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Term structure of interest rates: comparison of the Czech Republic and Germany

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Abstract. The aim of the paper is to demonstrate the possibility of using parametric and nonparametric methods for estimating the term structure of interest rates. First, the principles of several yield curves construction are described and later, advantages and disadvantages of parametric models (Nelson-Siegel and Svensson class model) and nonparametric models are presented. The nonparametric methods based on a kernel estimation are emphasized and closely described, demonstrated and analysed. The both methods (parametric and nonparametric) are used to estimate the term structure of interest rates based on the market price of the Czech government bonds and German government bonds.

Keywords: Yield curves construction, Nelson-Siegel model, Svensson model, Nonparametric models, Czech government bonds, German government bonds

JEL classification: C51, G12
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1 Introduction

The relationship between risk-free rate zero-coupon asset and its maturity is denoted as a time structure of interest rates and a graphical interpretation of such relationship is called a yield curve. Many authors are interested in the time structure of interest rates mainly from economical point of view. They are focused on questions: What information is hidden in the yield curve and what can we find out from the shape of the yield curve in the present or future economical state. There are many hypotheses developed during years which explain the shape of a yield curve, detailed overview can be found in [4]. The idea about the time structure of interest rates provides very important information especially to investors, who use information about interest rates for the risk management, pricing the underlying assets and derivates. It is also important to central banks who are interested in historical yield curves and these curves are the source of information for setting up the correct monetary politics. Many studies dedicated to estimation of the interest rate time structure come from the environment of central banks and other central financial institutions. For example, European central bank *https://www.ecb.europa.eu/stats/money/yc* publishes actual spot, forward and par yield curves estimated by Svensson model [15]. This model is used for the estimation of American yield curves published in [1]. Historical information about the interest rates time structure for the US is also available as par yield curves denoted by the method of quasi-cubic splines which can be found on http:*//www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/Historic-Yield-Data-Visualization.aspx.*

In the last couple of years the estimations of interest rates time structure is one of the classical topics solved in economical, econometric and financial studies. There are two possible approaches for these estimations from the methodical point of view – nonparametrical and parametrical methods. Nonparametrical models derived by using spline principles were used in nowadays classical text [6]. In this text the estimation of discount function is derived by using polynomial splines. In [14] the exponential splines are used for the estimation of discount function and authors show that this approach provides better yield curve estimation than the polynomial approach. Besides the type of spline functions for nonparametrical estimations also the choice of minimizing criterion is important as it also contains the element defining smoothness of the function.

The parametrical approaches are represented by classical methods as Nelson-Siegel [7] and Swenson

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[13] and its different versions [11], [1]. Recently, dynamical models derived from mentioned models are becoming more and more popular. Since parametric models are non-linear and we use methods based on a minimization of non-linear criterion functions we usually need to deal with the problems of sensitivity of these algorithms on the initial conditions and also with other problems connected to numerical stability of estimations [5]. The modern theoretical concepts of interest rate time structures assume that yield curves and recent economical activities expressed for example by macroeconomic indicators (inflation, GDP, ..) are strongly connected. The expected future inflation and expected future real economy progression might be important factors that affect the shape of the yield curve. On the other hand the yield curve can be used as predictor. The strong relationship between yield curves and macroeconomic indicators is a topic of many theoretical and empirical studies. One of the main approaches used for mutual relationship modelling uses dynamical linear and non-linear models.

The foregoing empirical studies show that it is impossible to classify which approach to modelling is better than the others. Every method used has its advantages and disadvantages and the model choice mainly depends on the author's subjective opinion. Particular application of methods which are used to estimate the interest rate time structure in the Czech environment based on actual prices of Czech assets was possible to use after sufficient development of the market with these assets. Similarly, as in other countries, numerous modelling approaches are used. The most often estimation is probably parametrical approach based on Nelson-Siegel model or Svensson model [9]. The complex estimate of daily coefficients for Czech yield curves modelled by Nelson-Siegel approach is available in [2]. In [4] three parametrical functions are chosen for time structure models. These functions combine polynomial and exponential functions. In [8] the estimation of interest rate time structure is based on interest swaps and for its estimation the bootstrap method is used. Modern approaches to interest rate time structures modelling which use dynamical Nelson-Siegel model and that include latent regional factors are mentioned in [12]. In this article, the regional dynamics of yield curves for the Czech Republic, Hungary, Poland and the Slovak Republic is investigated.

2 Yield curve relationship in continous time

The term structure of interest rates is a central concept in monetary and financial economies. The estimation of a yield curves is based on an assumed functional relationship between either par yields, spot rates y, forward rates f or discount factor d on the one hand and maturities on the other. The key term in construction of yield curves is discount factor that is defined as the quantities used at a given point in time to obtain the present value of future cash flow. A discount function $d\,(t_1, t_2)$ is the collection of discount factor at the time t_1 for all maturities t_2.

The forward rates f are defined as the instantaneous rates that are derived from the concept of the discount factor.

$$f\,(t) = \lim_{h\to 0}\left(\frac{d(t)/d(t+h)-1}{h}\right) = \frac{-d\,(t)'}{d\,(t)}. \tag{1}$$

On the other hand the discount rate can be computed as $d\,(t) = \exp\left(-\int\limits_0^t f\,(\tau)\;\mathrm{d}\tau\right)$. The yield of the spot rates is formulated as mean forward rates in interval $(0, t)$ and the spot yield is computed as

$$y\,(t) = \frac{1}{t}\int\limits_0^t f\,(\tau)\;\mathrm{d}\tau = \frac{1}{t}\int\limits_0^t \frac{-d\,(\tau)'}{d\,(\tau)}\;\mathrm{d}\tau = \frac{-\ln\left(d\,(t)\right)}{t}. \tag{2}$$

These relations can be also inverted to express forward and spot rate $f(t) = t\,y(t)' + y(t)$ and $d\,(t) = \exp\left(-t\,y(t)\right)$, with initial condition $f(0) = y(0) = 0$. From the above relationships, it is clear that relations between the discount rate of d, the forward rate of f and the spot rate of return y are unambiguous and knowledge in the discount rate d is equivalent knowledge of f or y.

The construction of these curves poses several problems for applied research. Many governments do not issue longer term (i.e. greater then 1–2 years) zero-coupon bonds. Hence the yield curve must be inferred from other instruments.

One possibility is the estimate of the time structure from bond prices, we assume that there is absence of arbitrary and financial flows associated with the bond issue are known: coupons c in time $t_1, t_2, \ldots, t_J$

and payment of the nominal value of R in time t_J. Arbitrage in the bond markets will cause the price P of the bond is equal to the present value of expected cash flows. The second problem is that, in practice, small pricing errors perhaps trading, taxation, illiquidity spreads necessitate adding an error term to present value of expected cash flows. The statistical problem is how estimate the function d from a sample of coupon paying bonds.

3 Yield curve models and estimation methods

The aim of the estimates in the term structure of interest rates is to find a model curves that give us the most reliable and useful estimates, not just on a specific day, but also in a longer time frame. It is necessary to take into consideration several requests and their corresponding criteria, when estimating the yield curve. The basic requirements for "good" estimates of yield curves include

- the requirement to the smoothness: the estimated yield curves should be relatively smooth, because the yield can be also affected by other random factors;
- the requirement for flexibility of curves: the estimation technique should provide sufficient flexibility at shorter maturities;
- the requirement for stability: the estimates of the yield curve should remain stable in the sense that small changes in data in a single maturity have a disproportionate impact on forward rates at other maturities.

3.1 Methods of estimations

The constructions of yield curves are usually based on the market prices of assets that have the same risk and liquidity. The selection of appropriate assets for the estimation the time structure is therefore usually restricted to the actual market price of government bonds.

Given the discount function we can price any coupon bond by summing the price of its individual cash flows. Let suppose we have an information on K bonds, every bond is represented by a vector $B_k = [\mathbf{b}_k, \mathbf{t}_k, p_k], k = 1, 2, 3, ..., K$, where vector $\mathbf{b}_k = (b_{1k}, b_{2k}, \ldots, b_{nk})$ denotes the payments returned to the owner of the bond k at times $\mathbf{t}_k = (t_{1k}, t_{2k}, \ldots, t_{nk})$ and p_k is the market price of the bond. Knowledge of discount rate can be used to express price of bonds as a sum of discounted cash flows $p_k^M = \sum_{j=1}^{n_k} b_{jk}\left(t_{jk}\right) d\left(t_{jk}\right).$

The yield rates curve can be estimated by minimising the sum of differences between the model prices and observed prices of a set of bonds: $\min \sum_k \left(p_k - p_k^M\right)^2$, where the superscript M stands for "model" price, or the model can be estimated by minimising the sum of differences between the model rates and observed rates of a set of bonds: $\min \sum_k \left(y_k - y_k^M\right)^2$.

3.2 Parametric models

Various parametric models can be used to determine the yield curve. The method developed by Nelson and Siegel [7] assumes explicitly the following function form for the spot rates

$$y(t) = \beta_1 + \beta_2 \left(\frac{1 - \exp\left(-\frac{t}{\lambda}\right)}{\frac{t}{\lambda}} \right) + \beta_3 \left(\frac{1 - \exp\left(-\frac{t}{\lambda}\right)}{\frac{t}{\lambda}} - \exp\left(-\frac{t}{\lambda}\right) \right). \tag{3}$$

The popularity and practical usability of the model is mainly based on the fact that this model is able to describe the time structure with a small number of parameters, while the three components, which are included into the model, allow to describe various theoretical and empirical observed shapes of the yield curve, starting from a monotonic curves through S-shaped curve to curve with one local maximum.

To improve the flexibility of the curves and fit, Svensson [13] extended Nelson and Siegel's function by adding a further term that allows for a second "jump". The extra precision in achieved at the cost of adding two parameters β_4 and λ_2 which have to be estimated. The spot forward rates curve thus becomes

$$y(t) = \beta_1 + \beta_2 \left(\frac{1 - \exp\left(-\frac{t}{\lambda_1}\right)}{\frac{t}{\lambda_1}} \right) + \beta_3 \left(\frac{1 - \exp\left(-\frac{t}{\lambda_1}\right)}{\frac{t}{\lambda_1}} - \exp\left(-\frac{t}{\lambda_1}\right) \right) + \\ + \beta_4 \left(\frac{1 - \exp\left(-\frac{t}{\lambda_2}\right)}{\frac{t}{\lambda_2}} - \exp\left(-\frac{t}{\lambda_2}\right) \right). \qquad (4)$$

But the model calibration and validation, i.e. obtaining parameters values such that the model yields accord with the market yields is difficult, many authors have reported "numerical difficulties". There are two reasons for these difficulties. First, the objective function is non convex and exhibits several local minima, so the optimization algorithm is also sensitive to the initial parameters value. The second problem is collinearity and this stems from the model specification. Both models can be interpreted as a factor models but factors in model are not independent. For example the Nelson-Siegel's model includes three factors, which can be interpreted as level (β_1), steepness (β_2) and curvature (β_3).

3.3 Nonparametric models

Rather than specifying a single functional form over the entire maturity range, nonparametric methods can be used. Over a closed interval, a given continuous function can be approximated by selection an arbitrary function, however quite frequently it displays insufficient smoothing properties.

In this article we focused on nonparametric estimations based on kernel regression smoothing. The inspiration for this article was the work [3], which included also asymptotic properties of kernel estimates of the yield curve and this paper follows on our previous published paper [10].

Assume the following regression model $Y_i = m(X_i) + \varepsilon_i$, $\varepsilon_i \sim iid, E(\varepsilon_i) = 0, i = 1, 2, \ldots, n,$. The Nadaraya–Watson kernel regression algorithm for estimation of the conditional expectation of a random variable $E(Y|X = x) = m(x)$ is based on local weighting mean

$$\widehat{m}(x) = \sum_i w_i(x)\, Y_i, \qquad \text{with } \; w_i(x) = \frac{K\left(\frac{x - X_i}{h}\right)}{\sum_i K\left(\frac{x - X_i}{h}\right)} I_{\left[\sum_i K\left(\frac{x - X_i}{h}\right) \neq 0\right]}, \qquad (5)$$

where $K(\cdot)$ is a kernel as a weighting function and h is a bandwidth. Basic calculus shows that $\widehat{m}_n(x)$ is the solution to a weighted least square problem, being the minimizer $\sum_i (Y_i - \beta_0)^2 K\left(\frac{x - X_i}{h}\right)$.

It is known, that the significant effect to the quality of kernel estimation have the choices of kernel function and bandwidth. In our study, we used three types of kernel function: Gaussian $K(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)$, exponential $K(u) = \frac{1}{2} \exp(-|u|)$ and quartic (biweight) $K(u) = \frac{15}{16}\left(1 - u^2\right)^2 I_{[|u| \leq 1]}$ form of kernel function and the optimal window width choice was founded by cross validation algorithm. These three types were selected because of their qualities and behaviour in the simulation studies.

The estimation of term structure with using nonparametric regression was based on local weighted minimization differences between model price and market price which is calculated as sum of discounted cash flows

$$\min_d \sum_k w_k(t) \left(p_k - \sum_{j=1}^{J_k} b_k(t_{jk}) d(t_{jk}) \right)^2. \qquad (6)$$

4 Data and empirical findings

The parametric and nonparametric approaches, described in Section 3, were applied to real data from the Czech Republic and Germany. The final data set for estimating the German term structure of interest rates comprises standard bonds issued by the Federal Republic of Germany, five-year special federal bonds and Federal Treasury notes. The final data for estimating the Czech term structure contains government bonds. For short end of yield curve we used PRIBOR and PRIBID, resp. EURIBOR. The data were obtained on May 10, 2015 and these data can be obtained at *http://www.boerse-frankfurt.de/en/bonds* and *http://www.patria.cz/kurzy/online/dluhopisy.html*. We adopt the standard approach and consider all bonds with a remaining time to maturity above six month. The yield of bonds with residual maturity below six month are excluded because they appear to be more liquid. In running the programming we choose several initial values for each model and then we used the best fitting model. That approach can eliminat problems with several local minimum of criterion function and dependence on choosing initial value. The quality of models were compared by mean square error $MSE_P = 1/K \sum_k \left(p_k - p_K^M\right)^2$ and $MSE_y = 1/K \sum_k \left(ytm_k - ytm_K^M\right)^2$, where ytm_k is the calculated yield to maturity of bonds.

The estimated parameters of the Nelson-Siegel's model (NS model) formulated in Eq. (3), Svensson model (S model) formulated in Eq. (4) and nonparametric model formulated in Eq. (6) are summarized in Table 1

Parametric models								
	β_1	β_2	β_3	β_4	λ_1	λ_2	MSE_p	MSE_y
NS model CZ	0.0167	-0.0155	0.0687		39.9977		1.4764	0.0457
S model CZ	0.0059	-0.0044	-0.0044	20.9269	11.3985	12896.2185	1.3589	0.0461
NS model GE	0.0117	-0.0120	0.0568		39.9999		2.6407	0.0259
S model GE	0.0000	-0.0001	0.0043	15.0001	10.3986	12721.9560	3.4148	0.0266
Nonparametric models								
			h				MSE_p	MSE_y
Gaussian kernel CZ			0.1594				4.9881	0.1014
Exponential kernel CZ			0.1000				5.1373	0.1033
Quartic kernel CZ			0.9187				5.1686	0.0847
Gaussian kernel GE			0.0913				5.9345	0.0520
Exponential kernel GE			0.2255				5.9334	0.0515
Quartic kernel GE			0.1841				5.9106	0.0530

Table 1 Estimated parameters of the Nelson-Siegel model, Svensson model and nonparametric model for the Czech Republic (CZ) and Germany (GE)

Our results demonstrate the problem with numerical solutions of optimization problem with several parameters. Although the Svensson model is extension form of Nelson Siegel model, the obtained estimates of parameters are completely different. Also the interpretation of value can be problematic especially for parameters lambda in both models. There is the same problem with estimation of bandwidth for nonparametric estimation but the values of criterion MSE are close. An analysis of the choice of kernel function is one of the possible directions for our future work.

The estimated yield curves based on the Nelson-Siegel model, Svensson model and nonparametric kernel approach for the Czech Republic and Germany are in Figure 1.

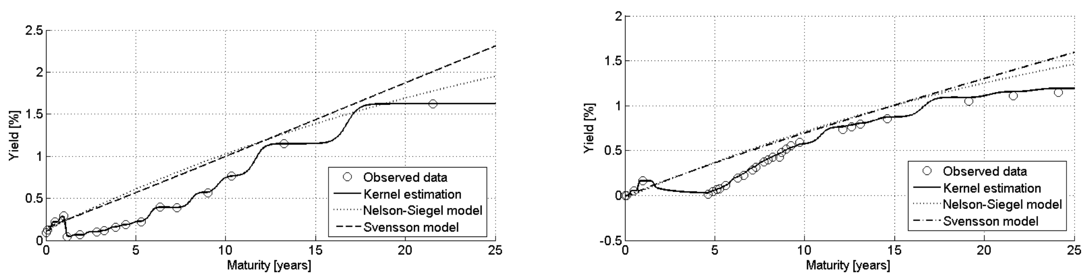

Figure 1 Estimated curves for the Czech Republic (left) and Germany (right)

The observed and modelled prices for bonds are shown in Figure 2. It shows more precise estimates for short end but there are not conspicuous differences between parametric and nonparametric approach.

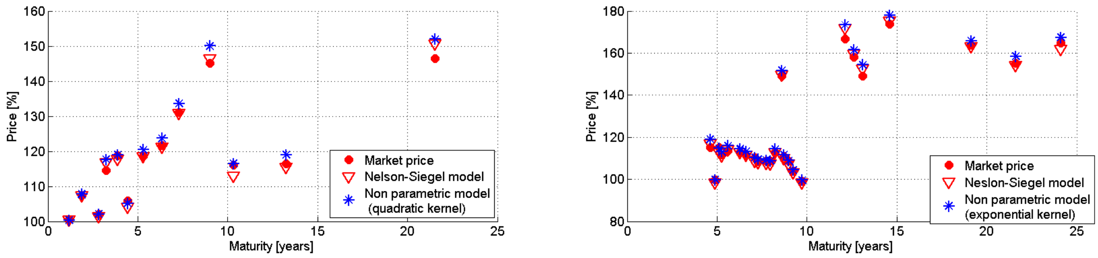

Figure 2 Market and estimated prices of bonds for the Czech Republic (left) and Germany (right)

5 Conclusions

The Nelson–Siegel and Svensson model are parametric models that have four or six parameters. Within these models, it is very difficult to estimate parameters. A different approach for estimating the time structure of interest rates is based on nonparametric methods. Our studies shown that the results obtained from parametric and nonparametric methods are comparable and both can be used for estimation of the market bond prices. The advantage of nonparametric methods are quickness of algorithm and the fact, that the results are more robust and do not depend on initial values. The disadvantage can be seen in its huge flexibility and sensitivity to data set, especially in the case there are only small number of market bond prices available. The choice of optimal kernel function is also the open problem.

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