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NUMERICAL INVERSION OF $\mathcal{Z}$ TRANSFORMS AND ITS APPLICATION IN LINEAR SYSTEM ANALYSIS

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Abstract: In the paper a method of numerical inversion of $\mathcal{Z}$ transforms is presented and a possibility of its application for the analysis of some linear systems is shown. The method under consideration has been programmed and verified by means of the universal scientific–technical language Matlab.

Keywords: $\mathcal{Z}$ transform, numerical inversion, linear system, Matlab language

1 Introduction

The $\mathcal{Z}$ transformation is one of the best–known functional transformation of sequences widely used in the theory of discrete signals and systems. Similarly as the Laplace transformation enables to solve differential equations describing continuous systems effectively the $\mathcal{Z}$ transformation can serve as a tool making possible to find a solution of difference equations describing discrete systems. In the work [1] a method called as "numerical" is presented to find objects to the $\mathcal{Z}$ transforms of the form of fractional rational functions. It is based on dividing two polynomials to directly obtain the Laurent series coefficients. Such an algebraic method is rather "semi–analytical", and can be applied only on this class of functions. In the paper an attempt to find objects to given $\mathcal{Z}$ transforms numerically is done. A computation is based on the numerical integration of the Cauchy inverse $\mathcal{Z}$ transformation integral. For this purpose a FFT algorithm is to advantage used which enables to get the whole object sequence in a single calculating step, and what is more it is thereby very fast. In the method no poles are needed to search when the $\mathcal{Z}$ transform is a fractional rational function, and moreover it can also have the form of an irrational or transcendental function.

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2 Inverse $\mathcal{Z}$ transformation

2.1 Brief theory description

Consider a one–sided sequence, generally complex, as

$$\{f_n\} = \{f_0, f_1, f_2, \cdots\} \; , \tag{1}$$

i.e. $f_n = 0$ for $n < 0$ is satisfied. Then the $\mathcal{Z}$ transform (one–sided) is defined by the formula [1,2]

$$F(z) = \sum_{n=0}^{\infty} f_n z^{-n} \tag{2}$$

under the supposition that the series is convergent. A sequence (1) is called as $\mathcal{Z}$ transformable if the series (2) converges for at least one finite complex z. It is necessary and sufficient that the sequence (1) be of the exponential type [1,2].

The inverse $\mathcal{Z}$ transform can generally be based on the application of the Cauchy integral theorem [1]. If $F(z)$ is a regular function in the region $|z| > 1/R$ (R as a radius of convergence at $z = \infty$) then a unique sequence $\{f_n\}$ for which $\mathcal{Z}\{f\} = F(z)$ is given by

$$f_n = \frac{1}{2\pi j} \oint_C F(z) z^{n-1} dz \qquad \text{for} \qquad n = 0, 1, 2, \cdots, \tag{3}$$

$$f_n = 0 \qquad \text{for} \qquad n < 0 \; .$$

Contour integration is performed along any simple closed path on or outside of which the $F(z)$ is regular. In the case of a fractional rational function $F(z)$ the classic approach consists in applying residue theorem to evaluate this integral.

2.2 Approximate inversion formula

Consider the integration path C as a circle stated by $z = re^{j\varphi}$, with $r > 1/R$ and $\varphi \in \langle 0; 2\pi \rangle$. We can after substituting write

$$f_n = \frac{r^n}{2\pi} \int_0^{2\pi} F(re^{j\varphi}) e^{jn\varphi} d\varphi \; . \tag{4}$$

Let us choose the trapezoidal method of the numerical integration and divide the range of integration into N intervals. Thus after replacing $d\varphi$ by $\Delta\varphi = 2\pi/N$, and the integral by a sum, we get the terms of an approximate sequence $\{\tilde{f}_n\}$ as

$$\widetilde{f}_n = \frac{r^n}{N}\sum_{k=0}^{N-1} F(re^{j2\pi k/N})e^{j2\pi kn/N} \tag{5}$$

Note: Here the trapezoidal rule of the numerical integration has the same form as the rectangular one because of the summed terms for $k=0$ and $k=N$ have the identical value $F(r)$.

It is evident that the formula (5) can be regarded as the inverse discrete Fourier transformation of the complex sequence { $F(re^{j2\pi k/N})$ }, when multiplied corresponding terms by r^n. We can thus write

$$\{\widetilde{f}_n\} = \{r^n\} \circ IDFT\{F(z_k)\} \ , \tag{6}$$

where $z_k = re^{j2\pi k/N}$, with $k, n = 0,1,2,\cdots,N-1$, and $\circ$ means Hadamard product (element–by–element product in terms of Matlab language). We can also rewrite the last formula using the forward *DFT* operation as

$$\{\widetilde{f}_n\} = \{r^n/N\} \circ \{DFT\{F^*(z_k)\}\}^* \ , \tag{7}$$

where $*$ means the complex conjugation.

2.3 Matlab language implementation

To be able to utilize a fast radix–2 FFT algorithm which is included in the built–in **fft** Matlab function the length of the input sequence is taken as $N = 2^m$, m integer. Therefore N can be chosen relatively big to be sufficiently accurate, and a resultant sequence of the desired length is extracted from (7). To avoid rounding errors when n is gaining big values it is suitable to choose $r=1$ if possible (if all the poles of the $F(z)$ lie inside the unit circle).

```
%***************************************************************************************************%
function fn=iztr(F,nmax,r);
N=4096; n=0:N-1;                                        % N is adjustable
z=r*exp(j*2*pi*n/N); Fz=feval(F,z);
S=conj(fft(conj(Fz))); fn=r.^n/N.*S;
stem(0:nmax,real(fn(1:nmax+1)),'filled','k'); grid on;
%***************************************************************************************************%
```

For example, the command iztr(´step´,10,1.1); where ´step´ is the name of the function defining $F(z)$, see below, solves first 11 terms of the discrete unit step.

```
%***************************************************************************************************%
function f=step(z)
    f=z./(z-1);                                         % definition of F(z)
%***************************************************************************************************%
```

3 Examples and conclusion

Consider a discrete linear system in [3], with a difference equation

$$y(n+2)-1.25y(n+1)+0.78125y(n)=x(n+2)-x(n), \tag{8}$$

and under initial conditions $y(0)=2$, $y(1)=1$. The input signal has the form

$$x(n)=2\cos(2\pi f_v n/f_{vz}+\phi),\ f_v=62.5Hz\ ,\ f_{vz}=1000Hz\,,\ \phi=\pi/5\,. \tag{9}$$

From (8) we can derive the $\mathcal{Z}$ transform of the total response $y(n)$ as

$$Y(z)=\frac{z^2-1}{z^2-1.25z+0.78125}X(z)+\frac{2z^2-1.5z}{z^2-1.25z+0.78125}\ , \tag{10}$$

where $$X(z)=\mathcal{Z}\{x(n)\}=\frac{2z[z\cos\phi-\cos(\beta-\phi)]}{z^2-2z\cos\beta+1}\ ,\ \beta=2\pi\frac{f_v}{f_{vz}}\ . \tag{11}$$

In the work [3] the solution has been found step by step performing two inverse $\mathcal{Z}$ transformations using residue theorem and utilizing convolution theorem. Here the total response of the system can be found directly from (10) using the numerical method under consideration. The results are shown in Fig. 1.

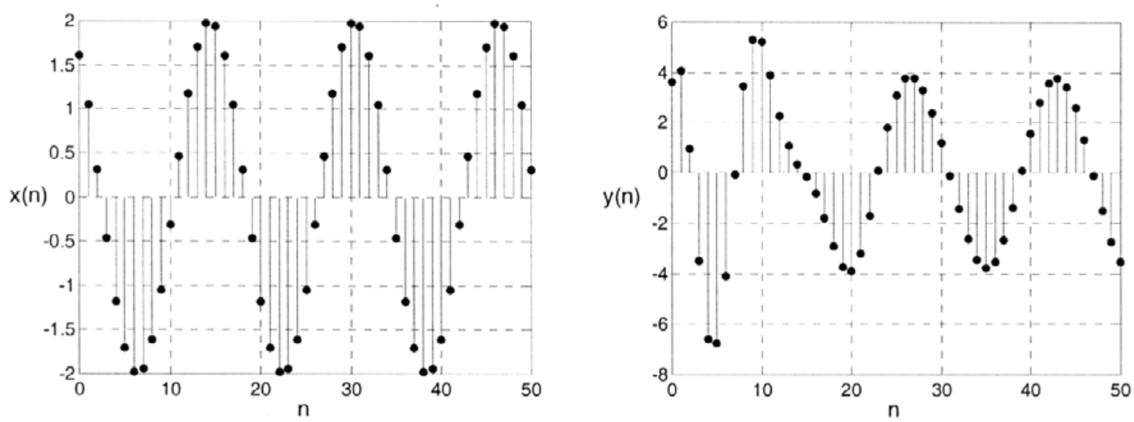

Fig. 1 Input and output signal of linear discrete system

A root–mean–square error achieved in this example was about 10^{-14}. The future works will be addressed to a general error analysis and some other applications.

References

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