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Numerical experiments in spatial patterning

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1 Introduction

In general, we study a system of reaction-diffusion equations with certain functions $f(u,v), g(u,v)$ describing reaction kinetics of two chemical substances. Let's consider the system

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \Delta u + b_{1,1}(u-\overline{u}) + b_{1,2}(v-\overline{v}) + n_1(u,v), \\ \frac{\partial v}{\partial t} &= d_2 \Delta v + b_{2,1}(u-\overline{u}) + b_{2,2}(v-\overline{v}) + n_2(u,v), \end{aligned} \tag{1}$$

with zero Neumann boundary conditions. The numbers d_1, d_2 are positive diffusion parameters, $b_{i,j}$ are constant elements of Jacobi matrix of the functions f, g in the constant stationary solution $[\overline{u}, \overline{v}]$ and $n_{1,2}$ are higher order terms of Taylor expansion around $[\overline{u}, \overline{v}]$. It was proposed by Turing (1952) that under some conditions on $b_{i,j}$ the stationary solution of the system (1) without diffusion ($d_1 = d_2 = 0$) is stable, but with diffusion it is unstable. Such effect was later called "diffusion driven instability". The loss of the stability of the constant stationary solution gives rise to the spatially non-homogeneous stationary solutions. These solutions describe patterns, which have application as patterns on animal coat, for example.

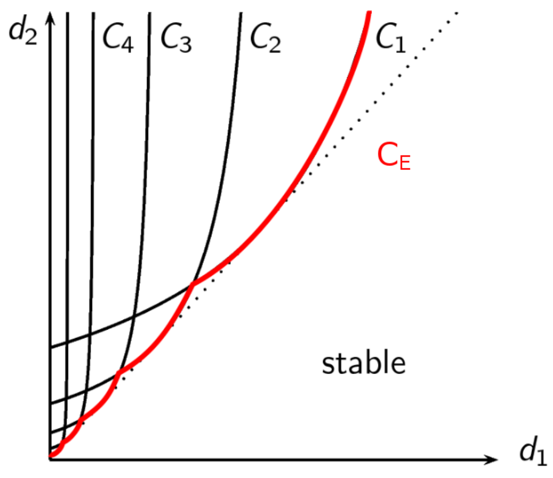

Figure 1: The envelope C_E of hyperbolas in the plane $[d_1, d_2]$

The positive quadrant of parameters $[d_1, d_2] \in \mathbb{R}^2_+$ can be divided by a curve C_E on two regions, i.e. region of stability and instability. The region of instability is the set of points $[d_1, d_2]$, for which the patterns exist. The curve C_E is an envelope of certain hyperbolas $C_i, i \in \mathbb{N}$ illustrated on Figure 1.

2 Problem modified by unilateral terms

In this classical case, there must be $d_1 \ll d_2$ for spatial patterns to appear, i.e. the portion $D = \frac{d_1}{d_2}$ is essentially less than one. However, this requirement seems to be pretty unrealistic. We would like to overcome this issue by adding source terms to the second equation of the system. Even though we base our research on Turing, we focus on the creation of patterns,

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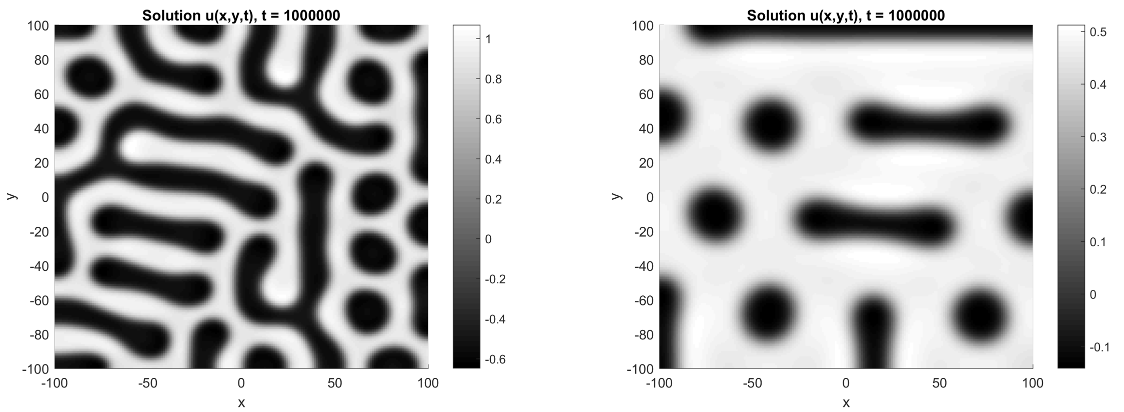
instead of the stability of the corresponding constant stationary state. Let's consider a model with specific reaction kinetics (Liu et al. (2006)):

$$\begin{aligned} \frac{\partial u}{\partial t} &= D\delta\Delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2, \\ \frac{\partial v}{\partial t} &= \delta\Delta v - \alpha u + \beta v + r_2 uv + \alpha r_3 uv^2. \end{aligned} \tag{2}$$

This model generates patterns for $D < D_{crit} \approx 0.53$. Vejchodský et al. (2015) experimented with the unilateral source term τv^- $\left(v^- = \frac{1}{2}\left(|v| - v\right)\right)$ and were able to push D_{crit} to the value $D_{crit} \approx 0.71$. Also, they observed that this term breaks the regularity of patterns, which leads to new interesting shapes of patterns.

We will present the unilateral term with saturation $s := \frac{\tau v^-}{1+\varepsilon v^-}$. This type of source is bounded, which seems to be more natural, and it is also more flexible, because it is two-parametric. Clearly $s \to \tau v^-$ as $\varepsilon \to 0$ and $s \to 0$ as $\varepsilon \to \infty$. This behaviour can be observed on the shape of patterns for different values of ε. The example of the pattern generated by the model (2) with s is on Figure $2a$. For the term s with parameters $\tau = 0.1$, $\varepsilon = 0.27$ we were able to find $D_{crit} \approx 0.85$. Hence, we achieved quite an improvement in this sense. The pattern for D close to critical value D_{crit} is illustrated on Figure $2b$.

We also performed experiments related to the source term τu^- in the first equation of the model. We already proved that this unilateral term has the opposite effect on the creation of patterns, i.e. the region of parameters for which patterns appear is smaller. Numerical experiments are in concert with these theoretical results.



(a) A pattern typical for $\frac{\tau v^-}{1+\varepsilon v^-}$

(b) The pattern for $D = 0.84$, $\tau = 0.1$, $\varepsilon = 0.27$

Figure 2: Examples of irregular patterns degenerated by an unilateral term $\frac{\tau v^-}{1+\varepsilon v^-}$

References

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