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Qualitative study of problems for elliptic (possibly also parabolic) equations with measure data – solvability, bifurcation, approximation of solutions

Diploma thesis

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I do hereby declare that the entire diploma thesis is solely my original work and that I have used only the cited sources and the consultations with supervisor.

Pilsen, 12. května 2016 Author's signature: ....

Abstract

This work concerns the solvability of the semi-linear elliptic partial differential equations with measure data in the very weak sense, i.e., the solution is an element of the space $L^1(\Omega)$. Particularly,

$$\begin{cases} -\Delta u - \lambda u = g(u) + \mu & \text{in} \quad \Omega\,, \\ u = 0 & \text{on} \quad \partial\Omega\,, \end{cases}$$

where Ω is bounded domain in $\mathbb{R}^N$ with C^2 boundary $\partial\Omega$, g is a continuous function and μ is a bounded real Radon measure on Ω such that $|\mu|(\partial\Omega) = 0$. To the best of the author's knowledge, the original contributions to the topic are: the solvability of the problem with $\lambda = 0$ and $g = 0$ for the dimension $N = 2$, the Fredholm alternative for the Laplace's operator with homogeneous Dirichlet boundary conditions in the very weak sense and the solvability of the problem out of and at resonance. The latter is obtained through posing conditions of Landesman-Lazer type on the measure μ.

Contents

Notation of number sets

$\mathbb{N}$	...	The set of all natural numbers, i.e. $\{1, 2, 3, \ldots\}$.
$\mathbb{N}_0$	...	The set of all natural numbers containing zero, i.e. $\{0, 1, 2, 3, \ldots\}$.
$\mathbb{Z}$	...	The set of all integers, i.e. $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
$\mathbb{R}^N$	...	The N-dimensional space of real numbers.
$\mathbb{R}^+ (\mathbb{R}^-)$	...	The set of all positive (negative) real numbers.
$\mathbb{R}_0^+ (\mathbb{R}_0^-)$	...	The set of all positive (negative) real numbers containing zero.
$\mathbb{C}$	...	The set of all complex numbers.
$\overline{x}$	...	The complex conjugate of the number $x \in \mathbb{C}$.

Notation of general sets

$X \subset Y$	...	The set X is subset of the set Y and possibly $X = Y$.
$X \subset\subset Y$	...	The set X is a compact subset of Y.
$\|x\|$	...	The Eucleidian norm of $x \in \mathbb{R}^N$.
$\rho(X, Y)$	...	The distance of two sets $X, Y \subset \mathbb{R}^N$ defined by $\rho(X, Y) := \inf\{\|x - y\| : x \in X, y \in Y\}$.
Ω	...	A domain of $\mathbb{R}^N$, that is an open and connected $\Omega \subset \mathbb{R}^N$; note that Ω has the same dimension as $\mathbb{R}^N$.
$\operatorname{diam} \Omega$	...	The diameter of the domain Ω defined by $\operatorname{diam} \Omega := \sup\{\rho(x, y),\ x, y \in \Omega\}$.
$\delta(x)$	...	The distance of the point $x \in \Omega$ from the boundary $\partial\Omega$ defined by $\delta(x) := \rho(x, \partial\Omega)$.
$\mathbf{n}$	...	The unit outer normal of the domain boundary $\partial\Omega$.
$B_x(a)$	...	An open ball centred at $x \in \mathbb{R}^N$ with the radius $a > 0$.
$\mathcal{B}(X)$	...	A Borel $\sigma-$algebra generated by open sets of the topological space X.
λ	...	Lebesgue measure defined on $\mathbb{R}^N$. Not to be confused with and eigenvalue of an operator.

Notation of the multivariate calculus

$u \cdot v$	...	The dot product of two vectors $u, v \in \mathbb{R}^N$.
∇u	...	The gradient of a differentiable function u.
$\frac{\partial}{\partial x} u$	...	The derivative of the function u with respect to $x \in \mathbb{R}^N$.
$\nabla_x G(x, y)$	...	The gradient of the function G with respect to $x \in \mathbb{R}^N$.
Δu	...	The Laplacian of a twice differentiable function u.
dS_x	...	The surface element used in integration with respect to $x \in \mathbb{R}^N$.
$f * g$	...	The convolution of two suitable objects (functions, measures and distributions) f and g.
$D^i u$	...	The weak first derivative of u with respect to i-th coordinate of $\mathbb{R}^N$.
$D^\alpha u$	...	The weak derivative of u with respect to the multiindex α.
Du	...	The weak gradient of u.
$D_x G(x, y)$	...	The weak gradient of the function G with respect to $x \in \mathbb{R}^N$.
$D^\Delta u$	...	The weak Laplacian of u.

Notation of the functional analysis

X^*	...	The space of all bounded linear functionals on X.
$\langle f, x \rangle_{X^*, X}$	...	The duality between $x \in X$ and $f \in X^*$.
$(x, y)_H$	...	The scalar product of $x, y \in H$, where H is a Hilbert space.
I	...	The identity mapping defined by $Ix = x$ for all $x \in X$.
$\operatorname{dom}(T)$	...	The domain of the operator $T : X \to Y$, i.e., the set of all $x \in X$ such that $T(x)$ is defined.
$\operatorname{ran}(T)$	...	The range of the operator $T : X \to Y$, i.e., the set of all $y \in Y$ such that there exists $x \in X$ and $T(x) = y$ holds.
$\operatorname{Ker}(T)$	...	The kernel of the operator $T : X \to Y$, i.e., the set of all $x \in X$ such that $T(x) = o$.
$C(X, Y)$	...	The set of all continuous mappings from topological space X to topological space Y.
$\mathcal{L}(X, Y)\ \ (\mathcal{L}(X))$	...	The space of all continuous linear mappings from linear space X to linear space Y (X).
$\mathcal{C}(X, Y)\ \ (\mathcal{C}(X))$	...	The set of all continuous and compact mappings from normed linear space X to normed linear space Y (X).

Notation of spaces of functions, measures and distributions

$C(\Omega)$	...	The space of continuous functions defined on Ω.
$C(\overline{\Omega})$	...	The space of continuous functions continuous up to the boundary $\partial\Omega$.
$C_0(\overline{\Omega})$	...	The space of functions $u \in C(\overline{\Omega})$ such that $u = 0$ on $\partial\Omega$.
$C^k(\overline{\Omega})$	...	The space of functions $u \in C(\overline{\Omega})$ such that their partial derivatives up to the order k belong to $C(\overline{\Omega})$.
$C^k(\Omega)$	...	The set of functions $u \in C(\Omega)$ such that their partial derivatives up to the order k belong to $C(\Omega)$.
$C_0^k(\overline{\Omega})$	...	The space of functions $u \in C_0(\overline{\Omega})$ such that their partial derivatives up to the order k belong to $C(\overline{\Omega})$; possibly $k = \infty$.
$C^{k,\lambda}(\overline{\Omega})$	...	The space of $\lambda-$Hölder continuous functions with partial derivatives up to the order k belonging to $C^{0,\lambda}(\overline{\Omega})$ with $0 \leq \lambda \leq 1$; possibly $k = \infty$.
$C_c(\Omega)$	...	The space of $u \in C(\Omega)$ with compact support; supp $f = \overline{\{x : u(x) \neq 0\}}$.
$L^p(\Omega)$	...	The space of Lebesgue integrable functions with the exponent p; $1 \leq p < +\infty$.
$W^{k,p}(\Omega)$	...	The space of functions $u \in L^p(\Omega)$ such that their weak partial derivatives up to order k belong to $L^p(\Omega)$ with $1 \leq p < +\infty$.
$W_0^{1,p}(\Omega)$	...	The space of functions $u \in W^{1,p}(\Omega)$ such that $u = 0$ on $\partial\Omega$ in the sense of traces.
$\mathcal{M}(\Omega)$	...	The Banach space of bounded real Radon measures μ on Ω such that $\lvert\mu\rvert(\partial\Omega) = 0$ endowed with the norm $\lVert\mu\rVert_{\mathcal{M}(\Omega)} = \lvert\mu\rvert(\Omega)$.
$\mathcal{D}(\Omega)$	...	The space of $u \in C_c(\Omega)$ such that all their partial derivatives belong to $C_c(\Omega)$ endowed with topology (see [19, p.136–137]); called the space of *test functions*.
$\mathcal{D}'(\Omega)$	...	The space of *distributions*; the dual of $\mathcal{D}(\Omega)$.

Chapter 1

Preface

Non-linear problems of the type

$$\begin{cases} -\Delta u + g(x,u) = f & \text{in} \quad \Omega\,, \\ \qquad\qquad\quad u = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{1.1}$$

arise in mathematical models and in applications of natural phenomena. A standard tool in examining the solvability of elliptic problems including non-linearities is the theory of monotone operators in Hilbert spaces. There are various generalisations of the problem (1.1) in the literature to problems that cannot be treated by the monotone operator theory. Relevant generalisations will be mentioned in this introduction.

In the book [16], authors generalise the problem in the following way. They search for a very weak solution of the problem

$$\begin{cases} -\Delta u + g(x,u) = \mu & \text{in} \quad \Omega\,, \\ \qquad\qquad\quad u = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{1.2}$$

where $\mu \in \mathcal{M}(\Omega)$. The nature of the problem (1.2) requires some conditions to be posed on the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$. Namely

1. $g \in C(\Omega \times R)$, $g(x,0) = 0$ and $g(x,\,\cdot\,)$ is non-decreasing;
2. $g(\,\cdot\,,t) \in L^1(\Omega,\delta)$ for all $t \in \mathbb{R}$.

The solvability of the problem (1.2) cannot be treated with the monotone operator theory in Hilbert spaces since μ is a Radon measure (which does not belong to the dual of $W_0^{1,2}(\Omega)$ for $N \geq 2$). It turns out, that the problem (1.2) does not possess a solution for every $\mu \in \mathcal{M}(\Omega)$ (the solvability of the problem is conditioned by the existence of the weak sub- and supersolutions, see [16, Section 2.2]).

Another generalisation of the problem (1.1) was developed by Landesman and Lazer in [14] where the problem

$$\begin{cases} -Lu - \lambda u = g(u) + f & \text{in} \quad \Omega\,, \\ \qquad u = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{1.3}$$

is considered in the weak sense at the resonance when λ is an eigenvalue. The operator L is a second order, self-adjoint and uniformly elliptic operator, $f \in L^2(\Omega)$ and g is a real-valued bounded continuous function such that

1′. the limits

$$g(+\infty) := \lim_{t\to+\infty} g(t), \quad g(-\infty) := \lim_{t\to-\infty} g(t)$$

exist and are finite;

2′. the inequalities

$$g(-\infty) \geq g(t) \geq g(+\infty)$$

hold for every $t \in \mathbb{R}$.

In general, this resonant case cannot be treated by the method of monotone operators.

The necessary conditions on the solvability of the problem (1.3) are so-called the Landesman-Lazer conditions

$$g(+\infty) \int_\Omega \varphi^+ \, dx - g(-\infty) \int_\Omega \varphi^- \, dx < (f, \varphi) < g(-\infty) \int_\Omega \varphi^+ \, dx - g(+\infty) \int_\Omega \varphi^- \, dx$$

where φ is the eigenfuction of the elliptic operator L with homogeneous Dirichlet conditions. These results are further developed, e.g., [10].

In this thesis, we fuse these generalisations by considering a non-monotone g and measure data

$$\begin{cases} -\Delta u - \lambda u = g(u) + \mu & \text{in} \quad \Omega\,, \\ u = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{1.4}$$

where $\mu \in \mathcal{M}(\Omega)$. The function $g : \mathbb{R} \to \mathbb{R}$ is continuous (i.e., it satisfies the Carathéodory condition). For the non-resonant case, it is sufficient to consider g possessing a sub-linear growth, i.e. there exists constants $0 \le \alpha < 1$, and $b, c > 0$ such that

$$|g(t)| \le b + c|t|^\alpha$$

for all $t \in \mathbb{R}$. If λ is an eigenvalue, then we assume g to be bounded function such that the limits

$$g(+\infty) := \liminf_{x \to +\infty} g(x), \quad g(-\infty) := \limsup_{x \to -\infty} g(x)$$

are finite and satisfy

$$g(-\infty) < g(+\infty)\,.$$

The thesis is organised as follows. Chapter 2 contains preliminaries regarding functional analysis (abstract spaces, spectral theory and fixed point theory), measure theory, integration, function spaces and elliptic PDE. Chapter 3 contains important theorems regarding linear elliptic problems with measure data. The results are mainly reproduced from [16] with the author's contribution for the 2-dimensional case. Chapter 4 develops the Fredholm alternative for the Laplace's operator with homogeneous Dirichlet conditions in the very weak sense for the problem with measure data. In Chapter 5, we study the solvability of the problem (1.4) at non-resonant case and in Chapter 6, we study the solvability of the same problem at resonance. Chapter 7 contains an example of the application of the main theorem of Chapter 6 in modelling the kinetics of the chemical reactions. Finally, Chapter 8 concludes results of this thesis and stated open questions and possible directions of the further research. Appendix contains an idea of numerical approach for solving elliptic PDE on a ball in $\mathbb{R}^2$ with measure data. Due to the time difficulty of this thesis, the convergence of approximation sequences was not proved.

I would like to express my gratitude to the supervisor of this thesis, doc. Ing. Petr Girg, Ph.D. for the scientific and moral guidance, patience, linguistic support and the example in Chapter 7. Many thanks comes also to my family, girlfriend L. Š. and close friends of mine for the patience and a constant support.

Keywords: elliptic PDE, Radon measure, measure data, Green's function, Landesman-Lazer.

Chapter 2

Preliminaries

2.1 Functional analysis

We assume, that the reader is familiar with the concepts of functional, operator, metric, linear, Banach and Hilbert space and dual space. Convenient study material for this section is, e.g., [21].

Here, we present basic results of the theory of topological spaces, functionals, operators, spectral analysis in Hilbert spaces and fixed point theory. Usually, general spaces are denoted by X and Hilbert spaces are denoted by H.

2.1.1 Abstract spaces

Definition 2.1 ([6], p.25)**.** A set X with collection $\mathcal{T}$ of its subsets is called a *topological space* denoted by $(X, \mathcal{T})$ if and only if $\mathcal{T}$ possesses following properties

1. $\emptyset, X \in \mathcal{T}$,
2. an intersection of a finite number of sets of $\mathcal{T}$ belongs to $\mathcal{T}$,
3. a union of any subcollection of $\mathcal{T}$ belongs to $\mathcal{T}$.

Elements of $\mathcal{T}$ are called *open sets*.

Definition 2.2. Let $(X, \mathcal{T})$ be a topological space. A *neighbourhood of point* $x \in X$ is any open set $A \in \mathcal{T}$ containing $x \in A$.

The counterpart of open sets are *closed sets*: a subset $A \subset X$ of a topological space $(X, \mathcal{T})$ is closed if $X \setminus A$ is open and the *closure of the set* $A \subset \mathcal{T}$ is the intersection of all closed sets containing A. Since the class of all topological spaces is very wide and the conditions any topological space must satisfy are quite weak, we define further properties in order to the topological spaces behave "nicely". For example, two points in general topological space can be "indistinguishable" from each other; i.e., we cannot find disjoint neighbourhoods of the points. This holds true, e.g., for metric spaces.

Definition 2.3 ([4], Definition 6.1.2., p.4)**.** Let $(X, \mathcal{T})$ be a topological space. $(X, \mathcal{T})$ is called *Hausdorff* if and only if every two distinct points $x, y \in X$ possess disjoint neighbourhoods.

Compactness is an important property of topological space. We can define various types of compactnesses in topological spaces which further coincide in the case of more special spaces (metric, normed linear spaces etc.). Here, we define only one regarding a covering of a set.

Definition 2.4. Let $(X, \mathcal{T})$ be a topological space and A subset of X. An *open cover* of A is a collection of open sets $\{A_\gamma\}_{\gamma\in\Gamma} \subset \mathcal{T}$ such that $\bigcup_{\gamma\in\Gamma} A_\gamma \supset A$. If Γ is countable set, the cover $\{A_\gamma\}_{\gamma\in\Gamma}$ is called a *countable cover* and if Γ is finite, the cover is called a *finite cover.*

Definition 2.5. Let $(X, \mathcal{T})$ be a topological Hausdorff space and A subset of X. Then A is called

1. *compact*, if and only if every open cover of A contains a finite subcover,
2. *relatively compact*, if and only if the closure of A s compact.

The space $(X, \mathcal{T})$ is called compact if and only if X is compact.

With the structure of topology, we are able to define a continuity of mapping between two topological spaces.

Definition 2.6 ([6], p.26)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ mapping. Then f is said to be *continuous* on X if and only if the preimage of any open set $B \in \mathcal{T}_Y$ is open set $A \in \mathcal{T}_X$.

This notion of continuous mapping is consistent with the definition of convergence through the metric (since every metric space can be endowed with topology). Last, we mention two theorems regarding linear operators in a narrower class of spaces – normed linear spaces. The uniform boundedness principle provides a connection between pointwise boundedness and boundedness in the norm of a sequence of bounded linear operators.

Theorem 2.7 ([6], Theorem 2.1.4., p.57, Uniform boundedness principle)**.** *Let X be a Banach space and Y normed linear space. If $\{A_\gamma\}_{\gamma\in\Gamma} \subset \mathcal{L}(X, Y)$ is such that the sets $\{\|A_\gamma x\|\}_Y : \gamma \in \Gamma$ are bounded for all $x \in X$, then $\left\{\|A_\gamma\|_{\mathcal{L}(X,Y)} : \gamma \in \Gamma\right\}$ is also bounded.*

If an injective linear operator $T : X \to Y$ such that $\mathrm{dom}(T) = X$, $\mathrm{ran}(T) = Y$ and where X and Y are linear spaces then there is defined an linear inverse $T^-1 : Y \to X$. Moreover, a stronger claim holds.

Theorem 2.8 ([21], Theorem 4.2-H, p.180)**.** *Let X and Y be complete metric linear spaces. Let T be a linear operator whose domaini is X and whose range is all of Y. Suppose that T is continuous and that T^{-1} exists. Then T^{-1} is continuous*

2.1.2 Spectral theory

Given a linear operator $T : H \to H$, we are interested in the solvability of the operator equation $Tx - \lambda x = f$ (i.e., for given $\lambda \in \mathbb{C}$ and $f \in H$, we try to find $x \in H$). Alternatively, we study the existence of the inverse operator $(T - \lambda I)^{-1}$ and the properties of its domain: for which $f \in H$, $f \in \mathrm{dom}((T - \lambda I)^{-1})$ holds? The values $\lambda \in \mathbb{C}$, for which bounded linear operator $(T - \lambda I)^{-1}$ exists are called *regular values* of the operator T. If the $\lambda \in \mathbb{C}$ is not a regular value, it belongs to the *spectrum* of the operator T denoted by $\sigma(T)$. If the inverse operator $(T - \lambda I)^{-1}$ does not exist, we call λ an *eigenvalue.*

Adjoint and self-adjoint operators take very important role in the spectral theory. Here, we recall basic definitions used in this thesis.

Definition 2.9 (see [21], p.249–250)**.** Let $T : H \to H$ be a linear operator with $\mathrm{dom}(T)$ being dense in H. Denote $\mathrm{dom}(T^*)$ the set of all $x \in H$ for which there exists $z \in H$ such that for all $y \in H$ $(x, Ty) = (z, y)$ holds. We then write $T^*x = z$ for each $x \in \mathrm{dom}(T^*)$ and the operator $T^* : H \to H$ assigning $z \in H$ to each $x \in \mathrm{dom}(T^*)$ is called *adjoint operator* to T.

Definition 2.10. Let $T : H \to H$ be a linear operator defined on a dense subset $\mathrm{dom}(T)$ of H. We say that the operator T is *symmetric* if and only if $(x, Ty) = (Tx, y)$ holds for all $x, y, \in \mathrm{dom}(T)$.

Definition 2.11. A linear operator $T : H \to H$ defined on a dense subset $\mathrm{dom}(T) \subset H$ is called *self-adjoint* if and only if it is symmetric and $\mathrm{dom}(T) = \mathrm{dom}(T^*)$.

The characterisation of the spectrum of a self-adjoint operator is provided by the following theorem.

Theorem 2.12 ([21], Theorem 6.2.-B, p.330)**.** *Suppose T is a bounded self-adjoint operator. Then the spectrum of T lies on the closed interval $[m_T, M_T]$* [1]*. The endpoints of this interval belong to the spectrum.*

The range and the kernel of the linear operator $T : H \to H$ are linear subspaces (if T is continuous, kernel is in addition closed) of H, which is a trivial consequence of linearity. A linear operator $T : H \to H$ is *normal*, if $AA^* = A^*A$. It can be shown, that for given $\lambda \in \mathbb{C}$ the operator $(T - \lambda I)$ is normal. Therefore, we can deduce the following relation between $\mathrm{ran}(T - \lambda I)$ and $\mathrm{Ker}(T - \lambda I)$.

Theorem 2.13 ([21], Theorem 6.2-G, p.332)**.** *If S is normal, $\overline{\mathrm{ran}(S)}$ and $\mathrm{Ker}(S)$ are orthogonal complements, so that $H = \overline{\mathrm{ran}(S)} \oplus \mathrm{Ker}(S)$.*

If we assume a compactness of the operator $T : H \to H$ and if $\lambda \neq 0$, then the $\mathrm{Ker}(T - \lambda I)$ is a finite dimensional linear subspace of H. The dimension of $\mathrm{Ker}(T - \lambda I)$ is called the *multiplicity* of the eigenvalue λ. If $\lambda \neq 0$ belongs to the spectrum, then the inverse operator $(T - \lambda I)^{-1}$ does not exist, but if $\lambda = 0$ belongs to the spectrum, the inverse operator may exist but it does not need to be bounded or the domain may not be dense in H. The eigenvalues of the compact self-adjoint operator are clustered at zero (i.e. there is only

[1] The values m_T, M_T are defined

$$m_T = \inf_{\|x\|_H=1} (Tx, x), \quad M_T = \sup_{\|x\|_H=1} (Tx, x).$$

a finite number of eigenvalues outside each open interval containing zero). We can order the eigenvalues of T such that $|\lambda_1| \geq |\lambda_2| \geq \ldots > 0$. Each $\lambda_n \neq 0$ is assigned a finite number of linearly independent $x_n \in H$ called the *eigenvectors*. Eigenvectors can be considered orthonormal. The eigenvectors form a closed linear subspace $H_1 \subset H$ and the orthogonal decomposition $H = H_1 \oplus H_2$ can be made. Moreover, $T : H_1 \to H_1$ and $T : H_2 \to H_2$, where all elements of H_2 are the eigenvectors of the eigenvalue 0. Finally, we formulate the Fredholm alternative, the main result of this section.

Theorem 2.14 ([11], Theorem 5.11, p.21, reformulated for self-adjoint operators)**.** *Let H be a Hilbert space and T a compact self-adjoint mapping of H into itself. Then there exists a countable set $\Lambda \subset \mathbb{R}$ having no limit points except possibly $\lambda = 0$, such that if $\lambda \neq 0$, $\lambda \neq \sigma(T)$ the equation*

$$Tx - \lambda x = f \tag{2.1}$$

have uniquely determined solutions $x \in H$ for every $f \in H$, and the inverse mapping $(T - \lambda I)^{-1}$ is bounded. If $\lambda \in \sigma(T)$, the null space of the mapping $T - \lambda I$ is of a positive finite dimension and the equation (2.1) *is solvable if and only if f is orthogonal to the null space of $T - \lambda I$.*

2.1.3 Fixed point theory

We briefly recall well-known fixed point theorems used in the non-linear analysis. Given an operator $T : X \to X$ mapping space X to itself the point $x \in X$ such that $T(x) = x$ is called a *fixed point* of the operator T. The fundamental result is the Schauder fixed point theorem and the existence of the *Leray-Schauder degree* of compact perturbation of identity.

Theorem 2.15 ([6], Theorem 5.2.5, p.254)**.** *Let K be a nonempty, closed, convex and bounded subset of a normed linear space X. Assume that $F \in \mathcal{C}(K, X)$ and $F(K) \subset K$. Then there is a fixed point of F in K.*

Theorem 2.16 ([6], Theorem 5.8.2, p.315)**.** *Let D be a bounded open subset of a Banach space X. There exists a mapping* $\deg(I - F, D, y_0)$ *defined for all $F \in \mathcal{C}(D, X)$ and $y_0 \in X$ such that*

$$x - F(x) \neq y_0 \quad \textit{for all} \quad x \in \partial D.$$

This mapping has the following properties:

1. $\deg(I, D, y_0) = \begin{cases} 1 & \textit{if} \quad y_0 \in D\,, \\ 0 & \textit{if} \quad y_0 \notin D\,. \end{cases}$.
2. $\deg(I - F, D, y_0) = \deg(I - F - y_0, D, o)$.
3. *If* $\deg(I - F, D, y_0) \neq 0$*, then the equation*

   $$x - F(x) = y_0$$

   has a solution in D.
4. *If $D_1, \ldots, D_k$ are pairwise disjoint open subsets of D and $x - F(x) \neq y_0$ for each $x \in \overline{D} \setminus \bigcup_{j=1}^{k} D_j$, then*

   $$\deg(I - F, D, y_0) = \sum_{j=1}^{k} \deg(I - F, D_j, y_0).$$

5. *If $F, G \in \mathcal{C}(D, X)$ and*

   $$\sup_{x \in \partial D} \|F(x) - G(x)\|_X < \inf_{x \in \partial D} \|x - F(x) - y_0\|_X\,,$$

   then

   $$\deg(I - F, D, y_0) = \deg(I - G, D, y_0)\,.$$

6. *(homotopy invariance property) If $F, G \in \mathcal{C}(D, X)$ and*

   $$H(t, x) = (1 - t)F(x) + tG(x), \quad t \in [0, 1], \quad x \in D\,,$$

   are such that

   $$x - H(t, x) \neq y_0 \quad \textit{for every} \quad x \in \partial D \quad \textit{and} \quad t \in [0, 1]\,,$$

   then $\deg(I - H(t, \cdot), D, y_0)$ *is constant on* $[0, 1]$*. In particular,*

   $$\deg(I - F, \Omega, y_0) = \deg(I - G, D, y_0)\,.$$

The last theorem gives the sufficient conditions for the set of fixed point of a compact mapping to be connected with respect to a real parameter.

Theorem 2.17 ([25], Theorem 14.C, p.629, Global continuation principle of Leray-Schauder). *Let the operator $F : [a, b] \times \overline{D} \to X$ be compact, where D is a bounded open set in the Banach space X. If the equation*

$$x - F(c, x) = 0, \quad c \in \mathbb{R},\, x \in X \tag{2.2}$$

has no solutions on $[a, b] \times \partial D$ and $\deg(I - F(a, \cdot), D, o) \neq 0$. Then the equation (2.2) has a continuum Σ of solutions in $\mathbb{R} \times X$ which connects the set $\{a\} \times D$ with the set $\{b\} \times D$.

2.2 Measure and integration theory

We assume, that the reader is familiar with the concepts of $\sigma-$algebra, Borel $\sigma-$algebra, measurable function, simple function, non-negative measure and Lebesgue measure. Convenient study material for this section is, e.g., [3], [4] or [9].

In this section, we recall the definition of the Lebesgue integral, signed measure and extension of the definition of Lebesgue integral for signed measures. For the sake of simplicity, the non-negative measures will be called just measures. The couple $(X, \mathcal{A})$ denotes a measurable space with $\sigma-$algebra $\mathcal{A}$.

Definition 2.18 ([3], Definition 1.3.1., p.9). A real-valued set function μ on a class of sets $\mathcal{A}$ is called *countably additive* if

$$\mu\left(\bigcup_{n=1}^{+\infty} A_n\right) = \sum_{n=1}^{+\infty} \mu\left(A_n\right)$$

for all pairwise disjoint sets A_n in $\mathcal{A}$ such that $\bigcup_{n=1}^{+\infty} A_n \in \mathcal{A}$. A countably additive set function defined on an algebra is called a *measure*.

Definition 2.19 ([3], Definition 2.4.1., p.118). Let a function f be defined and finite μ a.e. (i.e., f may be undefined or infinite on a set of measure zero). The function f is called *Lebesgue integrable* with respect to the measure μ (or μ-integrable) if there exists a sequence of simple functions f_n such that $f_n(x) \to f(x)$ almost everywhere and the sequence $\{f_n\}$ is fundamental in the mean[2]. The finite value

$$\lim_{n\to\infty} \int_X f_n(x)\, d\mu(x)\,,$$

which exists (see [3]), is called the *Lebesgue integral* of the function f and is denoted by

$$\int_X f(x)\, d\mu(x)\,.$$

The previous definition considered measure $\mu : \mathcal{A} \to [0, +\infty)$. However, it can be extended to a possibly infinite measure μ of the form $\mu : \mathcal{A} \to [0, +\infty]$. However, we restrict ourselves only on finite measures in this thesis. The countably additive mapping $\mu : \mathcal{A} \to (-\infty, +\infty)$ is called a *signed measure*.

Theorem 2.20 ([3], Theorem 3.1.1., p.175). *Let μ be a countably additive real-valued measure on a measurable space $(X, \mathcal{A})$. Then, there exist disjoint sets $X^-, X^+ \in \mathcal{A}$ such that $X^- \cup X^+ = X$ and for all $A \in \mathcal{A}$, one has*

$$\mu(A \cap X^-) \leq 0, \quad \textit{and} \quad \mu(A \cap X^+) \geq 0\,.$$

Lemma 2.21 ([3], Corollary 3.1.2., p.176). *Under the hypotheses of Theorem 2.20 let*

$$\mu^+(A) := \mu(A \cap X^+), \quad \mu^-(A) := -\mu(A \cap X^-), \quad A \in \mathcal{A}\,.$$

Then μ^+ and μ^- are a non-negative countably additive measures and one has equality $\mu = \mu^+ - \mu^-$.

[2] A sequence $\{f_n\}$ of simple functions is called fundamental in the mean if, for every $\varepsilon > 0$ there exists a number n such that

$$\int_X f_i(x) - f_j(x)\, d\mu(x) < \varepsilon \quad \text{for all} \quad i, j \geq n\,.$$

See [3, Definition 2.3.1., p.116].

The Lebesgue integral can be simply extended to signed measures by putting

$$\int_X f(x)\, d\mu(x) := \int_X f(x)\, d\mu^+(x) - \int_X f(x)\, d\mu^-(x)$$

where both integrals are finite.

Definition 2.22 ([3], Definition 3.1.4., p.176)**.** The measures μ^+ and μ^- constructed above are called the positive and negative parts of μ, respectively. The measure

$$|\mu| = \mu^+ + \mu^-$$

is called the total variation of μ.

Following result is known as the Radon-Nikodým theorem. In this thesis, we will use the fact, that every μ integrable function f defines a measure of certain properties.

Theorem 2.23 ([3], Theorem 3.2.2., p.178)**.** *Let μ and ν be two finite measures on a space $(X, \mathcal{A})$. The measure ν is absolutely continuous with respect to the measure μ*[3] *precisely when there exists a μ-integrable function f such that ν is given by*

$$\nu(A) := \int_A f(x)\, d\mu(x)$$

for each $A \in \mathcal{A}$.

Let $(X, \mathcal{A}_X, \mu)$ and $(X, \mathcal{A}_Y, \nu)$ be two spaces with finite measures, i.e., $\mu(X) < +\infty$ and $\nu(Y) < +\infty$. The $\sigma-$algebra generated by all rectangles of the form $\mathcal{A}_X \times \mathcal{A}_Y$ is denoted by $\mathcal{A}_X \otimes \mathcal{A}_Y$. Let $\mu \times \nu(\mathcal{A}_X \times \mathcal{A}_Y) := \mu(A_X)\nu(A_Y)$ for all $A_X \in \mathcal{A}_X$ and $A_Y \in \mathcal{A}_Y$. Then, the Lebesgue completion of the algebra $\mathcal{A}_X \otimes \mathcal{A}_Y$ with respect to the measure $\mu \times \nu$ is denoted by $\mathcal{A}_X \overline{\otimes} \mathcal{A}_Y$. The unique extension of finitely additive function $\mu \times \nu$ (defined on $\mathcal{A}_X \times \mathcal{A}_Y$) to $\mathcal{A}_X \overline{\otimes} \mathcal{A}_Y$ is denoted by $\mu \otimes \nu$. The set function $\mu \otimes \nu$ is countably additive. Hence, $(X \times Y, \mathcal{A}_X \overline{\otimes} \mathcal{A}_Y, \mu \otimes \nu)$ is a uniquely given space with a complete measure.[4] For more detailed proofs and definitions see [3, 3.3 Products of measure spaces].

Following theorems are frequently used results of the theory of integration.

Theorem 2.24 ([9], Theorem 2.18, p.52, Fatou's lemma)**.** *Let $(X, \mathcal{A}, \mu)$ be a space with measure. If $\{f_n\}$ is any sequence of non-negative measurable functions defined on X, then*

$$\int_X \left(\liminf_{n\to+\infty} f_n \right) d\mu \leq \liminf_{n\to+\infty} \int_X f_n\, d\mu\,.$$

Theorem 2.25 ([3], Theorem 3.4.4., p.185, Fubini's theorem)**.** *Let μ and ν be $\sigma-$finite non-negative measures on the measurable spaces $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$. Suppose that a function f in $X \times Y$ is integrable with respect to the product measure $\mu \otimes \nu$. Then, the function $y \mapsto f(x, y)$ is integrable with respect to ν for μ-a.e. x, the function $x \mapsto f(x, y)$ is integrable with respect to μ for ν-a.e. y, the functions*

$$x \mapsto \int_Y f(x, y)\, d\nu(y) \quad \textit{and} \quad y \mapsto \int_X f(x, y)\, d\mu(x)$$

are integrable on the corresponding spaces, and one has

$$\int_{X\times Y} f\, d(\mu \otimes \nu) = \int_Y \int_X f(x, y)\, d\mu(x) d\nu(y) = \int_X \int_Y f(x, y)\, d\mu(y) d\nu(x)\,.$$

Theorem 2.26 ([3], Theorem 3.4.5., p.185, Tonelli's theorem)**.** *Let f be non-negative $\mu\otimes\nu$-measurable function on $X \times Y$, where μ and ν are $\sigma-$finite measures. Then $f \in L^1(\mu \otimes \nu)$ provided that*

$$\int_Y \int_X f(x, y)\, d\mu(x) d\nu(y) < +\infty\,.$$

[3]Let μ and ν be defined on a measurable space $(X, \mathcal{A})$. A measure ν is absolutely continuous with respect to the measure μ precisely if $|\nu|(A) = 0$ for every set A with $|\mu|(A) = 0$ (see [3, p.178]).

[4]Let $(X, \mathcal{A}, \mu)$ be a space with measure. The measure μ is called complete if for any $A \in \mathcal{A}$ the identity $\mu(A) = 0$ holds, then $B \in \mathcal{A}$ for every $B \subset A$ and $\mu(B) = 0$.

2.3 Function, measure and distribution spaces

Definition 2.27 ([3], Definition 5.3.1., p.337)**.** A function f on an interval $[a,b]$ is called absolutely continuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon$$

for every finite collection of pairwise disjoint intervals (a_i, b_i) in $[a,b]$ with $\sum_{i=1}^{n} |b_i - a_i| < \delta$.

We remark that a continuously differentiable function in an interval $[a,b]$ is absolutely continuous in this interval. The notation of various sets and spaces are mentioned in the preface. We only recall, that the spaces of *continuous* functions $C^k(\overline{\Omega})$ and $C^k_0(\overline{\Omega})$ are Banach with respect to the norm

$$\|u\|_{C^k_0(\Omega)} = \|u\|_{C^k(\Omega)} := \sum_{|\alpha|\le k} \max_{x\in\overline{\Omega}} \left| \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}} u(x) \right|,$$

where $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is the *multiindex* and $|\alpha| := \sum_{i=1}^{N} \alpha_i$. Naturally, α_i are non-negative. The space of functions from $C^k(\overline{\Omega})$ which are $\lambda-$*Hölder continuous* functions[5] is denoted by $C^{k,\lambda}(\overline{\Omega})$ and is Banach when equipped with the norm

$$\|u\|_{C^{k,\lambda}(\overline{\Omega})} := \|u\|_{C^k(\overline{\Omega})} + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\overline{\Omega}\\ x\neq y}} \frac{\left| \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}} (u(x) - u(y)) \right|}{|x-y|^{\lambda}}.$$

Next, we introduce the sets of integrable functions.

Definition 2.28. Let $(X, \mathcal{A}, \mu)$ be a space with nonnegative measure and let $p \in [0, +\infty)$ be given constant. The set $\mathcal{L}^p_\mu(X)$ is the set of all $\mu-$measurable functions with $|f|^p$ being $\mu-$integrable. If $p = +\infty$, then the set $\mathcal{L}^\infty_\mu(X)$ is set of functions which are essentially bounded [6].

We introduce the relation of equivalence: two functions $f, g \in \mathcal{L}^p_\mu(X)$ are equivalent ($f \sim g$) if they differ on the set of μ measure zero.

Definition 2.29. Let $(X, \mathcal{A}, \mu)$ be a space with nonnegative measure and let $p \in [0, +\infty]$ be given constant (possibly infinite). The set $L^p_\mu(X)$ is the set of all equivalence classes of $\mathcal{L}^p_\mu(X)$ with respect to the relation $\sim$.

If the measure μ from the previous definitions is Lebesgue measure, the subscript in $\mathcal{L}^p(X)$ and $L^p(X)$ will be omitted. Given $p \in [0, \infty)$, we define $\|\cdot\|_{L^p_\mu(X)} = \left(\int_X |\cdot|^p \, d\mu\right)^{1/p}$ and $\|\cdot\|_{L^\infty_\mu(X)} = \operatorname{esssup}_X |\cdot|$, where essential supreme of $\mu-$measurable function f is equal to K if and only if $\mu(U) = 0$, where $U = \{x : f(x) > K\}$ and $\operatorname{esssup}_X |f| = \sup_{X\setminus U} |f| = K$. The spaces $L^p_\mu(X)$ and $L^\infty_\mu(X)$ are Banach spaces with respect to their respective limits.

Theorem 2.30 ([3], Theorem 2.11.12, Hölder inequality)**.** *Suppose that* $1 < p < \infty, q = p(p-1)^{-1}, f \in L^p_\mu(X), q \in L^q_\mu(X)$. *Then* $fg \in L^1_\mu(X)$ *and* $\|fg\|_{L^1_\mu(X)} \le \|f\|_{L^p_\mu(X)} \|g\|_{L^q_\mu(X)}$, *i.e., one has*

$$\int_X |fg| \, d\mu \le \left(\int_X |f|^p \, d\mu\right)^{1/p} \left(\int_X |g|^q \, d\mu\right)^{1/q}.$$

Functions satisfying Carathédory condition are important in non-linear analysis.

Definition 2.31 ([6], Definition 3.2.22, p.136)**.** Let Ω be an open set in $\mathbb{R}^N$. A function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to have the *Carathéodory condition* if

1. for all $t \in \mathbb{R}$ the function $x \mapsto f(x,t)$ is Lebesgue measurable on Ω;
2. for a.a. $x \in \Omega$ the function $t \mapsto f(x,t)$ is continuous on $\mathbb{R}$.

Theorem 2.32 ([6], Theorem 3.2.24, p.136)**.** *Let f satisfy the Carathéodory condition and $p, q \in [1, +\infty)$. Let there exist $g \in L^q(\Omega)$ and $c \in \mathbb{R}$ such that*

$$|f(x,t)| \le g(x) + c|t|^{\frac{p}{q}}, \quad \text{for a.a.} \quad x \in \Omega \quad \text{and all} \quad y \in \mathbb{R}.$$

Then

[5] A function u is $\lambda-$Hölder continuous if there exists a constant $C > 0$, such that $|u(x) - u(y)| \le C|x-y|^\lambda$ holds for all $x, y \in \overline{\Omega}$

[6] For each function $f \in \mathcal{L}^\infty_\mu(X)$, there exists a constant K such that $\mu(U) = 0$, where $U = \{x : f(x) > K\}$.

1. *$F(\varphi) \in L^q(\Omega)$ for all $\varphi \in L^p(\Omega)$;*

2. *F is a continuous mapping from $L^p(\Omega)$ to $L^q(\Omega)$;*

3. *F maps bounded sets in $L^p(\Omega)$ into bounded sets in $L^q(\Omega)$.*

Since the integrable functions can not be derived in the classical sense. Therefore, we introduce *weak derivative.*

Definition 2.33. Given $u \in L^1_{\mathrm{loc}}(\Omega)$[7] a function $w \in L^1_{\mathrm{loc}}(\Omega)$ is called a weak $\alpha-$derivative of the function u if it satisfies

$$\int_\Omega u(x)\, \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}} \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega w(x)\, \varphi(x)\, dx\,.$$

The following result by L. Schwartz provides us with the connection of the weak and the classical derivative of the function of more variables.

Theorem 2.34 ([20], Theoréme V, p.57, translated from French)**.**

1. *If the locally integrable function f is absolutely continuous in the variable x_i on almost all lines parallel to the axis x_i and can be differentiated (in the classical sense) almost everywhere to a locally integrable function $g = \frac{\partial}{\partial x_i} f$ a.e., then $g = D^i f$ in the sense of distributions.*

2. *If the function f admits derivative $g = D^i f$ in the sense of distributions, function f has a representative which is absolutely continuous in x_i on almost all lines parallel to the axis x_i, then f can be differentiated almost everywhere in the classical sense and $g = \frac{\partial}{\partial x_i} f$ almost everywhere. If in addition f, g are continuous in some open domain Ω then $g = \frac{\partial}{\partial x_i} f$ everywhere in the classical sense.*

The set of measurable functions $L^p(\Omega)$ such that their weak derivatives up to the order k belong to the space $L^p(\Omega)$ is called the *Sobolev space* is denoted by $W^{k,p}(\Omega)$ and is Banach space with respect to the norm

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p(\Omega)}\,.$$

In general, we cannot distinguish boundary values of integrable function $u \in L^p$ since the Lebesgue measure of the boundary $\partial\Omega$ is zero. The structure of Sobolev spaces is more strict and given a domain Ω with "nice" boundary we can find an equivalent notion of boundary values of the functions in $W^{1,p}(\Omega)$.[8]

Theorem 2.35 ([6], Theorem 5.5.1, p.275, Trace theorem)**.** *Let Ω be such that $\partial\Omega \in C^{0,1}$ be a bounded domain in $\mathbb{R}^N$. There exists one and only one continuous linear operator T which assigns to every function $u \in W^{1,p}(\Omega)$ a function $Tu \in L^p(\partial\Omega)$ and has the following property:*

$$\text{"For } u \in C^\infty(\overline{\Omega}) \text{ we have } Tu = u|_{\partial\Omega}.\text{"}$$

The following identity holds:

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) \,:\, Tu = o \quad in \quad L^p(\partial\Omega) \right\}\,.$$

By the virtue of the previous theorem we can say, that a function $u \in W^{1,p}(\Omega)$ is zero at the boundary $\partial\Omega$ in the sense of traces (or generalized sense) if $u \in W_0^{1,p}(\Omega)$.

The following theorems summarize embeddings of function spaces.

Theorem 2.36 ([8], Theorem 6., Section 5.7, p.270)**.** *Let Ω be a bounded open subset of $\mathbb{R}^N$ with a C^1 boundary. Assume $u \in W^{k,p}(\Omega)$.*

1. *If $k < n/p$. then $u \in L^q(\Omega)$, where $1/q = 1/p - k/N$. We have in addition the estimate*

$$\|u\|_{L^q(\Omega)} \le C\, \|u\|_{W^{k,p}(\Omega)}\,,$$

the constant C depending only on k, p, N and Ω.

[7] A function which is integrable on each compact subset of Ω.

[8] The following theorem uses an alternative definition of Sobolev spaces. The space $W^{k,p}(\Omega)$ (resp., $W_0^{k,p}(\Omega)$) is a closure of the space of functions $u \in C^\infty(\Omega)$ (resp., $C_c^\infty(\Omega)$) such that $\|u\|_{W^{k,p}(\Omega)} < +\infty$ with respect to the norm $\|\cdot\|_{W^{k,p}(\Omega)}$.

2. If $k > n/p$, then $u \in C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\overline{\Omega})$, where

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, \text{ if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number} < 1, \text{ if } \frac{n}{p} \text{ is not an integer.} \end{cases}$$

We have in addition the estimate

$$\|u\|_{C^{k-\left[\frac{n}{p}\right]-1}(\overline{\Omega})} \leq C\, \|u\|_{W^{k,p}(\Omega)}\,,$$

the constant C depending only on k, p, N, γ and Ω.

Theorem 2.37 ([8], Theorem 1, p.272, Rellich-Kondrachov Compactness Theorem). *Assume Ω is a bounded open subset of $\mathbb{R}^N$, and $\partial\Omega$ is C^1. Suppose $1 \leq p < n$. Then*

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$$

for each $1 \leq q < p^* := \dfrac{Np}{N-p}$.

Having claims regarding integrable functions at hand, we proceed to more general objects – measures[9]. Our main goal is to show, that the space $\mathcal{M}(\Omega)$ is the dual space of $C_0(\overline{\Omega})$, where Ω is a bounded domain in $\mathbb{R}^N$. Since we follow the stream of ideas in [9], the claims are stated in full generality working with a topological space $(X, \mathcal{T})$. We immediately compare the general situation to the situation of metric space $(\Omega, |\cdot|)$ whose open set in the metric $|\cdot|$ induce a topology $(\Omega, \mathcal{B}_\Omega)$.

The following lemma shows a close relation of continuity and measurability of mappings defined on a topological space.

Lemma 2.38 ([9], Corollary 2.2, p.44). *If X and Y are metric (or topological) spaces, every continuous $f : X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable ($\mathcal{B}_X$ and $\mathcal{B}_Y$ are Borel $\sigma-$algebras defined on the space X and Y respectively).*

In the following text, we work with a general topological space $(X, \mathcal{T})$ which is locally compact (every $x \in X$ has a compact neighbourhood) and Hausdorff (every two points $x, y \in X$ such that $x \neq y$ posses disjoint neighbourhoods) unless otherwise stated. Given $f \in C(X) := C(X, \mathbb{R})$ the *support* of f is defined by

$$\operatorname{supp}(f) := \overline{\{x \,:\, f(x) \neq 0\}}$$

and we say, that $f \in C(X)$ *vanishes at infinity* if for every $\varepsilon > 0$ the set $\{x \,:\, |f(x) \geq \varepsilon|\}$ is compact. We define $C_c(X)$ as the set of all $f \in C(X)$ such that $\operatorname{supp}(f)$ is compact and $C_0(X)$ as the set of all $f \in C(X)$ such that f vanishes at infinity. Every function in $C_c(X)$ and $C_0(X)$ is bounded, since any continuous functional defined on a compact space is bounded. The counterpart of $C(X)$ in $(\Omega, \mathcal{B}_\Omega)$ is simply $C_c(\Omega)$. The only functions defined on an open domain Ω which vanish at infinity are the restriction of the functions in $C_0(\overline{\Omega})$ to Ω. Hence, we can identify $C_0(X)$ with $C_0(\overline{\Omega})$.

Theorem 2.39 ([9], Proposition 4.35, p.132). *If X is a locally compact Hausdorff space, $C_0(X)$ is the closure of $C_c(X)$ in the uniform metric*[10].

Let μ be a Borel measure defined on $(X, \mathcal{B}_X)$. The measure μ is called *outer regular* on $A \in \mathcal{B}_X$ if $\mu(A) = \inf\{\mu(U) \,:\, U \supset A,\ U \text{ open}\}$ and *inner regular* on A if $\mu(A) = \sup\{\mu(K) \,:\, K \subset A,\ K \text{ compact}\}$. If μ is inner and outer regular on all $A \in \mathcal{B}_X$, then it is called *regular.* A Borel measure $(X, \mathcal{B}_X)$ that is finite on all compact sets, outer regular on all Borel sets and inner regular on all open sets is called a *Radon measure.* The case of $(\Omega, \mathcal{B}_\Omega)$ is slightly less complicated, since the following theorem holds.

Theorem 2.40 ([9], Proposition 7.5, p.216). *Every Radon measure is inner regular on all of its $\sigma-$finite sets.*

Since every set $A \in \mathcal{B}_X$ is $\sigma-$finite, Radon measure defined on $(\Omega, \mathcal{B}_\Omega)$ automatically regular. Every non-negative bounded linear functional on $C_c(X)$ can be represented by a Radon measure μ. Since $C_0(X)$ is a uniform closure of $C_c(X)$, then every μ can be extended to a bounded linear functional on $C_0(X)$ if and only if it is bounded with respect to the uniform norm. Moreover, we define $\mathcal{M}(\Omega)$[11] the set of all bounded Radon measures (i.e., $|\mu|(X) < +\infty$) on Ω. Naturally, if $\mu \in \mathcal{M}(\Omega)$ then we define $|\mu|(\partial\Omega) = 0$. The space $\mathcal{M}(X)$ is Banach with respect to the norm

$$\|\mu\|_{\mathcal{M}(X)} := |\mu|(X)\,.$$

Finally, we can formulate the Riesz representation theorem for continuous functions.

[9] By Radon-Nikodým theorem (see Theorem 2.23), every $\mu-$integrable function f on X defines a measure on X.

[10] Given $f, g \in C_0(X)$, the uniform metric is a mapping $m : X \times X \to \mathbb{R}_0^+$ such that $m(f, g) := \sup_{x \in X} |f(x) - g(x)|$.

[11] In [9], these statements are formulated for the space of all complex signed measures. We restrict ourselves to the real case. Thus, we formulate the statements on the linear subspace $\mathcal{M}(X)$ of the space of signed complex measures.

Theorem 2.41 ([9], Theorem 7.17, p.223)**.** *Let X be a locally compact Hausdorff space and for $\mu \in \mathcal{M}(X)$ and $f \in C_0(X)$ let $I_\mu(f) = \int_X f\, d\mu$. Then the map $\mu \mapsto I_\mu$ is an isometric isomorphism form $\mathcal{M}(X)$ to $(C_0(X))^*$.*

Since $\mathcal{M}(\Omega)$ is the dual of $C_0(\overline{\Omega})$, we can define an equivalent norm for $\mu \in \mathcal{M}(\Omega)$ by

$$\|\mu\|_{\mathcal{M}(\Omega)} := \sup\left\{ \int_\Omega f\, d\mu \,:\, f \in C_0(\overline{\Omega}),\, \|f\|_{C_0(\overline{\Omega})} = 1 \right\}.$$

Indeed, without loss of generality let $\mu \in \mathcal{M}(\Omega)$ be non-negative . Then, we can define a sequence $\{f_n\} \subset C_0(\overline{\Omega})$ such that $\|f_n\|_{C_0(\overline{\Omega})}$ for all $n \in \mathbb{N}$ and $f_n \nearrow 1$ on Ω and by monotone convergence theorem

$$\|\mu\|_{\mathcal{M}(\Omega)} = \int_\Omega 1\, d\mu = |\mu|(\Omega)\,.$$

By Radon-Nikodým theorem, every integrable function $L^1(\Omega)$ defines a measure in $\mathcal{M}(\Omega)$ (regularity is the consequence of the continuity of the Lebesgue integral with respect to the domain of integration and Lebesgue $\sigma-$algebra on Ω is "wider" than Borel $\sigma-$algebra on Ω). For each $u \in L^1(\Omega)$ representing a signed measure $\widetilde{u} \in \mathcal{M}(\Omega)$ the equality

$$\|u\|_{L^1(\Omega)} = \int_\Omega |u|\, dx = \int_\Omega u^+\, dx + \int_\Omega u^-\, dx = \int_\Omega d\widetilde{u^+}(x) + \int_\Omega d\widetilde{u^-}(x) = \int_\Omega d|\widetilde{u}|(x) = |\widetilde{u}|(\Omega)\,.$$

holds. Hence, there is a continuous embedding of the spaces $L^1(\Omega) \hookrightarrow \mathcal{M}(\Omega)$.

The next theorem states, that convolution has the nice property of "regularising" objects (such as measure) even to the integrable functions. Since we will need to use this theorem in slightly different form, we include the sketch of proof as in [9] and a further adjustment.

Theorem 2.42 ([9], Proposition 8.49., p.271)**.** *If $f \in L^p(\mathbb{R}^N)$ for $1 \le p \le \infty$ and $\mu \in \mathcal{M}(\mathbb{R}^N)$, then the integral $f * \mu(x) = \int f(x-y)d\mu(y)$ exists for a.e. x, $f * \mu \in L^p$, and $\|f * \mu\|_p \le \|f\|_p \|\mu\|_{\mathcal{M}}$. (Here "$L^p$" and "a.e." refer to Lebesgue measure.)*

Proof. [Sketch of the proof as in [9]] If f and μ are non-negative, then $f * \mu(x)$ exists (possibly being equal to ∞) for every x, and by the Minkowski's inequality for integrals,

$$\|f * \mu\|_{L^p(\mathbb{R}^N)} \le \int \|f(\cdot - y)\|_{L^p(\mathbb{R}^N)}\, d\mu(y) \le \|f\|_{L^p(\mathbb{R}^N)} \|\mu\|\,. \tag{2.3}$$

In particular $f * \mu < \infty$ for a.e. x. In the general case this argument applies to $|f|$ and $|\mu|$, and the result follows easily. □

We can see, that in (2.3) the shift invariance of the Lebesgue norm $\|\cdot\|_{L^p(\mathbb{R}^N)}$ of the function $f(\cdot - y)$ in the variable y in the whole space $\mathbb{R}^N$ is used, so the norm of the convolution can be estimated independently on y. Theorem 2.42 can be also (under further assumptions) formulated for a bounded domain $\Omega \subset \mathbb{R}^N$, see the proof of the inequality (3.2).

Finally, we include a lemma by which for two locally integrable functions $u, v \in L^1_{\text{loc}}(\Omega)$ it is necessary and sufficient to $\int_K u - v\, dx = 0$ hold for each compact $K \subset\subset \Omega$ in order to u and v be equal a.e.

Lemma 2.43 ([24], p. 72, Du Bois-Reymond lemma)**.** *In order that the function $f(x)$, locally integrable in Ω, should become zero in the region Ω in the sense of generalized functions, it is necessary and sufficient that $f(x) = 0$ almost everywhere in Ω.*

2.4 PDE theory

The classical results of the PDE solvability stated in the first section will be reproduced mainly from [11], provided some minor labelling changes were made for the sake of consistency. Statements will be formulated without proofs since are not in the main scope of the thesis. The Green's function G will be derived as a solution operator of the Poisson's equation. For given bounded domain Ω the conditions on the smoothness of the boundary $\partial\Omega$ will be posed in order for G to exist. The upper estimates for the Green's function G will be crucial in the further development of this thesis.

2.4.1 Green's function for the Laplace's operator

A function u satisfying $\Delta u = 0$ will be called *harmonic*. The problem of finding the function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ defined on domain $\Omega \subset \mathbb{R}^N$ satisfying

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega\,, \\ \quad u = g & \text{on} \quad \partial\Omega\,, \end{cases}$$

will be called the classical *Dirichlet problem* for Laplace's equation ($f \equiv 0$) of Poisson's equation (f nontrivial). The solution u of this problem satisfies the equations in every point $x \in \overline{\Omega}$
Let $y \in \mathbb{R}^{\mathbb{N}}$ be fixed. We search for a harmonic function radially symmetric with respect to y. The homogeneous partial differential equation reduces to an ordinary differential equation with the solution

$$\Gamma(x-y) = \Gamma(|x-y|) = \begin{cases} \dfrac{|x-y|^{2-N}}{N(N-2)\omega_N} & \text{for} \quad N > 2\,, \\ -\dfrac{\log|x-y|}{2\pi} & \text{for} \quad N = 2\,, \end{cases} \tag{2.4}$$

where ω_N is the volume of a unit ball in $\mathbb{R}^N$. The function Γ is called the *fundamental solution of the Laplace's equation.* It can be shown, that the fundamental solution Γ is harmonic at every point $x \in \mathbb{R}^N, x \neq y$. We recall the Green's identities. Let $u, v \in C^2(\Omega)$ be two functions defined in the bounded domain Ω with C^1 smooth boundary $\partial\Omega$ then the following identities hold

$$\int_\Omega v\,\Delta u\,dx + \int_\Omega \nabla u\,\nabla v\,dx = \int_{\partial\Omega} v\,\frac{\partial}{\partial\mathbf{n}} u\,dS_x\,,$$

(*Green's first identity*) and

$$\int_\Omega (v\,\Delta u - u\,\Delta v)\;dx = \int_{\partial\Omega} \left(v\,\frac{\partial}{\partial\mathbf{n}} u - u\,\frac{\partial}{\partial\mathbf{n}} v \right)\;dS_x$$

(*Green's second identity*). The first identity is derived from the divergence theorem and the second identity is obtained by interchanging u and v in the first identity and subtracting. Now, we use the Green's second identity for the function Γ and a twice differentiable function u The integral is taken over the region $\Omega \setminus \overline{B_y(r)}$ for some positive diameter $r > 0$. We obtain

$$u(y) = \int_{\partial\Omega} u\,\frac{\partial}{\partial\mathbf{n}}\Gamma(x-y) - \Gamma(x-y)\,\frac{\partial}{\partial\mathbf{n}} u\,dS_x + \int_\Omega \Gamma(x-y)\,\Delta u\,dx \tag{2.5}$$

for $r \to 0$ as a limit process. The formula (2.5) cannot be used to solve the Poisson's equation directly. The directional derivative $\frac{\partial}{\partial\mathbf{n}} u$ is a priori unknown. We try to find some "version" of the function Γ which is identically zero at the boundary $\partial\Omega$. Suppose, we can find a harmonic function $h \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ such that $\Gamma = -h$ on $\partial\Omega$. Then, by the Green's second identity

$$-\int_{\partial\Omega} u\,\frac{\partial}{\partial\mathbf{n}} h - h\,\frac{\partial}{\partial\mathbf{n}} u\,dS_x = \int_\Omega h\,\Delta u\,dx\,. \tag{2.6}$$

Setting $G := \Gamma + h$ and thus adding (2.5) and (2.6), we obtain the formula

$$u(y) = \int_{\partial\Omega} u\,\frac{\partial}{\partial\mathbf{n}} G\,dS + \int_\Omega G\,\Delta u\,dx\,.$$

where G is called the *Green's function for the Laplace's operator.* For the given problem, the choice of the function G is dependent on the domain Ω. It has been shown, e.g., in [24], that $G(x,y) = G(y,x)$ and $G(x,y) > 0$ for all $x, y, \in \Omega$. However, the function h can not be found for every domain Ω. In [11, Chapter 2] is shown, that such function h can be found for domains Ω satisfying the *exterior sphere condition* This is, for every $x \in \partial\Omega$, there exists a ball B such that $x = \overline{B} \cap \overline{\Omega}$. Finally, we recall [11, Lemma, p. 22] which gives us the interior estimates of the derivatives for harmonic functions.

Lemma 2.44. *Let u be harmonic in Ω and $B = B_y(R) \subset\subset \Omega$ be a ball strictly contained in Ω. Then*

$$|\nabla u(y)| \leq \frac{N}{R} \sup_{x \in \partial\Omega} |u(x)|\,. \tag{2.7}$$

C^2 **boundary** As will be promptly shown, we will be interested in a particular type of boundaries. We present a definition from [16] of a bounded domain Ω possessing a C^2 boundary $\partial\Omega$.

Definition 2.45. A bounded domain $\Omega \subset \mathbb{R}^N$ is of class C^2 if there exists a positive number r_0 such that, for every $x \in \partial\Omega$, there exists a set of Cartesian coordinates $\xi = \xi^x$, centred at x, and a function $F_x \in C^2(\mathbb{R}^{N-1})$ such that $F_x(0) = 0, \nabla F_x(0) = 0$ and

$$\Omega \cap B_x^N(r_0) = \{\xi \ : \ |\xi| < r_0, \xi_1 > F_x(\xi_2, \dots, \xi_N)\} \ .$$

The set of coordinates ξ^x is called a *normal set of coordinates* at x and F_x is called the *local defining function* at x.

A domain Ω with a C^2 boundary $\partial\Omega$ satisfies the interior and exterior sphere condition, since the boundary can be described by a twice differentiable function. Furthermore, the ball radius $r_0 > 0$ can be chosen independent of the choice of x and since Ω is bounded, the boundary can be covered by a finite number of balls. Amongst the functions $F_x \in C^2(\mathbb{R}^{N-1})$ defined in the balls, we can find the one with the maximum curvature. We conclude, that the domain Ω satisfies the *uniform interior and exterior sphere condition.* That is, the maximal balls constructed in the study of regularity of the boundary points have some lower bound on their size.

2.4.2 Estimates for the Green's functions

It is helpful to estimate the Green's function G and its gradient ∇G by some function which asymptotic behaviour is easy to analyse. Since the Green's function G is a sum of Γ and a bounded (by maximal principle) function h we would expect to be able to restrict the function G from above by some expression proportional to the negative integral power of the distance from the point of singularity. Such estimates are mentioned e.g. in [23].

Lemma 2.46. *Assuming that Ω is bounded with a C^2 boundary, then there exist constants $K_i, i = 1, \dots, 6$ such that*

$$\begin{aligned}
&1.\ G(x,y) \le K_1(\Omega)|x-y|^{2-N} && \textit{for} \quad N \ge 3\,,\\
&2.\ G(x,y) \le K_2(\Omega,\alpha)|x-y|^{-\alpha} && \textit{for} \quad N = 2,\ \alpha > 0\,,\\
&3.\ G(x,y) \le K_3(\Omega)\delta(x)|x-y|^{1-N} && \textit{for} \quad N \ge 3\,,\\
&4.\ G(x,y) \le K_4(\Omega,\alpha)\delta(x)|x-y|^{-1-\alpha} && \textit{for} \quad N = 2,\ \alpha > 0\,,\\
&5.\ |\nabla G(x,y)| \le K_5(\Omega)|x-y|^{1-N} && \textit{for} \quad N \ge 3\,,\\
&6.\ |\nabla G(x,y)| \le K_6(\Omega,\alpha)|x-y|^{-1-\alpha} && \textit{for} \quad N = 2,\ \alpha > 0\,,
\end{aligned} \tag{2.8}$$

for all $x, y \in \Omega$, $x \neq y$.

Proof.

1. Let $y \in \Omega$ be fixed, the Green's function $G(x,y)$ is defined by $G := \Gamma + h$. We can thus estimate

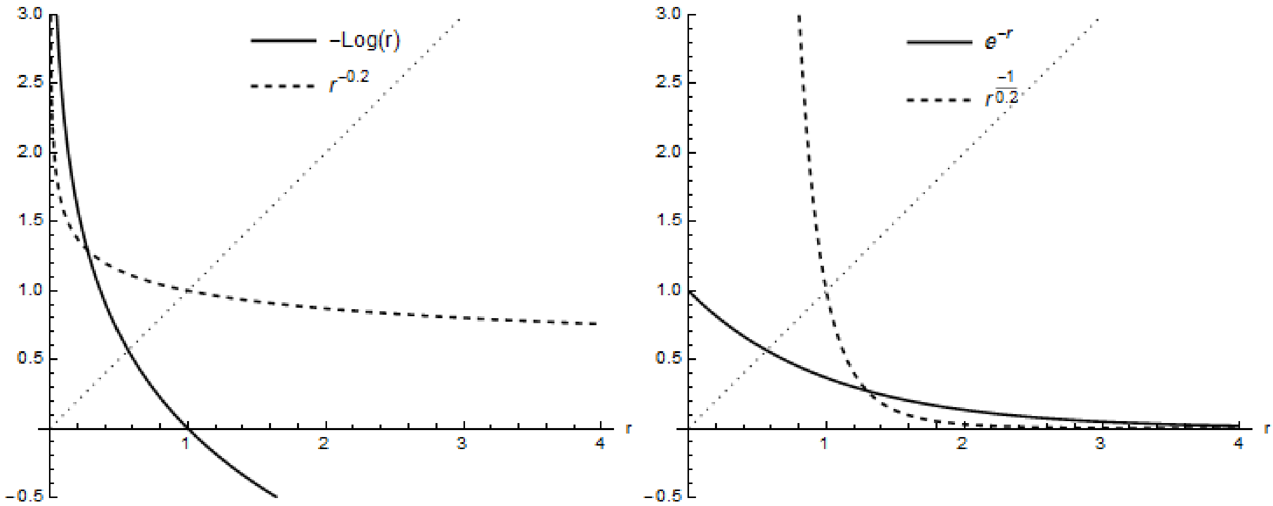
$$|G(x,y)| \le |\Gamma + h| \le |\Gamma| \le \frac{1}{N(2-N)\omega_N}|x-y|^{N-2} \le K(\Omega)|x-y|^{2-N}.$$

   for all $x \in \Omega$, $x \neq y$. Since the constant K does not depend on the choice of y, the inequality holds for all $x, y \in \Omega$, $x \neq y$.

2. Let $y \in \Omega$. First, we explore the asymptotic behaviour of G near the point of singularity y. Let us restrict to the ball $B_y(r)$ centred at y with $r > 0$ sufficiently small that we does not take in account the behaviour of the function h (G is "almost" radially symmetric there). The situation is depicted in the Figure 2.1a where we compared the function $\Gamma(r)$ with function $r^{-\alpha}$ for arbitrary positive $\alpha > 0$. We can express the inverse of both functions on the positive part of real line and we get $\Gamma^{-1}(r) = e^{-r}$ and $(r^{-\alpha})^{-1} = r^{-\frac{1}{\alpha}}$. The exponential function has more rapid decrease than polynomial function as the argument r approaches the infinity and thus

$$\frac{e^{-r}}{r^{-\frac{1}{\alpha}}} \to 0 \qquad \text{as} \qquad r \to +\infty\,.$$

   There exists a constant K such that the fundamental solution Γ and hence the Green's function G can be estimated from above by rational function $r^{-\alpha}$ for $\alpha > 0$ near the point y. Defining $G_0(x,y) :=$



(a) The original situation near point y.

(b) Inverse of the functions from Figure 2.1a

Figure 2.1: Behaviour of the fundamental solution Γ near the point of the singularity compared to the function $r^{-\alpha}$ with $\alpha = 0.2$.

$\Gamma(x,y) + \frac{1}{2\pi}\log(\operatorname{diam}\Omega)$ we obtain a function G_0 which is non-negative and $G_0 \geq G$ holds by the maximum principle. The fraction $M(x,y) := \frac{G_0(x,y)}{|x-y|^{-\alpha}}$ is a positive bounded function $x \mapsto M(x,y)$ in $\overline{\Omega}$ with one point of non-continuity $x = y$. The value of M at point (y,y) can be by the previous discussion defined $M(y,y) = 0$ preserving the continuity. Therefore, for given $y \in \Omega$, we are able to find $K_y = \max_{x\in\Omega} M$. The function G_0 is translation invariant in the second argument y in the plane $\mathbb{R}^2$ and there exists $K_2 := K(\Omega, \alpha)$, such that

$$G(x,y) \leq K_2 |x-y|^{-\alpha}, \qquad \alpha > 0\,.$$

holds for all $x, y \in \Omega$.

3. The proof is carried out in [12] using the first inequality in (2.8).

4. The inequality is the consequence of the second inequality in (2.8) in the same manner as in the case of higher dimension.

5. The function $G(x,y)$ is harmonic for $x \in \Omega$, $x \neq y$ and the interior derivative estimate (2.7) in Ω for subdomain not containing the point $x \neq y$ holds. Suppose, that $\delta(x) \leq |x-y|$. Then each point $x \in \Omega$ can be a centre of the ball $B_x(\frac{1}{2}\delta(x))$ which is strictly contained in Ω and does not contain the point y. Thus, using the third estimate from (2.8) we get

$$|\nabla G(x,y)| \leq \frac{N}{\frac{1}{2}\delta(x)} \sup_{x' \in B_x(\frac{1}{2}\delta(x))} G(x,y) \leq \frac{NK_3}{\frac{1}{2}\delta(x)} \sup_{x' \in B_x(\frac{1}{2}\delta(x))} \delta(x')|x'-y|^{1-N}. \tag{2.9}$$

Bounding the supreme from above, we consider "worst case" scenarios. The distance from the boundary can be maximally $\frac{3}{2}\delta(x)$ and choosing the nearest point at the ball boundary to y in the other case we get

$$|\nabla G(x,y)| \leq NK_3 \frac{\frac{3}{2}\delta(x)\left[|x-y| - \frac{1}{2}\delta(x)\right]^{1-N}}{\frac{1}{2}\delta(x)} \leq 3NK_3 \left(\frac{1}{2}\right)^{1-N} |x-y|^{1-N}, \tag{2.10}$$

utilizing the fact that $\delta(x) \leq |x-y|$. Now suppose that $\delta(x) > |x-y|$. Let the ball be $B_x(\frac{1}{2}|x-y|)$, using the same argumentation as before and the estimate 1. from (2.8)

$$|\nabla G(x,y)| \leq \frac{N}{\frac{1}{2}|x-y|} \sup_{x' \in B_x(\frac{1}{2}|x-y|)} G(x',y) \leq \frac{NK_1}{\frac{1}{2}|x-y|} \sup_{x' \in B_x(\frac{1}{2}|x-y|)} |x'-y|^{2-N}, \tag{2.11}$$

$$\leq NK_1 \frac{\left[|x-y| - \frac{1}{2}|x-y|\right]^{2-N}}{\frac{1}{2}|x-y|} \leq 2NK_1 \left(\frac{1}{2}\right)^{2-N} |x-y|^{1-N}. \tag{2.12}$$

Finally, define $K_5(\Omega) := \max\left\{3NK_3\left(\frac{1}{2}\right)^{1-N}, 2NK_1\left(\frac{1}{2}\right)^{2-N}\right\}$.

6. The proof is same as in the preceding case. The only natural consequence is dependence of K_6 on the domain Ω and the parameter $\alpha > 0$.

$\square$

2.4.3 Other PDE concepts

For the sake of clarity, we included the definitions of strong, weak and very weak formulations and formulation in the sense of distributions of the Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega\,, \\ \quad u = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{2.13}$$

Definition 2.47. A function $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a *strong solution* of the equation (2.13) if it satisfies the equation almost everywhere in Ω and the boundary condition holds in the sense of traces.

Definition 2.48. A function $u \in W^{1,2}(\Omega)$ is a *weak solution* of the equation (2.13) if it satisfies

$$\int_\Omega Du \cdot \nabla v \, dx = \int_\Omega f\, v \, dx$$

for all $v \in C_0^1(\overline{\Omega})$ and $u \in W_0^{1,2}(\Omega)$.

Definition 2.49 ([16], Definition 1.2.1, p.4)**.** Let $\mu \in \mathcal{M}(\Omega)$. A function $u \in L^1(\Omega)$ is a *very weak solution* of the equation

$$\begin{cases} -\Delta u = \mu & \text{in} \quad \Omega\,, \\ \quad u = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{2.14}$$

if it satisfies

$$-\int_\Omega u\, \Delta v \, dx = \int_\Omega v \, d\mu$$

for all $v \in C_0^2(\overline{\Omega})$.

Definition 2.50. A distribution $u \in \mathcal{D}'(\Omega)$ is the solution in the sense of distributions of the equation

$$-\Delta u = f$$

if it satisfies

$$-\langle u, \Delta v\rangle = \langle f, v\rangle \ .$$

for all $v \in \mathcal{D}(\Omega)$.

The equivalent of the Green's first identity holds even for weakly differentiable functions.

Theorem 2.51 ([18], Theorem 1.30, p.20, integration by parts)**.** *If $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$, $1 = 1/p+1/q$, then*

$$\int_\Omega \left(u\, D^i v + D^i u\, v\right) \, dx = \int_{\partial\Omega} u \cdot v\, \mathbf{n}_i \, dS \tag{2.15}$$

holds for all $i = 1, \ldots, N$.

The last two theorems specifies the regularity of the solutions of equations in the strong sense and in the sense of distributions respectively.

Theorem 2.52 ([8], Theorem 6, Section 6.3., p.317)**.** *Assume $f \in L^2(\Omega)$. Suppose that $u \in W_0^{1,2}(\Omega)$ is a weak solution of the boundary value problem*

$$\begin{cases} -\Delta u = f & \textit{in} \quad \Omega\,, \\ \quad u = 0 & \textit{on} \quad \partial\Omega\,. \end{cases}$$

Assume finally $\partial\Omega \in C^2$. Then $u \in W^{2,2}(\Omega)$.

Theorem 2.53 ([17], Lemma 2.85, p.76)**.** *Suppose that $u \in L^1(D)$ for each $D \subset\subset \Omega$ and satisfies*

$$\int_\Omega u\, \Delta v \, dx = 0$$

for all $v \in \mathcal{D}(\Omega)$. Then u is equivalent to a harmonic function.

Chapter 3

Elliptic PDE with measure data

In the sequel, we assume that Ω is a bounded a domain in $\mathbb{R}^N$ with the boundary $\partial\Omega$ of the class C^2. Theorem 3.1 was published in [16, Chapter 1] considering $\Omega \subset \mathbb{R}^N$ only for $N \geq 3$. Here, we provide a more detailed proof altogether with considering the possibility $N = 2$.

Theorem 3.1 ([16], Theorem 1.2.2, p.4)**.** *Assume $\mu \in \mathcal{M}(\Omega)$, then the problem* (2.14) *has a unique very weak solution u given by*

$$u(x) = \int_\Omega G(x, y)\, d\mu(y)\,. \tag{3.1}$$

Furthermore,

$$\|u\|_{L^p(\Omega)} \leq C_1(p, \Omega)\, \|\mu\|_{\mathcal{M}(\Omega)} \quad \textit{for} \quad 1 \leq p < \frac{N}{N-2}, \quad N \geq 3\,, \tag{3.2}$$

$$\|u\|_{L^p(\Omega)} \leq C_2(p, \Omega)\, \|\mu\|_{\mathcal{M}(\Omega)} \quad \textit{for} \quad 1 \leq p < +\infty, \quad N = 2\,, \tag{3.3}$$

and

$$\|u\|_{W^{1,p}(\Omega)} \leq C_3(p, \Omega)\, \|\mu\|_{\mathcal{M}(\Omega)} \quad \textit{for} \quad 1 \leq p < \frac{N}{N-1}, \quad N \geq 2\,, \tag{3.4}$$

where C_1, C_2, C_3 are constants depending only on p and Ω.

The following theorem characterises a boundary behaviour of the very weak solution of (2.14). The proof is omitted since it is independent of the dimension by the inequality (3.4).

Theorem 3.2 ([16], Assertion 1.3.7.a, p.15)**.** *Let $\mu \in \mathcal{M}(\Omega)$. If*

$$v(x) := \int_\Omega G(x, y)\, d\mu(y), \quad \forall x \in \Omega \tag{3.5}$$

then $v \in W_0^{1,p}(\Omega)$ for $1 \leq p < N/(N-1)$.

Proof. [Proof of Theorem 3.1] The proof will be carried out in several steps. At first, we show the uniqueness of the very weak solution. Then, we show that any function u given by the formula (3.1) satisfies (3.2) or (3.3) and (3.4). Finally, we show that the function u given by (3.1) is a very weak solution of (2.14).

1. *Uniqueness:* Let $u, v \in L^1(\Omega)$ be the solutions of (2.14) and thus satisfying

$$-\int_\Omega u\, \Delta\phi\, dx = \int_\Omega \phi\, d\mu\,,$$
$$-\int_\Omega w\, \Delta\phi\, dx = \int_\Omega \phi\, d\mu\,,$$

for each $\phi \in C_0^2(\overline{\Omega})$. Subtracting both identities we get

$$\int_\Omega (u - v)\Delta\phi\, dx = 0\,,$$

for each $\phi \in C_0^2(\overline{\Omega})$ and $w := u - v \in L^1(\Omega)$ is a very weak solution of the equation $\Delta w = 0$. Using Lemma 2.53 we deduce that w is equivalent to a harmonic function in Ω in the sense of classes in $L^1(\Omega)$.

Thus, applying the Green's first identity twice and the harmonic property of w we obtain

$$
\begin{aligned}
0 = \int_\Omega w\,\Delta\phi\,dx &= -\int_\Omega \nabla w\,\nabla\phi\,dx + \int_{\partial\Omega} w\,\frac{\partial}{\partial\mathbf{n}}\phi\,dS_x = \\
&= \int_\Omega \Delta w\,\phi\,dx + \int_{\partial\Omega}\frac{\partial}{\partial\mathbf{n}}w\,\phi\,dS_x + \int_{\partial\Omega} w\,\frac{\partial}{\partial\mathbf{n}}\phi\,dS_x = \\
&= \int_{\partial\Omega} w\,\frac{\partial}{\partial\mathbf{n}}\phi\,dx
\end{aligned}
$$

for all $\phi \in C_0^2(\overline{\Omega})$ and therefore, $w = 0$ on $\partial\Omega$. By the maximum principle for harmonic functions, w has a trivial representative $w \equiv 0$ in $\overline{\Omega}$ and therefore $u = v$ a.e. in Ω. Hence, the problem (2.14) has at most one solution.

2. *Estimate* (3.2), $N \geq 3$*:* First, we modify the form of solution u.

$$
\begin{aligned}
|u(x)| = \left|\int_\Omega G(x,y)\,d\mu(y)\right| &= \left|\int_\Omega G(x,y)\,d\mu^+(y) - \int_\Omega G(x,y)\,d\mu^-(y)\right| \leq \\
\leq \left|\int_\Omega G(x,y)\,d\mu^+(y)\right| + \left|\int_\Omega G(x,y)\,d\mu^-(y)\right| &\leq \int_\Omega G(x,y)\,d\mu^+(y) + \int_\Omega G(x,y)\,d\mu^-(y) = \\
&= \int_\Omega G(x,y)\,d|\mu|(y)\,,
\end{aligned}
$$

since the Green's function is non-negative. The first inequality in (2.8) is used to derive

$$
|u(x)| \leq \int_\Omega G(x,y)\,d|\mu|(y) \leq K_1(\Omega)\int_\Omega |x-y|^{2-N}\,d|\mu|(y) = K_1(\Omega)\left(|x-y|^{2-N} * |\mu|\right).
$$

The expression on the right hand-side is a convolution of function and a bounded measure on Ω. For $1 \leq p < \frac{N}{N-2}$ the function $|x-y|^{2-N}$ belongs to the space $L^p(\Omega)$ and using the properties of convolution, we observe $|x-y|^{2-N} * |\mu| \in L^p(\Omega)$ and further

$$
\left\| |x-y|^{2-N} * |\mu| \right\|_{L^p(\Omega)} \leq \left\| |\cdot - y|^{2-N} \right\|_{L^p(\Omega)} \|\mu\|_{\mathcal{M}(\Omega)} \leq K_1'(\Omega,p)\,\|\mu\|_{\mathcal{M}(\Omega)}\,.
$$

Defining $C_1 := K_1 K_1'$, we obtain

$$
\|u\|_{L^p(\Omega)} \leq C_1(\Omega,p)\,\|\mu\|_{\mathcal{M}(\Omega)}\,, \qquad 1 \leq p < \frac{N}{N-2}\,.
$$

3. *Estimate* (3.3), $N = 2$*:* The proof is similar to the one of the previous case $N \geq 3$. We choose an arbitrary $p \in [1,+\infty)$ and $\alpha := \alpha(p) > 0$ close to zero such that $|x-y|^{-\alpha} \in L^p(\Omega)$ (i.e. $\alpha p < 2$). The constant K_2 from the second inequality in (2.8) is dependent on $\alpha(p)$, we can therefore set $K_2 := K_2(p,\Omega)$. Finishing the proof as in the previous case gives (3.3).

4. *Estimate* (3.4), $N \geq 3$*:* We have shown, that the function u given by the kernel integral (3.1) satisfies (3.2). Now, if the function u possesses a weak derivative $D^i u$ in the direction x_i, it must satisfy

$$
\int_\Omega D^i u\,\psi(x)\,dx = -\int_\Omega u\,\frac{\partial}{\partial x_i}\psi(x)\,dx = -\int_\Omega\left[\int_\Omega G(x,y)\,d\mu(y)\right]\frac{\partial}{\partial x_i}\psi(x)\,dx \tag{3.6}
$$

for all $\psi \in \mathcal{D}(\Omega)$. The inner integral taken with the norm variation $|\mu|$ instead of μ can be estimated by a $L^1(\Omega)$ function. The derivative $\frac{\partial}{\partial x_i}\psi$ is a bounded function on $\overline{\Omega}$. The integral (3.6) with the variation norm $|\mu|$ satisfies the assumptions of Tonelli's theorem (Theorem 2.26) and is less than $+\infty$. Thus $G(x,y)\,\frac{\partial}{\partial x_i}\psi(x) \in L^1_{\lambda\otimes|\mu|}(\Omega\times\Omega)$ and therefore $G(x,y)\,\frac{\partial}{\partial x_i}\psi(x) \in L^1_{\lambda\otimes\mu}(\Omega\times\Omega)$. The assumptions of Fubini's theorem (Theorem 2.25) are satisfied and we can interchange the order of the integration and find a weak derivative of G

$$
\begin{aligned}
-\int_\Omega\left[\int_\Omega G(x,y)\,d\mu(y)\right]\frac{\partial}{\partial x_i}\psi(x)\,dx &= -\int_\Omega\left[\int_\Omega G(x,y)\,\frac{\partial}{\partial x_i}\psi(x)\,dx\right]d\mu(y) = \\
&= \int_\Omega\left[\int_\Omega D^i G(x,y)\,\psi(x)\,dx\right]d\mu(y)
\end{aligned}
$$

Let $y \in \Omega$ be fixed. Let H be a function defined by $H : t \mapsto G(x+tx_i, y)$ with $t \in \mathbb{R}$ such that $x + tx_i \in \Omega$. Then, the function H is continuously differentiable (and hence also absolutely continuous) on every line

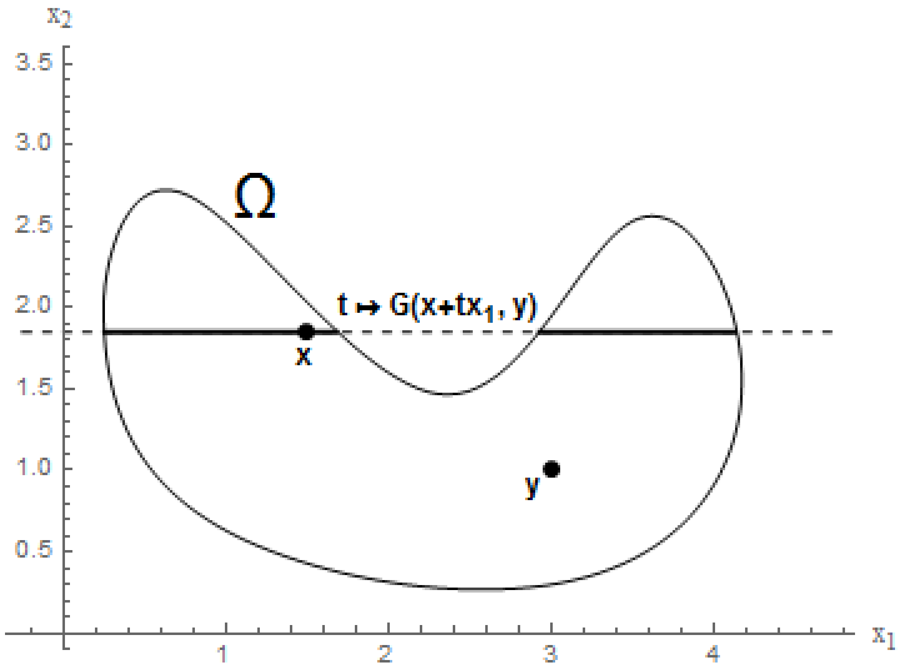


Figure 3.1: An example of the domain Ω for dimension $N = 2$. The points $x, y \in \Omega$ are fixed. Since the point y does not lie on the thick line segments of $x + tx_1$, the function $H : t \mapsto G(x + tx_i, y)$ is continuously differentiable on the both thick lines. For any given $x \in \Omega$, there is only a finite number of line segments (intersections of the line $x + tx_i$ with the domain Ω) as we assume the domain Ω is bounded with C^2 boundary. The differentiability of H is examined for each line segment separately.

parallel to the axis x_i with the exception of the line going through the point y, see Figure 3.1. Thus, G satisfies the assumptions of the first part of Theorem 2.34 and the classical and the weak derivatives are equal $D^i G = \frac{\partial}{\partial x_i} G$ a.e. in Ω. Next, we observe the inequalities $\left|\frac{\partial}{\partial x_i} G\right| \leq |\nabla_x G|$ (since the absolute value of the gradient is the sum of squared absolute values for the Euclidean norm) and $|\nabla G| \leq K_5 |x-y|^{1-N}$ (from the fifth inequality in (2.8)). Using the above stated estimates and the previous argumentation, we can use the Fubini's theorem again and proceed

$$\int_\Omega \left[\int_\Omega D^i G(x,y)\,\psi(x)\,dx\right] d\mu(y) = \int_\Omega \left[\int_\Omega \frac{\partial}{\partial x_i} G(x,y)\,\psi(x)\,dx\right] d\mu(y) =$$
$$= \int_\Omega \left[\int_\Omega \frac{\partial}{\partial x_i} G(x,y)\,d\mu(y)\right] \psi(x)\,dx \quad (3.7)$$

for all $\psi \in \mathcal{D}(\Omega)$. Getting back to (3.6), the weak derivative $D^i u \in \mathcal{D}'(\Omega)$ is generally a distribution but equating (3.6) and (3.7), we have

$$\int_\Omega D^i u\,\psi(x)\,dx = \int_\Omega \left[\int_\Omega \frac{\partial}{\partial x_i} G(x,y)\,d\mu(y)\right] \psi(x)\,dx$$

for all $\psi \in \mathcal{D}(\Omega)$. By the Du Bois-Raymond lemma (Lemma 2.43), $D^i u = \int_\Omega \nabla_{x_i} G(x,y)\,d\mu(y)$ a.e. in Ω. Applying the gradient estimates (2.8) for the Green's function and using same approach as in the part two of this proof, we get

$$\left|D^i u(x)\right| = \left|\int_\Omega \frac{\partial}{\partial x_i} G(x,y)\right| d\mu(y) \leq K(\Omega) \int_\Omega |x-y|^{1-N}\,d|\mu|(y)\,,$$

a.e. in Ω. Using the properties of convolution, we get

$$\left\|D^i u\right\|_{L^p(\Omega)} \leq C(\Omega, p)\,\|\mu\|_{\mathcal{M}(\Omega)}\,, \qquad 1 \leq p < \frac{N}{N-1}\,.$$

Since the direction x_i was chosen arbitrarily, this inequality holds for all $i = 1, 2, \ldots, N$ and all first weak derivatives are p integrable functions for $1 \leq p < N/(N-1)$, defining $C_3 := \max\{NC, C_1\}$, the estimate (3.4) holds.

5. *Estimate* (3.4), $N = 2$*:* The argument on the change of the integration and weak derivative holds for $N = 2$ as in the previous case. Again, using the sixth inequality in the estimate (2.8)

$$\left|D^i u(x)\right| = \left|\int_\Omega \frac{\partial}{\partial x_i} G(x,y)\right| d\mu(y) \leq K(\Omega,\alpha) \int_\Omega |x-y|^{-1-\alpha} d|\mu|(y)\,,$$

holds for all $\alpha > 0$. Note, that the function $|x-y|^{-1-\alpha}$ is p-integrable if $(1+\alpha)p < 2$ and therefore only for $p \in [1,2) = [1, N/(N-1))$ a relevant $\alpha := \alpha(p) > 0$ such that $|x-y|^{-1-\alpha} \in L^p(\Omega)$ can be found. Repeating the same argumentation as in the previous case and recalling, that the dependence of the constant $K(\Omega,\alpha)$ is actually dependence on Ω and p. Hence, the estimate (3.4) holds.

6. *The measure μ approximation and the convergence result:* For the sake of simplicity, we do not further specify the suitable range for p. It is sufficient to consider $p = 1+\epsilon$ ($\epsilon > 0$ being arbitrarily small) regardless of the dimension $N \geq 2$. Any distribution $g \in \mathcal{D}'(\Omega)$ can be approximated by a sequence of test functions $\{g_n\} \subset \mathcal{D}(\Omega)$ weakly relative to $\mathcal{D}(\Omega)$. Since $\mu \in \mathcal{M}(\Omega) \subset \mathcal{D}'(\Omega)$, even μ can be approximated in such manner by the sequence $\{f_n\} \subset \mathcal{D}(\Omega)$, such that

$$\int_\Omega f_n v\,dx \to \int_\Omega v\,d\mu\,,$$

for all $v \in \mathcal{D}(\Omega)$ as $n \to +\infty$. The space of test functions $\mathcal{D}(\Omega)$ is dense in $C_0(\overline{\Omega})$ with respect to the supreme norm and the limit $\mu \in \mathcal{M}(\Omega)$ belongs (by isomorphism) to the dual space of $C_0(\overline{\Omega})$, therefore

$$\int_\Omega f_n v\,dx \to \int_\Omega v\,d\mu\,, \tag{3.8}$$

holds for all $v \in C_0(\overline{\Omega})$ as $n \to +\infty$. Let u_n denote the solution of

$$\begin{cases} -\Delta u = f_n & \text{in} \quad \Omega\,, \\ \quad u = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{3.9}$$

The solution u_n can be expressed (see e.g. [11])

$$u_n(x) = \int_\Omega G(x,y)\,f_n(y)\,dy\,. \tag{3.10}$$

The real sequence $\int_\Omega f_n v\,dx$ is convergent for all $v \in C_0(\overline{\Omega})$ and is therefore bounded. The sequence $\{f_n\}$ can be interpreted as a sequence of continuous linear functionals on $C_0(\overline{\Omega})$ which is pointwise bounded and is by the uniform boundedness principle (Theorem 2.7) bounded in $\|\cdot\|_{\mathcal{M}(\Omega)}$ norm ($C_0(\overline{\Omega})$ equipped with the supreme norm is a Banach space). By the already proven estimate (3.4), the sequence $\{u_n\}$ is bounded in $\|\cdot\|_{W^{1,p}(\Omega)}$ norm. By the Rellich-Kondrachov theorem the space $W^{1,p}(\Omega)$ is compactly embedded in the space $L^p(\Omega)$ [1] and there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ weakly convergent in $W^{1,p}(\Omega)$ which is convergent in $\|\cdot\|_{L^p(\Omega)}$ norm. Since the classical solution of the equation (3.9) is also a very weak solution, the identity

$$-\int_\Omega u_{n_k}\,\Delta v\,dx = \int_\Omega f_n\,\phi\,dx$$

holds for all $\phi \in C_0^2(\overline{\Omega})$. Let $w \in L^p(\Omega)$ be a limit of the sequence $\{u_{n_k}\}$ in the norm $\|\cdot\|_{L^p(\Omega)}$, thus

$$\lim_{n\to+\infty} \int_\Omega |u_{n_k} - w|^p\,dx = 0\,.$$

The embedding of the spaces $L^p(\Omega) \hookrightarrow L^1(\Omega)$ is continuous and we can conclude

$$\lim_{n\to+\infty} \int_\Omega (u_{n_k} - w)\,\Delta\phi\;dx \;\leq\; K(\phi) \lim_{n\to+\infty} \int_\Omega |u_{n_k} - w|\;dx \;\leq\; K(\phi,p) \lim_{n\to+\infty} \int_\Omega |u_{n_k} - w|^p\;dx \;=\; 0\,,$$

for all $\phi \in C_0^2(\overline{\Omega})$. By the previous argumentation and the convergence of the sequence $\{f_n\}$ to the bounded Radon measure $\mu \in \mathcal{M}(\Omega)$, we have

$$-\int_\Omega w\,\Delta\phi\,dx = -\lim_{n\to+\infty} \int_\Omega u_{n_k}\,\Delta\phi\,dx = \lim_{n\to+\infty} \int_\Omega f_n\,\phi\,dx = \int_\Omega \phi\,d\mu$$

[1]This assertion follows from Remark in [8] on the page 274, after the proof of the Rellich-Kondrachov theorem.

for all $\phi \in C_0^2(\overline{\Omega}) \subset C_0(\overline{\Omega})$. Thus, w is a solution of (2.14). Furthermore, w is a unique solution by the first part of this proof. Therefore, the limit of the sequence $\{u_n\}$ does not depend on the choice of the subsequence $\{u_{n_k}\}$ [2].

7. *Equivalence of the limit* $w = \int_\Omega G(x,y)\, f_n(y)\, dy$ *and* (3.1)*:* Finally, we show, that $u = w$, i.e.

$$\int_\Omega G(x,y)\, d\mu(y) = \lim_{n\to+\infty} \int_\Omega G(x,y) f_n(y) dy\,. \tag{3.11}$$

Here, we assume that $\mu \geq 0$ and therefore the approximating sequence $\{f_n\}$ can be considered non-negative $f_n \geq 0$. Given parameter $\varepsilon > 0$, let $\psi_\varepsilon \in C^\infty(\mathbb{R}^N)$ be a function such that $0 \leq \psi_\varepsilon \leq 1$ with $\psi_\varepsilon = 0$ in $B_0(\varepsilon/2)$ and $\psi_\varepsilon = 1$ in $\mathbb{R}^N \setminus B_0(\varepsilon)$. Distinguishing the behaviour of the sequence of solutions $\{u_n\}$ strictly inside the domain Ω and near the boundary $\partial\Omega$ we can write

$$\begin{aligned} u_n(x) = \int_\Omega G(x,y)\, f_n(y)\, dy &= \\ &= \int_\Omega G(x,y)\, \psi_\varepsilon(|x-y|)\, f_n(y)\, dy + \int_\Omega G(x,y)\, (1-\psi_\varepsilon(|x-y|))\, f_n(y)\, dy\,. \end{aligned} \tag{3.12}$$

For any fixed $x \in \Omega$, the function $G(x,y)\, \psi_\varepsilon(|x-y|)$ belongs to the space $C_0(\overline{\Omega})$ (the singularity in G at the point $x = y$ is removed by the zero value of ψ_ε) and therefore

$$\int_\Omega G(x,y)\, \psi_\varepsilon(|x-y|)\, f_n(y)\, dy \to \int_\Omega G(x,y)\, \psi_\varepsilon(|x-y|)\, d\mu(y)\,, \tag{3.13}$$

holds for $n \to +\infty$ from the convergence of the sequence $\{f_n\}$, (3.8). Now, subtracting $w - u$ altogether with applying the identity (3.12) and limits (3.13), $u_n \to w$ results in

$$w(x) - u(x) = \lim_{n\to+\infty} \int_\Omega G(x,y)(1-\psi_\varepsilon(|x-y|)) f_n(y)\, dy - \int_\Omega G(x,y)(1-\psi_\varepsilon(|x-y|))\, d\mu(y)\,. \tag{3.14}$$

By (3.13), the convergence result (3.11) holds near the boundary $\partial\Omega$ for arbitrary $\varepsilon > 0$. We show, that $w = u$ a.e. on arbitrary compact subdomain of Ω by showing the right hand-side of (3.14) converging to zero for $\varepsilon \to 0$. For given compact subset $F \subset \Omega$ with $\varepsilon < \frac{1}{4}\rho(F, \partial\Omega)$ we define

$$F_\varepsilon = \left\{x \in \mathbb{R}^N \,:\, \rho(x,F) < \varepsilon\right\}\,.$$

Using this notation and the Fubini's theorem

$$\begin{aligned} \int_F \left[\int_\Omega G(x,y)(1-\psi_\varepsilon(|x-y|)) f_n(y)\, dy\right] dx &= \\ &= \int_\Omega \left[\int_F G(x,y)(1-\psi_\varepsilon(|x-y|)) f_n(y)\, dy\right] dx \leq \int_\Omega f_n(y)\, dy \sup_{y\in F_\varepsilon} \int_{B_y(\varepsilon)} G(x,y)\, dx\,, \end{aligned}$$

where the inequality

$$0 \leq 1 - \psi_\varepsilon(|x-y|) \leq 1\,,$$

was used. Since $f_n \in \mathcal{D}(\Omega)$ the norm is given by $\|f_n\|_{\mathcal{M}(\Omega)} = \int_\Omega f_n\, dx$ by the dual characterisation. The sequence $\{f_n\}$ can be viewed as sequence of continuous linear functionals on $C_0(\overline{\Omega})$. By (3.8), $\{f_n\}$ is point-wise bounded and by uniform boundedness principle, there exists $K > 0$ such that $\|f_n\|_{\mathcal{M}(\Omega)} \leq K$ holds for $n \in \mathbb{N}$. Thus

$$\begin{aligned} \limsup_{n\to+\infty} \int_F \left[\int_\Omega G(x,y)(1-\psi_\varepsilon(|x-y|)) f_n(y)\, dy\right] dx &\leq \\ &\leq \limsup_{n\to+\infty} \int_\Omega f_n(y)\, dy \sup_{y\in F_\varepsilon} \int_{B_y(\varepsilon)} G(x,y)\, dx \leq \\ &\leq K \sup_{y\in F_\varepsilon} \int_{B_y(\varepsilon)} G(x,y)\, dx\,. \end{aligned}$$

[2] Assume by contradiction, that there exists a subsequence $\{u_{n_l}\} \subset \{u_n\}$ such that $\|u_{n_l} - w\|_{L^p(\Omega)} > \varepsilon$ holds for each $n > n_0$ for some $\varepsilon > 0$ and $n_0 \in \mathbb{N}$. But there exists a subsequence $\left\{u_{n_{l_m}}\right\} \subset \{u_{n_l}\}$ which is convergent to w by the compact embedding of $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$. This is a contradiction with the assumption of a non-convergence of the sequence $\{u_{n_l}\}$.

The Green's function G is integrable and for $\varepsilon \to 0$, the area of integration vanishes and therefore the expression on the right hand-side tends to zero. Analogous to the previous case

$$\left|\int_F \left[\int_\Omega G(x,y)(1-\psi_\varepsilon(|x-y|))\ d\mu(y)\right]\ dx\right| =$$

$$= \left|\int_\Omega \left[\int_F G(x,y)(1-\psi_\varepsilon(|x-y|))\ dx\right]\ d\mu(y)\right| \le \left|\int_\Omega d\mu(y) \sup_{y\in F_\varepsilon} \int_{B_y(\varepsilon)} G(x,y)\ dx\right| \le$$

$$\le \left|\mu(\Omega) \sup_{y\in F_\varepsilon} \int_{B_y(\varepsilon)} G(x,y)\ dx\right|$$

and the last expression tends to zero as $\varepsilon \to 0$. Obtained estimates give altogether with the Fatou's lemma and the fact that the limit $\lim_{n\to+\infty} \int_\Omega G(x,y)(1-\psi_\varepsilon(|x-y|))f_n(y)\ dy$ exists

$$\begin{aligned}
0 &\le \left|\int_F w - u\ dx\right| = \\
&= \left|\int_F \left[\lim_{n\to+\infty} \int_\Omega G(x,y)(1-\psi_\varepsilon(|x-y|))f_n(y)\ dy - \int_\Omega G(x,y)(1-\psi_\varepsilon(|x-y|))\ d\mu(y)\right]\ dx\right|, \\
&\le \left|\int_F \left[\liminf_{n\to+\infty} \int_\Omega G(x,y)(1-\psi_\varepsilon(|x-y|))f_n(y)\ dy\right]\ dx\right| + \left|\int_F \left[\int_\Omega G(x,y)(1-\psi_\varepsilon(|x-y|))\ d\mu(y)\right]\ dx\right|, \\
&\le \left|\liminf_{n\to+\infty} \int_F \left[\int_\Omega G(x,y)(1-\psi_\varepsilon(|x-y|))f_n(y)\ dy\right]\ dx\right| - \left|\int_F \left[\int_\Omega G(x,y)(1-\psi_\varepsilon(|x-y|))\ d\mu(y)\right]\ dx\right|, \\
&\le \left|\limsup_{n\to+\infty} \int_F \left[\int_\Omega G(x,y)(1-\psi_\varepsilon(|x-y|))f_n(y)\ dy\right]\ dx\right| - \left|\int_F \left[\int_\Omega G(x,y)(1-\psi_\varepsilon(|x-y|))\ d\mu(y)\right]\ dx\right|.
\end{aligned}$$

The expression on the right hand-side tends to zero as $\varepsilon \to 0$ and therefore $u = v$ a.e. in arbitrary compact $F \subset\subset \Omega$. Thus, $u = v$ a.e. in Ω (by the Du Bois-Raymond lemma). If the Radon measure μ is not non-negative, we prove the convergence for the positive μ^+ and negative μ^- part respectively.

□

Chapter 4

Fredholm alternative

The main goal of this chapter is to prove the Fredholm alternative of the Laplace's operator in the sense of Definition 2.49 (the very weak sense). First, we define the solution operator $S' : \mathcal{M}(\Omega) \to W_0^{1,p}(\Omega)$ of the problem (2.14). Using the Rellich-Kondrachov Compactness theorem, we show that the image space of the operator S' is a subset of the Hilbert space $L^2(\Omega)$ and the operator $S : \mathcal{M}(\Omega) \to L^2(\Omega)$ is compact. Furthermore, the operator restriction $S : L^2(\Omega) \to L^2(\Omega)$ is self-adjoint and the Fredholm alternative for the operator S can be proved.

By Theorem 3.1, the linear operator $S' : \mathrm{dom}(S') = \mathcal{M}(\Omega) \to W_0^{1,p}(\Omega)$ for $p \in [1, N/(N-1))$ defined by

$$S' : \mu \mapsto \int_\Omega G(x,y)\, d\mu(y)\,. \tag{4.1}$$

is bounded. Moreover, it assigns the unique very weak solution of the problem (2.14) to any measure $\mu \in \mathcal{M}(\Omega)$. We start with a simple claim which will be implicitly used throughout the following chapters.

Lemma 4.1. *Let $u \in W_0^{1,p}(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$, then*

$$\begin{cases} -\Delta u = \mu & \text{in} \quad \Omega\,, \\ \quad u = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{4.2}$$

holds in the very weak sense if and only if

$$u = S\mu\,. \tag{4.3}$$

Proof. The implication from "right to left" is a consequence of Theorem 3.1. The reverse implication will be proved by a contradiction. Suppose, that (4.2) holds and $u \neq S\mu$. Then there exists $v := S\mu \in W_0^{1,p}(\Omega)$ such that $u \neq v$ on a set of positive (Lebesgue) measure. But, the equality $v = S\mu$ implies that v satisfies (4.2) in the very weak sense. By the uniqueness of the solution, $u = v$ holds a.e. in Ω which is a contradiction. □

Using Theorem 2.37 (the Rellich-Kondrachov compactness theorem) we will prove the following lemma.

Lemma 4.2. *The operator $S : \mathcal{M}(\Omega) \to L^2(\Omega)$ defined by the same formula as S', i.e. (4.1), is compact for $N = 2, 3$.*

Proof. The boundary $\partial\Omega$ of the domain Ω is of the class C^2 and the inequality $1 \leq p < N$ holds. Both conditions are in accordance with the assumptions of Theorem 2.37. Substituting the upper bound $N/(N-1)$ of the interval for $p \in [1, N/(N-1))$ into the formula of the critical exponent $p^* = Np/(N-p)$ in Theorem 2.37, the compact embedding

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$$

holds for $p < N/(N-1)$ and $q < N/(N-2)$. Therefore, the embedding for $q = 2$

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$$

holds only for $N \leq 3$. Thus, the operator S is compact since it is a composition of a continuous operator S' and a compact embedding operator. □

Remark 4.3. Since $L^2(\Omega) \hookrightarrow \mathcal{M}(\Omega)$, the restriction of the operator S to the space $L^2(\Omega)$ is well-defined. Furthermore, the restriction of S is in fact a composition of a continuous embedding operator from the space $L^2(\Omega)$ into $\mathcal{M}(\Omega)$ and a compact linear operator $S : \mathcal{M}(\Omega) \to L^2(\Omega)$. Therefore, $S : L^2(\Omega) \to L^2(\Omega)$ is a compact linear operator. For the sake of simplicity, we denote the restriction of the operator S to the space $L^2(\Omega)$ also by S since the linearity, the continuity and the compactness are preserved.

The proof of self-adjointness of the operator S consists of two parts. It must be shown, that S is a symmetric operator and the domains of S and S^* are equal, $\mathrm{dom}(S) = \mathrm{dom}(S^*)$.

Lemma 4.4. *The operator $S : L^2(\Omega) \to L^2(\Omega)$ is self-adjoint.*

Proof. Let $u, v \in L^2(\Omega)$. Then, by definition

$$(u, Sv) = \int_\Omega u(x) \left[\int_\Omega G(x,y)\, v(y)\, dy \right] dx = \int_\Omega \left[\int_\Omega u(x)\, v(y)\, G(x,y)\, dy \right] dx\,.$$

The term $u(x)\, v(y)\, G(x,y)$ can be decomposed to its positive and negative part and for both, the integral $\int_\Omega G(x,y)\, v(y)\, dy \in L^2(\Omega)$ (from the properties of S). The outer integral is just a scalar product of two $L^2(\Omega)$ functions. Therefore, using the Tonelli's theorem, the assumptions of Fubini's theorem are satisfied.Thus, we can interchange the order of the integration

$$(u, Sv) = \int_\Omega u(x) \left[\int_\Omega G(x,y)\, v(y)\, dy \right] dx = \int_\Omega \left[v(y) \int_\Omega G(x,y)\, u(x)\, dx \right] dy = (Su, v). \tag{4.4}$$

The operator S is symmetric. Next, we observe that $\mathrm{dom}(S) = L^2(\Omega)$ and for each $u \in L^2(\Omega)$ there exists $w \in L^2(\Omega)$ such that for all $v \in \mathrm{dom}(S) = L^2(\Omega)$ the equality $(u, Tv) = (w, v)$ holds. Such $w \in L^2(\Omega)$ is defined by $w := Su$. The operator S is symmetric and $\mathrm{dom}(S) = \mathrm{dom}(S^*) = L^2(\Omega)$. Thus, the operator S is self-adjoint. $\square$

Lemma 4.2 is useful in two ways. Obviously, it shows the compactness of the operator S, but it also shows that $\mathrm{ran}(S)$ is a linear subspace of $L^2(\Omega)$ under certain conditions. The latter will be used now.

Lemma 4.5. *Let Ω be a domain with $\partial\Omega \in C^2$, and $N \in \{2, 3\}$. Then the very weak solution v of the equation*

$$\begin{cases} -\Delta u = \lambda u & \text{in} \quad \Omega\,, \\ \quad u = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{4.5}$$

has a continuous representative in $C_0(\overline{\Omega})$.

Proof. This lemma will be proved by the so called "bootstrap argument". Lemma 4.2 implies, that the very weak solution of (4.5) belongs to space $L^2(\Omega)$. Therefore, we can assume, that the right hand-side λu is element of the space $L^2(\Omega)$. Using Theorem 2.52, the solution v is in addition an element of the space $W^{2,2}(\Omega)$. Every weak solution is also very weak solution and there exists at most one very weak solution. This justifies the use of Theorem 2.52 for very weak solutions. Now, we use Theorem 2.36 with parameters $k = 2$ (v is twice weakly differentiable), $p = 2$ (v and its derivatives are square integrable) and $N \in \{2, 3\}$ (dimensions for which S maps measures from $\mathcal{M}(\Omega)$ into the space $L^2(\Omega)$). The inequality $k > n/p$ holds in both cases $N = 2$ or $N = 3$ an the space $W^{2,2}(\Omega)$ is embedded into the space of Hölder continuous functions on $\overline{\Omega}$ and therefore it is embedded in the space of continuous functions $C_0(\overline{\Omega})$. Hence $v \in C_0(\overline{\Omega})$. $\square$

The previous lemma enables us to apply a measure $\mu \in \mathcal{M}(\Omega)$ as a continuous linear functional to the solution of (4.5). We can now prove the following claim connected to the symmetry of the operator S.

Lemma 4.6. *Given $\mu \in \mathcal{M}(\Omega)$ and $\phi \in C_0(\overline{\Omega})$, the equality $(S\mu, \phi) = \langle \mu, S\phi \rangle$ holds.*

Proof. By definition

$$(S\mu, \phi) = \int_\Omega \left[\int_\Omega G(x,y)\, d\mu(y) \right] \phi(x) dx = \int_\Omega \left[\int_\Omega G(x,y)\, \phi(x)\, d\mu(y) \right] dx\,.$$

Integrating by the measure variation $|\mu|$ (note that $|\mu| \in \mathcal{M}(\Omega)$) and considering the absolute value $|\phi|$ we see, that the whole integral is finite. Now, the order of the integration can be exchanged by Fubini's theorem obtaining

$$(S\mu, \phi) = \int_\Omega \left[\int_\Omega G(x,y)\, d\mu(y) \right] \phi(x) dx = \int_\Omega \left[\int_\Omega G(x,y)\, \phi(x)\, dx \right] d\mu(y) = \langle \mu, S\phi \rangle\ .$$

Moreover, $S\phi \in C_0(\overline{\Omega})$ by Lemma 4.5 and the duality $\langle \mu, S\phi \rangle$ is well-defined. □

The main result of this chapter is the Fredholm alternative for the operator S.

Theorem 4.7. *The equation*

$$\begin{cases} -\Delta u - \lambda u = \mu & in \quad \Omega \,, \\ \qquad\quad u = 0 & on \quad \partial\Omega \,, \end{cases} \tag{4.6}$$

has a uniquely defined solution $v \in L^2(\Omega)$ if and only if $\langle \mu, \phi \rangle = 0$ for all solutions of the equation (4.5)*.*

Proof. Every solution ϕ of (4.5) is a continuous function $\phi \in C_0(\overline{\Omega})$ by Lemma 4.5. Since ϕ is the solution of the homogeneous problem (4.5), then the equality $\phi = \lambda S\phi$ holds from Lemma 4.1. Applying Lemma 4.6 and the linearity of μ as a real bounded linear functional we obtain

$$\langle \mu, \phi \rangle = \langle \mu, \lambda S\phi \rangle = \lambda (S\mu, \phi)$$

and therefore $\langle \mu, \phi \rangle = 0$ if and only if $(S\mu, \phi) = 0$. It is now sufficient to prove the following claim.
The equation

$$u - S\lambda u = S \tag{4.7}$$

has a uniquely determined solution $v \in L^2(\Omega)$ if and only if $(S\mu, \phi) = 0$ holds for all solutions ϕ of the equation

$$u - S\lambda u = 0\,. \tag{4.8}$$

.

Trivially, if $\lambda = 0$, then the equation (4.6) has a uniquely determined solution $v \in L^2(\Omega)$ which is orthogonal to the unique trivial solution of the homogeneous equation (4.5); this is all the consequence of Theorem 3.1. Assuming $\lambda \neq 0$ the equations (4.7), (4.8) can be divided by λ. Now, both equations satisfy the assumptions of the Fredholm alternative with the compact self-adjoint operator $S : L^2(\Omega) \to L^2(\Omega)$. □

Remark 4.8. Since $\lambda = 0$ is regular value of the operator S, then all eigenfunctions $\{\phi_i\}_{n=1}^{+\infty}$ of the operator S form the orthonormal basis (after the normalisation) of the space $L^2(\Omega)$.

Chapter 5

Semi-linear equation with sub-linear nonlinearity – non-resonant case

In this chapter, we study the solvability of the problem

$$\begin{cases} -\Delta u - \lambda u = g(u) + \mu & \text{in} \quad \Omega\,, \\ u = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{5.1}$$

in the very weak sense where $\mu \in \mathcal{M}(\Omega)$. Let us assume that

1. $g : \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition;
2. g has a sub-linear growth, i.e. there exists constants $0 \le \alpha < 1$, and $b, c > 0$ such that
$$|g(t)| \le b + c|t|^{\alpha} \tag{5.2}$$
for all $t \in \mathbb{R}$.

We show, that the conditions stated above are sufficient for the solvability of the problem (5.1) provided the constant λ is not an eigenvalue of the Laplace's operator with the homogeneous Dirichlet conditions.

For the sake of consistency, we prove that the function g generates a correctly defined Nemytsky operator $G : u \mapsto g(u)$.

Lemma 5.1. *The operator $G : u \mapsto g(u)$ is a continuous mapping $G : L^2(\Omega) \to L^2(\Omega)$.*

Proof. The function g satisfies the Carathéodory condition (see Definition 2.31). The sub-linear growth (5.2) of g implies the existence of constants $b, c > 0$ such that $|g(t)| \le b + c|t|$ for all $t \in \mathbb{R}$ and therefore, the assumptions of Theorem 2.32 are satisfied (constant function b is trivially $b \in L^2(\Omega)$). The operator G maps the space $L^2(\Omega)$ to the space $L^2(\Omega)$. Moreover, G is a continuous mapping by Theorem 2.32. □

Since the equivalent operator equation of the problem (5.1) is well defined, the main result of this chapter can be proved.

Theorem 5.2. *Let $\lambda \in \mathbb{R}$ such that λ is not an eigenvalue of the Laplace's operator with a homogeneous Dirichlet condition be fixed. Let $g : \mathbb{R} \to \mathbb{R}$ satisfy the Carathéodory condition and let g possess a sub-linear growth* (5.2)*. Then the problem* (5.1) *possesses at least one very weak solution.*

Proof. By Lemma 4.1, it is necessary and sufficient to study the solvability of the operator equation

$$(I - \lambda S)u = Sg(u) + S\mu\,.$$

Lemma 4.7 implies, that the operator $(I - \lambda S)$ is invertible since the equation $I + \lambda S = o$ has only trivial solution (λ is not an eigenvalue). Therefore, we search for the fixed point of

$$u = (I - \lambda S)^{-1} S(g(u) + \mu)\,.$$

We define $T : u \mapsto (I + \lambda S)^{-1} S(g(u) + \mu)$. The operator T is composed of the continuous operator $G : L^2(\Omega) \to L^2(\Omega)$, the compact linear operator $S : \mathcal{M}(\Omega) \to L^2(\Omega)$ and the bounded linear operator

$(I+\lambda S)^{-1}: L^2(\Omega) \to L^2(\Omega)$. Therefore, the operator T is compact. Moreover,

$$0 \leq \frac{\|T(u)\|}{\|u\|} = \frac{\left\|(I-\lambda S)^{-1} S(g(u)+\mu)\right\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}} \leq \frac{\left\|(I-\lambda S)^{-1} S\right\|_{\mathcal{L}(\mathcal{M}(\Omega), L^2(\Omega))} \|g(u)+\mu\|_{\mathcal{M}(\Omega)}}{\|u\|_{L^2(\Omega}} \leq$$
$$\leq \left\|(I-\lambda S)^{-1} S\right\| \left[\frac{K \|g(u)\|_{L^2(\Omega)}}{\|u\|} + \frac{\|\mu\|_{\mathcal{M}(\Omega)}}{\|u\|}\right], \quad (5.3)$$

where we used the continuous embedding $L^2(\Omega) \hookrightarrow \mathcal{M}(\Omega)$. The sublinear growth of g (5.2) implies

$$\frac{\|g(u)\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}} \leq \frac{b}{\|u\|_{L^2(\Omega)}} + \frac{c\,\|u\|^\alpha_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}}$$

for some $0 \leq \alpha < 1$ and $c > 0$. Using this fact, we see, that $\|T(u)\|_{L^2(\Omega)} \to 0$ approaches to zero as $\|u\| \to +\infty$. Therefore, we can find $C > 0$ sufficiently large such that the mapping T maps the ball $B_o(C) \subset L^2(\Omega)$ into itself and there is no fixed point of T at the boundary $\partial B_o(C)$. The assumptions of the Shauder Fixed Point theorem are satisfied and there is at least one fixed point of T and hence the solution of (5.1). □

Chapter 6

Semi-linear equation with sub-linear nonlinearity – resonant case

Here, we are interested in the solvability of the problem

$$\begin{cases} -\Delta u - \lambda_i u = g(u) + \mu & \text{in} \quad \Omega\,, \\ \qquad\quad u = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{6.1}$$

where λ_i is any simple eigenvalue of the Laplace's operator with homogeneous Dirichlet conditions. Particularly, we are interested in the case where the function $g : \mathbb{R} \to \mathbb{R}$ is such that

1. g is continuous and bounded;
2. the limits

$$g(+\infty) := \liminf_{x\to+\infty} g(x), \quad g(-\infty) := \limsup_{x\to-\infty} g(x)$$

are finite and satisfy

$$g(-\infty) < g(+\infty)\,. \tag{6.2}$$

The example of such functions g are in the Figure 6.1. The functions satisfying the conditions stated above form the set denoted by $\mathcal{G}$[1]. To prove the solvability of the problem (5.1) we pose the conditions of Landesmann-Lazer type. We start with auxiliary lemmas which will be used in the proof of the main theorem of this chapter.

Let λ_i be a simple eigenvalue of the Laplace's operator with homogeneous Dirichlet conditions and let φ_i be a corresponding eigenfunction. The space of all functions $u \in L^2(\Omega)$ orthogonal to φ_i will be denoted by

$$L^2_\perp(\Omega) := \left\{ u \in L^2(\Omega) \,:\, (u, \varphi_i) = 0 \right\}$$

and similarly

$$\mathcal{M}_\perp(\Omega) := \{ \mu \in \mathcal{M}(\Omega) \,:\, \langle \mu, \varphi_i \rangle = 0 \}$$

be the set of all bounded real Radon measures which are "orthogonal" to φ_i in the sense of the duality $\langle \cdot\,,\,\cdot \rangle_{\mathcal{M}(\Omega),C(\overline{\Omega})}$. Naturally, $L^2_\perp(\Omega)$ and $\mathcal{M}_\perp(\Omega)$ are closed linear subspaces of the respective spaces $L^2(\Omega)$ and $\mathcal{M}(\Omega)$.

Lemma 6.1. *The spaces $L^2_\perp(\Omega)$ and $\mathcal{M}_\perp(\Omega)$ closed subspaces of $L^2(\Omega)$ and $\mathcal{M}(\Omega)$ respectively.*

Proof. Let $\{u_n\} \subset L^2(\Omega)$ be a sequence such that $u_n \to u \in L^2(\Omega)$ in the norm $\|\cdot\|_{L^2(\Omega)}$ and $(u_n, \varphi_i) = 0$ for all $n \in \mathbb{N}$. Then

$$0 \le |(u, \varphi_i)| = |(u, \varphi_i) - (u_n, \varphi_i)| = \left| \int_\Omega (u - u_n)\varphi_i \, dx \right| \le \|u - u_n\|_{L^2(\Omega)} \|\varphi_i\|_{L^2(\Omega)}\,.$$

Since $\|u - u_n\|_{L^2(\Omega)} \to 0$ for $n \to +\infty$, the identity $(u, \varphi_i) = 0$ holds.

[1] Note that every function $g \in \mathcal{G}$ satisfies the sufficient conditions for the non-resonant problem to be solvable (compare with Chapter 5)

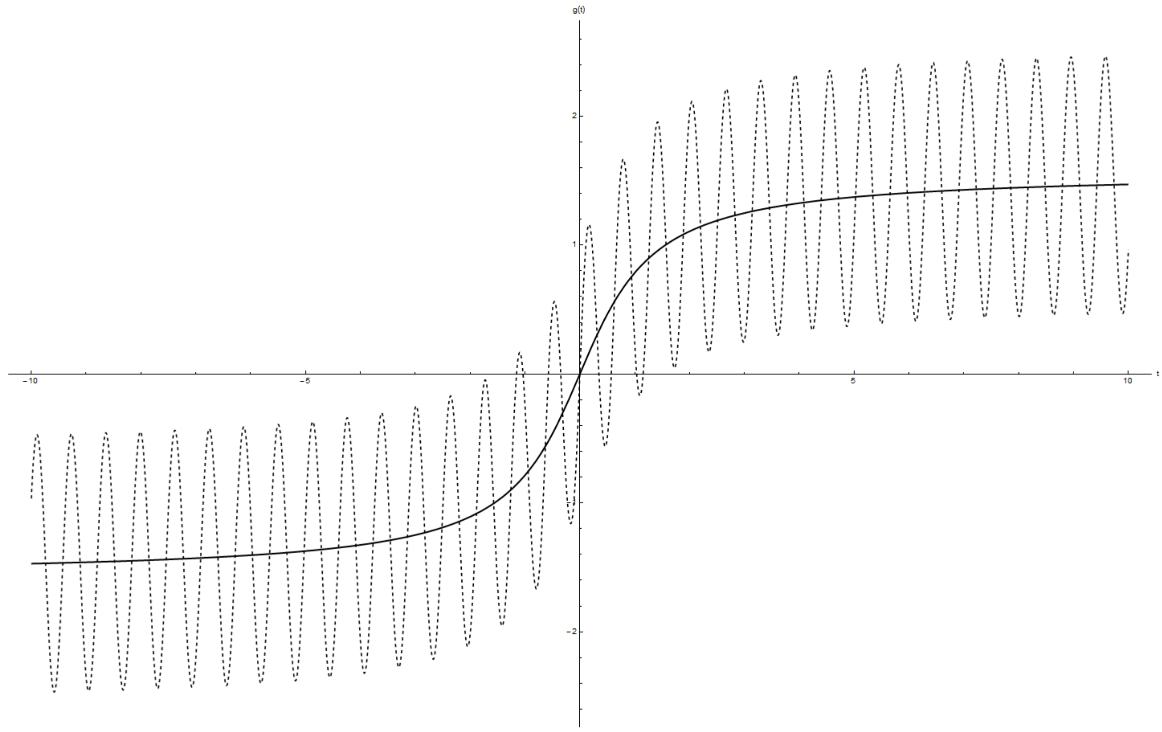


Figure 6.1: Examples of suitable functions $g \in \mathcal{G}$. Function arctangent (thick) possesses even finite limits at $\pm\infty$. The perturbed arctangent (dotted) possesses only finite limes inferior and superior at $+\infty$ and $-\infty$ respectively. Note, if any function $g \in \mathcal{G}$, then $g + \beta \in \mathcal{G}$ for any $\beta \in \mathbb{R}$.

Similarly, let $\{\mu_n\} \subset \mathcal{M}(\Omega)$ be a sequence such that $\mu_n \to \mu \in \mathcal{M}(\Omega)$ in the norm $\|\cdot\|_{\mathcal{M}(\Omega)}$ and $\langle \mu_n, \varphi_i \rangle = 0$ for all $n \in \mathbb{N}$. Then

$$0 \le |\langle \mu, \varphi_i \rangle| = |\langle \mu, \varphi_i \rangle - \langle \mu_n, \varphi_i \rangle| = |\langle \mu - \mu_n, \varphi_i \rangle| \le \|\mu - \mu_n\|_{\mathcal{M}(\Omega)} \|\varphi_i\|_{C_0(\overline{\Omega})} .$$

Since $\|\mu - \mu_n\|_{\mathcal{M}(\Omega)} \to 0$ for $n \to +\infty$, the identity $\langle \mu, \varphi_i \rangle = 0$ holds. □

Lemma 6.2. *The restriction of the operator S to the space $L^2_\perp(\Omega)$ denoted by $S_\perp$ is a compact linear operator $S_\perp : \mathcal{M}_\perp(\Omega) \to L^2_\perp(\Omega)$. Moreover, the operator $S_\perp : L^2_\perp(\Omega) \to L^2_\perp(\Omega)$ is self-adjoint.*

Proof. Let $\mu \in \mathcal{M}_\perp(\Omega)$, then by Lemma 4.6 and the fact that φ_i is the eigenfunction of S, the identity

$$0 = \langle \mu, \varphi_i \rangle = \lambda_i \langle \mu, S_\perp \varphi_i \rangle = \lambda_i (S_\perp \mu, \varphi_i) = 0$$

holds. Since, $\lambda_i \neq 0$, then $(S_\perp \mu, \varphi_i) = 0$ holds and therefore $S_\perp \mu \in L^2_\perp(\Omega)$. The compactness and self-adjointness of the operator $S_\perp$ arise from the same properties of the operator S.

The operator $S_\perp : L^2_\perp(\Omega) \to L^2_\perp(\Omega)$ is symmetric and $\mathrm{dom}(S_\perp) = L^2_\perp(\Omega)$. Moreover, $\mathrm{dom}(S^*_\perp) = L^2_\perp(\Omega)$ since the corresponding $w = Su \in L^2_\perp(\Omega)$. The operator $S_\perp : L^2_\perp(\Omega) \to L^2_\perp(\Omega)$ is self-adjoint. □

For the sake of simplicity, we denote the operator $S_\perp$ by S with special attention to fact, that in the resonance case, the operator $(I - \lambda_i S) : L^2(\Omega) \to L^2(\Omega)$ is not invertible but $(I - \lambda_i S_\perp) : L^2_\perp(\Omega) \to L^2_\perp(\Omega)$ already is.

Every $L^2(\Omega)$ function defines a Radon measure and therefore a bounded linear functional on the space $C_0(\overline{\Omega})$ by integration. Therefore, for given function $\varphi \in C_0(\overline{\Omega})$ such that $\|\varphi\|_{L^2(\Omega)} = 1$ we define a projection operator $P : \mathcal{M}(\Omega) \to \mathrm{span}\{\varphi\}$ by

$$P\mu = \langle \mu, \varphi \rangle \varphi$$

and its complement $P^C : \mathcal{M}(\Omega) \to \mathcal{M}_\perp(\Omega)$

$$P^C \mu = (I - P)\mu = \mu - \langle \mu, \varphi \rangle \varphi .$$

For the sake of clarity, we prove the following lemma.

Lemma 6.3. *Given $\varphi \in C_0(\overline{\Omega})$ such that $\|\varphi\|_{L^2(\Omega)} = 1$, the operator P^C maps the space $\mathcal{M}(\Omega)$ to the subspace $\mathcal{M}_\perp(\Omega)$.*

Proof. Let $\mu \in \mathcal{M}(\Omega)$. Then,

$$\left\langle P^C\mu, \varphi \right\rangle = \langle \mu - \langle \mu, \varphi\rangle \varphi, \varphi\rangle = \langle \mu, \varphi\rangle - \langle \mu, \varphi\rangle \langle \varphi, \varphi\rangle = \langle \mu, \varphi\rangle - \langle \mu, \varphi\rangle \int_\Omega \varphi^2 \, dx = \langle \mu, \varphi\rangle - \langle \mu, \varphi\rangle = 0 .$$

Therefore, $P^C\mu \in \mathcal{M}_\perp(\Omega)$ for all $\mu \in \mathcal{M}(\Omega)$. □

Theorem 6.4. *Let $g \in \mathcal{G}$, let λ_i be a simple eigenvalue of the Laplace's operator with homogeneous Dirichlet conditions. Then the problem (6.1) possesses a very weak solution for each $\mu \in \mathcal{M}(\Omega)$ provided*

$$g(-\infty) \int_\Omega \varphi_i^+ \, dx - g(+\infty) \int_\Omega \varphi_i^- \, dx < \langle \mu, \varphi_i\rangle < g(+\infty) \int_\Omega \varphi_i^+ \, dx - g(-\infty) \int_\Omega \varphi_i^- \, dx \tag{6.3}$$

holds, where φ_i is the corresponding eigenfunction of λ_i.

Proof. Again, it is sufficient and necessary to study the solvability of the operator equation

$$(I - \lambda_i S)u = Sg(u) + S\mu . \tag{6.4}$$

The operator $(I + \lambda_i S) : L^2(\Omega) \to L^2(\Omega)$ is not invertible in general. But, by the Fredholm alternative, the operator $(I - \lambda_i S_\perp) : L^2_\perp(\Omega) \to L^2_\perp(\Omega)$ is invertible. We decompose (6.4) using the projection operators P and P^C projecting to the space spanned by φ_i and its orthogonal complement respectively obtaining the equivalent system of Lyapunov-Schmidt equations

$$\begin{aligned} P^C(I - \lambda_i S)u &= P^C Sg(u) + P^C S\mu , \\ P(I - \lambda_i S)u &= PSg(u) + PS\mu , \end{aligned}$$

Since the operator S maps $S : \operatorname{span}\{\varphi_i\} \to \operatorname{span}\{\varphi_i\}$ and $S : \mathcal{M}_\perp(\Omega) \to \mathcal{M}_\perp(\Omega)$, the projection operators P, P^C and the solution operator S are commutative. Therefore

$$\begin{aligned} (I - \lambda_i S)P^C u &= SP^C g(u) + SP^C \mu , \\ (I - \lambda_i S)Pu &= SPg(u) + SP\mu . \end{aligned}$$

Denoting $u = Pu + P^C u := c\varphi_i + u_\perp$ where $u_\perp \in L^2_\perp(\Omega)$, we rewrite

$$\begin{aligned} (I - \lambda_i S)u_\perp &= SP^C g(c\varphi_i + u_\perp) + SP^C\mu , \\ 0 &= SP(g(c\varphi_i + u_\perp) + \mu) . \end{aligned} \tag{6.5}$$

The right hand-side of the first equation in (6.5) belongs to the space $\mathcal{M}_\perp(\Omega)$ and therefore, there exists a bounded linear inverse of $(I - \lambda_i S)$. The operator P maps to a one-dimensional space. Thus,

$$\begin{aligned} o &= u_\perp - (I - \lambda_i S)^{-1} SP^C \left[g(c\varphi_i + u_\perp) + \mu\right] , \\ 0 &= \langle g(c\varphi_i + u_\perp) + \mu, \varphi_i\rangle . \end{aligned} \tag{6.6}$$

The equations in (6.6) can be rewritten

$$\begin{aligned} o &= u_\perp - F(c, u_\perp) , \\ 0 &= H(c, u_\perp) , \end{aligned} \tag{6.7}$$

where $F(c, u_\perp) : (c, u_\perp) \mapsto (I - \lambda_i S)^{-1} SP^C \left[g(c\varphi_i + u_\perp) + \mu\right]$ is a compact operator (see Proof of Theorem 5.2) and $H : (c, u_\perp) \mapsto \langle g(c\varphi_i + u_\perp) + \mu, \varphi_i\rangle$ is a continuous operator (see Lemma 5.1). By boundedness of g, there exist $C_1, C_2 > 0$ such that

$$\begin{aligned} \|F(c, u_\perp)\|_{L^2(\Omega)} &< C_1 , \\ |H(c, u_\perp)| &< C_2 \end{aligned} \tag{6.8}$$

for all $c \in \mathbb{R}, u_\perp \in L^2_\perp(\Omega)$. Let $c \in \mathbb{R}$ be fixed. We define a homotopy operator

$$F_\tau(c, u_\perp) := \tau F(c, u_\perp) \quad \text{for} \quad \tau \in [0, 1] .$$

Naturally, $F_0 = 0$ and $F_1 = F$. By (6.8), there is no zero of $I - F_\tau(c,\cdot)$ (or equivalently fixed point of $F_\tau(c,\cdot)$) on the boundary of the ball $B_o(C_1) \subset L^2_\perp(\Omega)$. Therefore, the Leray-Schauder degree of $I(\cdot) - F(c,\cdot)$ is defined and

$$\deg(I(\cdot) - F(c,\cdot), B_o(C_1), o) = \deg(I(\cdot), B_o(C_1), o) = 1$$

holds. Thus, there exists a zero of $I - F(c,\cdot)$ (denoted by $\Sigma_c \subset L^2_\perp(\Omega)$) for each $c \in \mathbb{R}$. Since the degree is constant, the set $\bigcup_{c\in\mathbb{R}} \Sigma_c$ is connected with respect to the parameter $c \in \mathbb{R}$ (particularly, it is connected on each interval $[c_1, c_2] \subset \mathbb{R}$ by Theorem (2.17))

Now, the second equation in (6.6) can be rewritten

$$0 = \int_\Omega g(c\varphi_i + u_\perp)\,\varphi_i\,dx + \langle\mu,\varphi_i\rangle = \int_\Omega g(c\varphi_i^+ + u_\perp)\,\varphi_i^+\,dx - \int_\Omega g(-c\varphi_i^- + u_\perp)\,\varphi_i^-\,dx + \langle\mu,\varphi_i\rangle\,. \tag{6.9}$$

From now on, we will freely use the super- and sub-additivity of the limes inferior and limes superior respectively and the equality $\liminf_{n\to+\infty} -x_n = -\limsup_{n\to+\infty} x_n$. Considering $c \to +\infty$, we use the Fatou's lemma[2]

$$\begin{aligned}
&\liminf_{c\to+\infty}\left[\int_\Omega g(c\varphi_i^+ + u_\perp)\,\varphi_i^+\,dx - \int_\Omega g(-c\varphi_i^- + u_\perp)\,\varphi_i^-\,dx\right] \geq \\
&\qquad \geq \liminf_{c\to+\infty}\int_\Omega g(c\varphi_i^+ + u_\perp)\,\varphi_i^+\,dx + \liminf_{c\to+\infty}\int_\Omega -g(-c\varphi_i^- + u_\perp)\,\varphi_i^-\,dx \geq \\
&\qquad \geq \int_\Omega \liminf_{c\to+\infty} g(c\varphi_i^+ + u_\perp)\,\varphi_i^+\,dx + \int_\Omega \liminf_{c\to+\infty} -g(-c\varphi_i^- + u_\perp)\,\varphi_i^-\,dx \geq \\
&\qquad\qquad \geq g(+\infty)\int_\Omega \varphi_i^+\,dx - g(-\infty)\int_\Omega \varphi_i^-\,dx\,.
\end{aligned}$$

If there exists the limit as $c \to +\infty$ of the expression on the right hand-side of (6.9), it is equal to the limes inferior. Therefore,

$$\begin{aligned}
&\lim_{c\to+\infty}\int_\Omega g(c\varphi_i + u_\perp)\,\varphi_i\,dx + \langle\mu,\varphi_i\rangle = \liminf_{c\to+\infty}\int_\Omega g(c\varphi_i + u_\perp)\,\varphi_i\,dx + \langle\mu,\varphi_i\rangle \geq \\
&\qquad \geq g(+\infty)\int_\Omega \varphi^+\,dx - g(-\infty)\int_\Omega \varphi^-\,dx + \langle\mu,\varphi_i\rangle > 0
\end{aligned}$$

by the first inequality in (6.3). If the limit as $c \to +\infty$ of the expression on the right hand-side of (6.9) does not exist, then there must exist $c_+ > 0$ great enough such that

$$\int_\Omega g(c_+\varphi_i + u_\perp)\,\varphi_i\,dx + \langle\mu,\varphi_i\rangle > 0$$

holds. Considering $c \to -\infty$ we use the Fatou's lemma again

$$\begin{aligned}
&\limsup_{c\to-\infty}\left[\int_\Omega g(c\varphi_i^+ + u_\perp)\,\varphi_i^+\,dx - \int_\Omega g(-c\varphi_i^- + u_\perp)\,\varphi_i^-\,dx\right] \leq \\
&\qquad \leq \limsup_{c\to-\infty}\int_\Omega g(c\varphi_i^+ + u_\perp)\,\varphi_i^+\,dx + \limsup_{c\to-\infty}\int_\Omega -g(-c\varphi_i^- + u_\perp)\,\varphi_i^-\,dx \leq \\
&\qquad \leq -\liminf_{c\to-\infty}\int_\Omega -g(c\varphi_i^+ + u_\perp)\,\varphi_i^+\,dx - \liminf_{c\to-\infty}\int_\Omega g(-c\varphi_i^- + u_\perp)\,\varphi_i^-\,dx \leq \\
&\qquad \leq -\int_\Omega \liminf_{c\to-\infty} -g(c\varphi_i^+ + u_\perp)\,\varphi_i^+\,dx - \int_\Omega \liminf_{c\to-\infty} g(-c\varphi_i^- + u_\perp)\,\varphi_i^-\,dx \leq \\
&\qquad\qquad \leq g(-\infty)\int_\Omega \varphi_i^+\,dx - g(+\infty)\int_\Omega \varphi_i^-\,dx\,.
\end{aligned}$$

If there exists the limit as $c \to -\infty$ of the expression on the right hand-side of (6.9), it is equal to the limes superior. Therefore,

$$\begin{aligned}
&\lim_{c\to-\infty}\int_\Omega g(c\varphi_i + u_\perp)\,\varphi_i\,dx + \langle\mu,\varphi_i\rangle = \limsup_{c\to+\infty}\int_\Omega g(c\varphi_i + u_\perp)\,\varphi_i\,dx + \langle\mu,\varphi_i\rangle \leq \\
&\qquad \leq g(-\infty)\int_\Omega \varphi^+\,dx - g(+\infty)\int_\Omega \varphi^-\,dx + \langle\mu,\varphi_i\rangle < 0
\end{aligned}$$

[2] The Fatou's lemma as is can be used only on non-negative function sequences. By boundeness of g, we can find an integrable minorant $-h < 0$ of the integrand and use the Fatou's lemma on the sum of integrand and h which is non-negative.

by the second inequality in (6.3). If the limit as $c \to -\infty$ of the expression on the right hand-side of (6.9) does not exist, then there must exist $c_- < 0$ small enough such that

$$\int_\Omega g(c_- \varphi_i + u_\perp)\, \varphi_i \, dx + \langle \mu, \varphi_i \rangle < 0$$

holds.

Therefore, we are able to find $c_-, c_+ \in \mathbb{R}$ such that

1. trivially by (6.8), $\Sigma_c \subset B_o(C_1)$ (equivalently: the zeros of $I - F(c, \cdot)$ belong to $B_o(C_1)$) for all $c \in [c_-, c_+] =: \mathcal{I}$;
2. $H(c_-, u_\perp) < 0 < H(c_+, u_\perp)$ for each $c \in \mathcal{I}$ and $u_\perp \in B_o(C_1)$.

Since I is a connected interval and $\Sigma' := \bigcup_{c\in\mathcal{I}} \Sigma_c$ is a connected subset of $L^2_\perp(\Omega)$ and H is a continuous mapping $H : \mathbb{R} \times L^2_\perp(\Omega) \mapsto \mathbb{R}$, the set $H(\mathcal{I}, \Sigma')$ is also connected (see e.g. [1, Example 3]). Therefore, there exists $c_0 \in \mathcal{I}$ such that $H(c_0, u_\perp) = 0$ holds for some $u_\perp \in \Sigma'$. Thus, there exists a solution of (6.6) and hence of (6.1).

$\square$

Remark 6.5. Theorem 6.4 can be also proved in the same manner with the inequalities (6.3) and (6.2) reversed.

Chapter 7

Example – Heat Generation

Heat generation by exothermic reaction driven by the Arrhenius reaction term with the preexponential factor given by Transition state theory (see e.g. [15]) is modelled by semi-linear PDE with non-linear term given by

$$k(u) := c_1 u \exp\left(-\frac{c_2}{u}\right) \quad \text{for } u > 0\,,$$

where $c_1, c_2 > 0$ are positive parameters. The function u represents the thermodynamic temperature in this model. In our analytical treatment, we define $k(0) := 0$ and consider an odd extension of the function k by inserting an absolute value $|u|$. Furthermore, we expand

$$k(u) = c_1 u \exp\left(-\frac{c_2}{|u|}\right) = c_1 u + c_1 u\left(\exp\left(-\frac{c_2}{|u|}\right) - 1\right)\,. \tag{7.1}$$

The heat generation is then described by the PDE

$$\begin{cases} -\Delta u = \lambda k(u) + \mu & \text{in} \quad \Omega\,, \\ \quad u = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{7.2}$$

Plugging in the expansion (7.1) of the function k, the problem (7.2) has the form

$$\begin{cases} -\Delta u - \lambda c_1 u = \lambda c_1 u \left(\exp\left(-\frac{c_2}{|u|}\right) - 1\right) + \mu & \text{in} \quad \Omega\,, \\ \qquad u = 0 & \text{on} \quad \partial\Omega\,. \end{cases}$$

Setting

$$\lambda c_1 := \lambda_i$$

where λ_i is a simple eigenvalue of the Laplace's operator with homogeneous Dirichlet conditions and

$$g(u) := \lambda_i u\left(\exp\left(-\frac{c_2}{|u|}\right) - 1\right)\,,$$

The problem (7.2) can be now rewritten as

$$\begin{cases} -\Delta u - \lambda_i u = g(u) + \mu & \text{in} \quad \Omega\,, \\ \qquad u = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{7.3}$$

The limits of the non-linear function $g : \mathbb{R} \to \mathbb{R}$ satisfy

$$g(+\infty) := \liminf_{t\to+\infty} g(t) = \lim_{t\to+\infty} g(t) = -\lambda c_1 c_2, \quad g(-\infty) := \limsup_{t\to-\infty} g(t) = \lim_{t\to-\infty} g(t) = \lambda c_1 c_2$$

The problem (7.3) posses a very weak solution by Theorem 6.4 provided (6.3) holds. In the application of the results, we must consider only positive solution u, since u represents the thermodynamic temperature.

We remark, that for example heating of the substance at one single point by laser can be represented by setting $\mu := \delta_{x_0}$ being the Dirac measure concentrated at point $x_0 \in \Omega$ (see e.g. [2]).

Chapter 8

Conclusion

In this thesis, we focused mainly on the solvability of the elliptic PDE (1.4) in the very weak sense. Our main results are the Fredholm alternative for the Laplace's operator with the homogeneous Dirichlet conditions, the proof of the solvability of (1.4) out of resonance and at resonance. The latter led to the sufficiency of the Landesman-Lazer type of conditions to be posed on the measure $\mu \in \mathcal{M}(\Omega)$.

The example in Chapter 7 shows, that the problems considered in this thesis can not be considered artificial. The functions $g \in \mathcal{G}$ are used in modelling of chemical reaction kinetics (see, e.g., [15] and other books focused on kinetics of chemical reactions).

Using the results from Chapters 4–6 the bifurcation of the system (1.4) can be studied in a similar manner as in [13]. The compactness results and the estimates of the integrals from Chapter 6 are the key ingredients to prove analogous results to those in [13].

The appendix contains a possible approximation approach by which the solution of the simpler form of (1.4) can be approximated by the series of the Bessel functions. Since the topic of this thesis was very time demanding, we were lacking time to prove the convergence of the series. However, we still considered the appendix suitable to include because of the lightness of the underlying idea. A convenient study material for proving the convergence of the Fourier series of the Bessel functions is [22].

Appendix A

Elliptic PDE involving measures on a unit disk

A.1 Homogeneous Helmholtz equation

In this section, we find a radially symmetric solution of the homogeneous Helmholtz equation. The equation with the Dirichlet boundary conditions is formulated

$$\begin{cases} -\Delta u - \lambda u = 0 \quad \text{in} \quad \Omega\,, \\ \qquad\quad u = 0 \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{A.1}$$

where we consider $\Omega = B_0(1)$. This equation can be solved by the separation of variables. First, consider the equation (A.1) in polar coordinates

$$\begin{cases} -u_{rr}(r,\theta) - \frac{1}{r}u_r(r,\theta) - \frac{1}{r^2}u_{\theta\theta}(r,\theta) - \lambda u(r,\theta) &= 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad u(1,\theta) &= 0, \end{cases} \tag{A.2}$$

with $r \in [0,1]$ be the radius and $\theta \in [0, 2\pi)$ be the angle. Now, we suppose the solution of (A.2) is of the form $u(r,\theta) = R(r)\Theta(\theta)$ with derivatives

$$\begin{aligned} u_r(r,\theta) &= R'(r)\Theta(\theta)\,, \\ u_{rr}(r,\theta) &= R''(r)\Theta(\theta)\,, \\ u_{\theta\theta}(r,\theta) &= R(r)\Theta''(\theta)\,. \end{aligned}$$

and conditions on R

$$R'(0) = 0, \quad R(1) = 0\,,$$

and the periodic boundary condition for Θ

$$\Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi)\,.$$

Assuming the non-trivial solution, we multiply the first equation in (A.2) by $r^2/(R\Theta)$ and collect the terms with R and Θ obtaining

$$\frac{1}{R(r)}\left[-r^2R''(r) - rR'(r) - \lambda r^2 R(r)\right] = \frac{\Theta''(\theta)}{\Theta(\theta)}\,. \tag{A.3}$$

Expressions on the left- and right-hand side can equal only if they are both equal to some constant m. Considering only the expression containing Θ we have the ODE

$$\Theta''(\theta) = m\Theta(\theta)\,,$$

which has only non-trivial solutions for $\sqrt{m} \in \mathbb{N}_0$ of the form

$$\Theta(\theta) = c_1 \cos\sqrt{m}\theta + c_2 \sin\sqrt{m}\theta,$$

but the radially symmetrical solution can be achieved only with Θ being constant function and therefore $m = 0$. This gives us the equation derived from (A.3) after dividing by r

$$\begin{cases} -r^2R''(r) - rR'(r) - \lambda r^2 R(r) = 0\,, \\ \qquad\qquad R'(0) = 0, \quad R(1) = 0\,. \end{cases} \tag{A.4}$$

A sufficiently smooth function R can be expressed by the Maclauran series

$$R(r) = \sum_{k=0}^{+\infty} a_k r^k \,,$$
$$R'(r) = \sum_{k=0}^{+\infty} a_k k r^{k-1} \,,$$
$$R''(r) = \sum_{k=0}^{+\infty} a_k k(k-1) r^{k-2} \,.$$

inserting into the first equation of (A.4)

$$\sum_{k=0}^{+\infty} a_k k(k-1) r^k + \sum_{k=0}^{+\infty} a_k k r^k + \lambda \sum_{k=2}^{+\infty} a_{k-2} r^k = 0 \,.$$

We first handle odd exponents of r by induction. Comparison of the coefficients of r^1 yields simply $a_1 = 0$. Now, let us take arbitrary odd k, suppose $a_{k-2} = 0$ and compare the coefficients of r^k which yields

$$k(k-1)a_k + k a_k + \lambda a_{k-2} = 0 \,, \tag{A.5}$$
$$k^2 a_k = 0 \,, \tag{A.6}$$

and therefore $a_k = 0$. By induction, $a_k = 0$ for arbitrary odd k. Taking $k = 0$, we can see that a_0 can be an arbitrary real number $a_0 \in \mathbb{R}$. The higher even indices k can be expressed as follows

$$\begin{array}{lrcll}
\mathbf{k=2}: & 2a_2 + 2a_2 + \lambda a_0 = 0 & \rightarrow & a_2 = -\frac{\lambda a_0}{4} \,, & \\
\mathbf{k=4}: & 12a_4 + 4a_4 + \lambda a_2 = 0 & \rightarrow & a_4 = -\frac{\lambda a_2}{16} & = \frac{\lambda^2 a_0}{2^2 4^2} \,, \\
\mathbf{k=6}: & 30a_6 + 6a_6 + \lambda a_4 = 0 & \rightarrow & a_6 = \frac{\lambda a_4}{36} & = -\frac{\lambda^3 a_0}{2^2 4^2 6^2} \,, \\
\vdots & & \vdots & & \\
\mathbf{k=2n}: & 2n(2n-1)a_{2n} + 2n a_{2n} + \lambda a_{2n-2} = 0 & \rightarrow & a_2 n = -\frac{\lambda a_{2n-2}}{(2n)^2} & = (-1)^n \frac{\lambda^n a_0}{2^2 4^2 6^2 \ldots (2n)^2} \,.
\end{array}$$

Therefore, the power series expansion of R yields

$$R(r) = \sum_{k=0}^{+\infty} a_k r^k = a_0 \sum_{k=0}^{+\infty} (-1)^k \frac{\lambda^k r^{2k}}{2^2 4^2 6^2 \ldots (2k)^2} = a_0 \sum_{k=0}^{+\infty} (-1)^k \frac{\left(\sqrt{\lambda} r\right)^{2k}}{2^2 4^2 6^2 \ldots (2k)^2} \,,$$

which is the Bessel function of the first kind of the order zero (see, e.g., [5])

$$R(r) = a_0 J_0(\sqrt{\lambda} r) \,.$$

This expression of the solution satisfies the initial condition on derivative implicitely. In order to satisfy the condition $R(1) = 0$ we must take $\sqrt{\lambda}$ to be a root of the function J_0. This function has a countable number of roots λ_n, so we have countably many nontrivial solutions of (A.4) denoted by

$$\varphi_n(r) = J_0(\sqrt{\lambda_n} r) \,.$$

Please note, that the solutions φ_n are radially symmetric and defined on $B_0^2(1)$. Solutions of the homogeneous Helmholtz equation are depicted in Figure A.1

A.2 Non-homogeneous Helmholtz Equation with Dirac measure

Now, we consider a non-homogeneous Helmholtz equation containing Dirac Measure centred at origin $\delta_0(x)$ as a source term

$$\begin{cases} -\Delta u - \lambda_n u = \delta_0 - c_n \varphi_n & \text{in} \quad B_0(1), \\ u(x) = 0 & \text{on} \quad \partial B_0(1), \end{cases} \tag{A.7}$$

where c_n is suitably chosen so that $\delta_0 - c_n \varphi_n$ is orthogonal to φ_n and λ_n is squared n-th root of J_0. The solution of (A.7) u and Dirac measure δ_0 can be expressed in the Bessel series form as

$$u(r) = \sum_{k=1}^{+\infty} b_k \varphi_k(r), \quad \delta_0(r) = \sum_{k=1}^{+\infty} c_k \varphi_k(r) \,.$$

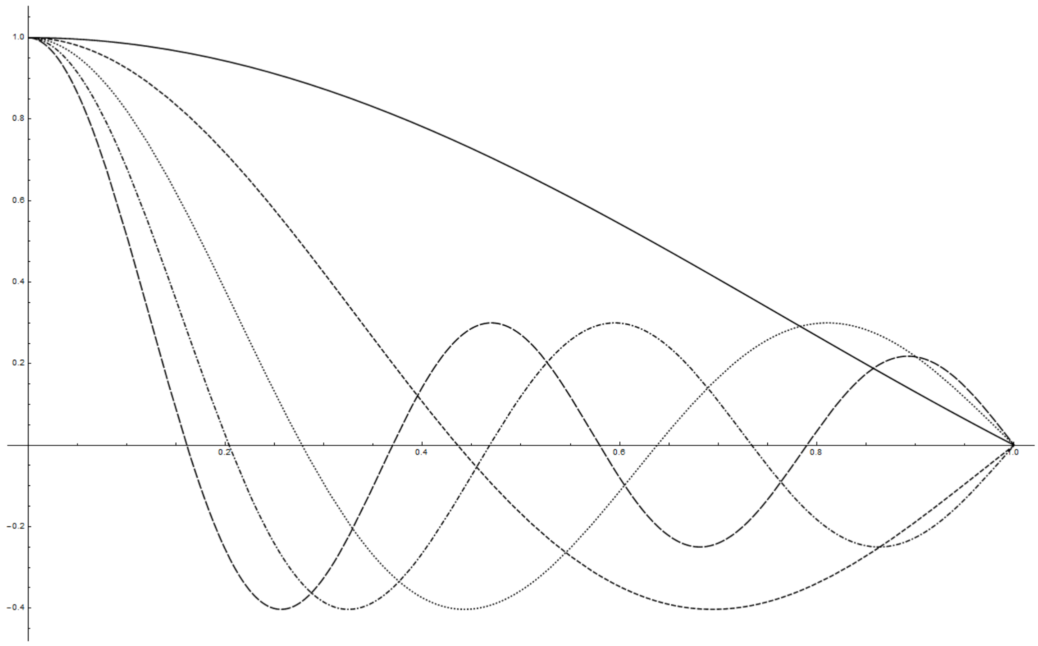


Figure A.1: Five radially symmetric solutions of homogeneous Helmholtz equation. Precisely, slices of the solutions for $r \in [0,1]$.

Knowing the functions φ_n are the eigenfunctions of the Laplace operator Δ on the unit ball $B_0^2(1)$ (i.e. $\Delta\varphi_k = -\lambda_k\varphi_k$) we can reformulate (A.7)

$$-\Delta\left[\sum_{k=1}^{+\infty} b_k\varphi_k(r)\right] - \lambda_n \sum_{k=1}^{+\infty} b_k\varphi_k(r) = \sum_{\substack{k=1\\k\neq n}}^{+\infty} \frac{1}{\pi J_1^2\left(\sqrt{\lambda_k}\right)}\varphi_k(r)\,,$$

$$\sum_{k=1}^{+\infty} b_k\lambda_k\varphi_k(r) - \lambda_n \sum_{k=1}^{+\infty} b_k\varphi_k(r) = \sum_{\substack{k=1\\k\neq n}}^{+\infty} \frac{1}{\pi J_1^2\left(\sqrt{\lambda_k}\right)}\varphi_k(r)\,.$$

Equating the coefficients, the expression

$$\lambda_k b_k - \lambda_n b_k = c_k\,,$$

must hold for every $k \in \mathbb{N}$, $k \neq n$. Thus

$$b_k = \begin{cases} c_k/\left(\lambda_k - \lambda_n\right) & k \neq n\,,\\ 0 & k = n\,. \end{cases}$$

The coefficients c_k are defined in the terms of the inner product of the space $L^2 := L^2(B_0(1))$

$$c_k = \frac{(\varphi_k, \delta_0)_{L^2}}{(\varphi_k, \varphi_k)_{L^2}} = \frac{\int_{B_0^2(1)} \varphi_k(x)\, d\delta_0(x)}{\int_{B_0^2(1)} \varphi_k(x)\varphi_k(x)\, dx} = \frac{1}{2\pi\int_0^1 \varphi_k^2(r)\, dr}\,.$$

From [5, p.11], the integral in the denominator can be computed

$$\int_0^1 r\varphi_k^2(r)\, dr = \int_0^1 rJ_0^2\left(\sqrt{\lambda_k}r\right)\, dr = \frac{1}{2}\left[J_0^2\left(\sqrt{\lambda_k}\right) + J_1^2\left(\sqrt{\lambda_k}\right)\right] = \frac{1}{2}J_1^2\left(\sqrt{\lambda_k}\right)\,.$$

Therefore

$$c_k = \frac{1}{\pi J_1^2\left(\sqrt{\lambda_k}\right)}\,.$$

and

$$u(r) = \sum_{\substack{k=1\\k\neq n}}^{+\infty} \frac{1}{\pi J_1^2\left(\sqrt{\lambda_k}\right)\left(\lambda_k - \lambda_n\right)}\varphi_k(r)\,.$$

Remark A.1. The function u is not bounded at the origin $r = 0$. The inequality

$$\frac{1}{\pi J_1^2\left(\sqrt{\lambda_k}\right)(\lambda_k - \lambda_n)}\varphi_k(0) \geq \frac{1}{\pi(\lambda_k - \lambda_n)} > \frac{1}{\pi(k-n)} > 0\,,$$

holds for $k > n$. The second inequality was estimated using [7, Conjecture 1.4.]. Since the series $\sum_{k>n}^{+\infty} \frac{1}{\pi(k-n)}$ diverges, the function is not bounded at $r = 0$.

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