Adaptive parameter estimations of Markowitz model for portfolio optimization

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Abstract. This article is focused on a stock market portfolio optimization. The used method is a modification of traditional Markowitz model which extends the original one for adaptive approaches of parameter estimations. One of the basic factors which significantly influence optimal portfolio is the method of estimations of return on assets, risk and covariance between them. Since the stock market processes tend to be not stationary we can expect that prioritization of recent information will lead to improvement of these parameter estimations and thus to better results of the entire model. For this purpose a modified algorithm was design to estimate expected return and correlation matrix more stable. For implementation and verification of this algorithm we needed to build a program which was able to download historical stock market data from the internet and compute optimal portfolio using either traditional Markowitz model and its modified approach. Obtained results will be compared to the traditional Markowitz model provided real data.

Keywords: Markowitz model, estimation of parameters, adaptive method.

JEL classification: G11
AMS classification: 91G10

1 Introduction

There are many articles about optimal portfolio in the science of mathematics. The reason is that investing on a stock market with potential of profit is interesting for a large amount of people in the world. Thanks to this fact, there is also many approaches how mathematicians try to model the stock market. Some of them tend to believe that there is no relationship between the history and the future. This group use methods of random walk and tries to simulate large number of scenarios to forecast the future. The other group of scientists believe that there is a strong relation between historical prices and the future ones. In this article will focus on this approach and try to treat one of the biggest milestones of these methods - stability of the model provided the parameter estimation. We will introduce two different ways to improve the stability of optimization - matrix cleaning and data weighting.

2 Portfolio theory: basic results

Suppose we have a set of N financial assets characterized by their random return in chosen time period, so the random vector is the vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_N)$ of random returns on the individual assets.

The distribution of vector $\mathbf{X}$ is characterized by vector of expected value with elements $\mathbb{E}X_i = r_i$ and by covariance matrix $\boldsymbol{V}$ whose i, j^{th} element is the covariance between the X_i^{th} and the X_j^{th} random variables. The elements on diagonal σ_i^2 of matrix $\boldsymbol{V}$ represent variances of asset i.

The Markowitz's theory of optimal portfolio is focused on the problem to find optimal weight of each assets such that overall portfolio provides the best return for a fixed level of risk, or conversely the smallest risk for a given overall return [5]. More precisely, the average return R_p of a portfolio P of N assets is defined as $R_p = \sum_{i=1}^{N} w_i r_i$ where w_i is the amount of capital invested in the assets i and r_i are expected returns of the individual assets. Similarly the risk of a portfolio P can by associated with the total variance $\sigma_P^2 = \sum_{i,j=1}^{N} w_i V_{ij} w_j$ or in alternative form $\sigma_P^2 = \sum_{i,j=1}^{N} w_i \sigma_i C_{ij} \sigma_j w_j$ where σ_i^2 is the

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variance of asset i and C is the correlation matrix. The optimal portfolio which minimizes σ_P^2 for a given value of R_p can be easily found introducing a Lagrange multiplier and leads to a linear problem where the matrix $\boldsymbol{C}$ has to be invertible [3],[2].

The resulting from a Markowitz optimization scheme, which gives the portfolio with the minimum risk for a given return $R_P = \sum w_i r_i$

$$w_i \sigma_i = R_p \frac{\sum_j C_{ij}^{-1} r_j/\sigma_j}{\sum_{i,j} r_i/\sigma_i C_{ij}^{-1} r_j/\sigma_j} \tag{1}$$

By redefining w_i as $w_i\sigma_i$ the σ_i is absorbed in r_i and w_i and the equations (3) can be write in matrix notation

$$\boldsymbol{w}_C = R_p \frac{\boldsymbol{C}^{-1}\boldsymbol{r}}{\boldsymbol{r}^T\boldsymbol{C}^{-1}\boldsymbol{r}} \tag{2}$$

and the corresponding risk of the portfolio over the period using this construction is

$$\sigma_P^2 = \frac{R_p^2}{\boldsymbol{r}^T\boldsymbol{C}^{-1}\boldsymbol{r}} \tag{3}$$

From mathematical equation (2) is obvious that usability of Markowitz model strongly depends on input data which are used for asset mean return estimations and the dominant role for stability is given by quality of estimation of the covariance matrix.

3 Parameter estimations - stability of the model

3.1 Empirical correlation matrix

Suppose we have N stock return series with T elements each. If we want to measure and optimize the risk of this portfolio, it is necessary to use a reliable estimate of the covariance matrix $\boldsymbol{V}$ or correlation matrix $\boldsymbol{C}$.

If r_i^t is the daily return of stock i at time t, the empirical variance of each stock is given by

$$\sigma_i^2 = \frac{1}{T}\sum_t \left(r_i^t - \overline{r}_i\right)^2 \tag{4}$$

and can be assumed for simplicity to be perfectly known. We also suppose, as usual, the daily return of stock r_i^t is demeaned ($\overline{r}_i = 0$). The empirical correlation matrix is obtained as

$$E_{ij} = \frac{1}{T}\sum_t x_i^t x_j^t, \quad \text{where } x_i^t = r_i^t/\sigma_i \tag{5}$$

or in matrix form $\boldsymbol{E} = (1/T)\,\boldsymbol{X}^T\boldsymbol{X}$, where $\boldsymbol{X}$ is the normalization $T \times N$ matrix of return $X_{it} = r_i^t/\sigma_i$.

3.2 Random matrix theory and matrix cleaning

For a set of N different assets, the correlation matrix contains $N(N-1)/2$ entries, which must be determined from N time series of length T. If T is not very large compared to N, we can expect that the determination of covariances is noisy and therefore that the empirical correlation matrix is to a large extent random. Because a covariance matrix is positive semidefinite, that the structure of it can by describe by real eigenvalues and corresponding eigenvectors. Eigenvalues of the covariance matrix that are small (or even zero) correspond to portfolios of stocks that have non-zero returns, but extremely low or vanishing risk; such portfolios are invariably related to estimation errors resulting from insufficient data. One of the approaches used to eliminate the problem of small eigenvalues in the estimated covariance matrix is the so-called random matrix technique. Random matrix theory (RMT), first developed by authors such as Dyson [4] and Mehta [9] for physical application, but there are also many results of interest in a financial context [7], [1], [11].

The spectral properties of $\boldsymbol{C}$ may be compared to those of random correlation matrix. As described by [7], [11] and others, if $\boldsymbol{R}$ is any matrix defined by $\boldsymbol{R} = (1/T)\, \boldsymbol{A}^T \boldsymbol{A}$, where $\boldsymbol{A}$ is an $N \times T$ matrix whose elements are i.i.d random variables with mean zero and fixed variance σ^2, than in the limit $T, N \to \infty$ keeping ratio $Q = T/N \geq 1$ constant, the density of eigenvalues of $\boldsymbol{R}$ is given by

$$P(\lambda) = \frac{Q}{2\pi\sigma^2} \frac{\sqrt{(\lambda_{max} - \lambda)(\lambda - \lambda_{min})}}{\lambda}, \quad \lambda_{min} \leq \lambda \leq \lambda_{max}, \tag{6}$$

where the maximum and minimum eigenvalues are given by

$$\lambda_{max/min} = \sigma^2 \left(1 \pm \sqrt{\frac{1}{Q}}\right)^2. \tag{7}$$

The distribution $P(\lambda)$ are known as the Marčenko-Pastur density [8] and the theoretical maximum and minimum eigenvalues determined the bounds for random matrix. If the eigenvalues of matrix are beyond these bounds, it is said that they deviate from random. If we apply this theoretical background of RMT to the correlation matrix we can separate the noise and non-noise parts of $\boldsymbol{E}$. We cleaned the matrix by following procedure: 1. to construct the empirical correlation matrix as (5), 2. separate the noisy eigenvalues from non-noisy eigenvalues as (6), 3. to keep the non-noisy eigenvalues the same and to take average of the noisy eigenvalues, 4. to replace each eigenvalue associated with the noisy part by average of the eigenvalues, 5. to reconstruct correlation matrix. The simple repair mechanism, based on the spectral decomposition of the correlation matrix, is described for example in [6].

3.3 Exponential weights - parameter estimation

Another method how we can minimize the non-stability of the model is weighting. Since we suppose that the most recent data are the most relevant and the older ones influence the future less, we designed model, which weights the data exponentially to the history. This idea was introduced in [10], where we can also find more details. The parameters we need to estimate are:

- Return - estimate of expected return on asset X_i is in this case calculated by weighted mean:

$$\widehat{r}_i = \left(\sum_{t=1}^{T} r_i^t \cdot \delta^t\right) / \left(\sum_{t=1}^{T} \delta^t\right) \tag{8}$$

- Risk - the estimated expected return of asset X_i is in this case calculated by sample weighted variance:

$$\widehat{\sigma}_i = \sqrt{\frac{\sum\limits_{t=1}^{T} \delta^t \cdot (r_i^t - \overline{r}_i)^2}{\sum\limits_{t=1}^{T} \delta^t} \cdot \frac{T}{T-1}}, \tag{9}$$

- Empirical correlation matrix is computed using exponentially weighted by

$$E_{ij} = \left(\sum_{t=1}^{T} \delta^t x_i^t x_j^t\right) / \sum_{t=1}^{T} \delta^t, \quad \text{where } x_i^t = r_i^t / \widehat{\sigma}_i \tag{10}$$

  where $\delta \in (0, 1\rangle$ is weighting parameter. This parameter δ is sometimes called a "forgetting coefficient" and should be close to 1. The smaller it is, the faster older data get unsignificant.

This approach is implemented and tested in created program StockMaTT, where we can either analyse our data using traditional Markowitz model (uses linear data weights) or using this adaptive approach with exponential weighting. The results of these two methods vary and depend on its parameters - on data history length for linear model and on λ for the adaptive model.

4 Data analysed

4.1 Program StockMaTT

So that we can verify the model and figure out it's results we needed to build a programme tool. A satisfying environment needs to have sufficient mathematical and statistical background and also needs to be fast enough to implement quite a complicated algorithm on complex data.

A tool that fitted our needs was the programming language MATLAB® in combination with its GUI environment that makes the user's controlability comfortable. The main screen where we update data and compute the optimal portfolio can be found in the Figure 1.

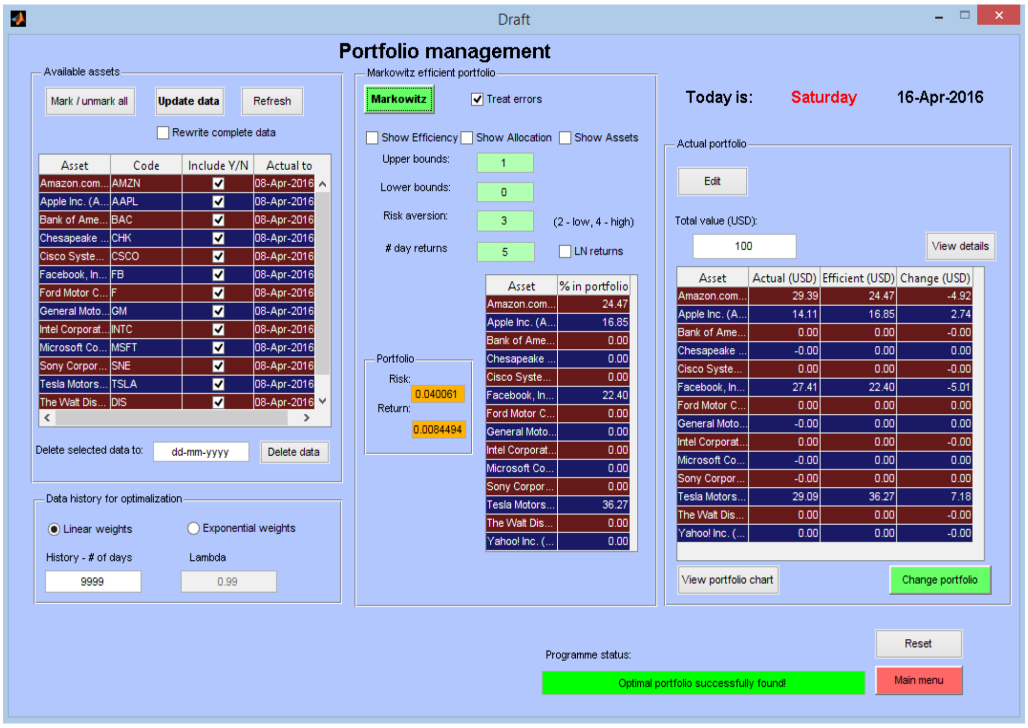

Figure 1 Window for data update and portfolio optimization

4.2 Real data

The model was also tested on real data. Since the focus of this article is on portfolio optimization we chose to use the data from the stock market. As a prefered server was chosen server *yahoo.finance.com.*

For our analysis we used the time series of daily closing prices of stocks available mainly on *NASDAQ*, which is the biggest stock market in the USA with over 3900 assets from about 39 countries. The financial instruments that can be traded here besides stocks are also options and futures.

Asset split and dividends

Downloaded data also needed to be "smoothened" for unexpected jumps caused by splitting the stocks and also for dividends. The stock spilt is a phenomenon that happens usually for expensive assets when the stakeholders want to support the stock liquidity. Usually they decide to split the stock in the rate of X:1 which means that suddenly all stock holders have X times more assets with $\frac{1}{X}$ of its value. This phenomenon causes this obvious jumps that can bee seen for example on the Apple Inc. stock in the Figure 2.

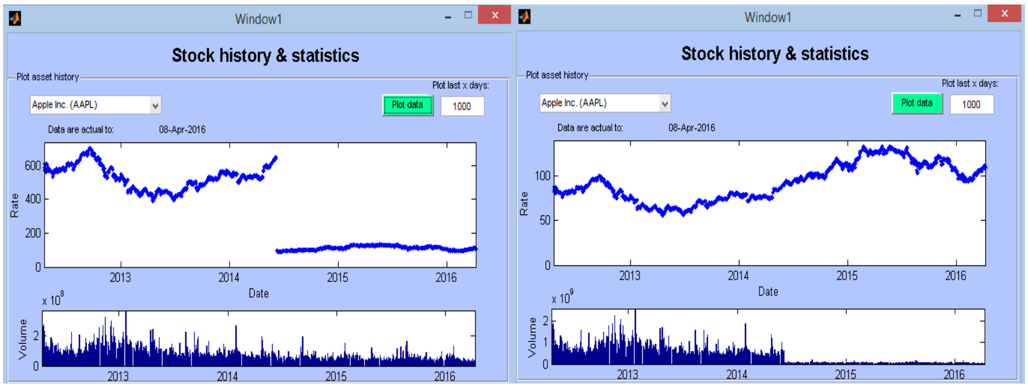

Figure 2 Sample asset split - Apple Inc. - before and after smoothening

4.3 Matrix cleaning based on random matrix theory

For better understanding of the data we applied the method described in subsection 3.2. Firstly we constructed the empirical measured correlation matrix $\boldsymbol{E}$ by using several different numbers of observations (T) and we analysed the distribution of the eigenvalues. We compared the empirical distribution of the eigenvalues of the correlation matrix with the theoretical prediction given by (6) based on assumption that the correlation matrix is purely random. The results are summarized in Table 1.

# observation	% of $\lambda < \lambda_{min}$	% of $\lambda_{min} < \lambda < \lambda_{max}$	% of $\lambda_{max} < \lambda$
15	7%	91%	3%
30	20%	75%	5%
50	33%	60%	7%
100	54%	38%	8%
200	74%	19%	8%

Table 1 Comparing eigenvalues of empirical correlation matrix and Marčenko- Pastur bounders.

From these results we can see that the important information about asset mutual connections is carried by 3 to 7% of eigenvalues of the correlation matrix. By increasing the number of observations on which is based the correlation matrix estimation, slightly increases the number of non-random correlations, but also increases the instability of the correlation matrix since its eigenvalues are very small. As an optimal number of observations in this case seems to be number between 30 and 50.

Figure 3 shows the results of our experiments on the data with 50 observations used for estimation of the correlation matrix. There is histogram of eigenvalues of the empirical correlation matrix and for comparison we plotted the density of (6) for $Q = 3.8462$ and $\sigma = 0.660$. A better fit can be obtained with a smaller value of $\sigma = 0.462$ (dashed blue line).

This result is in accordance with similar articles, for example [7] or [11] and shows problems related to correct market risk estimation.

5 Conclusion

The goal of this article was to develop an advance approach for stability treatment of portfolio optimization. We have developed two methods to minimize the effects of unstability and tested these methods on real data. For purposes of comparison of traditional Markowitz model and the weighting modification we created a SW solution StockMaTT, where we could try to simulate investments with both methods and different parameters. As we expected the traditional Markowitz model is very sensitive on input data and using the weighting we obtained different results. To sum up, as a possible treatment of the unstability can be recommended both methods described in this article. A potential topic for the following studies could be the correct estimation of weighting parameter.

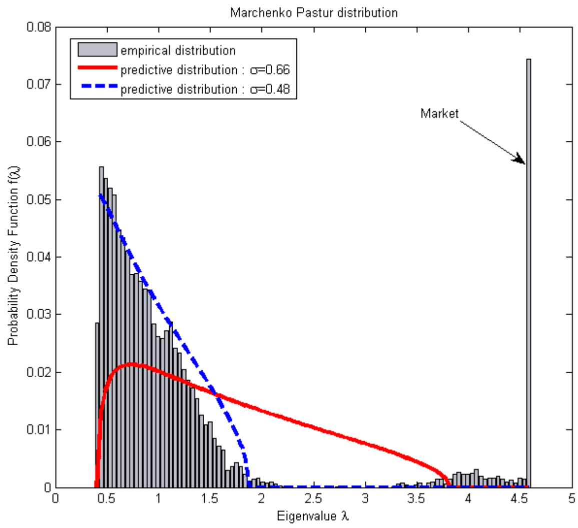

Figure 3 Empirical and prediction distribution of eigenvalues for $T = 50$.

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References

[1] Daly, J., Crane, M. and Ruskin, H. J.: Random matrix theory filters in portfolio optimisation: a stability and risk assessment. *Physica A: Statistical Mechanics and its Applications*, 387.16 (2008), pp. 4248–4260.

[2] Danielsson, J.: *Financial risk forecasting: the theory and practice of forecasting market risk with implementation in R and Matlab* (Vol. 588). John Wiley & Sons, 2011.

[3] Dupačová, J.: Markowitzův model optimální volby portfolia - předpoklady, data, alternativy *(n.d.). Retrieved April 25, 2016, from http://msekce.karlin.mff.cuni.cz/ dupacova/downloads/Markowitz.pdf*

[4] Dyson, Freeman J.: Correlations between eigenvalues of a random matrix. *Comm. Math. Phys.* 19, no. 3 (1970), pp. 235–250.

[5] Elton, E. J., Gruber, M. J., Brown, S. J. and Goetzmann, W. N.: *Modern portfolio theory and investment analysis.* John Wiley & Sons, 2009.

[6] Gilli, M., Maringer, D. and Schumann, E.: *Numerical methods and optimization in finance.* Academic Press, 2011.

[7] Laloux, L., Cizeau, P., Potters, M. and Bouchaud, J. P.: Random matrix theory and financial correlations. *International Journal of Theoretical and Applied Finance*, 3.03 (2000), pp. 391–397.

[8] Marčenko, V. A. and Pastur, L. A.: Distribution of eigenvalues for some sets of random matrices. *Mathematics of the USSR-Sbornik*, 1.4 (1967), 457.

[9] Mehta, M. L.: *Random matrices* (Vol. 142). Academic Press, 2004.

[10] Pavelec, J.: *Programový nástroj pro volbu optimálního portfolia.* Diplomová práce. Západočeská univerzita v Plzni. Fakulta aplikovaných věd, 2013.

[11] Potters, M., Bouchaud, J. P. and Laloux, L.: Financial applications of random matrix theory: Old laces and new pieces. *arXiv preprint physics/0507111* (2005).