in the sharp bend prior to the straightening. On the other hand, in case b), the pressure gradient was reduced considerably, in addition to a more uniform flow in Ω_C.

initial design

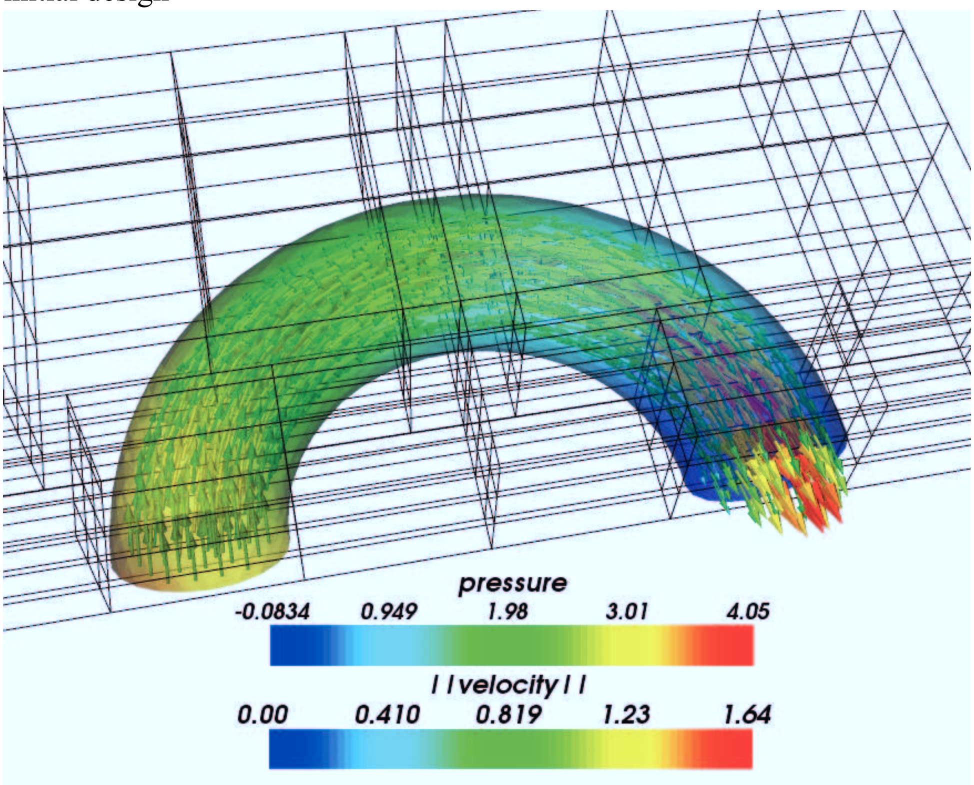
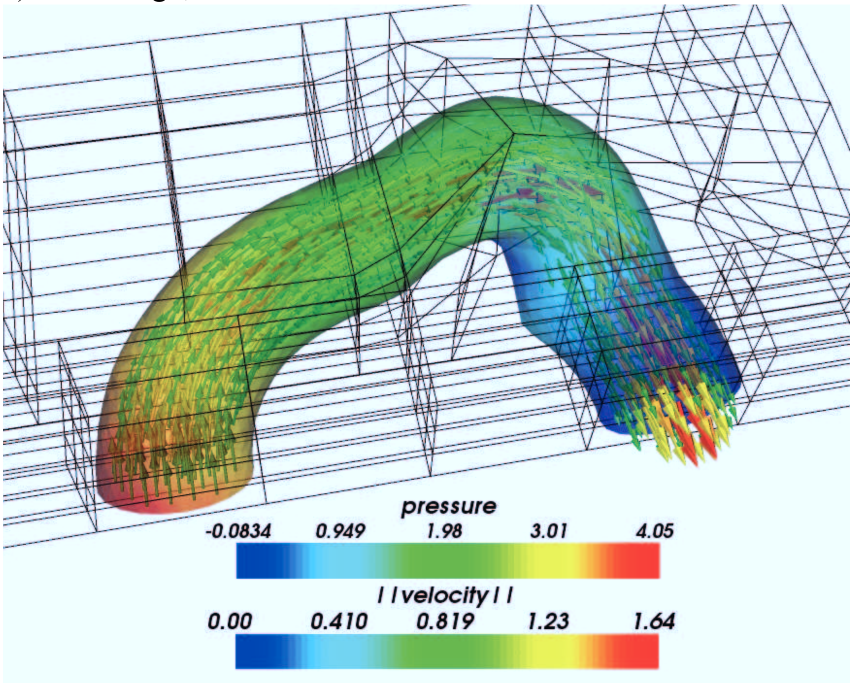


Fig. 2. Flow and domain control boxes. Control domain Ω_C for Ψ_1 next to the outlet.

a) final design, Ψ_1



b) final design, $0.9\Psi_1 + 0.1\Psi_2$

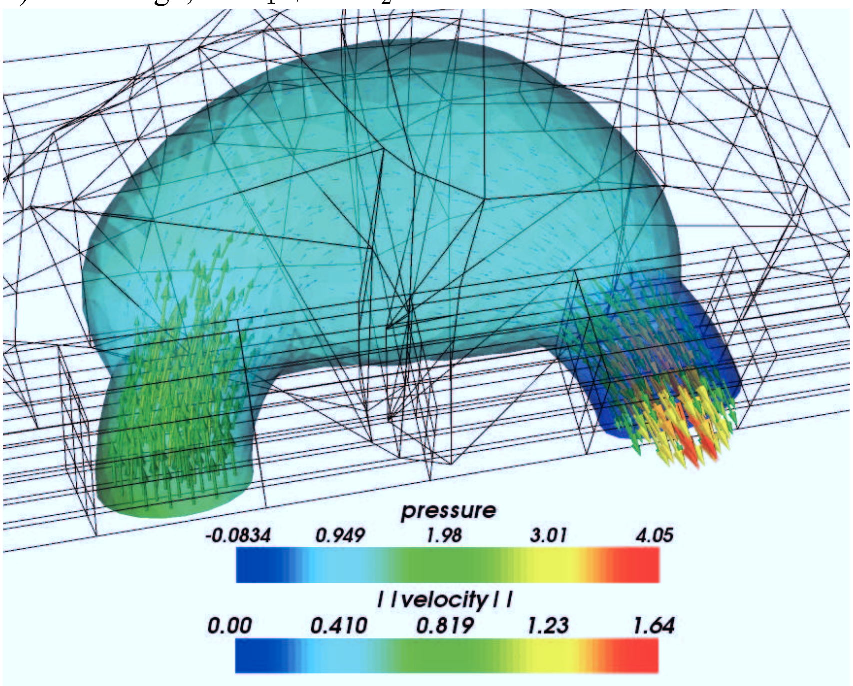
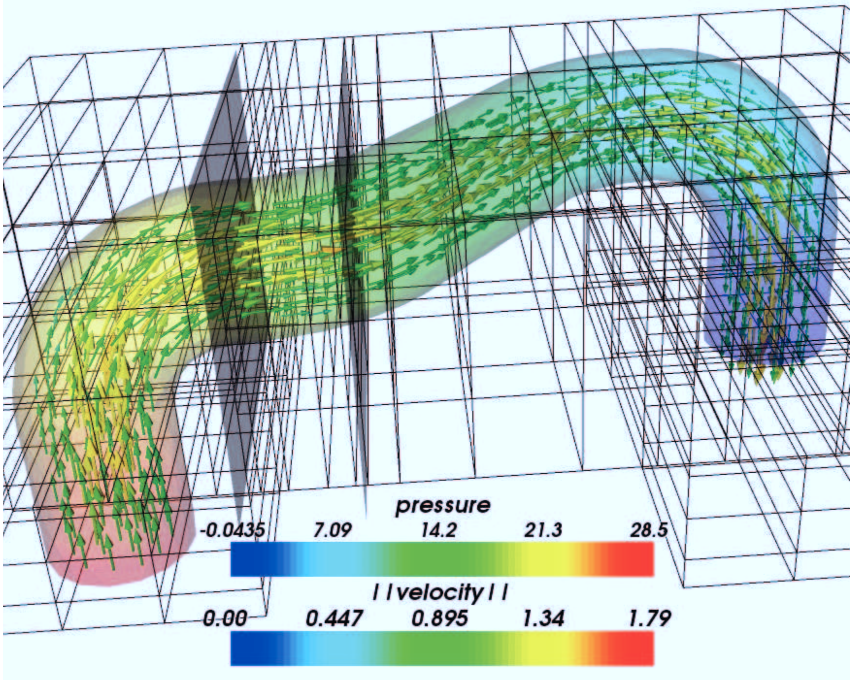


Fig. 3. Flow and domain control boxes. Control domain Ω_C for Ψ_1 next to the outlet.

3.2. More complex shape

In Fig. 4 a computation with $\Psi(\boldsymbol{u}) = \Psi_1(\boldsymbol{u})$ is shown on a more complex tube geometry (diameter: 1 cm). Again, the flow uniformity in Ω_C, denoted by two grey planes in the figure, was forced by straightening the tube. Note, however, that the shape changes of the domain are local — only the control boxes relevant to the objective improvement move.

initial design



final design, Ψ_1

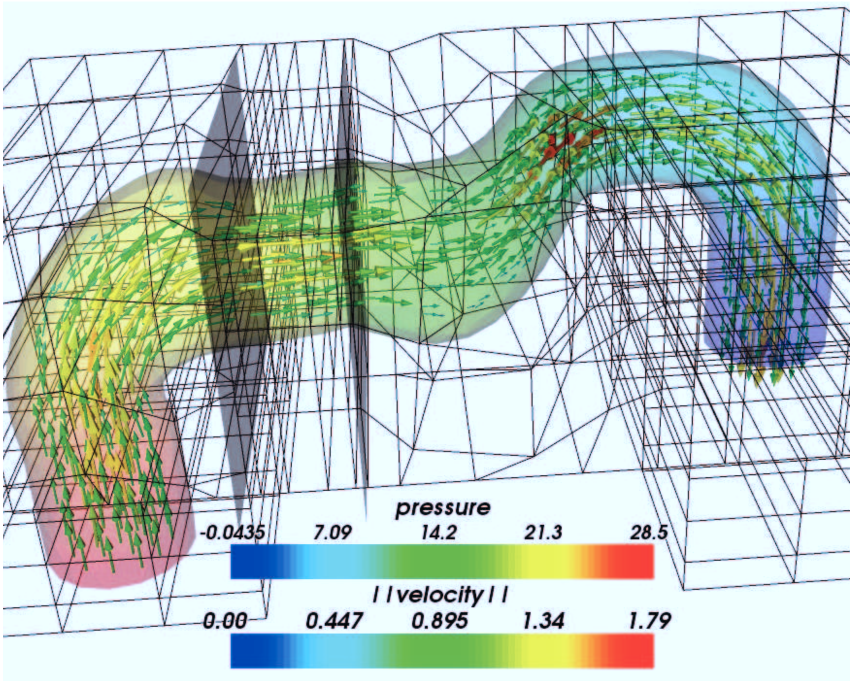


Fig. 4. Flow and domain control boxes. Control domain Ω_C between two grey planes.

In both examples the final designs were better than the initial designs w.r.t. the objective functions used. In practice, however, more constraints need to be added to the FFD control boxes to enforce, e.g. higher degree of continuity of the boundary, or to prevent excessive bloating of the structure, as in case b) in Fig. 3.

4. Stabilization of solution

In order to be able to solve low viscosity problems (air flow in a channel, $\nu \approx 10^{-5}$, a stabilization of the finite element solution is required. In [2] a promising approach was published recently, combining both the inf-sup stable discretization (fulfilling the Babuška-Brezzi condition) and convection stabilization strategies. As our software implements those ideas, we recall here briefly the main results for the sake of completeness.

4.1. Generalized Oseen problem

The nonlinear Navier-Stokes equations (2) can be solved by a fixed-point or Newton-type iteration. This leads to a generalized Oseen problem, where the convective term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ is replaced by $\boldsymbol{b} \cdot \nabla \boldsymbol{u}$ with the convection velocity $\boldsymbol{b}$ known (e.g. from the previous iteration step), $\boldsymbol{f}$ are volume forces (not present in our computations):

$$\begin{aligned} -\nu \nabla^2 \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{u} + \sigma \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad & \text{in } \Omega \;, \\ \nabla \cdot \boldsymbol{u} = 0 \quad & \text{in } \Omega \;. \end{aligned} \tag{10}$$

The term $\sigma \boldsymbol{u}$ originates from time discretization of the nonstationary Navier-Stokes problem, $\sigma \sim \frac{1}{\Delta t}$. In the stationary case $\sigma = 0$. Let us denote

$$\begin{aligned} A\left((\boldsymbol{u}, p), (\boldsymbol{v}, q)\right) &:= a_\Omega(\boldsymbol{u}, \boldsymbol{v}) + c_\Omega(\boldsymbol{b}, \boldsymbol{u}, \boldsymbol{v}) - b_\Omega(\boldsymbol{v}, p) + b_\Omega(\boldsymbol{u}, q) + \sigma(\boldsymbol{u}, \boldsymbol{v})_\Omega \;, \\ L\left((\boldsymbol{v}, q)\right) &:= (\boldsymbol{f}, \boldsymbol{v})_\Omega + g_{\Gamma_{\text{out}}}(\boldsymbol{v}) \;, \\ (\boldsymbol{u}, \boldsymbol{v})_G &:= \int_G \boldsymbol{u} \cdot \boldsymbol{v} \quad \ldots L^2 \text{ inner product on } G \;. \end{aligned} \tag{11}$$

4.2. Grad-div, SUPG and PSPG stabilization

The weak form of the problem (10) is discretized by finite elements using inf-sup stable elements (for example Taylor-Hood P_2/P_1 elements on simplices) leading to the discrete weak formulation of the generalized Oseen problem: find $\boldsymbol{u}_h \in \boldsymbol{X}_h$ and $p \in M_h$ such that

$$A\left((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)\right) = L\left((\boldsymbol{v}_h, q_h)\right) \quad \forall (\boldsymbol{v}_h, q_h) \in (\boldsymbol{X}_h, M_h) \;, \tag{12}$$

where $\boldsymbol{X}_h, M_h$ are appropriate finite element spaces. The authors in [2] now introduce a modified forms

$$\begin{aligned} A_S\left((\boldsymbol{u}, p), (\boldsymbol{v}, q)\right) &:= A\left((\boldsymbol{u}, p), (\boldsymbol{v}, q)\right) + \gamma(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v})_\Omega \\ &\quad + \sum_{K \in \mathcal{T}_h} \left(-\nu \nabla^2 \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{u} + \sigma \boldsymbol{u} + \nabla p, \delta_K (\boldsymbol{b} \cdot \nabla \boldsymbol{v}) + \tau_K \nabla q\right)_K \\ L_S\left((\boldsymbol{v}, q)\right) &:= L\left((\boldsymbol{v}, q)\right) + \sum_{K \in \mathcal{T}_h} \left(\boldsymbol{f}, \delta_K (\boldsymbol{b} \cdot \nabla \boldsymbol{v}) + \tau_K \nabla q\right)_K \;, \end{aligned} \tag{13}$$

where $\bigcup_{K \in \mathcal{T}_h} \overline{K} = \overline{\Omega}$ is a triangulation of Ω. The γ term realizes the *grad-div* stabilization, the terms with δ_K correspond to the *streamline-diffusion (SUPG)* stabilization and the terms with τ_K mean the *pressure (PSPG)* stabilization.

4.3. Choice of stabilization parameters

Assuming scaling of the Oseen problem such that $b_\infty := ||\boldsymbol{b}||_\infty \sim 1$, and denoting $C_F \sim \mathrm{diam}(\Omega)$ the Friedrichs constant for Ω, the stabilization parameters are chosen as follows:

$$\gamma = \nu + b_\infty C_F \;, \tag{14}$$

and there exists a constant C such that

$$0 \leq \tau_K \leq \delta_K \leq C \frac{\min(1; \frac{1}{\sigma}) h_K^2}{\nu + b_\infty C_F + \sigma C_F^2 + b_\infty^2 \min(\frac{C_F^2}{\nu}; \frac{1}{\sigma})} \;. \tag{15}$$

The theoretical considerations in [2] require σ to be a positive constant bounded away from zero. However, in practice, the stabilization may work even for stationary problems with $\sigma = 0$.

5. Conclusion

We have briefly presented a variational form of an optimal flow problem in the context of internal stationary laminar incompressible flows of Newtonian liquid. Feasibility of our approach has been demonstrated on numerical examples featuring two kinds of optimization objectives: enhancing flow uniformity in a control region and minimizing pressure losses. A steepest descent algorithm was used for the optimization with gradients w.r.t. design computed by the adjoint equation technique, see [1], [4]. The shape changes of the computational domain were governed by means of the FFD approach, cf. [3], [6].

We have also summarized main points of the finite element solution stabilization found in [2] in Section 4, as it is required for low viscosity flows. Our next step will be to derive sensitivity formulae for the stabilized Navier-Stokes problem (13), analogously to the procedure presented in [5] for the classical Navier-Stokes problem (6).

Furthermore, to deal with real world fluid dynamics problems, some additional constraints on domain parametrization need to be added, to allow a better control of the domain shape obeying space and boundary continuity requirements.

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