Adaptivity techniques in *hp*-finite element method

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Abstract **Sophisticated methods of automatic adaptivity in finite element methods of higher order of accuracy are presented. The main attention is devoted to *hp*-adaptivity techniques that exhibit the highest level of flexibility and exponential convergence of results.**

Keywords: **automatic adaptivity; higher-order finite element method; convergence problems**

I. INTRODUCTION

All advanced techniques for numerical solution of physical fields contain special algorithms for automatic adaptivity of discretization meshes (see, for example [1–3]). These algorithms are applied at the moment when some local error of solution is higher than the acceptable tolerance. This error defined as the difference between the current numerical solution and exact solution is usually caused by locally rougher mesh, presence of one or more singular points, curvilinear boundaries or interfaces approximated by polygonal lines, etc. In all these cases, such errors must be identified in the course of computation and appropriate measures have to be taken for their fixing.

This technology is also implemented in the codes Hermes [4] and Agros [5] that have been developed for a couple of years in our group. Hermes is a library of numerical algorithms for monolithic and full adaptive solution of systems of generally nonlinear and nonstationary partial differential equations (PDEs) based on the finite element method of higher order of accuracy, while Agros is a powerful user interface serving for pre- and postprocessing of the problems solved.

The paper describes sophisticated adaptive techniques implemented in 2D version of both codes – Hermes 2D and Agros2D.

II. BASIC FEATURES OF HERMES2D AND AGROS2D

Both codes are based on a finite element method of higher order of accuracy and employ the most advanced adaptivity techniques that are not restricted to specific classes of PDEs. Therefore, they allow solving any system of PDEs in monolithic formulations, which can be used for solution of complex multiphysics problems (electromagnetic processing of solid and molten metals, electromechanical problems of any kind, induction heating-produced thermoelasticity, etc).

The algorithms of automatic adaptivity implemented in them may be divided into the following principal groups:

- Refinement of elements in regions exhibiting unacceptable errors. This way is called *h*-adaptivity – while the magnitude of the finite elements changes, the degree of the polynomials replacing the real distribution of the investigated quantity in them remains the same,
- Improvement of approximation of the investigated quantity in an element. This way is called *p*-adaptivity – the shapes of elements in the region do not change, but we increase the orders of the approximating polynomials.
- The combination of both above ways is called *hp*-adaptivity, which belongs to the most flexible and powerful techniques characterized by an extremely fast convergence of results.
- Curvature of edges of selected elements. This technique is quite original and we do not know any commercial SW that would use it (although, for example, ANSYS works with curvilinear elements that are, however, generated in a different manner).
- Combination of triangular and quadrilateral elements (in selected cases leads to reduction of the degrees of freedom – DOFs).

The *h*-adaptivity is the simplest one and is implemented in practically all existing codes. The elements burdened with high errors are divided into several smaller elements, but there exist even other possibilities. The principal problems accompanying this type of adaptivity are the hanging nodes appearing on the interfaces between the refined elements and elements without refinements. These nodes must be handled with particular care, otherwise they may significantly contribute to the growth of degrees of freedom of the problem solved [6].

The *p*-adaptivity is even simpler to implement, because the mesh remains unchanged. Only in the selected elements we enlarge the order of the corresponding approximating polynomials.

The *hp*-adaptivity represents the most complicated method and its implementation is highly nontrivial. On the other hand, it exhibits the exponential convergence of results and it was proven to be an extremely powerful tool just in the finite element methods of higher orders of accuracy.

The curved elements are applied in case of curvilinear boundaries and interfaces.

Algorithms of adaptivity start to be applied at the moment when some local error of solution is higher than the acceptable tolerance. Consider an equation

$$Lf = 0\,, \tag{1}$$

where L denotes a differential operator and f a function whose distribution over some domain Ω is to be found. If f' is its approximation obtained by numerical solution of (1), the absolute and percentage relative errors δ and η are defined by the relations

$$\delta = f - f', \quad \eta = 100|\delta / f|. \tag{2}$$

Other quantities that can be checked in this way are the norms. Hermes2D works with the basic energetic norm given by the expression

$$\|e\| = \left|\int_{\Omega} \delta(L\delta)\,\mathrm{d}\Omega\right|^{1/2}, \tag{3}$$

L^2 norm defined by the relation

$$\|e\|_{L^2} = \left|\int_{\Omega} \delta^2\,\mathrm{d}\Omega\right|^{1/2}, \tag{4}$$

and H^1 norm given by the expression

$$\|e\|_{H^1} = \left|\int_{\Omega}\left(\delta^2 + (\operatorname{grad}\delta)\cdot(\operatorname{grad}\delta)\right)\mathrm{d}\Omega\right|^{1/2}. \tag{5}$$

Unfortunately, the exact solution f is known only in very simple analytically solvable cases. Moreover, there exists no general method that would provide a good estimation of the error for an arbitrary PDE. That is why we work [7] with the reference solution f_{ref} instead, that is obtained either by a refinement of the whole mesh (*h*-adaptivity), by enlargement of the polynomial degree (*p*-adaptivity) or by both above techniques (*hp*-adaptivity). In this manner we get the candidates for adaptivity even without knowledge of the exact solution f. The library Hermes2D works with very sophisticated and subtle tools based on the above considerations.

A convergence curve typical for the *hp*-adaptivity (in comparison with the classical *h*-adaptivity or *p*-adaptivity) is depicted in Fig. 1. The dependence of the relative error and number of DOFs is exponential.

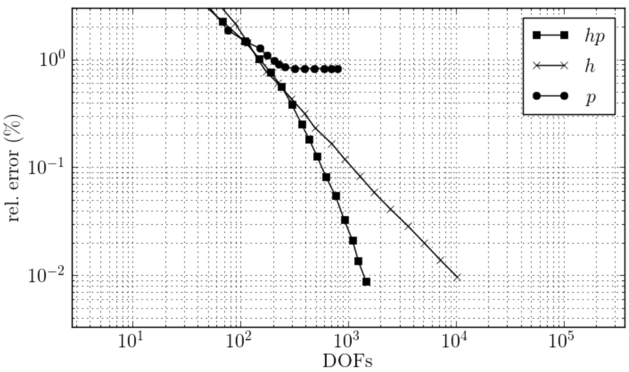

Fig. 1. A typical comparison of adaptive low-order FEM and *hp*-FEM

Other strongly advanced features characterizing Hermes 2D and Agros 2D are:

- Support of hanging nodes of any level. Usually, the hanging nodes bring about a considerable increase of the number of DOFs. The code contains higher-order algorithms for respecting these nodes without any need of an additional treatment.
- Multimesh technique. Each physical field can be solved on quite a different mesh that best corresponds to its particulars. Special powerful higher-order techniques of mapping are then used to avoid any numerical errors in the process of assembly of the final stiffness matrix. In nonstationary processes, moreover, every mesh can change in time, in accordance with the real evolution of the corresponding physical quantities.

A typical mesh containing hanging nodes is shown in Fig. 2, while Fig. 3 shows two meshes used (in one task) for computation of the distribution of two different physical quantities.

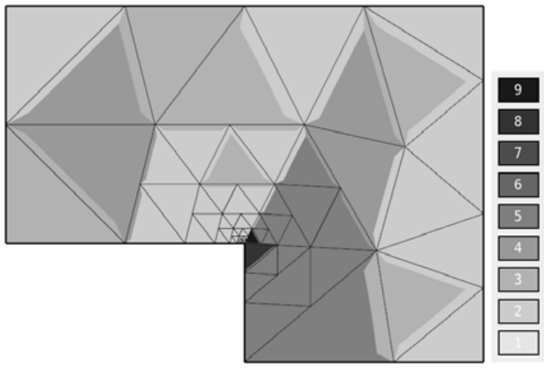

Fig. 2. A mesh containing hanging nodes (the numbers in rectangles on the right side denote the orders of polynomials in particular elements)

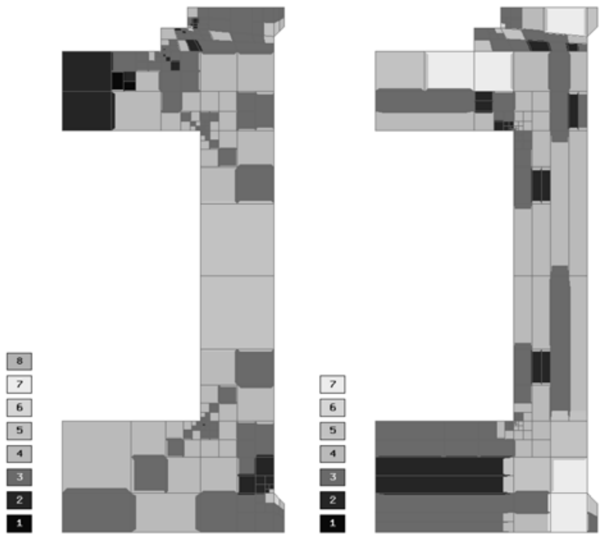

Fig. 3. Two meshes used for computation of the distribution of two different physical quantities in one problem.

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