THE BUSBAR OF A RECTANGULAR CROSS-SECTION - THE MAGNETIC FIELD

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Abstract: This analysis is going to present the example of determining the magnetic field in a busbar of a rectangular cross-section with a finite length and in its surroundings using *Mathematica* programme. The solution obtained is going to be used, by *Mathematica* programme, to describe the magnetic field in the surroundings of a ribbon busbar with a finite length.

Mathematica programme is a good and convenient tool for analytical, by the integration function and conversion of the analytical solutions, determination of the field quantities and for graphical visualisation of the obtained final solutions.

The components of magnetic field intensity in a busbar of a rectangular cross-section with a finite length for different values of cross-section can be determined by applying Biot-Savart law. Analytical equation describing magnetic field intensity in a busbar of rectangular cross-section with a finite length as well as its graphical visualisation have been obtained by using *Mathematica* programme in solving the integral calculi and in cancellation of algebraic expressions. *Mathematica* programme also enabled to describe the magnetic field in the analytical form for the case of a ribbon busbar.

The derived formulas describing the magnetic field in the busbar of rectangular cross-section and finite length are complicated and it is difficult to predict the change of magnetic field distribution after the change of ratio of the transverse dimensions of this busbar, for example, on the basis of these formulas. However *Mathematica* programme provides a ready-made instrument to get a quick visualisation of the obtained solutions.

Key words: magnetic field, busbar, Mathematica

1. Introduction

In the process of teaching electromagnetic field theory one can often come across problems demanding long calculations not only in numerical solving of problems but also in their analytical solving – integration, differentiation, rearranging and abridging the obtained expressions and seeking the special solutions to the terminal values of certain parameters or terminal domains. What is important is also the quick visualisation of the obtained solutions, especially the spatial visualisation in the geometry changes of a considered configuration, the values of quantities of electrical input (current, voltage, external electric and magnetic fields, their frequency, etc.) as well as of electrical parameters of a configuration (conductivity, permittivity and magnetic permeability, etc.). The possession of a proper instrument to achieve the aforementioned aims is very instructive as it facilitates memorising the analysed issues and helps the students to understand and interpret the derived formulas. Thus it enables them to have deeper insight into the features of electromagnetic field.

QuickField or *Flux* are instruments specialised in solving the field problems. However, other mathematical programmes can be easily used in solving the electric field problems. *MatCad* and *Mathematica* ca be undoubtedly classified as the aforementioned programmes.

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2. The magnetic field intensity in a busbar of a rectangular cross section with a finite lenght

It is assumed that the direct or free-variable sinusoidal current with the intensity *I* flows along the rectangular busbar with the dimensions $a \times b$ (fig.1). Then the density of the current is constant in each point of the busbar and is

$$J = \frac{I}{a\,b} \,. \tag{1}$$

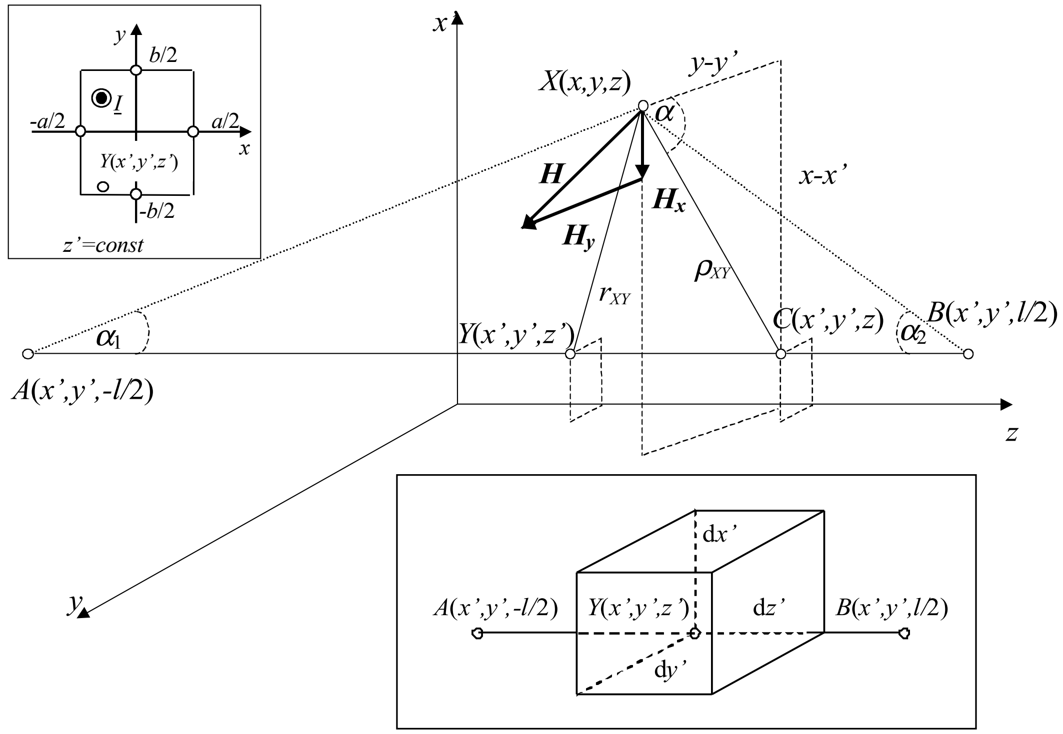


Fig.1. A busbar of a rectangular cross-section with the current *I* and length *l*

The magnetic field in each *X (x, y, z)* point, inside or outside of the busbar, is determined as the superposition of the field created by the currents at the source *Y (x', y', z')* points which are distant from the *X (x, y, z)* point of the r_{xy} value – fig. 1. For the *Y (x', y', z')* point and assuming *z'* = const., the infinitesimal domain

$$\mathrm{d}s = \mathrm{d}x'\,\mathrm{d}y' \tag{2}$$

and then, at the constant current density in a rectangular cross-section of a busbar, the current of the current tube

$$\mathrm{d}I = J\,\mathrm{d}s = \frac{I}{a\,b}\,\mathrm{d}x'\,\mathrm{d}y' \,. \tag{3}$$

According to the Biot-Savart law [4, 6] the infinitesimal vector of the magnetic field intensity at the *X (x, y, z)* point

$$\mathrm{d}\boldsymbol{H} = \frac{\mathrm{d}I}{4\pi\, r}\,\mathbf{1}_z \times \mathbf{1}_r = \mathrm{d}H_x\,\mathbf{1}_x + \mathrm{d}H_y\,\mathbf{1}_y \,, \tag{4}$$

where $\mathbf{1}_x$, $\mathbf{1}_y$ and $\mathbf{1}_z$ are unitary vectors of a Cartesian co-ordinate system, $\mathbf{1}_r$ is a unitary vector of $\boldsymbol{r}_{xy}$ vector that joins the *Y (x', y', z')* point with the *X (x, y, z)* point, $\mathbf{d}\boldsymbol{H}_x$ and $\mathbf{d}\boldsymbol{H}_y$ are the components of $\mathbf{d}\boldsymbol{H}$ vector and the length

$$r_{XY} = \sqrt{(x - x')^2 + (y - y')^2} \,. \tag{4a}$$

The module of a vector (4)

$$\mathrm{d}H = \frac{I}{4\pi ab}\frac{\mathrm{d}x'\,\mathrm{d}y'}{r_{XY}}\left[\cos\alpha_1 + \cos\alpha_2\right] \tag{5}$$

where: $\cos\alpha_1 = \dfrac{\frac{l}{2}+z}{\sqrt{r_{XY}^2 + \left(z+\frac{l}{2}\right)^2}}$ and $\cos\alpha_2 = \dfrac{\frac{l}{2}-z}{\sqrt{r_{XY}^2 + \left(z-\frac{l}{2}\right)^2}}$.

and its components are expressed with the following formulas (fig. 1):

$$\begin{aligned}\mathrm{d}H_x &= -\mathrm{d}H\cos\alpha = -\mathrm{d}H\frac{y-y'}{r_{XY}} = -\frac{I}{4\pi ab}\frac{\mathrm{d}x'\,\mathrm{d}y'}{r_{XY}}\left[\cos\alpha_1+\cos\alpha_2\right]\frac{y-y'}{r_{XY}} = \\ &= -\frac{I}{4\pi ab}\frac{\mathrm{d}x'\,\mathrm{d}y'}{r_{XY}}\left[\frac{\frac{l}{2}+z}{\sqrt{r_{XY}^2+\left(z+\frac{l}{2}\right)^2}} + \frac{\frac{l}{2}-z}{\sqrt{r_{XY}^2+\left(z-\frac{l}{2}\right)^2}}\right]\frac{y-y'}{r_{XY}}\end{aligned} \tag{6}$$

and

$$\begin{aligned}\mathrm{d}H_y &= \mathrm{d}H\sin\alpha = \mathrm{d}H\frac{x-x'}{r_{XY}} = \frac{I}{4\pi ab}\frac{\mathrm{d}x'\,\mathrm{d}y'}{r_{XY}}\left[\cos\alpha_1+\cos\alpha_2\right]\frac{x-x'}{r_{XY}} = \\ &= \frac{I}{4\pi ab}\frac{\mathrm{d}x'\,\mathrm{d}y'}{r_{XY}}\left[\frac{\frac{l}{2}+z}{\sqrt{r_{XY}^2+\left(z+\frac{l}{2}\right)^2}} + \frac{\frac{l}{2}-z}{\sqrt{r_{XY}^2+\left(z-\frac{l}{2}\right)^2}}\right]\frac{x-x'}{r_{XY}}\end{aligned} \tag{6a}$$

Taking into consideration the aforementioned H_x and H_y components of a complete vector of a magnetic field in a rectangular busbar with a finite length $\boldsymbol{H} = H_x\,\mathbf{1}_x + H_y\,\mathbf{1}_y$ what is going to be determined by the following integrations is:

$$H_x(x,y) = -\frac{I}{4\pi ab}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}}\frac{y-y'}{(x-x')^2+(y-y')^2}\left[\cos\alpha_1+\cos\alpha_2\right]\mathrm{d}x'\,\mathrm{d}y' \tag{7}$$

and

$$H_y(x,y) = \frac{I}{4\pi ab}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}}\frac{x-x'}{(x-x')^2+(y-y')^2}\left[\cos\alpha_1+\cos\alpha_2\right]\mathrm{d}x'\,\mathrm{d}y' \tag{7a}$$

Within the outside of the busbar, that is for $x > \frac{a}{2} \cup x < -\frac{a}{2} \cup y > \frac{b}{2} \cup y < -\frac{b}{2}$, the observation point *X (x, y, z)* is never going to cover the source point *Y (x', y', z')* and the above integrals are the proper ones. *Mathematica* programme enables to determine and abridge them directly by using the *Simplify, FullSimplify* or *PowerExpand* options. The solutions have been presented in the work [2], pp. 113-114.

Within the inside of the busbar, that is for $-\frac{a}{2} \le x \le \frac{a}{2} \cap -\frac{b}{2} \le y \le \frac{b}{2}$, the observation point *X (x, y, z)* may cover the source point *Y (x', y', z')* and the above integrals are improper but convergent.

The problem of the convergence of those integrals is seen in the second integration. However *Mathematica* programme enables to calculate the terminal values for upper and lower margin of integration. Having additionally used the *Simplify, FullSimplify* or *PowerExpand* options, the component of the magnetic field intensity along the *Ox* axis - formula A1 located in appendix.

Because of the denominators of inverse functions to trigonometrical functions in the A1 formula, that component must be additionally determined for the upper and lower side of the cross-section of a busbar (vide appendix A2 i A3):

$$H_x(x, y = \frac{b}{2}) = \underset{y \to \frac{b}{2}}{\mathrm{Limit}}\, H_x(x, y) \tag{8}$$

and

$$H_x(x, y = -\frac{b}{2}) = \underset{y \to -\frac{b}{2}}{\mathrm{Limit}}\, H_x(x, y) \tag{8a}$$

The component of the magnetic field intensity along the *Oy* axis is determined in a way similar to the above - formula A4 located in appendix.

In this case this component is also additionally determined for the right and left side of the cross-section of a busbar (vide appendix A5 i A6):

$$H_y(x = \frac{a}{2}, y) = \underset{x \to \frac{a}{2}}{\mathrm{Limit}}\, H_y(x, y) \tag{9}$$

and

$$H_y(x = -\frac{a}{2}, y) = \underset{x \to -\frac{a}{2}}{\mathrm{Limit}}\, H_y(x, y) \tag{9a}$$

The above formulas specify the components of the magnetic field intensity at each point *X(x, y, z)* inside the rectangular busbar with a finite length. The module of a complete magnetic field is determined from the following formula:

$$H(x, y) = \sqrt{H_x^2(x, y) + H_y^2(x, y)}\,. \tag{10}$$

When looking merely at the formulas A1 and A2 in appendix, it must be noticed that they also describe the magnetic fields in the external domain of a busbar with a finite length. When in the (6) and (6a) formulas $l \to \infty$, the amount of the cosines → 2 and the components of a magnetic field vector for a rectangular busbar with an infinite length are obtained [3].

The derived formulas describing the magnetic field in the busbar of rectangular cross-section and finite length are complicated and it is difficult to predict the change of magnetic field distribution after the change of ratio of the transverse dimensions of this busbar, for example, on the basis of these formulas. However *Mathematica* programme provides a ready, in our opinion very good, instrument to get a quick visualisation of the obtained solutions.

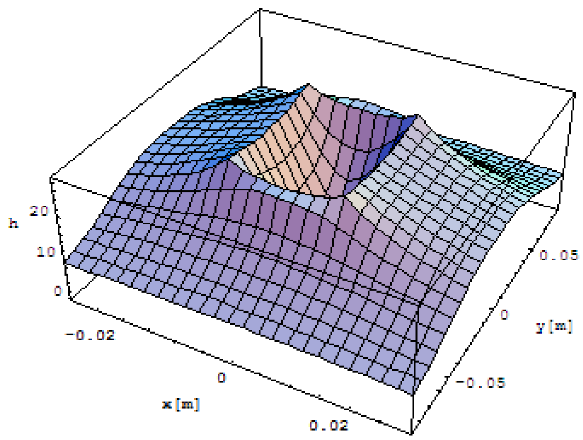
In the analysed case of a busbar the distribution of the magnetic field intensity module at each point *X (x, y, z)* inside and outside of the busbar is shown in the figure 2 whilst this field is expressed in relative units as a function

$$h(x, y) = \frac{H(x, y)}{H_0}\,, \tag{11}$$

where the intensities of the reference

$$H_0 = \frac{I}{2\,(a + b)}\,. \tag{11a}$$

The distribution of this field module on the plane of a cross-section of the busbar is presented in the figure 3.



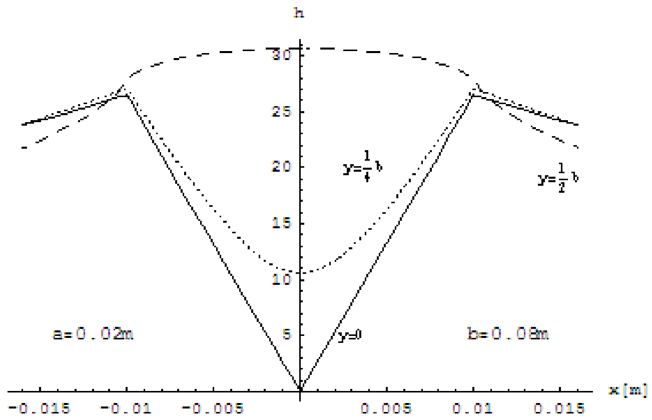


Fig.2. The distribution of magnetic field intensity module in the internal and external domain of a busbar with a rectangular cross-section and finite length; *a* = 0.02 m, *b* = 0.08 m, *l* =1 m

Fig.3. The distribution of the magnetic field module on the plane of a cross-section of a rectangular busbar with a finite length;*a* = 0.02 m, *b* = 0.08 m, *l* =1 m

The change of the ratios between *a* and *b* dimensions of a busbar changes the distribution of a magnetic field, what is presented in the figure 4.

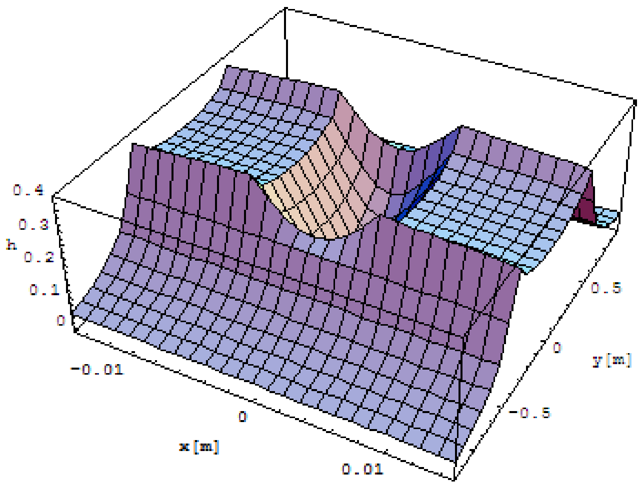


Fig.4. The distribution of a magnetic field intensity module in the internal and external domain of a busbar with a rectangular cross-section and finite length; *a* = 0.01 m, *b* = 0.9 m, *l* =1 m

3. The magnetic field of a ribbon busbar with a finite length

In case of a ribbon busbar with a finite length, that is a busbar with a cross-section, for which *b>>a,* the magnetic field at each point *X (x, y, z)* can be determined much easier than in the case of a rectangular busbar with a finite length as the solution is obtained by single integration – [2], pp. 111-112 and [5], pp. 110-112. What can be an additional facilitation is the limitation of a solution to the axis of busbar symmetry –[7], pp. 135-136 and [8], pp. 153-155.

However, if one is in possession of (8) and (9) solutions, *Mathematica* programme enables to determine the components of a magnetic field by determination of the limits of these functions at the transverse dimension of a busbar $a \to 0$. The following formulas are then obtained:

$$H_x^{(t)}(x,y)=\frac{1}{8\pi b}\left\{\begin{array}{l}2\ln\frac{2z-l+\sqrt{4x^2+(b-2y)^2+(l-2z)^2}}{2z-l+\sqrt{4x^2+(b+2y)^2+(l-2z)^2}}+\\ \ln\frac{\sqrt{4x^2+(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4x^2+(b+2y)^2+(l+2z)^2}}{\sqrt{4x^2+(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4x^2+(b-2y)^2+(l+2z)^2}}+\\ \ln\frac{-\sqrt{4x^2+(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4x^2+(b+2y)^2+(l+2z)^2}}{-\sqrt{4x^2+(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4x^2+(b-2y)^2+(l+2z)^2}}\end{array}\right\} \quad (12)$$

and

$$H_y^{(t)}(x,y)=0\,. \quad (12a)$$

In special cases the following may be obtained from the above formulas:

- field intensity on the *Ox* axis

$$H_{x,Ox}^{(t)}(x)=\operatorname*{Limit}_{y\to 0}H_x^{(t)}(x,y)=0 \quad (13)$$

and

$$H_{y,Ox}^{(t)}(x)=\operatorname*{Limit}_{y\to 0}H_y^{(t)}(x,y)=0 \quad (13a)$$

- field intensity on the *Oy* axis

$$H_{x,Oy}^{(t)}(y)=\operatorname*{Limit}_{x\to 0}H_x^{(t)}(x,y)=$$

$$=\frac{1}{8\pi b}\left\{\begin{array}{l}2\ln\frac{2z-l+\sqrt{(b-2y)^2+(l-2z)^2}}{2z-l+\sqrt{(b+2y)^2+(l-2z)^2}}+\\ \ln\frac{\sqrt{(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(b+2y)^2+(l+2z)^2}}{\sqrt{(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(b-2y)^2+(l+2z)^2}}+\\ \ln\frac{-\sqrt{(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(b+2y)^2+(l+2z)^2}}{-\sqrt{(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(b-2y)^2+(l+2z)^2}}\end{array}\right\} \quad (14)$$

and

$$H_{y,Oy}^{(t)}(x)=\operatorname*{Limit}_{x\to 0}H_y^{(t)}(x,y)=0\,. \quad (14a)$$

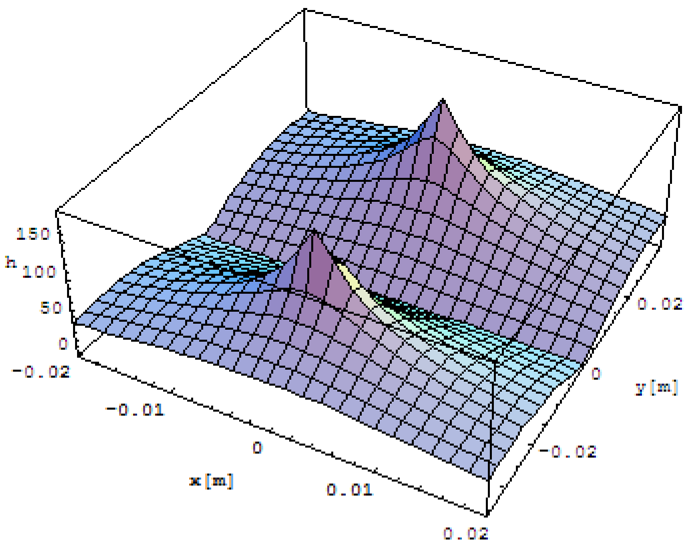


Fig.5. The distribution of the magnetic field intensity module in the surroundings of a ribbon busbar with a finite length; b = 0.04 m, l = 1 m

The distribution of the magnetic field module in the surroundings of a ribbon busbar with a finite length was shown in the figure 5, whilst this field is expressed in relative units as a function

$$h^{(t)}(x,y) = \frac{H^{(t)}(x,y)}{H_0^{(t)}}\,, \tag{15}$$

where the intensities of the reference

$$H_0^{(t)} = \frac{I}{2b}\,. \tag{15a}$$

4. Conclusions

On the basis of an example from the theory of electromagnetic field presented in this work it has been shown that *Mathematica* programme is a good and convenient tool for analytical, by the integration function and conversion (with no mistakes) of the analytical solutions, determination of the field quantities and for graphical visualisation of the obtained final solutions. This enables to perform a quick analysis of the field after the changes of geometrical or electrical parameters of the analysed systems what is an important didactic issue in the process of magnetic field theory teaching. This enables the students to understand, remember and physically interpret the analysed problem.

5. References

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APPENDIX

$$H_x(x,y)=\frac{1}{16\pi a b}\left\{\begin{array}{l}
2(b-2y)\left[\begin{array}{l}\operatorname{arctg}\dfrac{(a-2x)(l-2z)}{(b-2y)\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}}+\operatorname{arctg}\dfrac{(a+2x)(l-2z)}{(b-2y)\sqrt{(a+2x)^2+(b-2y)^2+(l-2z)^2}}+\\
\operatorname{arctg}\dfrac{(a-2x)(l+2z)}{(b-2y)\sqrt{(a-2x)^2+(b-2y)^2+(l+2z)^2}}+\operatorname{arctg}\dfrac{(a+2x)(l+2z)}{(b-2y)\sqrt{(a+2x)^2+(b-2y)^2+(l+2z)^2}}\end{array}\right]-\\
2(b+2y)\left[\begin{array}{l}\operatorname{arctg}\dfrac{(a-2x)(l-2z)}{(b+2y)\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}}+\operatorname{arctg}\dfrac{(a+2x)(l-2z)}{(b+2y)\sqrt{(a+2x)^2+(b+2y)^2+(l-2z)^2}}+\\
\operatorname{arctg}\dfrac{(a-2x)(l+2z)}{(b+2y)\sqrt{(a-2x)^2+(b+2y)^2+(l+2z)^2}}+\operatorname{arctg}\dfrac{(a+2x)(l+2z)}{(b+2y)\sqrt{(a+2x)^2+(b+2y)^2+(l+2z)^2}}\end{array}\right]+\\
2(l-2z)\left[\ln\dfrac{-a+2x+\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}}{-a+2x+\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}}+\ln\dfrac{a+2x+\sqrt{(a+2x)^2+(b+2y)^2+(l-2z)^2}}{a+2x+\sqrt{(a+2x)^2+(b+2y)^2+(l+2z)^2}}\right]+\\
2(l+2z)\left[\ln\dfrac{-a+2x+\sqrt{(a-2x)^2+(b-2y)^2+(l+2z)^2}}{-a+2x+\sqrt{(a-2x)^2+(b+2y)^2+(l+2z)^2}}+\ln\dfrac{a+2x+\sqrt{(a+2x)^2+(b+2y)^2+(l+2z)^2}}{a+2x+\sqrt{(a+2x)^2+(b-2y)^2+(l+2z)^2}}\right]+\\
(a-2x)\left[\ln\dfrac{l-2z+\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}}{l-2z+\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}}+\ln\dfrac{2z-l+\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}}{2z-l+\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}}\right]+\\
(a+2x)\left[\ln\dfrac{l-2z+\sqrt{(a+2x)^2+(b+2y)^2+(l-2z)^2}}{l-2z+\sqrt{(a+2x)^2+(b-2y)^2+(l-2z)^2}}+\ln\dfrac{2z-l+\sqrt{(a+2x)^2+(b-2y)^2+(l-2z)^2}}{2z-l+\sqrt{(a+2x)^2+(b+2y)^2+(l-2z)^2}}\right]+\\
(a-2x)\left[\begin{array}{l}\ln\dfrac{-\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a-2x)^2+(b+2y)^2+(l+2z)^2}}{\left(l-2z+\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}\right)(l+2z)^2}-\\
\ln\dfrac{-\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a-2x)^2+(b-2y)^2+(l+2z)^2}}{\left(l-2z+\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}\right)(l+2z)^2}+\\
\ln\dfrac{\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a-2x)^2+(b+2y)^2+(l+2z)^2}}{\left(2z-l+\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}\right)(l+2z)^2}-\\
\ln\dfrac{\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a-2x)^2+(b-2y)^2+(l+2z)^2}}{\left(2z-l+\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}\right)(l+2z)^2}\end{array}\right]+\\
(a+2x)\left[\begin{array}{l}\ln\dfrac{-\sqrt{(a+2x)^2+(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a+2x)^2+(b+2y)^2+(l+2z)^2}}{\left(l-2z+\sqrt{(a+2x)^2+(b+2y)^2+(l-2z)^2}\right)(l+2z)^2}-\\
\ln\dfrac{-\sqrt{(a+2x)^2+(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a+2x)^2+(b-2y)^2+(l+2z)^2}}{\left(l-2z+\sqrt{(a+2x)^2+(b-2y)^2+(l-2z)^2}\right)(l+2z)^2}+\\
\ln\dfrac{\sqrt{(a+2x)^2+(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a+2x)^2+(b+2y)^2+(l+2z)^2}}{\left(2z-l+\sqrt{(a+2x)^2+(b+2y)^2+(l-2z)^2}\right)(l+2z)^2}-\\
\ln\dfrac{\sqrt{(a+2x)^2+(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a+2x)^2+(b-2y)^2+(l+2z)^2}}{\left(2z-l+\sqrt{(a+2x)^2+(b-2y)^2+(l-2z)^2}\right)(l+2z)^2}\end{array}\right]
\end{array}\right\} \tag{A1}$$

$$H_x(x,y=\frac{b}{2})=\operatorname*{Limit}_{y\to\frac{b}{2}}H_x(x,y)=$$

$$=\frac{1}{16\pi a b}\left\{\begin{array}{l}
-4b\left[\begin{array}{l}\operatorname{arctg}\dfrac{(a-2x)(l-2z)}{2b\sqrt{4b^2+(a-2x)^2+(l-2z)}}+\operatorname{arctg}\dfrac{(a+2x)(l-2z)}{2b\sqrt{4b^2+(a+2x)^2+(l-2z)}}+\\
\operatorname{arctg}\dfrac{(a-2x)(l+2z)}{2b\sqrt{4b^2+(a-2x)^2+(l+2z)}}+\operatorname{arctg}\dfrac{(a+2x)(l+2z)}{2b\sqrt{4b^2+(a+2x)^2+(l+2z)}}\end{array}\right]+\\
2(a+2x)\left[\ln\dfrac{2z-l+\sqrt{(a+2x)^2+(l-2z)^2}}{2z-l+\sqrt{4b^2+(a+2x)^2+(l-2z)^2}}\right]+2(a-2x)\left[\ln\dfrac{2z-l+\sqrt{(a-2x)^2+(l-2z)^2}}{2z-l+\sqrt{4b^2+(a-2x)^2+(l-2z)^2}}\right]+\\
(a+2x)\left[\begin{array}{l}\ln\dfrac{-\sqrt{4b^2+(a+2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4b^2+(a+2x)^2+(l+2z)^2}}{-\sqrt{(a+2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a+2x)^2+(l+2z)^2}}+\\
\ln\dfrac{\sqrt{4b^2+(a+2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4b^2+(a+2x)^2+(l+2z)^2}}{\sqrt{(a+2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a+2x)^2+(l+2z)^2}}\end{array}\right]+\\
(a-2x)\left[\begin{array}{l}\ln\dfrac{-\sqrt{4b^2+(a-2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4b^2+(a-2x)^2+(l+2z)^2}}{-\sqrt{(a-2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a-2x)^2+(l+2z)^2}}+\\
\ln\dfrac{\sqrt{4b^2+(a-2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4b^2+(a-2x)^2+(l+2z)^2}}{\sqrt{(a-2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a-2x)^2+(l+2z)^2}}\end{array}\right]+\\
2(1-2z)\left[\ln\dfrac{-a+2x+\sqrt{(a-2x)^2+(l-2z)^2}}{-a+2x+\sqrt{4b^2+(a-2x)^2+(l-2z)^2}}-\ln\dfrac{a+2x+\sqrt{(a+2x)^2+(l-2z)^2}}{a+2x+\sqrt{4b^2+(a+2x)^2+(l-2z)^2}}\right]+\\
2(1+2z)\left[\ln\dfrac{-a+2x+\sqrt{(a-2x)^2+(l+2z)^2}}{-a+2x+\sqrt{4b^2+(a-2x)^2+(l+2z)^2}}-\ln\dfrac{a+2x+\sqrt{(a+2x)^2+(l+2z)^2}}{a+2x+\sqrt{4b^2+(a+2x)^2+(l+2z)^2}}\right]
\end{array}\right\} \tag{A2}$$

APPENDIX

$$
\begin{aligned}
&H_x\left(x, y=-\frac{b}{2}\right)=\operatorname*{Limit}_{y\to-\frac{b}{2}} H_x(x,y)= \\
&=\frac{1}{16\pi a b}\left\{
\begin{aligned}
&4b\left[\begin{aligned}&\operatorname{arctg}\frac{(a-2x)(l-2z)}{2b\sqrt{4b^2+(a-2x)^2+(l-2z)^2}}+\operatorname{arctg}\frac{(a+2x)(l-2z)}{2b\sqrt{4b^2+(a+2x)^2+(l-2z)^2}}+\\&\operatorname{arctg}\frac{(a-2x)(l+2z)}{2b\sqrt{4b^2+(a-2x)^2+(l+2z)^2}}+\operatorname{arctg}\frac{(a+2x)(l+2z)}{2b\sqrt{4b^2+(a+2x)^2+(l+2z)^2}}\end{aligned}\right]+\\
&2(a+2x)\left[\ln\frac{2z-l+\sqrt{4b^2+(a+2x)^2+(l-2z)^2}}{2z-l+\sqrt{(a+2x)^2+(l-2z)^2}}\right]+2(a-2x)\left[\ln\frac{2z-l+\sqrt{4b^2+(a-2x)^2+(l-2z)^2}}{2z-l+\sqrt{(a-2x)^2+(l-2z)^2}}\right]+\\
&(a+2x)\left[\begin{aligned}&\ln\frac{-\sqrt{(a+2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a+2x)^2+(l+2z)^2}}{-\sqrt{4b^2+(a+2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4b^2+(a+2x)^2+(l+2z)^2}}+\\&\ln\frac{\sqrt{(a+2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a+2x)^2+(l+2z)^2}}{\sqrt{4b^2+(a+2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4b^2+(a+2x)^2+(l+2z)^2}}\end{aligned}\right]+\\
&(a-2x)\left[\begin{aligned}&\ln\frac{-\sqrt{(a-2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a-2x)^2+(l+2z)^2}}{-\sqrt{4b^2+(a-2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4b^2+(a-2x)^2+(l+2z)^2}}+\\&\ln\frac{\sqrt{(a-2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a-2x)^2+(l+2z)^2}}{\sqrt{4b^2+(a-2x)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4b^2+(a-2x)^2+(l+2z)^2}}\end{aligned}\right]+\\
&2(1-2z)\left[-\ln\frac{-a+2x+\sqrt{(a-2x)^2+(l-2z)^2}}{-a+2x+\sqrt{4b^2+(a-2x)^2+(l-2z)^2}}+\ln\frac{a+2x+\sqrt{(a+2x)^2+(l-2z)^2}}{a+2x+\sqrt{4b^2+(a+2x)^2+(l-2z)^2}}\right]-\\
&2(1+2z)\left[\ln\frac{-a+2x+\sqrt{(a-2x)^2+(l+2z)^2}}{-a+2x+\sqrt{4b^2+(a-2x)^2+(l+2z)^2}}+\ln\frac{a+2x+\sqrt{(a+2x)^2+(l+2z)^2}}{a+2x+\sqrt{4b^2+(a+2x)^2+(l+2z)^2}}\right]
\end{aligned}
\right\}
\end{aligned}
\tag{A3}
$$

$$
H_y(x,y)=\frac{1}{16\pi a b}\left\{
\begin{aligned}
&2(a+2x)\left[\begin{aligned}&\operatorname{arctg}\frac{(b-2y)(l-2z)}{(a+2x)\sqrt{(a+2x)^2+(b-2y)^2+(l-2z)^2}}+\operatorname{arctg}\frac{(b+2y)(l-2z)}{(a+2x)\sqrt{(a+2x)^2+(b+2y)^2+(l-2z)^2}}+\\&\operatorname{arctg}\frac{(b-2y)(l+2z)}{(a+2x)\sqrt{(a+2x)^2+(b-2y)^2+(l+2z)^2}}+\operatorname{arctg}\frac{(b+2y)(l+2z)}{(a+2x)\sqrt{(a+2x)^2+(b+2y)^2+(l+2z)^2}}\end{aligned}\right]-\\
&2(a-2x)\left[\begin{aligned}&\operatorname{arctg}\frac{(b-2y)(l-2z)}{(a-2x)\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}}+\operatorname{arctg}\frac{(b+2y)(l-2z)}{(a-2x)\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}}+\\&\operatorname{arctg}\frac{(b-2y)(l+2z)}{(a-2x)\sqrt{(a-2x)^2+(b-2y)^2+(l+2z)^2}}+\operatorname{arctg}\frac{(b+2y)(l+2z)}{(a-2x)\sqrt{(a-2x)^2+(b+2y)^2+(l+2z)^2}}\end{aligned}\right]+\\
&2(l-2z)\left[\ln\frac{-b+2y+\sqrt{(a+2x)^2+(b-2y)^2+(l-2z)^2}}{-b+2y+\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}}+\ln\frac{b+2y+\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}}{b+2y+\sqrt{(a+2x)^2+(b+2y)^2+(l+2z)^2}}\right]+\\
&2(l+2z)\left[\ln\frac{-b+2y+\sqrt{(a+2x)^2+(b-2y)^2+(l+2z)^2}}{-b+2y+\sqrt{(a-2x)^2+(b-2y)^2+(l+2z)^2}}+\ln\frac{b+2y+\sqrt{(a-2x)^2+(b+2y)^2+(l+2z)^2}}{b+2y+\sqrt{(a+2x)^2+(b+2y)^2+(l+2z)^2}}\right]+\\
&(b-2y)\left[\ln\frac{l-2z+\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}}{l-2z+\sqrt{(a+2x)^2+(b-2y)^2+(l-2z)^2}}+\ln\frac{2z-l+\sqrt{(a+2x)^2+(b-2y)^2+(l-2z)^2}}{2z-l+\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}}\right]+\\
&(b+2y)\left[\ln\frac{l-2z+\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}}{l-2z+\sqrt{(a+2x)^2+(b+2y)^2+(l-2z)^2}}+\ln\frac{2z-l+\sqrt{(a+2x)^2+(b+2y)^2+(l-2z)^2}}{2z-l+\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}}\right]+\\
&(b-2y)\left[\begin{aligned}&\ln\frac{-\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a-2x)^2+(b-2y)^2+(l+2z)^2}}{\left(l-2z+\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}\right)l+2z)^2}-\\&\ln\frac{-\sqrt{(a+2x)^2+(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a+2x)^2+(b-2y)^2+(l+2z)^2}}{\left(l-2z+\sqrt{(a+2x)^2+(b-2y)^2+(l-2z)^2}\right)l+2z)^2}+\\&\ln\frac{\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a-2x)^2+(b-2y)^2+(l+2z)^2}}{\left(2z-l+\sqrt{(a-2x)^2+(b-2y)^2+(l-2z)^2}\right)l+2z)^2}-\\&\ln\frac{\sqrt{(a+2x)^2+(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a+2x)^2+(b-2y)^2+(l+2z)^2}}{\left(2z-l+\sqrt{(a+2x)^2+(b-2y)^2+(l-2z)^2}\right)l+2z)^2}\end{aligned}\right]+\\
&(b+2y)\left[\begin{aligned}&\ln\frac{-\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a-2x)^2+(b+2y)^2+(l+2z)^2}}{\left(l-2z+\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}\right)l+2z)^2}-\\&\ln\frac{-\sqrt{(a+2x)^2+(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a+2x)^2+(b+2y)^2+(l+2z)^2}}{\left(l-2z+\sqrt{(a+2x)^2+(b+2y)^2+(l-2z)^2}\right)l+2z)^2}+\\&\ln\frac{\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a-2x)^2+(b+2y)^2+(l+2z)^2}}{\left(2z-l+\sqrt{(a-2x)^2+(b+2y)^2+(l-2z)^2}\right)l+2z)^2}-\\&\ln\frac{\sqrt{(a+2x)^2+(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(a+2x)^2+(b+2y)^2+(l+2z)^2}}{\left(2z-l+\sqrt{(a+2x)^2+(b+2y)^2+(l-2z)^2}\right)l+2z)^2}\end{aligned}\right]
\end{aligned}
\right\}
\tag{A4}
$$

APPENDIX

$$
\begin{aligned}
H_x(x=\frac{a}{2},y) &= \operatorname*{Limit}_{x\to\frac{a}{2}} H_x(x,y) = \\
&= \frac{1}{16\pi a b}\left\{
\begin{aligned}
& 4a\left[\begin{aligned}
& \operatorname{arctg}\frac{(b-2y)(l-2z)}{2a\sqrt{4a^2+(b-2y)^2+(l-2z)}}+\operatorname{arctg}\frac{(b+2y)(l-2z)}{2a\sqrt{4a^2+(b+2y)^2+(l-2z)}}+\\
& \operatorname{arctg}\frac{(b-2y)(l+2z)}{2a\sqrt{4a^2+(b-2y)^2+(l+2z)}}+\operatorname{arctg}\frac{(b+2y)(l+2z)}{2a\sqrt{4a^2+(b+2y)^2+(l+2z)}}
\end{aligned}\right]+\\
& 2(b+2y)\left[\ln\frac{2z-l+\sqrt{4a^2+(b+2y)^2+(l-2z)^2}}{2z-l+\sqrt{(b+2y)^2+(l-2z)^2}}\right]+2(b-2y)\left[\ln\frac{2z-l+\sqrt{4a^2+(b-2y)^2+(l-2z)^2}}{2z-l+\sqrt{(b-2y)^2+(l-2z)^2}}\right]+\\
& (b+2y)\left[\begin{aligned}
& \ln\frac{-\sqrt{(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(b+2y)^2+(l+2z)^2}}{-\sqrt{4a^2+(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4a^2+(b+2y)^2+(l+2z)^2}}+\\
& \ln\frac{\sqrt{(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(b+2y)^2+(l+2z)^2}}{\sqrt{4a^2+(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4a^2+(b+2y)^2+(l+2z)^2}}
\end{aligned}\right]+\\
& (b-2y)\left[\begin{aligned}
& \ln\frac{-\sqrt{(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(b-2y)^2+(l+2z)^2}}{-\sqrt{4a^2+(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4a^2+(b-2y)^2+(l+2z)^2}}+\\
& \ln\frac{\sqrt{(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(b-2y)^2+(l+2z)^2}}{\sqrt{4a^2+(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4a^2+(b-2y)^2+(l+2z)^2}}
\end{aligned}\right]+\\
& 2(1-2z)\left[-\ln\frac{-b+2y+\sqrt{(b-2y)^2+(l-2z)^2}}{-b+2y+\sqrt{4a^2+(b-2y)^2+(l-2z)^2}}+\ln\frac{b+2y+\sqrt{(b+2y)^2+(l-2z)^2}}{b+2y+\sqrt{4a^2+(b+2y)^2+(l-2z)^2}}\right]+\\
& 2(1+2z)\left[-\ln\frac{-b+2y+\sqrt{(b-2y)^2+(l+2z)^2}}{-b+2y+\sqrt{4a^2+(b-2y)^2+(l+2z)^2}}+\ln\frac{b+2y+\sqrt{(b+2y)^2+(l+2z)^2}}{b+2y+\sqrt{4a^2+(b+2y)^2+(l+2z)^2}}\right]
\end{aligned}\right\}
\end{aligned}
\qquad \text{(A5)}
$$

$$
\begin{aligned}
H_x(x=-\frac{a}{2},y) &= \operatorname*{Limit}_{x\to-\frac{a}{2}} H_x(x,y) = \\
&= \frac{1}{16\pi a b}\left\{
\begin{aligned}
& -4a\left[\begin{aligned}
& \operatorname{arctg}\frac{(b-2y)(l-2z)}{2a\sqrt{4a^2+(b-2y)^2+(l-2z)}}+\operatorname{arctg}\frac{(b+2y)(l-2z)}{2a\sqrt{4a^2+(b+2y)^2+(l-2z)}}+\\
& \operatorname{arctg}\frac{(b-2y)(l+2z)}{2a\sqrt{4a^2+(b-2y)^2+(l+2z)}}+\operatorname{arctg}\frac{(b+2y)(l+2z)}{2a\sqrt{4a^2+(b+2y)^2+(l+2z)}}
\end{aligned}\right]+\\
& 2(b+2y)\left[\ln\frac{2z-l+\sqrt{(b+2y)^2+(l-2z)^2}}{2z-l+\sqrt{4a^2+(b+2y)^2+(l-2z)^2}}\right]+2(b-2y)\left[\ln\frac{2z-l+\sqrt{(b-2y)^2+(l-2z)^2}}{2z-l+\sqrt{4a^2+(b-2y)^2+(l-2z)^2}}\right]+\\
& (b+2y)\left[\begin{aligned}
& \ln\frac{-\sqrt{4a^2+(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4a^2+(b+2y)^2+(l+2z)^2}}{-\sqrt{(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(b+2y)^2+(l+2z)^2}}+\\
& \ln\frac{\sqrt{4a^2+(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4a^2+(b+2y)^2+(l+2z)^2}}{\sqrt{(b+2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(b+2y)^2+(l+2z)^2}}
\end{aligned}\right]+\\
& (a-2x)\left[\begin{aligned}
& \ln\frac{-\sqrt{4a^2+(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4a^2+(b-2y)^2+(l+2z)^2}}{-\sqrt{(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(b-2y)^2+(l+2z)^2}}+\\
& \ln\frac{\sqrt{4a^2+(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{4a^2+(b-2y)^2+(l+2z)^2}}{\sqrt{(b-2y)^2+(l-2z)^2}(l-2z)+8lz+(l+2z)\sqrt{(b-2y)^2+(l+2z)^2}}
\end{aligned}\right]+\\
& 2(1-2z)\left[\ln\frac{-b+2y+\sqrt{(b-2y)^2+(l-2z)^2}}{-b+2y+\sqrt{4a^2+(b-2y)^2+(l-2z)^2}}-\ln\frac{b+2y+\sqrt{(b+2y)^2+(l-2z)^2}}{b+2y+\sqrt{4a^2+(b+2y)^2+(l-2z)^2}}\right]+\\
& 2(1+2z)\left[\ln\frac{-b+2y+\sqrt{(b-2y)^2+(l+2z)^2}}{-b+2y+\sqrt{4a^2+(b-2y)^2+(l+2z)^2}}-\ln\frac{b+2y+\sqrt{(b+2y)^2+(l+2z)^2}}{b+2y+\sqrt{4a^2+(b+2y)^2+(l+2z)^2}}\right]
\end{aligned}\right\}
\end{aligned}
\qquad \text{(A6)}
$$