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September 10 - 12, 2007

Pilsen, Czech Republic

# MECHANISMS OF NONLINEAR AND CHAOTIC OSCILLATIONS OCCURRENCE

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**Abstract**: *In this paper three different mechanisms of nonlinear and chaotic oscillation occurrence are studied. The first model comes from antisymmetric system structure and oscillations are caused by impossibility to reach the equilibrium state or to diverge to infinity. The second case studies phenomena in Lorenz system of three differential equations. The third case comes from study of the electronic circuit with comparator with hysteresis. It is shown that switching with hysteresis can cause chaotic oscillations as well.*

**Key words**: *chaos, equilibrium state, hysteresis, nonlinear oscillations*

## INTRODUCTION

In mathematics and dynamical system s theory there are a lot of systems that can show the nonlinear periodic or chaotic oscillations. The first numerical experiments related to the mathematical models of fluid flow, but now we know that nonlinear resonance or chaotic oscillations can occur in energetic systems as well as in electronic circuits. Let us generally describe three different possibilities how the chaotic oscillations can occur.

## 1 SYSTEM OF 4. ORDER WITH ANTISYMMETRIC STRUCTURE

### 1.1 The system structure

The antisymmetric structure corresponds the energy conservation law as defined in [1]. A stability of the system is given by the sign of the dissipation parameter α1. Let us consider a system whose representation is given by

$$R\{S\}: \quad \dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} \tag{1}$$

$$y = \mathbf{C}\,\mathbf{x} \tag{2}$$

where matrices **A** and **C** are given by

$$\mathbf{A} = \begin{bmatrix} -a_1 & a_2 & 0 & 0 \\ -a_2 & 0 & a_3 & 0 \\ 0 & -a_3 & 0 & a_4 \\ 0 & 0 & -a_4 & 0 \end{bmatrix} \tag{3}$$

$$\mathbf{C} = \begin{bmatrix} g & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

and the parameters are

$$a_1 = k_0 + k_1 x_2^2, \qquad k_i \in \{-1 \quad 1\} \tag{5}$$

$$a_2 = a_3 = g = 1 \tag{6}$$

$$a_4 = \mathrm{var}. \tag{7}$$

The system has the only one equilibrium state at the beginning of the coordinate system

$$\mathbf{x}^* = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

The abstract signal energy and signal power are given by

$$E = \frac{1}{2}\|\mathbf{x}\|^2 = \frac{1}{2}\sum_{i=1}^{4} x_i^2 \tag{9}$$

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial E}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} = \mathrm{grad}E \cdot \dot{\mathbf{x}} = \\ = -a_1 x_1^2 = -k_0 x_1^2 - k_1 x_1^2 x_2^2 \tag{10}$$

The power can be decomposed into two separate parts and it is obvious that there are four technically different ways of the system behavior.

When both the constants $k_0$ and $k_1$ are positive, the equilibrium state is stable and the system is dissipative. Energy of the system decreases monotonously which corresponds with the energy conservation law and the system dissipativity (fig. 1).

When both the constants are negative, the equilibrium state is unstable, the power is anti-dissipative and the energy monotonously arises to infinity (fig. 2).

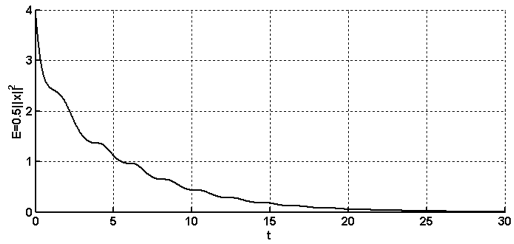


*Fig. 1.: The system energy evolution,* $k_0 = k_1 = 1$, $a_4 = 1$, $\mathbf{x_0} = [2, 0, 0, 2]^T$

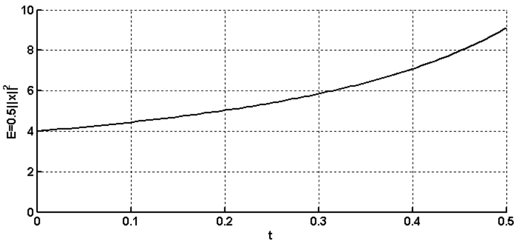


*Fig. 2.: The system energy evolution,* $k_0 = k_1 = -1$, $a_4 = 1$, $\mathbf{x_0} = [2, 0, 0, 2]^T$

When $k_0 = 1$ and $k_1 = -1$, the situation is different. The equilibrium state is stable but stability of the system depends on the actual position of the state vector in the state space, especially on the absolute value of the component $x_2$. When that component exceeds interval $\langle -1,1 \rangle$ the system becomes unstable. When the state vector comes near the equilibrium state the dissipative mode prevails and the system becomes stable (fig. 3).

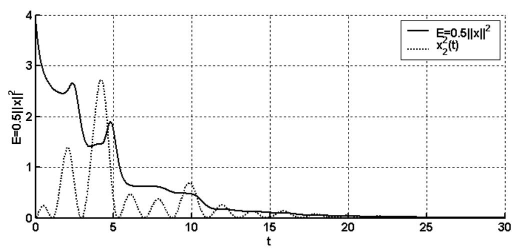


*Fig. 3.: The system energy evolution,* $k_0 = 1$, $k_1 = -1$, $a_4 = 1$, $\mathbf{x_0} = [2, 0, 0, 2]^T$

The fourth case brings most interesting situations. When $k_0 = -1$ and $k_1 = 1$ the equilibrium state is unstable. When the state vector comes near this point the system becomes unstable and the state vector trajectory starts diverging. But now the type of nonlinearity does not allow the state vector trajectory diverge to infinity. When the absolute value of the component $x_2$ exceeds 1 the system becomes stable, dissipative and starts converge to zero again. It means that now there are some intervals of stability and some intervals of instability. Their changing can be periodic or aperiodic which depends on the control parameter $\alpha_4$. Aperiodic changes of intervals of stability and instability can be called chaotic.

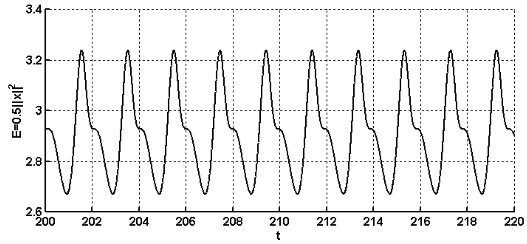


*Fig. 4.: The system energy evolution, the periodic case,* $k_0 = -1$, $k_1 = 1$, $a_4 = 1$, $\mathbf{x_0} = [2, 0, 0, 2]^T$

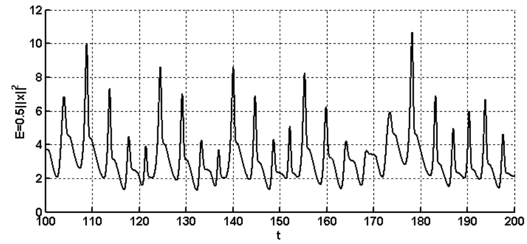


*Fig. 5.: The system energy evolution, the chaotic case,* $k_0 = -1$, $k_1 = 1$, $a_4 = 2$, $\mathbf{x_0} = [2, 0, 0, 2]^T$

## *1.2 Types of nonlinearities and chaotic oscillations*

Considering the possibility of chaotic oscillations occurrence, we can try to change the particular type of nonlinearity. The type of nonlinearity in the dissipation parameter $\alpha_1$ must satisfy the condition of instability the equilibrium state and must prevent the state vector trajectory to diverge to infinity.

These experiments were made with these types of nonlinearities in the dissipation parameter:

$$a_1^{(1)} = -1 + \left| x_2^3 \right| \tag{11}$$

$$a_1^{(2)} = -2 + e^{|x_2|} \tag{12}$$

$$a_1^{(3)} = -2\cos(2x_2) \tag{13}$$

All these functions including (5) are depicted in the fig. 6.

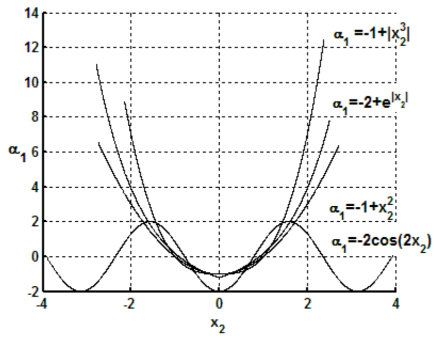


*Fig. 6.: Four types of nonlinearities used in experiments*

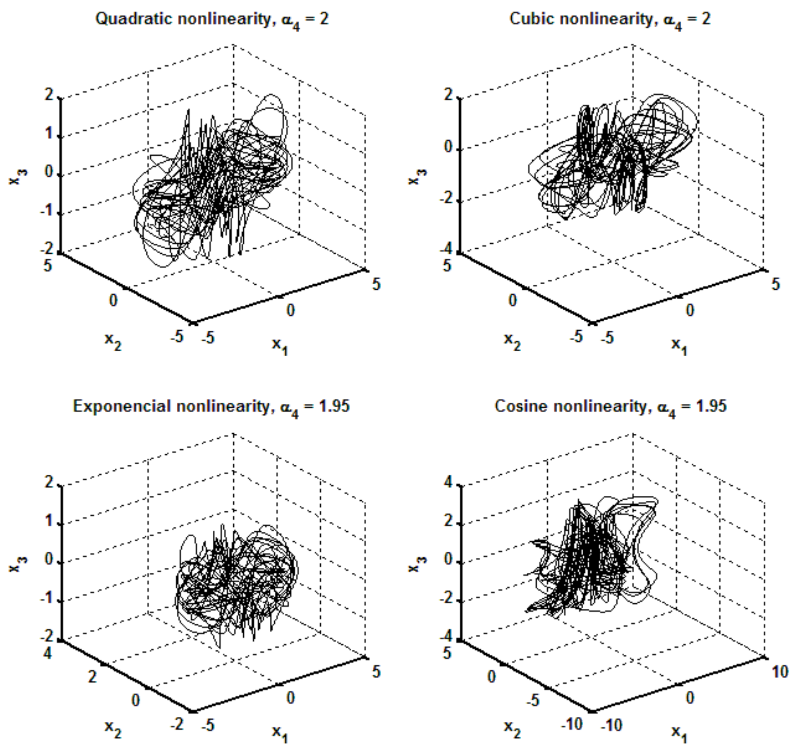


*Fig. 7.: Trajectories of the state vector in the state space, 3D projection. Depicted in time $t \in \langle 350,500 \rangle$*

The results of these experiments are that considering the possibility of chaotic oscillations occurrence the particular of nonlinearity is not critical. Periodical changes of intervals of stability and instability loose their periodicity with increasing the control parameter $\alpha_4$ and become chaotic for some value of that parameter in all those cases which is documented in the figure (7).

## 2 System with more equilibrium states

There are some systems whose structure is not anti-symmetrical and we are not supposed to say anything about their stability or instability only by one parameter. The nonlinearity is not present only in one parameter and there are more than one equilibrium state. Methods of investigating their stability are mire difficult. One of those systems is well known Lorenz system, given by these differential equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -s & s & 0 \\ r & -1 & -x_1 \\ & x_1 & -b \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (14)$$

The model came from approximation of partial differential equations describing incompressible fluid flow.

There are three equilibrium states (15) and their stability depends on values of parameters $\sigma$, $r$ and $b$.

$$\begin{aligned} \mathbf{x}_0^* &= [0 \quad 0 \quad 0]^T \\ \mathbf{x}_1^* &= [\sqrt{b(r-1)} \quad \sqrt{b(r-1)} \quad r-1]^T \\ \mathbf{x}_2^* &= [-\sqrt{b(r-1)} \quad -\sqrt{b(r-1)} \quad r-1]^T \end{aligned} \qquad (15)$$

At the figure (8) there is the evolution of these equilibrium states ($x_2$ component) and their stability in dependency of parameter $r$ value. For experiments $\sigma = 10$, $s = 5$ and $r$ = var. were chosen.

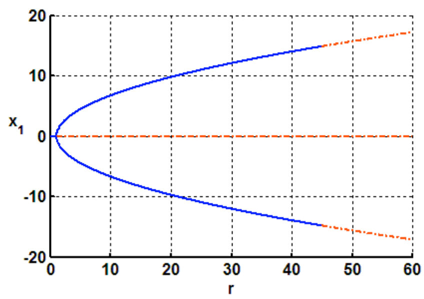


*Fig. 8.: Equilibrium states (blue line – stable, red – unstable) in dependency on the r parameter.*

In the figure (9) we can see how the stability of the three equilibrium states affects the state vector trajectory. For small values of the parameter $r$ there is only one steady state and the trajectory leads towards it very quickly. As the parameter grows at the point of 1 the bifurcation of the equilibrium states takes place. The first equilibrium state becomes unstable and the trajectory is attracted to one of the other stable states, the particular point depends on initial conditions (trajectories at fig. 9).

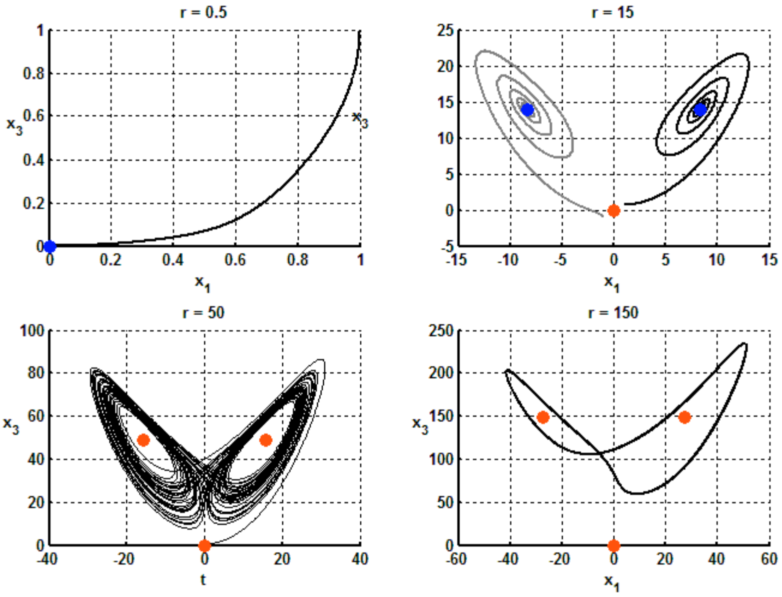


*Fig. 9.: Attractor of Lorenz system, 2D projection, blue/red points: stable/unstable equilibrium states*

Another growing of the $r$ parameter causes instability of all the equilibrium states. Then the trajectory becomes chaotic. It moves in diverging spiral with center in one of the equilibrium states. But when the $x_1$ component changes its sign the state vector trajectory starts to be affected by the second unstable equilibrium state. The state vector moves in diverging spiral again but the center of the spiral is the second equilibrium state. Both these states symmetrically placed in the state space changed their influence. This phenomenon repeats again and again, aperiodically (chaotically) or periodically which depends on the r parameter value.

The mechanism of chaotic oscillations occurrence is different from the first case. Results for so called geodynamo [2] are very similar. The system of three differential equations can show the chaotic oscillations by the similar mechanism – the state vector moves along two unstable equilibrium states in diverging spirals.

## 3 SYSTEM WITH HYSTERESIS AND SATURATION

The third mechanism leading to nonlinear and chaotic oscillations is hysteresis in system of two technically linear differential equations.

$$\dot{x}_1 = Ax_1 + Bx_2 \tag{16}$$

$$\dot{x}_2 = Cx_1 + Dx_2 \pm E(x_1) \tag{17}$$

The first nonlinearity is changing of constant $E$ sign depending on the first variable value (switching) and sign of its derivation (the hysteresis of width $\pm R$). Second nonlinearity is present in the saturation of the first variable. It can move only in given limits $\pm K$. When the $x_1$ variable comes to its limits the equations (16), (17) change into

$$\dot{x}_1 = 0 \tag{18}$$

$$\dot{x}_2 = \pm CK + Dx_2 \pm E \tag{19}$$

These nonlinearities are typical for magnetic circuits and for circuits with comparators, operational amplifiers and semiconductor components. The hysteresis is depicted in the figure (10).

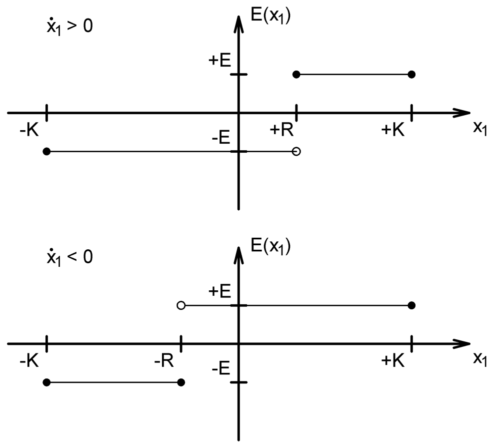


*Fig. 10: Hysteresis of the constant E sign switching and saturation of the $x_1$ variable*

This case is very interesting with its possibility to find the analytical solution of equations (16) – (19) . The resultant trajectory of the state vector arises as the connection of partial analytical solutions valid for given initial conditions that changes with the system evolution.

This system has only one equilibrium state given by

$$\mathbf{x}^*_{1,2} = \left[ \frac{BE}{AD-BC} \quad \pm \frac{AE}{AD-BC} \right]^T \tag{20}$$

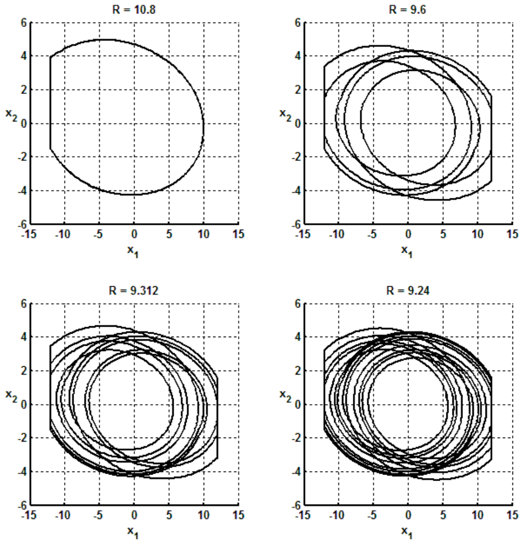


*Fig. 11: Trajectory of the state vector in the state space, periodic solution*

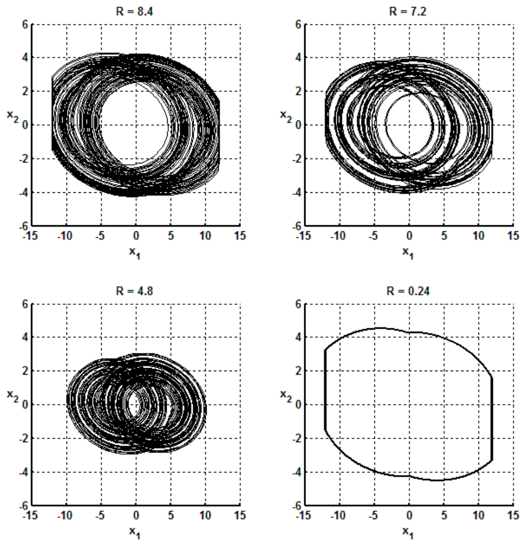


*Fig. 12: Trajectory of the state vector in the state space, chaotic and periodic solutions*

As the sign of $E$ constant changes, the equilibrium state changes its position in the state space. That is the reason why there seem to be two equilibrium states.

In following experiments the width of the hysteresis loop is chosen as the parameter. All the constants have these values:

$$\begin{aligned} &A=-303 \quad B=-10000 \quad C=1378 \\ &D=577 \quad E=-(+)3114 \quad K=+(-)12 \end{aligned} \tag{21}$$

and the parameter $R$ changes from 0 to $K$:

$$R \in \langle 0, K \rangle \tag{22}$$

In the figure (11) we can see periodic solutions. As the parameter $R$ declines it is possible to observe changes of the period. It suddenly becomes larger. The first periodic attractor disappears and the second one arises at that moment. The parameter changes are continuous but the period changes are discrete. This process repeats several times and then the attractor becomes chaotic as we can see in the figure (12).

The mechanism of these chaotic oscillations occurrence is little bit similar to the case studied at the Lorenz system. The state vector moves around the only one unstable equilibrium state in diverging spiral. At the moment of changing the sign of the $E$ constant the equilibrium state changes its position in the state space. The state vector continues its move in diverging spiral that has different center now. The system is unstable and it tends to divergence but switching in hysteresis holds the trajectory of the state vector in finite area.

## 4 CONCLUSION

Three different mechanisms leading to nonlinear and chaotic oscillations were studied. In the first case the chaotic oscillations were consequence of changing intervals of stability and instability of the system. It was experimentally shown that considering the possibility of the chaotic oscillations occurrence the type of the nonlinearity in the dissipation parameter is important but the particular form is not critical.

The second case is well known Lorenz system where the state vector moves in diverging spirals around two unstable equilibrium states. This mechanism can be found in systems of two coupled disk dynamos as well [2].

The third case can be found in electronic circuits with comparators or components with possibility of saturation [3]. Switching in hysteresis loop together with saturation of state variables can cause periodic or aperiodic oscillations as well.

## 5 REFERENCES

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