FACULTY OF ELECTRICAL ENGINEERING
KTE
September 10 - 12, 2003
Pilsen, Czech Republic

# INTEGRAL SOLUTION OF ELECTRO-STATIC FIELDS IN 3D ARRANGEMENTS

ING. ROMAN HAMAR, PH.D.[1]
PROF. ING. IVO DOLEŽEL, CSC.[2]
DOC. ING. BOHUŠ ULRYCH, CSC.[1]

**Abstract:** The paper deals with computation of 3D electrostatic fields (distribution of charges, electric potential and other derived quantities) by means of integral equations and their numerical solution. Selected are neither configurations that can be simplified to 2D problems and solved analytically, nor arrangements that might be processed by the FD or FE techniques. Analysed are fundamental mathematical aspects of the method, which is illustrated on a 3D field between two cubes in a general position.

**Keywords:** Integral equations, electric field, numerical solution, electric charge.

## 1 Introduction

A lot of tasks associated with mapping of electrostatic fields cannot be successfully solved by means of classic FD or FE techniques. We can mention, for example, problems characterised by the lack of boundary conditions, geometrical incommensurability of particular subdomains in the investigated area etc. In such cases alternative algorithms have to be applied, one of them being the method of moments using the first-kind Fredholm integral equation.

Although the method itself is well known [1], [2], its application brings specific problems. One of them is manipulation with fully populated matrices containing improper integrals in their main diagonals. Finite values of these integrals with weakly singular kernel functions exist only for 2D surface elements, which corresponds with the physical reality, but on the other hand, discretisation of large complicated surfaces often leads to a very high number of equations in the discretised model. Local problems with accuracy may also appear, associated with a high charge density near edges, corners etc.

[1] FEL ZČU, Sady Pětatřicátníků 14, 306 14 Plzeň, e-mail: {hamar, ulrych}@kte.zcu.cz
[2] FEL ČVUT, Technická 2, 166 27 Praha 6, e-mail: dolezel@fel.cvut.cz

The paper illustrates the method on computation of electric field near two charged cubes in general position.

## 2 Mathematical model of the problem

Consider a system of $n$ mutually electrically isolated well conductive metal conductors $C_1$, $C_2$,…, $C_n$ carrying constant electric potentials $\varphi_1$, $\varphi_2$,…, $\varphi_n$, see Fig. 1. The system is placed in a homogeneous medium of permittivity $\varepsilon_r$. Dimensions of the conductors are finite and their surfaces smooth by parts. It is necessary to map the electric field in the area of the conductors.

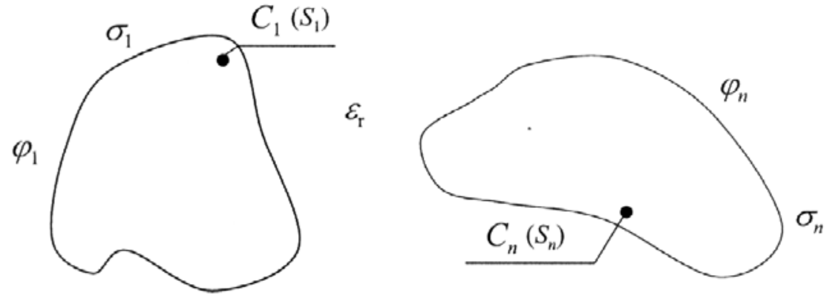


Fig. 1: Arrangement of the charged bodies

*The first step* is to find functions $\sigma_i(S_i)$, $i = 1, 2, \ldots, n$, describing the distribution of electric charges along surfaces $S_i$ of conductors $C_i$. This is realised by solution of a system of the first-kind Fredholm integral equations in the form

$$\varphi_i(P) = \frac{1}{4\pi\varepsilon_0\varepsilon_r} \cdot \sum_{j=1}^{n} \int_{S_j} G(\boldsymbol{r}_P, \boldsymbol{r}_Q) \cdot \sigma_j(S_j)\, \mathrm{d}S, \quad P \in S_i,\ Q \in S_j,\ i, j = 1, \ldots, n \quad (1)$$

with weakly-singular kernel functions

$$G(\boldsymbol{r}_P, \boldsymbol{r}_Q) = \frac{1}{|\boldsymbol{r}_P - \boldsymbol{r}_Q|} \quad (2)$$

where $P$ is the reference point and $Q$ is variable point that passes through all surfaces $S_j$, $j = 1, 2, \ldots, n$.

The basic advantage of the function $G(\boldsymbol{r}_P, \boldsymbol{r}_Q)$ is its integrability in 2D. Solvability of the system (1) and unambiguousness of the continuous model was proved in [3]. Discretisation is performed in the standard manner. The surfaces are approximated by triangular or quadrilateral meshes. Real distribution of electric charge in the cells may be substituted by a polynomial function (constant function may be disadvantageous within corner and edge cells, where variations of the charge are significant). Proper and improper integrals in the system matrix were calculated analytically for both triangular and rectangular cells in both Cartesian (inverse goniometric and logarithmic expressions) and cylindrical (expression containing elliptic and some other higher functions) coordinate systems.

*The second step* is to find the field quantities (potential, electric field strength, partial or total capacitances etc.) by means of relatively simple integral expressions. The method was programmed by the authors in Matlab code that is available at practically every Czech technical university.

## 3 Illustrative example

The method was tested on determining the electric charge distribution near two charged cubes in general position (see Fig. 2) placed in the air. The length of the edge of both cubes is 0.02 m and the distance of their centers 0.035 m. All walls were discretised uniformly. Geometrical convergence follows from Tab. 1. Selected results for subdivision of each edge by 12 cells are depicted in Figs. 3, 4 and 5.

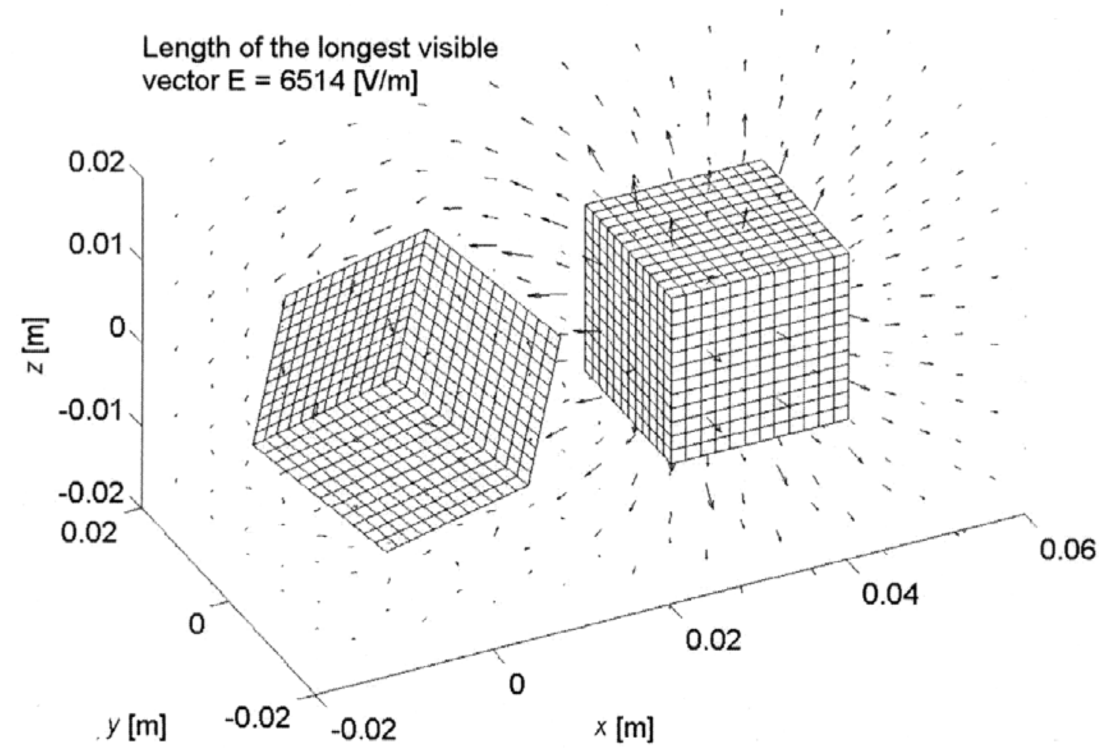


Fig. 2: Basic arrangement of the cubes and distribution of the electric field (potential of the left cube $\varphi_1$ = -50 V, right cube $\varphi_2$ = 50 V)

Tab. 1: Geometrical convergence for growing division of an edge (Athlon XP 2000+)

| elements in each edge | charge $Q_1$ (C) | charge $Q_2$ (C) | capacitance $C$ (F) | time of computation (s) | number of unknowns |
|---|---|---|---|---|---|
| 6 | -1.2086e-010 | 1.2119e-010 | 1.2103e-012 | 8.139 | 432 |
| 7 | -1.2122e-010 | 1.2156e-010 | 1.2139e-012 | 16.078 | 588 |
| 8 | -1.2148e-010 | 1.2183e-010 | 1.2166e-012 | 29.359 | 768 |
| 9 | -1.2168e-010 | 1.2203e-010 | 1.2186e-012 | 51.156 | 972 |
| 10 | -1.2184e-010 | 1.2219e-010 | 1.2202e-012 | 82.719 | 1200 |
| 11 | -1.2196e-010 | 1.2231e-010 | 1.2214e-012 | 131.547 | 1452 |
| 12 | -1.2206e-010 | 1.2241e-010 | 1.2224e-012 | 200.281 | 1728 |
| 20 | -1.2247e-010 | 1.2283e-010 | 1.2265e-012 | 2735.514 | 4800 |

Fig. 3 shows two planes A and B in which we calculated the distribution of potential. It was determined by means of analytically solved integral expressions based on the knowledge of the computed surface charges.

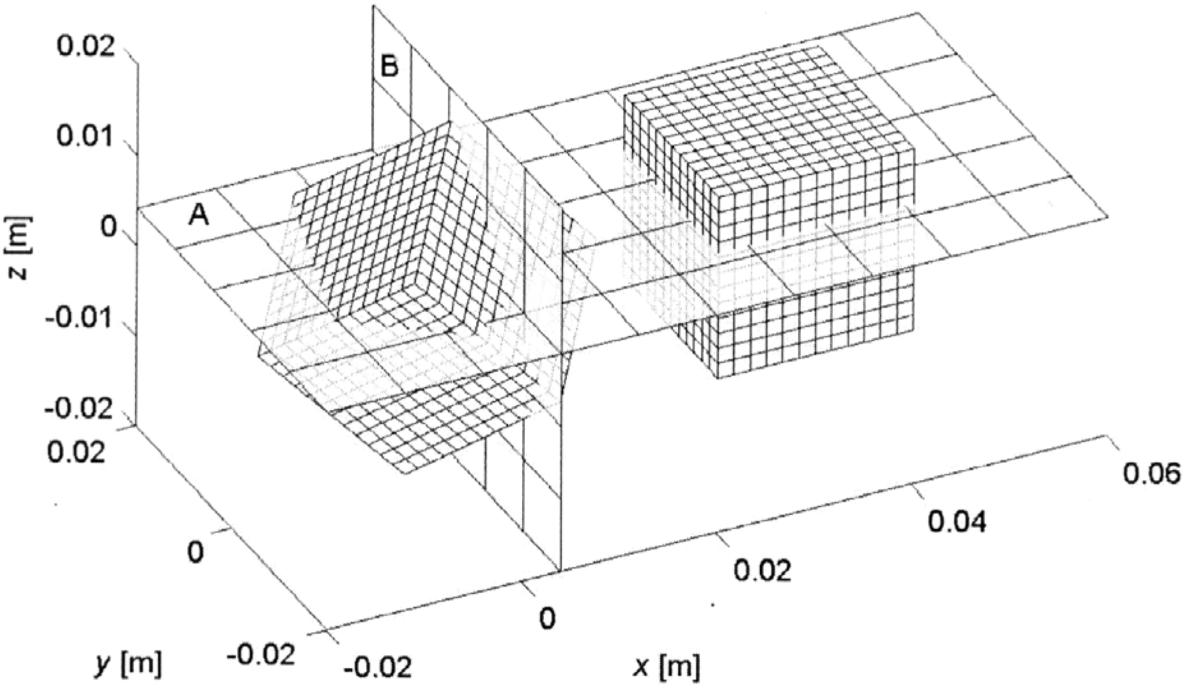


Fig. 3: Two planes with calculated potential distribution

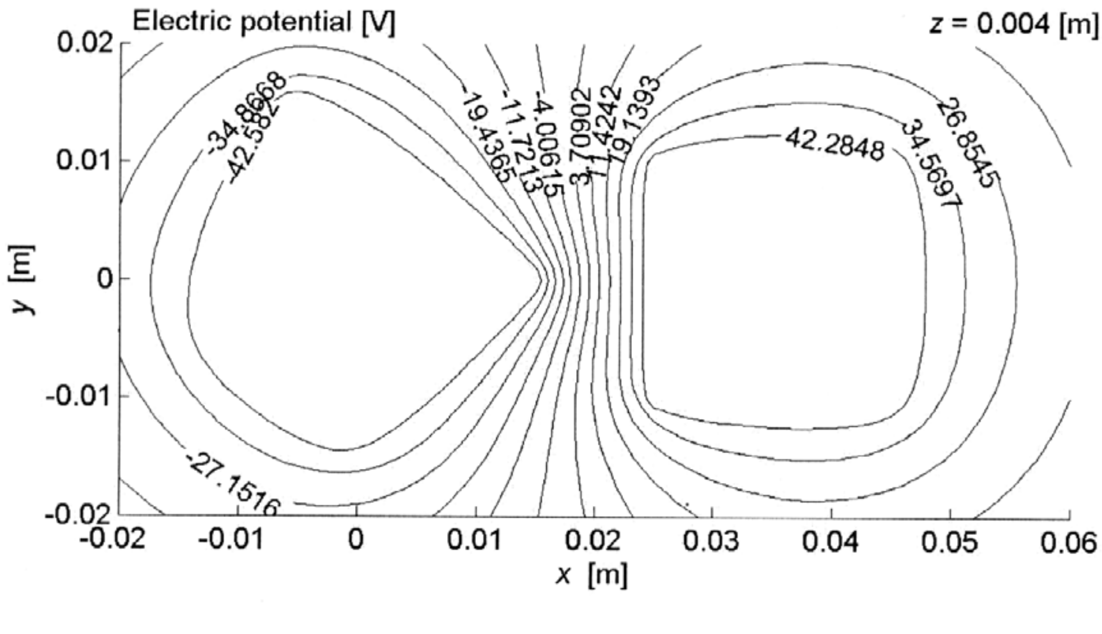


Fig. 4: Distribution of electric potential in plane A (Fig. 3)

The distribution in Fig. 4 was obtained from values of the potential at a relatively large set of selected points. Nevertheless, the equipotential lines consist of short straight lines obtained by means of a linear interpolation. An analogous distribution in plane B can be seen in Fig. 5.

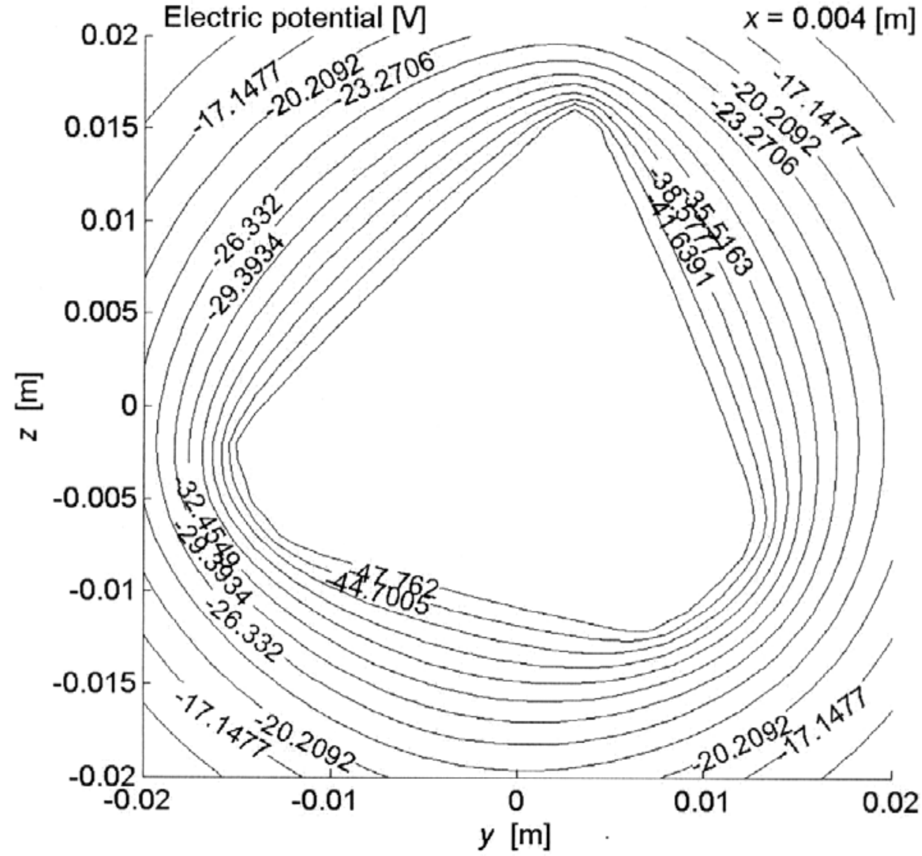


Fig. 5: Distribution of electric potential in plane B (Fig. 3)

Particular components of the electric field strength at any selected point were calculated directly from the distribution of charges, which provides much more accurate results in comparison with their calculation from the potential. However, computation of this quantity requires a lot of additional operations associated with necessary co-ordinate transformations and takes a considerable amount of time.

Fig. 6 shows distribution of the potential and module of the electric field strength along abscissa MN connecting the rightmost vertex of the left-hand cube with the centre of the nearest wall of the right-hand cube. The number of the cells is 20. Obvious is inaccuracy in potential at point M (there is about -38 V instead of -50 V). This error caused by considering constant surface charge in each cell and round-off errors could be reduced by using finer mesh.

## 4 Conclusion

The paper (that represents an organic continuation of [4]) shows that solution of even fairly complicated problems in 3D geometries can be reliable and relatively cheap when using own single-purpose user procedures. In the Czech Republic the Matlab is widely used both in research organisations and at universities and procedures like that may also be utilised via Internet.

As for problems characterised by curved boundaries, the accuracy of their solution depends only on its covering by sufficiently fine mesh consisting of triangular or rectangular cells, best in the Cartesian or cylindrical coordinates. For such elements it is possible to determine most of the definite integrals occurring in the system matrix analytically.

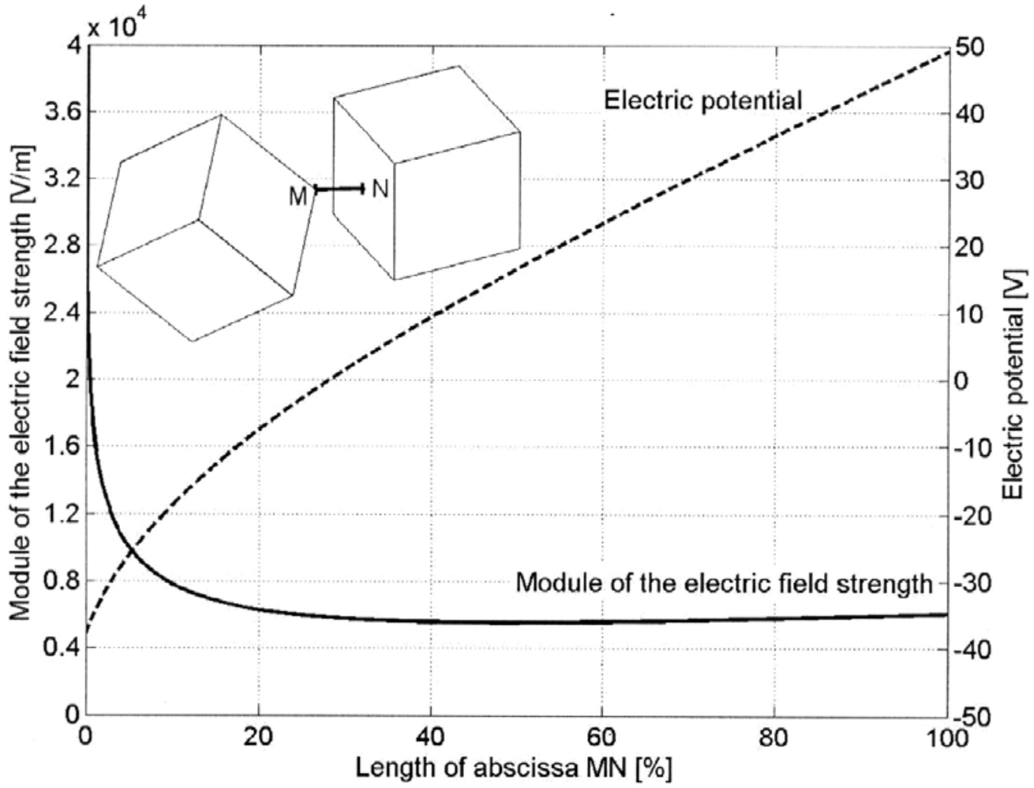


Fig. 6: Distribution of potential and electric field strength along abscissa MN

Next work in the field will be aimed at more sophisticated approximations of the charge density in particular cells by means of higher-order polynomials in order to increase accuracy of the results.

# 5 Acknowledgement

This work has been financially supported from the Research Plans MSM 232200008 and MSM 212300016.

# References

[1] Ida, N.: Engineering Electromagnetics. Springer-Verlag New York, Inc., 2001.

[2] Chari, M. V. K., Salon, S. J.: Numerical Methods in Electromagnetism, Academic Press, San Diego, 2000.

[3] Doležel, I., Kloucek, P., Šolín, P., Ulrych, B.: Distribution of Electrical Charge in a System of Finite Conductors. Acta Techn. CSAV 48, 2003, pp. 1–13.

[4] Hamar, R., Doležel, I., Ulrych, B.: Solution of 3D Electrostatic Problems in Homogeneous Media by Means of Integral Equations. Proc. IC SPETO'2003, Niedzyca, Poland, 2003, pp. 35–38.