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CHROMATICKÁ TEORIE GRAFŮ

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CHROMATIC GRAPH THEORY

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Declaration

I hereby declare that this Doctoral Thesis is the result of my own work and that all external sources of information have been duly acknowledged.

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Šárka Petříčková

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I would like to thank my advisor Doc. RNDr. Tomáš Kaiser, Ph.D. for his continuous support throughout my studies, for his kindness and encouragement. His guidance and ever optimistic attitude have been an inspiration to me. I could not have imagined having a better advisor and mentor for my Ph.D study.

Abstract

The goal of this thesis is to analyze several distinct problems regarding graph colorings. To motivate the topic, we first demonstrate that the theory of graph coloring provides useful tools for modeling a wide variety of scheduling and assignment problems. We then introduce basic notation and review fundamental results in the area of chromatic graph theory. The main part of the thesis is divided into four chapters devoted to the following topics: Online Ramsey theory, Chromatic number of fractional graph powers, List chromatic number of graph powers, and Game chromatic number. In each of these chapters we introduce the graph coloring problem, give an overview of known results, present new results, and state several open problems. The published/accepted papers for the first three problems are attached at the end of the thesis. The results stated and proved in Section 5.3 have not been published.

Keywords: graph coloring, Ramsey theory, online coloring, unavoidable graph, chromatic number, list coloring, total coloring, fractional power, graph powers, game chromatic number, game coloring.

Abstrakt

Cílem této práce je studium několika problémů v oblasti barvení grafů. V úvodu nejprve ukážeme, že teorie barvení grafů nabízí užitečné nástroje pro modelování široké škály problémů z praxe. Dále seznámíme čtenáře se základní terminologií a stěžejními výsledky v oblasti chromatické teorie grafů. Hlavní část disertační práce je rozdělena do čtyř kapitol podle následujících témat: Online Ramseyova teorie, Barevnost racionálních mocnin grafů, Seznamová barevnost mocnin grafů, a Herní barevnost. V každé z těchto kapitol představíme daný problém, uvedeme známé a nové výsledky, a navrhneme několik otevřených problémů. Publikace výsledků prvních třech kapitol jsou přiloženy na konci práce. Výsledky uvedené a dokázané v poslední kapitole nebyly publikovány.

Klíčová slova: barvení grafů, barevnost, Ramseyove teorie, online barvení, nevyhnutelný graf, seznamová barevnost, totální barevnost, racionální mocniny grafů, herní barevnost.

Contents

Chapter 1

Introduction

A graph is a structure consisting of vertices and edges between them. A graph coloring is an assignment of colors to vertices or edges that satisfies certain conditions. In the next section we discuss several applications of various types of graph colorings: scheduling, mobile network communication, and register allocation. In Section 1.2 we introduce basic terminology and notation used in graph theory. In Section 1.3 we review fundamental results in chromatic graph theory. Finally in Section 1.4 we give a brief description of the four problems studied in this thesis.

1.1 Motivation

What do final exams and sport tournaments have in common? Both require an enormous level of planning and scheduling. And, both can strongly benefit from a part of graph theory called graph coloring.

The first graph coloring problem, dating back to 1850's, dealt with coloring maps: How many colors are needed in order to color a map of the counties of England in such a way that no two neighboring counties receive the same color. According to Mitchem [47], a student at University College London Francis Guthrie discovered that four colors were sufficient to properly color the map of the counties of England and many other maps. When neither Guthrie nor his adviser De Morgan could prove that four colors are always sufficient, F. G. (likely Francis Guthrie or his brother; see F. G. [19], McKay [46]) proposed the following problem:

> "*In tinting maps, it is desirable for the sake of distinctness to use as few colours as possible, and at the same time no two conterminous divisions ought to be tinted the same. Now, I have found by experience*

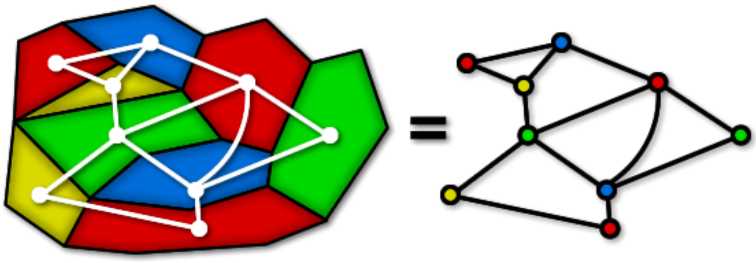


Figure 1.1: Modeling the Four color problem the using graph theory. (Source: [64])

> *that four colours are necessary and sufficient for this purpose, but I cannot prove that this is the case, unless the whole number of divisions does not exceed five. I should like to see (or know where I can find) a general proof of this apparently simple proposition, which I am surprised never to have met with in any mathematical work.*" F. G.

Suppose that we represent each county by a vertex, and we connect every two vertices whose corresponding counties share a boundary. The resulting structure is a graph. Moreover, this graph can be drawn in the plane with no edge-crossing, that is, this graph is planar. Using the terminology of graph coloring, we can restate the Four color conjecture as follows: Every planar graph is 4-colorable, meaning that the vertices of every planar graph can be colored with 4 colors in such a way that no two adjacent vertices receive the same color.

The Four Color problem remained open for more than 120 years until it was finally proved by Kenneth Appel and Wolfgang Haken at the University of Illinois at Urbana-Champaign in 1976. It is worth noting that the original proof was several hundred pages long and used 1200 hours worth of computer computation time (Mitchem [47]) .

Nowadays, graph coloring is a useful tool in modeling many real-life problems. We present a variety of applications and describe how the problems are modeled using graph coloring. Before proceeding further, we need to introduce some basic definitions.

A graph is a discrete structure consisting of vertices and edges, where every edge connects exactly two distinct vertices. Here we will only consider graphs with finitely many vertices. There are many types of graphs: for example, planar graphs are graphs that can be drawn on a paper without any two edges crossing, regular graphs are graphs in which every vertex is incident to the same number of edges, and directed graphs are graphs in which every edge has a direction (i.e. starts at one vertex and ends at another). Different classes of graphs model different problems and their properties can vary greatly. One of many parameters studied in graph theory is the graph chromatic number. In general, a graph coloring is a function that assigns colors to a set of objects in our graph G. The most common type of coloring is a (proper) vertex coloring, in which we color the vertices of G so that no two vertices connected by an edge receive the same color. The chromatic number of G is the smallest number of colors we need to properly color the vertices of G. Similarly, we can define a (proper) edge coloring, where we require every pair of edges which share a vertex to receive distinct colors. There are also many types of colorings where we do not require every two incident elements to receive different colors, but have some other condition on the coloring.

1.1.1 Scheduling

The first application of graph coloring discussed here is called the job scheduling problem: Given a set of jobs together with information about which jobs cannot be done at the same time, the goal is to find the shortest time in which all jobs can be finished. The problem can be modeled by a graph with vertices representing jobs and edges representing conflicts. So there will be an edge between two vertices if and only if the corresponding two jobs cannot be done at the same time. To find the shortest time in which all jobs can be completed, we need to determine the chromatic number of the associated graph. For example, if the graph can be colored with three colors, red, blue, and green, then all the jobs can be scheduled in three consecutive time slots: first all the jobs colored red, then the jobs colored blue, and then the jobs colored green. Since there is no edge between two vertices of the same color, any two jobs of the same color can be done at the same time.

Graph coloring is also useful for scheduling exams. Here, each vertex corresponds to a course, and there is an edge between two vertices if and only if there exists a student who is registered in both of the corresponding courses. If we can properly color the vertices of the associated graph by ten colors, then it is enough to reserve ten final exam slots. We are typically interested in the minimum number of such time slots (i.e., in the chromatic number of the graph). Of course, in case of large scale classes, for which conflict exams are typically scheduled, we can relax the definition of the associated graph. For example, we only put an edge between two vertices if there are at least 20 students registered for both of the two corresponding courses.

To determine the chromatic number of a given graph is computationally difficult. As shown by Garey et al. [21], finding the chromatic number is an NP-complete problem (unless the chromatic number is 1, or 2). This means that there is no polynomial-time algorithm that would correctly determine the chromatic number of any given graph. There are, however, many practical algorithms that approximate the chromatic number in polynomial time (see for example Leighton [44] or Malkawi et al. [45]).

Another application of graph coloring is the open-shop scheduling problem. This asks us to determine the shortest time in which a set of products, each of which must go through a certain set of processes on various machines, can be manufactured. If all the processes take the same amount of time and their order does not matter, then we can model this problem with a bipartite graph. In this graph, both products and processes are represented by vertices, and there is an edge between a product and a processes if and only if the product must go through that process. The minimum time needed to finish all processes is the minimum number of colors required for a proper edge-coloring of the associated graph, called the edge-chromatic number of this graph. If there are only two machines or only two products, then the problem of finding the shortest time can be solved in polynomial time. If, however, we have at least three machines and at least three products that need to be manufactured, then currently the best known algorithm is exponential. We refer the reader to the work of Anand and Panneerselvam [2] for an excellent review of known techniques, algorithms, and heuristics for the open-shop scheduling problems. For an example of an approximation algorithm for the open-shop problem, see Gandhi et al. [20].

1.1.2 Cellular networks

Chromatic graph theory is also used in mobile communication. The most common network used for mobile communication is the GSM network. GSM is a cellular network, which means that the land is divided into cells (typically hexagonal; see Figure 1.2) each of which has a transceiver connecting the mobile phones within the cell. To avoid signal interference, one channel should never be used by two neighboring cells. By the Four Color Theorem discussed above, we only need four different channels to ensure this condition can always be satisfied, independently of the shape of the cells.

In practice, the network is not so simple. First, users need to be able to move from one cell area to another without losing their signal. Second, there are many phone carriers. Third, the cells that use the same frequency must actually be in distance two or three to avoid interference. Hence, our mathematical model will be much more complicated. In particular, the associated graph will not be planar, and so it will typically not be four-colorable. We also need a more complicated type of graph coloring, called list coloring. In list coloring, each vertex of the graph has associated with it a list of available colors. A list-coloring is a coloring of the vertices such that each vertex is assigned a

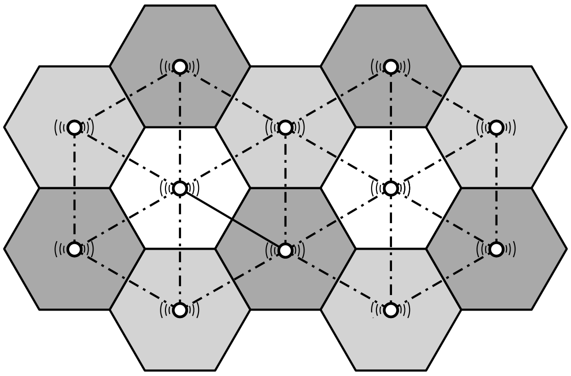


Figure 1.2: A naive model of cellular network. Only three frequencies are needed.

color from its list and the resulting coloring is proper (i.e., no two adjacent vertices receive the same color). For our cellular network problem, each vertex will play the role of a transceiver and its list of available colors will be the list of frequencies available at that transceiver. To determine the number of frequencies needed we thus want to find the minimum size of the lists that allows the vertices of the graph to be properly colored. This is called the list chromatic number of a graph and it is even more difficult to determine than the standard chromatic number. More information on this topic, including list coloring algorithms and practical frequency assignment, is available, for example, in the works of Park and Lee [51], Eisenblatter et al. [17], and Tamura et al. [57].

1.1.3 Register allocation

A particularly important application of graph coloring in computer science is register allocation. A typical computer program has a large number of variables, but a processor only has a small (say 32) number of fast registers to perform basic operations. The compiler must decide how to allocate these variables to the registers. Many variables can be assigned to one register, but variables that are used simultaneously cannot be assigned to the same register without corrupting their values.

In the graph theory model, vertices represent temporary variables and two variables are connected by an edge if they are employed simultaneously at some point in the program. The number of registers needed to run the program is then equal to the minimum number of colors required for a proper vertex coloring of this interference graph. It is of course possible that this number exceeds the fixed number of registers, which means that there exist variables which cannot be assigned to any register. These extra variables have to be moved to the RAM after every operation, a process called spilling. Since accessing RAM is slow, the goal is to minimize the number of spills.

The famous Chaitin's algorithm (see [11]) applies graph coloring of the interference graph for both register allocations and spilling. First, the interference graph is built. This graph is relatively sparse, so instead of using the adjacency matrix, an adjacency vector is kept for every vertex. Next, unnecessary register copy operations are eliminated by combining some vertices. Finally, all vertices of degree less than 32 are repeatedly removed. As Chaitin notes, this often results in the empty graph at the end of the process, in which case it is easy to assign colors for nodes by reversing the process and adding the removed vertices back. It remains to analyze the second scenario when every vertex in the reduced graph has at least 32 neighbors. In this case a spill code must be added. To determine which node to spill, the algorithm keeps a table with estimated costs of spills for nodes, and decides to spill a node whose cost divided by its current degree is smallest.

1.2 Notation and terminology

Most of the notation presented in this section is inspired by Diestel [14]. We use the symbol $[n]$ for the set of all natural numbers from 1 to n, that is $[n] = \{1,\dots,n\}$. A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a set of *vertices* and $E(G) \subseteq \binom{V(G)}{2}$ is a set of *edges*. We write uv for an edge $\{u, v\}$ of $E(G)$. Next, we use $|V(G)|$ and $|E(G)|$ to denote the number of vertices and edges of G, respectively. The number $|V(G)|$ is also called the *order* of G. The two vertices defining an edge are called the *endpoints* of that edge. We say that a vertex v is incident with an edge e if v is an endpoint of e. Two vertices u, v are *adjacent* if $uv \in E(G)$. Two edges are *adjacent* if they share one endpoint. A vertex adjacent to v is called a neighbor of v (in G). The set of all neighbors of v is called the neighborhood of v (in G), and is denoted by $N_G(v)$, or simply $N(v)$ if no confusion can arise. The *degree of* v, denoted by $d_G(v)$ or $d(v)$, is the number of edges adjacent to v (i.e., $d_G(v) = |N_G(v)|$). The *maximum degree of* G $\max_{v\in V(G)} d(v)$ is denoted by $\Delta(G)$. A graph G is called *r-regular graph*, if all vertices of G have degree r. In particular, a 3-*regular* is called *cubic*.

We say that two graphs *G and H are isomorphic*, and write $G \simeq H$, if there exists a bijective function $f : V(G) \to V(H)$ satisfying $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. For a fixed graph G, a *copy* H of G is a graph isomorphic to G with $V(G) \cap V(H) = \emptyset$.

The *union* $G \cup H$ of two graphs G and H is a graph with with vertex set

$V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $V(G) \cap V(H) = \emptyset$, then we call the union a *disjoint union*. If we say that some graph is a disjoint union of G and H, where the vertex sets $V(G)$ and $V(H)$ are not specified, then we automatically assume that they are disjoint.

A graph H is said to be a *subgraph* of G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If in addition, $E(H)$ contains all edges of $E(G)$ that have endpoints in $V(H)$, then H is called an *induced subgraph* of G or a *graph induced by* $V(H)$, and is denoted by $G[V(H)]$. For a vertex v and an edge e of G we define $G - v = G[V(G) \setminus \{v\}]$ and $G - e = (V(G), E(G) \setminus \{e\})$.

A *path* P is a graph isomorphic (for some $n \geq 1$) to a graph with vertex set $\{x_1, \ldots, x_n\}$ and edge set $\{x_i x_{i+1} : i \in [n-1]\}$. The *length of a path* is the number of its edges. The *distance* $d_G(u, v)$ (or $d(u, v)$) between two vertices u and v in G is the length of a shortest path connecting u and v. A non-empty graph G is *connected* if there is a path connecting any two vertices of G. Maximal connected subgraphs of G are called *components*. A cycle C_n is a graph isomorphic to a graph with vertex set $\{x_1, \ldots, x_n\}$ and edge set $\{x_i x_{i+1} : i \in [n-1]\} \cup \{x_n x_1\}$. A graph without cycles is called a *forest*. A *tree* is a forest with exactly one component.

A graph G is called a *complete graph* if $E(G) = \binom{V(G)}{2}$. A complete graph on n vertices is denoted by K_n. A *clique* of a graph G is a complete subgraph of G. A *maximum clique* of is a clique with the largest possible order, called *clique number* and denoted $\omega(G)$.

A graph is called *r-partite* if we can partition $V(G)$ into r subsets, called *parts*, so that each part induces a graph with no edges. A 2-partite graph is called *bipartite*. A *complete r-partite* graph is an r-partite graph with the maximum number of edges. A complete bipartite graph with the two parts of size r and s is denoted by $K_{r,s}$.

A *planar graph* is a graph that can be drawn in the plane such that its edges intersect only at their endpoints. An *outerplanar graph* is a planar graph that can be embedded so that all its vertices belong to the boundary of the outer face.

The *line graph* $L(G)$ of a graph G is a graph with vertex set $E(G)$ and an edge ef if and only if e and f are adjacent (edges) in G. The *total graph* $T(G)$ of G is a graph with vertex set $V(G) \cup E(G)$, where two vertices are adjacent if and only if their corresponding elements are adjacent or incident in G.

Coloring c	Objects O	Rules R	Parameter
vertex coloring	V	PROPER	χ
edge coloring	E	PROPER	χ'
total coloring	$V \cup E$	PROPER	χ''
list vertex coloring	V	PROPER & LIST	χ_ℓ
list edge coloring	E	PROPER & LIST	χ'_ℓ
list total coloring	$V \cup E$	PROPER & LIST	χ''_ℓ

Table 1.1: Examples of standard types of graph colorings.
PROPER: $\forall o_1, o_2 \in O : o_1 \sim o_2 \Rightarrow c(o_1) \neq c(o_2)$,
LIST : $\forall o \in O : c(o) \in L_o$.

1.3 Fundamental results in graph coloring

In this section we give an overview of the most important results in chromatic graph theory. In general, a graph coloring is a function

$$c : O \to S,$$

where O is a set of objects in G, and S is an arbitrary set called a *color set*, satisfying some set R of rules. Table 1.1 specifies O and R for the most common types of colorings, while Table 1.2 displays several more rare types of graph colorings. If $|S| = k$, then we replace the word 'coloring' with 'k-coloring'. For example, a *(proper) vertex k-coloring* of a graph G is a function $c : V(G) \to S$, where $|S| = k$, such that if $uv \in E(G)$, then $c(u) \neq c(v)$. 'Proper vertex coloring' is often shorten to 'vertex coloring' or just 'coloring'. Note that a proper vertex coloring is a special case of an H-free vertex coloring, where H is an edge. A graph is *k-vertex-colorable* or just *k-colorable* if there exists a proper vertex k-coloring of G. The least k such that G is k-colorable is called the *(vertex) chromatic number of G*, and is denoted $\chi(G)$. Proper vertex coloring is the most studied type of coloring. The most famous theorem regarding vertex coloring is Brooks' theorem.

Theorem 1.1 (Brooks [10], 1941)**.** *If G is connected, not complete, and not an odd cycle, then $\chi(G) \leq \Delta(G)$. Otherwise $\chi(G) = \Delta(G) + 1$.*

The next well-studied type of graph coloring is proper edge coloring, where we color edges instead of vertices and require every pair of adjacent edges to receive different colors. Every edge coloring of G can be transformed into a vertex coloring by considering the line graph $L(G)$ of G. Indeed, two edges e and f of G are adjacent in G if and only if the corresponding vertices e and

Coloring c	Objects	Rules R
equitable coloring	V	PROPER & all color classes same size ± 1
r-dynamic coloring	V	PROPER & every neighborhood $\geq r$ colors
acyclic coloring	V	PROPER & every cycle ≥ 3 colors
harmonious coloring	V	PROPER & no two edges same two colors
H-free vertex coloring	V	no monochromatic copy of H
H-free edge coloring	E	no monochromatic copy of H

Table 1.2: Examples of less common types of graph colorings.

f of $L(G)$ are adjacent in $L(G)$. As in the case of proper vertex coloring, a proper edge coloring is a special case of an H-free edge coloring, where H is now a path of length two. A graph is *k-edge-colorable* if there exists a proper edge k-coloring of G. The least k such that G is k-edge-colorable is called the *edge chromatic number of* G, and denoted $\chi'(G)$. The obvious lower bound on $\chi'(G)$ is $\Delta(G)$. Somewhat surprisingly, the upper bound on $\chi'(G)$ differs from the lower bound only by 1, as states the celebrated Vizing's theorem.

Theorem 1.2 (Vizing [59], 1964). *For any graph G either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.*

A graph G is said to be of *class 1* if $\chi'(G) = \Delta(G)$. Otherwise, G is said to be of *class 2*.

Total coloring is a combination of proper vertex and proper edge coloring. The almost 50 year old (still open) Total Coloring Conjecture states that the total chromatic number is either $\Delta(G) + 1$ or $\Delta(G) + 2$, classifying graphs in a similar way as Vizing's theorem. We give a brief exposition of this conjecture in Section 3.1.

One of the most studied class of graphs is the class of planar graphs. In Section 1.1 we already mentioned the famous Four color theorem.

Theorem 1.3 (Four color theorem). *Every planar graph is 4-colorable.*

Given the difficulty of the proof of the Four color theorem it is somewhat surprising that if we restrict our attention to triangle-free graphs, the upper bound on the number of colors drops by 1.

Theorem 1.4 (Grötzsch [23], 1959). *Every planar graph without triangles is 3-colorable.*

For various types of graph colorings we are often interested in their corresponding list coloring versions. Let $c : O \to S$ be a coloring of elements in O with colors from S satisfying a set R of rules. If, in addition, we are given a list $L_o \subset S$ of colors for each object $o \in O$, then a function $c : O \to S$ that maps every object o to a color from L_o is called *list coloring*. We then seek the smallest k such that for any assignment of lists of size k to the objects in O, there is a list coloring c_l of O. For example, the *list-chromatic number* $\chi_l(G)$ is the least k such that for any assignment of lists of size k to the vertices of G, there is a proper coloring of $V(G)$ where the color at each vertex is in that vertex's list. Similarly, the *list-chromatic index* $\chi'_\ell(G)$ is the least k such that for any assignment of lists of size k to the edges of G, there is a proper coloring of the edges from their lists.

Observe that if we can list-color G with k colors, then we can color G with k colors (consider assigning the list $\{1, 2 \ldots, k\}$ to every element). Hence, $\chi(G) \leq \chi_l(G)$ and $\chi'(G) \leq \chi'_l(G)$ for any graph G. The list chromatic number however cannot be bounded in terms of chromatic number. For example, the chromatic number of a complete bipartite graph is always at most 2, but the graph K_{r,r^r} is not r-list colorable. Indeed, suppose we assign list $\{i1, i2, \ldots, ik\}$ to the i-th vertex of the smaller part A. Then every choice of colors for the k vertices in k is in the form $\{1w_1, 2w_2, \ldots, kw_k\}$, where $w_1, \ldots, w_k \in [k]$. There are k^k such different sets, so if we assign them to the vertices in the large part, we will not be able to properly color the vertices from their lists.

Some graph coloring results can be easily generalized to the list coloring setting. For example, almost no change of the original proof is needed to show $\chi_l(G) \leq \Delta(G) + 1$ for any graph G and $\chi_l(G) \leq 5$ for any planar graph G. We will discuss more list coloring problems and results in Chapter 4, where we study the relation between the chromatic number and the list chromatic number of graph powers.

1.4 Description of problems

1.4.1 Online Ramsey theory

We consider a game with two players, Builder and Painter. In each turn, Builder draws an edge, and Painter then colors it red or blue. The goal of Builder is to force Painter to create a monochromatic copy of a given graph G. The only limitation for Builder is that the graph constructed so far belongs to some fixed class of graphs $\mathscr{H}$. Builder wins if, for any strategy of Painter, a monochromatic copy of the target graph G eventually occurs. We then say that G is unavoidable on $\mathscr{H}$. Otherwise, Painter wins. So, Painter wins if she can forever avoid creating a monochromatic copy of G.

In this thesis we are interested in the case when $\mathscr{H}$ is the class of planar graphs. In particular, we study the following conjecture:

> **Conjecture** (Grytczuk, Hałuszczak, and Kierstead [24], 2004)
> *The class of graphs unavoidable on planar graphs is exactly the class of outerplanar graphs.*

This conjecture was studied in the Master's Thesis of the author of this thesis (Petříčková [53]). It was shown there that the graph $K_{2,3}$ is unavoidable on planar graphs, which implies that the conjecture does not hold in full generality. On the other hand, it was proven that the class of triangle-free outerplanar graphs with all cycles of the same parity is unavoidable on the class of planar graphs, supporting the other direction of the conjecture.

Here we generalize the ideas of [53], and prove that the class of outerplanar graphs is a proper subclass of the class of graphs unavoidable on planar graphs. In particular, we present a winning strategy for Builder for forcing a monochromatic copy of any fixed outerplanar graph (such that the resulting graph is planar). We also find an infinite class of planar, but not outerplanar, graphs such that each of them is unavoidable on the class of planar graphs, and give the respective Builder's strategy.

The paper *Online Ramsey theory for planar graphs* [54] is attached in Appendix as it appeared in The Electronic Journal of Combinatorics.

1.4.2 Chromatic number of fractional graph powers

It is obvious that the chromatic number of a graph only increases when we take its powers, i.e., $\chi(G) \leq \chi(G^2)$. On the other hand, the chromatic number only decreases when we take its subdivisions. So, if the *n-th subdivision* $G^{\frac{1}{n}}$ of G is a graph obtained by replacing each edge in G with a path of length n, then $\chi(G) \geq \chi(G^{\frac{1}{n}})$. The question studied here is what happens with the chromatic number if we consider the m-th power of the n-subdivision of a given graph G. For this purpose, Iradmusa [31] introduced the notion of a *fractional power of* G, defined by $G^{\frac{m}{n}} = (G^{\frac{1}{n}})^m$ for some $m, n \in N$. He showed that $\chi(G^{\frac{2}{n}}) = \omega(G^{\frac{2}{n}})$ for all $n > 2$, and proposed the following conjecture.

> **Conjecture** (Iradmusa [31], 2010)
> *If G is a connected graph with $\Delta(G) \geq 3$, $n, m \in \mathbb{N}$ and $1 < m < n$, then $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$.*

Here we show the Conjecture does not hold in full generality by presenting a graph H for which $\chi(H^{\frac{3}{5}}) > \omega(H^{\frac{3}{5}})$. However, we prove that the conjecture is true if m is even and $\Delta(G) \geq 4$. We also study the case when m is odd, obtaining a general upper bound $\chi(G^{\frac{m}{n}}) \leq \omega(G^{\frac{m}{n}})+2$ for graphs with $\Delta(G) \geq 4$.

This is joint work with Stephen Hartke and Hong Liu. Our paper *Coloring fractional powers of graphs* has been accepted for publication in the Journal of Graph Theory [28].

1.4.3 List chromatic number of graph powers

A graph G is said to be chromatic-choosable if its chromatic number $\chi(G)$ is equal to its list chromatic number $\chi_\ell(G)$. Disproving the List Square Coloring Conjecture, Kim and Park found an infinite family of graphs whose squares are not chromatic-choosable. Xuding Zhu asked whether there exists a k such that all k-th power graphs are chromatic-choosable. We answer this question in the negative: we show that for any natural number k there is a family of graphs G such the $\chi_\ell(G^k) > \chi(G^k)$, where G^k denotes the k-th power of G. This is a joint work with Nicholas Kosar, Benjamin Reiniger, and Elyse Yeager. Our paper *A note on list-coloring powers of graphs* is attached as it appeared in Discrete Mathematics [41].

1.4.4 Game chromatic number

Here we investigate a game of two players, Alice and Bob, who alternately color vertices of a given graph G. Each player can only use a color that has not yet been used on the neighbors. If at some point either player is not able to color an uncolored vertex, the game ends, and Bob wins. Otherwise, all vertices of G are properly colored at the end, and Alice wins. The *game chromatic number* $\chi_g(G)$ is the smallest number of colors needed such that Alice can always win.

It seems that Alice does not gain much from being able to distinguish colors – the proofs of upper bounds on $\chi_g(G)$ typically only count the number of colored neighbors. For this reason, Zhu introduced the *game coloring number* $\mathrm{col}_g(G)$ of G as the number $1+k$ where k is the maximum back degree of a linear order produced by playing the game with both players using their optimal strategies. Faigle et al.[18] showed that for every interval graph G, $\chi_g(G) \leq 3\omega(G)-2$, and gave examples of graphs G for which $\chi_g(G) \geq 2\omega(G)-1$. Our new result is that for every $\omega \in \mathbb{N}$ there exists an interval graph G with clique number ω such that $\mathrm{col}_g(G) \geq 2.5\omega-3$. This is a joint result with Tomáš Kaiser.

Chapter 2

Online Ramsey theory

In this chapter we will be concerned with online Ramsey games defined by Grytczuk, Hałuszczak, and Kierstead [24] in 2004. According to them, the foundations of the online Ramsey theory were laid by Beck [4] and by Friedgut et al. [9]. In [9], the *Ramsey one-person triangle avoidance games against a random graph* consider one player, called Painter, whose aim is to avoid creating a monochromatic triangle when coloring the edges of H by 2 colors as long as possible. Three different versions are studied there: the offline, the online and the two-round game. In the *offline* version, the graph H is presented to Painter at the beginning of the game. On the other hand, if the game is played *online*, Painter receives edges of a graph H (chosen randomly by computer) step-wise, only receiving a new edge after coloring the previous one. The *two-round* version, is a mixture of the online and the offline version.

Online Ramsey games studied in Grytczuk et al. [24], are derived from the online version of games described above. In an online Ramsey game, there is a second player, called Builder, who generates the edges instead of the computer. Builder is typically not restricted to one graph H, but rather to a wider class of graphs $\mathscr{H}$ (for example forests, planar graph, k-colorable graphs). Finally, Painter may be required to avoid any fixed graph G, rather than a triangle.

So, for a fixed graph G and a class of graphs $\mathscr{H}$ such that $G \in \mathscr{H}$, the online Ramsey game $(G, \mathscr{H})$ proceeds as follows: There are two players in the game, Builder and Painter. Builder starts by drawing an edge, and Painter then colors it red or blue. In every subsequent step Builder adds exactly one edge, which is afterwards colored by Painter. The goal of Builder is to force Painter to create a monochromatic copy of the graph G. The only limitation for Builder is that the generated graph always has to belong to some predefined class $\mathscr{H}$. Builder *wins* if, playing on $\mathscr{H}$, there exists such a strategy for Builder that a monochromatic copy of G always arises, independently of moves of Painter. In this case we say that G is *unavoidable* for Painter on the class $\mathscr{H}$. Otherwise, we call G *avoidable* on the class $\mathscr{H}$. If any graph from $\mathscr{H}$ is unavoidable on $\mathscr{H}$, we say that $\mathscr{H}$ is *self-unavoidable*.

Note that without restriction to $\mathscr{H}$, Builder would always win the online Ramsey game. Indeed, according to the Ramsey's theorem ([56], [22]), for every $t \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that every 2-coloring of the edges of K_n contains a monochromatic copy of K_t. Thus, it would be enough for Builder to create a sufficiently large complete graph to win the game.

2.1 Online coloring

All colorings mentioned in Section 1.3 assume that the whole graph is known before it is colored. Such colorings are sometimes called *offline colorings*. However, there are many real world applications, where the graph is not available at the beginning, but appears piece-by-piece in a serial fashion. Such colorings are on the other hand called *online colorings*. In the most common settings, the input graph is given only one vertex at a time (along with the edges to previous vertices), and each vertex has to be colored before a new one is presented.

We present a precise definition by Kierstead and Trotter [35]. An *online graph* is a structure $G^< = (V, E, <)$, where $G = (V, E)$ is a graph and $<$ is a linear ordering of V. The graph $G^<$ is said to be an *online presentation* of G. Let $V = \{v_1, \ldots, v_n\}$ such that $v_i < v_j$ if and only if $i < j$. Then we define $V_i = \{v_j : j \leq i\}$ and $G_i^< = G^<[V_i]$. An *online algorithm* for coloring V is an algorithm that colors each v_i based solely on $G_i^<$. One of the simplest online algorithms is the *First-Fit algorithm*, which assigns to v_i the smallest integer (as color) that is not used on any neighbor of v_i in V_i. We refer the reader to [27] and [35] for more information about online coloring.

An online edge coloring is defined analogously and has the following connection to the online Ramsey theory. Let $(G, \mathscr{H})$ be an online Ramsey game, where G is any fixed graph and $\mathscr{H}$ is the class of all graphs. Then the minimum number of edges that Builder needs to draw in order to force a monochromatic copy of G is called an *online size Ramsey number* of G and is denoted by $\tilde{r}(G)$. So, $\tilde{r}(G)$ is one larger than the maximum number of edges of an online graph $G^<$ such that all online presentations of G are online G-free 2-edge colorable. Let $r(G)$ denote the ordinary size Ramsey number, defined as the smallest number of edges of a graph that contains a monochromatic copy of G for any 2-coloring of its edges. Obviously, for any graph G the number $\tilde{r}(G)$ is bounded from above by $r(G)$. On the other hand, we have $\tilde{r}(K^n) \geq \frac{1}{2} r(K^n)$ as proved in Beck [4].

2.2 Known results

In the following we summarize known results of online Ramsey theory proven by Grytczuk, Hałuszcak, and Kierstead [24].

- The class of k-colorable graphs is self-unavoidable.
- The class of forests is self-unavoidable.
- K_3 is avoidable on outerplanar graphs.
- K_3 is unavoidable on 2-degenerate graphs.
- Cycles are unavoidable on planar graphs.
- $K_4 - e$ is unavoidable on planar graphs.

The technique in the proof of the second result is important for our work. We will therefore sketch the basic idea. First, it is sufficient to prove that any tree is unavoidable on the class of forests. Indeed, for any given forest we can find a tree that contains it (only in graph theory!), and apply our strategy on this tree. Let T be a tree with n vertices. The proof is by induction on n. Let v be any leaf of T and u be a vertex adjacent to v. By the induction hypothesis, Builder can force a monochromatic copy of $T - v$. Clearly, he can repeat his strategy $(2n-1)$-times to obtain n of vertex disjoint copies of $T-v$ of the same color, say blue. It now remains for Builder to add edges between the vertices of these $T-v$'s that correspond to u so that they form a copy of T (see Figure 2.1). If any of the new edges is colored blue, a blue copy of T appears. Otherwise, the new edges form a red copy of T.

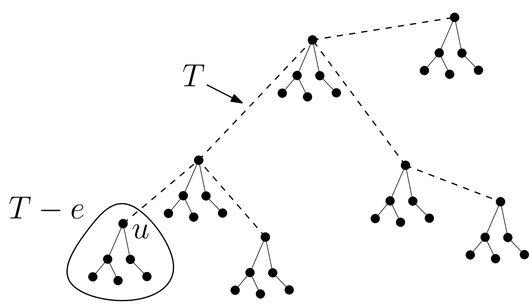


Figure 2.1: Forcing a monochromatic copy of a tree T.

Grytczuk et al. [24] also studied the multicolored version of this game. They showed that even if Painter is allowed to use c colors, Builder can force a monochromatic triangle when playing on the class of 3-colorable graphs.

Kierstead and Konjevod [34] vastly generalized the game to uniform hypergraphs. We present the definition of the game as it is defined in [34]. An *s-uniform hypergraph* or *s-graph* H is an ordered pair $(V(H), E(H))$, where $V(H)$ is a set of vertices and $E(H) \subseteq \binom{V(H)}{s}$ is a set of s-edges. The complete s-graph on t vertices is defined by K_t^s. The (c, s, t)-*Ramsey game* is a game between Builder and Painter that begins with a hypergraph $H_0 = (V, \emptyset)$, where V is determined by Builder. In the i-th round, Builder adds an s-edge e_i (not yet added) to the so far constructed hypergraph H_{i-1} and presents the resulting hypergraph H_i to Painter, who then colors e_i with one of the c colors. Builder wins if a monochromatic copy of K_t^s is created. Painter wins if no monochromatic copy of K_t^s appears at the end of play. The main result of Kierstead and Konjevod is that Builder wins the (c, s, t)-Ramsey game on $\chi(K_t^s)$-colorable graph for any $c, t \in \mathbb{N}$.

In the rest of this section we give a brief exposition of online degree Ramsey theory, introduced by Butterfield et al. [8]. Online degree Ramsey theory studies online Ramsey games played on the class $\mathscr{S}_k$ of graphs with maximum degree less than or equal to k. The *online degree Ramsey game* is thus any game (G, S_k), where G is any fixed graph that belongs to S_k. Let $\tilde{r}_\Delta(G)$ denote the *online degree Ramsey number*, the minimum k such that G is unavoidable in the class $\mathscr{S}_k$. We list the main results given in [8].

- $\tilde{r}_\Delta(G) \leq 3$ if and only if each component of G is a path or each component is a subgraph of $K_{1,3}$.
- $\tilde{r}_\Delta(G) \geq \Delta(G) + t - 1$, where $t = \max_{uv \in E(G)} \min\{d(u), d(v)\}$.
- $\tilde{r}_\Delta(T) \geq d_1 + d_2 - 1$, where $d_1, \ldots, d_n$ is a non-increasing degree list for T.
- $\tilde{r}_\Delta(C_n) \in \{4, 5\}$ for any cycle C_n.
- $\tilde{r}_\Delta(G) \geq 6$ if $\Delta(G) \leq 2$.

2.3 New Results

Grytczuk, Hałuszczak, and Kierstead [24] made the following conjecture.

Conjecture 2.1 ([24]). *The class of graphs unavoidable on planar graphs is exactly the class of outerplanar graphs.*

It was shown by Petříčková [53] that the non-outerplanar 1graph $K_{2,3}$ is unavoidable on planar graphs. So, Conjecture 2.1 does not hold in general.

On the other hand, some sub-classes of outerplanar graphs were shown in [53] to be unavoidable on planar graphs. In the paper *Online Ramsey theory on planar graphs* attached to this thesis we further generalize these results. Here we state the main two results of the paper. The first of them shows that the unavoidability condition is indeed necessary for outerplanar graphs.

Theorem 2.2. *Every outerplanar graph is unavoidable on planar graphs.*

The second result generalizes the proof of unavoidability of $K_{2,3}$ given in [53] to a subclass of theta graphs. A *theta-graph* $\theta_{i,j,k}$ is the union of three internally vertex disjoint paths of lengths i, j, k that share the same two end vertices.

Theorem 2.3. $\theta_{2,j,k}$ *is unavoidable on planar graphs for even* j *and* k.

2.4 Further research

There are closely related open problems concerning online Ramsey theory on planar graphs. The question whether the class of planar graphs is self-unavoidable is still open. In particular, we are interested in the following questions:

- Is K_4 unavoidable on planar graphs?
- For which n are the graphs $K_{2,n}$ unavoidable on planar graphs?
- Is there any other planar graph that is avoidable on the class of planar graphs?

Another problem that has not been addressed in our work is the length of the game. If a graph G is unavoidable on the class of graphs $\mathscr{H}$, then we could define an online Ramsey number $\tilde{r}(G, \mathscr{H})$ as the smallest number of edges Builder must draw to win the online Ramsey game $(G, \mathscr{H})$. Clearly, $\tilde{r}(G) \leq \tilde{r}(G, \mathscr{H})$, but it would be interesting to see far are these two functions for various classes of graphs $\mathscr{H}$. For the class of planar graphs studied here, how far from optimal are the given strategies?

Chapter 3

Chromatic number of fractional graph powers

In this chapter we study the relation between chromatic number and clique number of graphs of the form $G^{\frac{m}{n}} = (G^{\frac{1}{n}})^m$, where $G^{\frac{1}{n}}$ is the *n-subdivision* of G obtained by replacing each edge in G with a path of length n, and G^m is the m-th power of G. It was conjectured by Iradmusa [31] that if G is a connected graph with $\Delta(G) \geq 3$ and $1 < m < n$, then $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$. Here we show that the conjecture does not hold in full generality. In particular, we present a graph H for which $\chi(H^{\frac{3}{5}}) > \omega(H^{\frac{3}{5}})$. However, we prove that the conjecture is true if m is even. We also study the case when m is odd, obtaining a general upper bound $\chi(G^{\frac{m}{n}}) \leq \omega(G^{\frac{m}{n}}) + 2$ for graphs with $\Delta(G) \geq 4$.

The chapter is organized as follows. Section 3.1 is intended to motivate our investigation of fractional powers of graphs and their chromatic numbers. In section 3.2 we give a grief exposition of the known results on this topic. Our results are then stated in section 3.3. Finally, some open problems are stated in Section 3.4.

3.1 Total Coloring Conjecture

Recall that for a graph G, a *total coloring* of G is a function f that assigns colors to both the vertices and edges of G such that no two adjacent vertices, no two adjacent edges, and no edge and an incident vertex are assigned the same color by f. So, the underlying coloring of vertices $f|_{V(G)}$ is a proper vertex coloring of G and the underlying coloring of edges $f|_{E(G)}$ is a proper edge coloring of G. The *total chromatic number* $\chi''(G)$ is the least number of colors such that there is a total coloring of G. Hence $\chi''(G) = \chi(T(G))$, where $T(G)$ is the total graph of G.

The trivial lower bound is $\chi''(G) \geq \Delta(G)+1$. Indeed, $\Delta(G)$ colors have to be used on the edges incident to a vertex v with degree $\Delta(G)$, and the vertex v uses an additional color. There are many graphs with $\chi''(G) = \Delta(G)+2$, including very simple ones like K_2 or K_4. But no graphs with $\chi''(G)$ bigger than $\Delta(G)+2$ are known. In fact, it has been independently conjectured by Behzad (1965) and Vizing (1968), that such graphs do not exist.

Total Coloring Conjecture. $\chi''(G) \leq \Delta(G)+2$ *for any graph G.*

Proving the Total Coloring Conjecture (TCC) would provide us with a nice analogy to Vizing's Theorem on edge coloring, which states that for every graph either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G)+1$. If the TCC is true, then we have a similar classification: either $\chi''(G) = \Delta(G)+1$ or $\chi''(G) = \Delta(G)+2$.

It is easy to see that the total graph $T(G)$ is isomorphic to $G^{\frac{2}{2}}$. Therefore, $\chi''(G) = \chi(G^{\frac{2}{2}})$. Also, $\omega(G^{\frac{2}{2}}) = \Delta(G)+1$ for every graph G with $\Delta(G) \geq 2$. Thus we can rewrite the Total Coloring Conjecture as follows:

Total Coloring Conjecture' ([5, 59])**.** *If G is a simple graph with maximum degree at least 2, then* $\chi(G^{\frac{2}{2}}) \leq \omega(G^{\frac{2}{2}})+1$.

The TCC has been open for almost 50 years. Many classes of graphs were shown to satisfy the conjecture (graphs with $\Delta(G) \leq 4$, complete graphs, r-partite graphs, outerplanar graphs, ...). Probably the best result so far was given by Molloy and Reed [48, Theorem 1.1].

Theorem 3.1 ([48])**.** *If a simple graph G has maximum degree* Δ *at least as large as a particular constant then* $\chi''(G) \leq \Delta + C$*, where* $C = 10^{26}$.

The proof uses the probabilistic method. According to the authors, Theorem 3.1 holds for much smaller constants, like $C = 500$, when adding some additional constrains. The constant $C = 10^{26}$ however provides a simpler proof.

A fractional version of the Total coloring conjecture was investigated by Kilakos and Reed [37]. An $a:b$*-coloring* of G is a vertex coloring $c : V(G) \to \binom{S}{b}$ with $|S| = a$ such that for pair of adjacent vertices u and v, the intersection of the color sets $f(u)$ and $f(v)$ is empty. A fractional chromatic number $\chi_f(G)$ of G is defined as

$$\chi_f(G) = \lim_{n\to\infty} \frac{\min b \text{ such that an } a:b\text{-coloring exists}}{b}.$$

Theorem 3.2 ([37])**.** *For any graph G,* $\chi_f(T(G)) \leq \Delta(G)+2$.

3.2 Known results

Iradmusa [31] observed that

$$\chi(G) \leq \chi(G^2) \leq \chi(G^3) \leq \ldots, \quad \text{but} \quad \chi(G) \geq \chi(G^{\frac{1}{2}}) \geq \chi(G^{\frac{1}{3}}) \geq \ldots$$

He suggested to study the chromatic number of a graph $G^{\frac{m}{n}}$ with $m, n \in \mathbb{N}$, and in particular in the case when $m < n$. He conjectured the following.

Conjecture 3.3 ([31]). *If G is a connected graph with $\Delta(G) \geq 3$ and $1 < m < n$, then $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$.*

Iradmusa showed that Conjecture 3.3 is true for $m = 2$. This means that there exist graphs with $\chi(G^{\frac{2}{3}}) = \omega(G^{\frac{2}{3}})$ and at the same time $\chi(G^{\frac{2}{2}}) = \omega(G^{\frac{2}{2}}) + 1$.

Theorem 3.4 ([31]). *If G is a connected graph for which $\Delta(G) \geq 3$ and n is a positive integer greater than 2, then $\chi(G^{\frac{2}{n}}) = \Delta(G) + 1 = \omega(G^{\frac{2}{n}})$.*

The proof of Theorem 3.4 proceeds as follows: First, the clique number of $G^{\frac{2}{n}}$ is determined. Every maximal clique of $G^{\frac{2}{n}}$ is a subgraph induced a branch vertex and its neighbors, so $\omega(G^{\frac{2}{n}}) = \Delta(G)+1$. Then, a proper vertex coloring of $G^{\frac{2}{n}}$ using $\Delta(G) + 1$ colors is constructed. Since $\chi(H) \geq \omega(H)$ for any graph H, constructing such coloring implies $\chi(G^{\frac{2}{n}}) = \omega(G^{\frac{2}{n}})$. The coloring is defined inductively with base cases $G^{\frac{2}{3}}$, $G^{\frac{2}{4}}$, $G^{\frac{2}{5}}$ [31, Lemma 3 and Lemma 4]. The inductive step then uses the following result:

Lemma 3.5 ([31]). *Let G be a graph, and $m, n \in \mathbb{N}$ such that $m < n$. If $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$, then $\chi(G^{\frac{m}{n+m+1}}) = \omega(G^{\frac{m}{n+m+1}})$.*

Iradmusa provided a classification of clique number for $G^{\frac{m}{n}}$ for general $m, n \in \mathbb{N}$ with $m < n$. See Figure 3.1 for a "proof by picture".

Theorem 3.6 ([31]). *If G is a graph and $m, n \in \mathbb{N}$ such that $m < n$, then*

$$\omega(G^{\frac{m}{n}}) = \begin{cases} m+1 & \text{if } \Delta(G) = 1, \\ \frac{m}{2}\Delta(G) + 1 & \text{if } \Delta(G) \geq 2, m \text{ even}, \\ \frac{m-1}{2}\Delta(G) + 2 & \text{if } \Delta(G) \geq 2, m \text{ odd}. \end{cases}$$

Hence, to prove Conjecture 3.3 for $m < n$, we need to construct a coloring of $G^{\frac{m}{n}}$ using $\frac{m}{2}\Delta(G) + 1$ colors if m is even or $\frac{m-1}{2}\Delta(G) + 2$ colors if m is odd.

Several other results were proved in in [31]:

- Conjecture 3.3 is true for $n = k(m+1)$, for any $k \in \mathbb{N}$.
- Conjecture 3.3 is true if $\Delta(G) \geq 4$, m is even, and $n \geq 2m + 2$.
- Conjecture 3.3 is true if $\Delta(G) \geq 5$, m is odd, and $n \geq 2m + 2$.

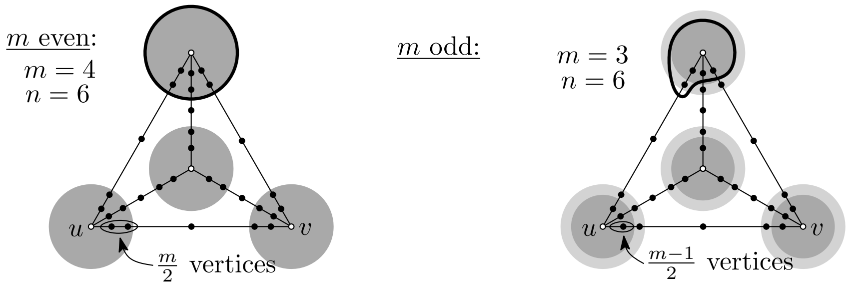


Figure 3.1: Maximal cliques of $G^{\frac{m}{n}}$, only the subgraph $G^{\frac{1}{n}}$ is depicted.

3.3 New results

Our first results is that Conjecture 3.3 is true if m is even and $\Delta(G) \geq 4$.

Theorem 3.7. *If G is a graph with $\Delta(G) \geq 4$ and $1 < m < n$ with m even, then $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$.*

Next, we study the case when m is odd. We show that the conjecture is not true in full generality. In particular, it is not true for the cartesian product $C_3 \square K_2$ of C_3 and K_2 (triangular prism), when $m = 3$ and $n = 5$. However, we prove that Conjecture 3.3 holds for infinitely many values of n if m is odd, and we give the following general bound:

Theorem 3.8. *If G is a graph with $\Delta(G) \geq 4$ and $1 < m < n$ with m odd, then $\chi(G^{\frac{m}{n}}) \leq \omega(G^{\frac{m}{n}}) + 2$.*

Additionally, we give sufficient conditions for Conjecture 3.3 to hold when m is odd using r-dynamic colorings.

Corollary 3.9. *If G is a Δ-regular graph with $\Delta \geq 39$, $\chi(G) \leq \frac{\Delta}{4}$, and $1 < m < n$ with m odd, then $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$.*

3.4 Further research

We showed that if m is odd and $\Delta(G) \geq 4$, then Conjecture 3.3 holds for about “half of the values” of n. It would be interesting to know if it also holds for the “second half”. Also, we do not know much about the case when $\Delta(G) = 3$, or equivalently, what is the situation for cubic graphs. The graph $G = C_3 \square K_2$ is the only known counterexample to Conjecture 3.3. We tried to find more counterexamples among cubic graphs using a computer, but we were unsuccessful. We make the following conjecture.

Conjecture 3.10. *Conjecture 3.3 holds except when* $G = C_3 \square K_2$.

Conjecture 3.3 assumes the $m < n$. A natural question is what happens when $m > n$. Is the chromatic number $\chi(G^{\frac{m}{n}})$ still bounded by $\omega(G^{\frac{m}{n}}) + c$ for some $c \in \mathbb{N}$? In particular, what is the chromatic number of the graph $G^{\frac{3}{2}}$ in terms of its clique number?

Chapter 4

List chromatic number of graph powers

4.1 Motivation and known results

Recall that the *list-chromatic number* of a graph G, denoted $\chi_\ell(G)$, is the least k such that for any assignment of lists of size k to the vertices of G, there is a proper coloring of $V(G)$ where the color at each vertex is in that vertex's list. A graph is said to be *chromatic-choosable* if $\chi_\ell(G) = \chi(G)$. The *k-th power* of a graph G, denoted G^k, is the graph on the same vertex set as G with an edge uv if and only if the distance from u to v in G is at most k. The *line graph* $L(G)$ of G is a graph on the vertex $E(G)$ where two vertices are adjacent if and only if their corresponding edges are incident in G. The *total graph* $T(G)$ of G is a graph on the vertex $V(G) \cup E(G)$ where two vertices are adjacent if and only if their corresponding elements are adjacent or incident in G.

Several conjectures on the chromatic-choosability of various classes of graphs have been made. The List-Edge-Coloring Conjecture (LECC) asserts that $\chi'_\ell(G) = \chi'(G)$ for every graph G. According to Jensen and Toft [32], the LECC first appeared in a paper by Bollobás and Harris [6], but it was thought of earlier by several other authors. Since $\chi'_\ell(G) = \chi_\ell(L(G))$ and $\chi'(G) = \chi(L(G))$, we can equivalently rewrite the LECC as follows:

LECC Conjecture (Bollobás–Harris[6]; Vizing; Gupta; Albertson–Collins). *$L(G)$ is chromatic-choosable for every graph G.*

A generalization of the LECC is the The List-Total-Coloring Conjecture (LTCC), which states that $\chi''_\ell(G) = \chi''(G)$ for every graph G. Since $\chi''_\ell(G) = \chi_\ell(T(G))$ and $\chi''(G) = \chi(T(G))$, we can rewrite the LTCC in the following form:

LTCC Conjecture (Borodin–Kostochka–Woodall 1997 [7]).
$T(G)$ is chromatic-choosable for every graph G.

The List-Square-Coloring Conjecture (LSCC) suggests that even more is true.

LSCC Conjecture (Kostochka–Woodall 2001 [42]).
G^2 is chromatic-choosable for every graph G.

The LSCC is stronger than the LTCC since, given a graph G, its total graph $T(G)$ can be obtained by subdividing each edge of G and taking the square. The LSCC was recently disproved by Kim and Park [39], who constructed an infinite family of counter examples to the conjecture, and showed that the value $\chi_\ell(G^2) - \chi(G^2)$ can be arbitrarily large. Let K_{r*s} denote the complete r-partite graph with each part of size s.

Theorem 4.1 ([39]). *For each prime $n \geq 3$, there exists a graph G such that G^2 is the complete multipartite graph $K_{(2n-1)*n}$.*

Since $\chi(K_{(2n-1)*n}) = 2n-1$ and $\chi_\ell(K_{(2n-1)*n}) \geq (n-1)\lfloor\frac{4n-3}{n}\rfloor$ by [58], the authors conclude that there exists a graph G such that $\chi_\ell(G^2) - \chi(G^2) \geq n-1$ for any prime $n \geq 3$. Xuding Zhu asked whether there is any k such that all k-th powers are chromatic-choosable [63]. In [41], we give a negative answer to Zhu's question, with a lower bound on $\chi_\ell(G^k)$ that matches that of Kim and Park for $k = 2$.

4.2 New results

In [41] we generalize the construction of Kim and Park [39]. We then apply a better lower bound on $\chi_\ell(K_{r*s})$ and obtain the following result.

Theorem 4.2. *There is a positive constant c such that for every $K \in \mathbb{N}$, there is an infinite family of graphs G with $\chi(G^K)$ unbounded such that*

$$\chi_\ell(G^K) \geq c\chi(G^K)\log\chi(G^K).$$

To prove Theorem 4.2, we use affine planes to construct a family of graphs G such that G^K is very close to a complete multipartite graph. An affine plane $(\mathscr{P}, \mathscr{L})$ is a system of n^2 points $\mathscr{P}$ and $n^2 + n$ lines $\mathscr{L}$ satisfying all of the following (see [12]):

- Each line is a set of n points.
- For each pair of points, there is a unique line containing them.

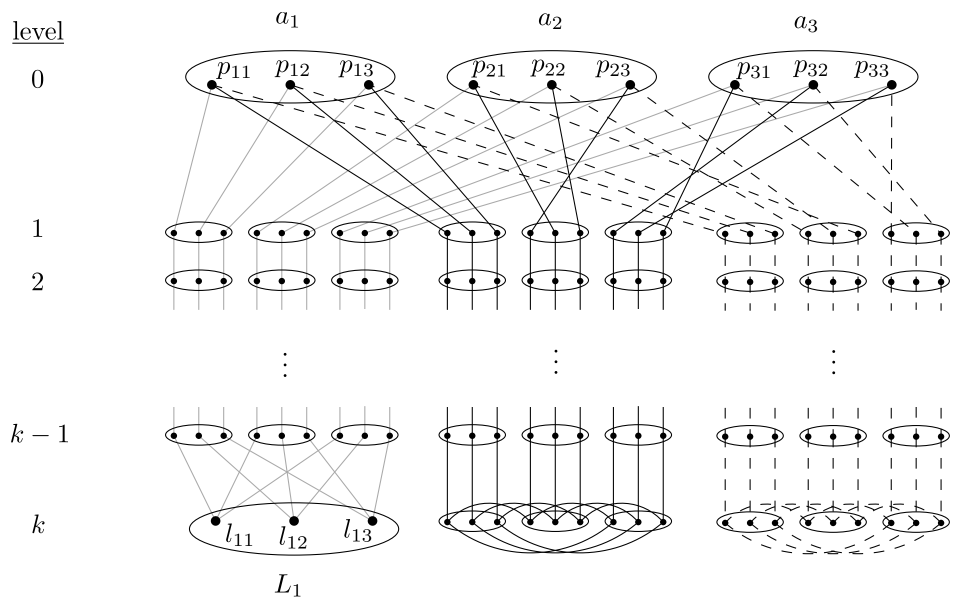


Figure 4.1: The graph G for $n = 3$.

- Two lines in the same parallel class do not intersect, and two lines in different parallel classes intersect in exactly one point.

Such a plane exists whenever n is a (positive) power of a prime. Note that the parallel class structure of an affine plane of order n may be used to construct a set of $n-1$ mutually orthogonal latin squares, which were used by Kim and Park [39] to construct their example.

We first construct a bipartite graph H on parts $\mathscr{P}$ and $\mathscr{L} - L_0$, where L_0 is any parallel class. We place an edge between $p \in \mathscr{P}$ and $\ell \in \mathscr{L} - L_0$ if and only if the point p lies on the line ℓ. Next, we refine the partition of the $V(H)$ as follows: the points $\mathscr{P}$ according to the partition of L_0 into n lines $a_1, \dots, a_n$, and the lines $\mathscr{L} - L_0$ into $L_1, \dots, L_n$. Observe that the subgraph of H induced by a_i and L_j is a matching for each i and j. Next, we subdivide the edges of H: edges adjacent to L_1 k-times and the rest $(k+1)$-times. Finally, for every $l \in \cup_{i=2}^{k} L_i$, we make the neighborhood of l a clique and eliminate l. The resulting graph G is shown on Figure 4.1.

The graph G^{4k} is multipartite with $kn^2 + 1$ parts, and so $\chi(G^{4k}) \le kn^2 + 1$. The subgraph of G^{4k} induced by vertices in levels 0 through $k-1$ is complete multipartite with $(k-1)n^2$ parts of size n. A result of Alon [1] implies $\chi_\ell(G^{4k}) \ge c'(k-1)n^2 \log n$ for some constant c'. Theorem 4.2 follows by taking n large enough and $c < c'/4$.

4.3 Further research

Noel [49] asked whether there is an $f(x) = o(x^2)$ such that $\chi_\ell(G^2) \leq f(\chi(G^2))$ for all G. The construction of Kim and Park [39] shows that such an f must satisfy $f(x) = \Omega(x \log x)$. We may ask the same question for larger k:

Problem 4.3. *Fix $k \geq 2$. Is there an $f_k(x) = o(x^2)$ such that $\chi_\ell(G^k) \leq f(\chi(G^k))$ for all G?*

Theorem 4.2 implies that such an f_k must satisfy $f_k(x) = \Omega(x \log x)$. The question whether this is tight is still open.

Problem 4.4. *Fix $k \geq 2$. Is there a constant c_k such that $\chi_\ell(G^k) \leq c_k \chi(G^k) \log(\chi(G^k))$ for every graph G? If so, can such c_k be found independent of k?*

Chapter 5

Game chromatic number

In this chapter we consider a game of Alice and Bob, who alternately color vertices of a given graph G, with Alice starting. Both players have to color the vertices properly, that is, such that no two adjacent vertices receive the same color. If at some point either player is not able to color an uncolored vertex, the game ends, and Bob wins. Otherwise, all vertices of G are eventually colored, which means that Alice wins the game. The *game chromatic number* $\chi_g(G)$ is the smallest number of colors needed such that Alice can always win, independently of Bob's play.

5.1 History and fundamental results

According to Bartnicki et al. [3], the game was first introduced for planar graphs as the *map-coloring game* by Steven J. Brams in 1980 as an attempt to prove the Four-Color theorem without using computers. It soon became clear that there are many planar graphs with game chromatic number more than 4. Indeed, consider coloring the faces of the cube, where Bob's strategy is to always use a new color on the opposite face to the face just colored by Alice. In her third move, Alice is unable to color either of the two remaining faces, and we conclude that the dual of the cube graph has game chromatic number at least 5. For a long time it was not even clear whether the game chromatic number for planar graphs is bounded. A natural idea is to try to ensure that at any point of the game, there is an uncolored vertex which has a small number of colored neighbors. As the next paragraph implies, this would be easy if Bob were willing to cooperate.

Given a graph G and an ordering $v_1, \ldots, v_n$ of $V(G)$, a neighbor v_k of v_i is called a *backward* (or *left*) *neighbor* if $k < i$ and a *forward* (or *right*) *neighbor* if $k > i$. In Figure 5.2, v_k is a forward neighbor of v_i. Following the literature,

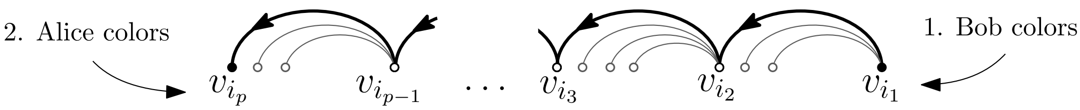


Figure 5.1: The jumping strategy of Alice.

we denote by $N^+(v)$ and $N^-(v)$ the left and right neighborhood of v. (Think about the orientation of the edges from right to the left.) We also use the subscripts A and B to distinguish between neighborhoods colored by Alice and Bob. For example, $N_B^+(v)$ denotes the set of the left neighbors of v that have been colored by Bob. The *coloring number* $\mathrm{col}(G)$ is the smallest k for which there is an ordering of vertices such that every vertex has less than k backwards neighbors. Equivalently, $\mathrm{col}(G)$ is he smallest k such that every subgraph of G has a vertex of degree less than k. Note that $\mathrm{col}(G)$ is also equal to the degeneracy of G plus 1 and that for every graph G we have $\chi(G) \leq \mathrm{col}(G)$. Every planar graph has a vertex of degree at most 5, and so for every planar graph G we have $\mathrm{col}(G) \leq 6$. Thus, 6 colors would be enough if both players were coloring the vertices from the left.

Bob however does not have to follow any order. We thus need to bound, for every uncolored vertex u,

1. the number of colors used on $N_A^-(u)$ by Alice, and
2. the number $|N_B^-(u)|$ of forward neighbors colored by Bob.

To address the first problem, we might consider the following *jumping strategy* for Alice: After Bob colors v_{i_1}, Alice 'jumps' to the leftmost uncolored neighbor v_{i_2} of v_{i_1} and then to the leftmost uncolored neighbor v_{i_3} of v_{i_2} and so on until reaching a vertex v_{i_p} that does not have any left uncolored neighbor, and she colors v_{i_p} with any available color. This strategy ensures that at any point of the game, the set $N_A^-(u)$ of forward neighbors colored by Alice is empty for every uncolored vertex u. The proof of Theorem 5.1 below provides another approach for bounding $|N_A^-(u)|$.

For the second part, the following *simple strategy* seems to be helpful: After Bob colors v, Alice colors its leftmost backward neighbor u. The important observation is that then, for any uncolored vertex v_i, every vertex $v_k \in N_B^-(v_i)$ must have a colored neighbor $v_{k'}$ to the left of v_i. Otherwise, Alice would have colored v_i after Bob colored v_k (confer Figure 5.2, left). Given a graph G and an ordering $v_1 \ldots, v_n$ of its vertices, the vertex v_k is a *loose backward neighbor* of v_i if $k < i$ and v_k is adjacent either to v_i or to a forward neighbor of v_i. The set of loose backwards neighbors of v_i will be denoted by $LN^+(v_i)$. The

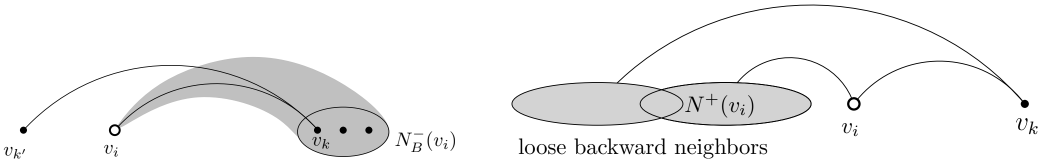


Figure 5.2: Every colored forward neighbor v_k of an unlored vertex v_i must have a backward neighbor v_k' before v_i (left picture). The set of such vertices v_k' together with backward neighbors of v_i are called loose backward neighbors of v_i.

2-*coloring number* $\mathrm{col}_2(G)$ of G is the smallest k for which there is an ordering of vertices such that every vertex has less than k loose backwards neighbors. An ordering of $V(G)$ in which every vertex has less than $\mathrm{col}_2(G)$ backwards neighbors is called a col_2-*ordering* of G. If Alice follows the simple strategy on a col_2-*ordering* of a graph G, then for at any point of the game after the move of Alice, $|N_B^-(u)| \leq \mathrm{col}_2(G) - 1$ for any uncolored vertex u.

In the process of proving the Burr-Erdős conjecture (stating that the Ramsey number of n-vertex graphs with $O(n)$ edges is linear in n) for planar graph, Chen et al. showed that the 2-coloring number of planar graphs is bounded. This fact allowed Kierstead and Trotter [36] to finally show that the game chromatic number for every planar graph is bounded.

Theorem 5.1 (Kierstead-Trotter [36], 1994). *For every graph G,* $\chi_g(G) \leq \chi(G)(col_2(G)+1)$.

Proof. Let $v_1, \ldots, v_n$ be an ordering of $V(G)$ such that every vertex has less than $\mathrm{col}_2(G)$ loose backwards neighbors. Consider a proper vertex coloring $f : V(G) \to \{c_1, \ldots, c_{\chi(G)}\}$. For each vertex u Alice colors, her goal is to use one of $\mathrm{col}_2(G) + 1$ shades of the color $f(u)$, i.e. a *shade of the correct color*. Her strategy goes as follows: After Bob colors a vertex v, Alice colors the leftmost uncolored backward neighbor u of v with one of the $\mathrm{col}_2(G) + 1$ shades of the color $f(u)$. If there is no such uncolored backward neighbor, she colors the leftmost uncolored vertex with a shade of the correct color. First observe that since Alice has been using only shades of the correct color, none of $N_A(u)$ is colored with a shade of $f(u)$. It is thus sufficient to bound the number of neighbors $N_B(u)$ of u that have been colored by Bob in his first i turns. Since u is uncolored, when Bob colored a vertex in $N^-(u)$, Alice responded by coloring a loose backward neighbor of u. Therefore, before Bob's i-th turn,

$$N_B(u) = N_B^+(u) + N_B^-(u) < \mathrm{col}_2(G).$$

In the i-th turn, Bob might color an extra forward neighbor of u using the $\mathrm{col}_2(G)$-th shade of $f(u)$, in which case Alice colors u with the remaining $(\mathrm{col}_2(G)+1)$-st shade of $f(u)$. □

Kierstead and Trotter [36] improved the upper bound on $\mathrm{col}_2(G)$ from the original bound of 761 by Chen et al. to 10. Using this fact together with some additional improvements, they were able to show that $\chi_g(G) \leq 33$ for every planar graph G.

Few years later, Dinski and Zhu [15] found a new strategy, lowering the upper bound on the game chromatic number for planar graphs even further. The key parameter here is the *acyclic chromatic number* $\chi_a(G)$ of G, which is the smallest number of colors in a proper vertex coloring of G with no bi-colored cycles.

Theorem 5.2 (Dinski-Zhu, 1999). *For every graph G,* $\chi_g(G) \leq \chi_a(G)(\chi_a(G)+1)$.

By a theorem of Borodin from 1979, $\chi_a(G) \leq 5$ for any planar graph G. Therefore, Theorem 5.2 implies $\chi_g(G) \leq 30$.

The most powerful strategy currently known is the *activation strategy*, which combines the jumping strategy and the simple strategy. Alice first fixes a col_2-ordering of $V(G)$. Suppose that at some step, Bob colors a vertex v. Alice first activates v, and then she jumps to the leftmost uncolored neighbor u of v. If this is the first time she visited u, she does not color it, but only activates it, and then she jumps to the leftmost uncolored vertex of u and repeats the same process. The process ends when Alice jumps to an already activated vertex, in which case she marks it with any available color. If such a vertex does not exist, Alice colors the leftmost uncolored vertex among all uncolored vertices.

Observe that the activation strategy works even if Alice is color-blind. She only needs to ensure that at each step of the game, every vertex has small number of colored neighbors. Such a game is called a *marking game*, and the *game coloring number* $\mathrm{col}_g(G)$ is the corresponding smallest k such that Alice can always ensure that every vertex has strictly less than k marked neighbors. Clearly, $\chi_g(G) \leq col_g(G)$, but the difference between the two numbers can be unbounded (consider the graph $K_{n,n}$). Using the activation strategy, Bartnicki et al. [3] were able to show the following theorem, which implies that $\chi_g(G) \leq 29$ for every planar graph G.

Theorem 5.3. *For every graph G,* $\chi_g(G) \leq 3\, col_2(G) - 1$.

Proof. Alice first fixes a col_2-ordering of $V(G)$, and then follows the activation strategy. We count the maximum number of active neighbors $AN(u)$ of an uncolored vertex u at any round of the game after Alice's turn. For the number of active backward neighbors $AN^+(u)$ we clearly have $|AN^+(u)| \leq |N^+(u)| \leq \text{col}_2(G) - 1$. We now bound the number of active forward neighbors $AN^+(u)$. Since Alice can only jump to each vertex twice, $|AN^-(u)| \leq 2|LN^+(u)| \leq 2(\text{col}_2(G) - 1)$. Together we have $|AN(u)| \leq 3(\text{col}_2(G) - 1)$. Counting the next move of Bob, there are at most $3k - 2$ (i.e., less than $3k - 1$) colored neighbors of u when u is colored. □

5.2 Known bounds for various classes of graphs

5.2.1 Planar Graphs

Currently, the best known bounds on the game chromatic number of planar graphs are

$$7 \leq \chi_g(\mathscr{P}) \leq 17.$$

To show that $7 \leq \chi_g(\mathscr{P})$, Kierstead of Trotter give a simple construction with a clever strategy for Bob that forces Alice to use more than 6 colors. It the same paper the authors claim that they found a planar graph with $\chi_g(G) \geq 8$, but this result has not been published.

There is a short proof of the upper bound $\chi_g(\mathscr{P}) \leq 18$ in Bartnicki et al. [3]. It uses the activation strategy together with a clever recursively defined vertex ordering. Further improvements of the activation strategy by Zhu lead to the best known upper bound of 17. In both strategies, Alice bounds the number of colored neighbors (not just colors) by 16, and thus $\text{col}_g(\mathscr{P}) \leq 17$. For the lower bound on the game coloring number, Wu and Zhu [60] constructed a planar graph G with $\text{col}_g(G) \geq 11$. Together,

$$11 \leq \text{col}_g(\mathscr{P}) \leq 17.$$

There are many results for planar graphs of a fixed girth $g(G)$. The key observation is that every planar graph G has a partition into a forest G_1 and a graph G_2 of bounded maximum degree (depending only on $g(G)$). He et al. [29] bounded the maximum degree of G_2 for several values of $g(G)$, and applied these results to the observation of Guan and Zhu [26]:

Theorem 5.4. *Suppose $G = (V, E)$ and $E = E_1 \cup E_2$. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. Then $col_g(G) \leq col_g(G_1) + \Delta(G_2)$.*

Theorem 5.5 (He et al. [29]). *Let G be a planar graph of girth $g(G)$.*

- *If $g(G) \geq 5$, then $col_g(G) \leq 8$.*
- *If $g(G) \geq 7$, then $col_g(G) \leq 6$.*

- *If* $g(G) \geq 11$*, then* $col_g(G) \leq 5$.
- *If G does not contain 4-cycles, then* $col_g(G) \leq 11$.

5.2.2 Outerplanar Graphs

The best known bounds on game chromatic number are for an outerplanar graph G are

$$6 \leq \chi_g(G) \leq \mathrm{col}_g(G) \leq 7.$$

The lower is by Kierstead and Trotter [36]. The upper bound is by Guan and Zhu [26]. They applied Theorem 5.4 with $G_1 := T$ and $G_2 := G - T$, where T is a tree constructed as follows: We can assume that G is triangulation. First, find an ordering $v_1, v_2, \ldots, v_n$ of vertices such that $v_1 v_2$ is an edge incident to the outer face, and each v_i with $i \geq 3$ has two backwards neighbors. From G, delete for each $i \geq 3$ the larger of the two backward neighbors, and call the resulting tree T. Finally, observe that for each vertex we deleted at most 3 incident edges, and so $\Delta(G - T) \leq 3$. It follows that $\mathrm{col}_g(G) \leq \mathrm{col}_g(T) + \Delta(G - T) \leq 7$.

5.2.3 Partial k-trees

A graph G is a *k-tree* if $G \simeq K_k$, or if G can be constructed from a k-tree by adding a vertex adjacent to exactly k vertices that form a clique. Alternatively, k-trees are the maximal graphs with a treewidth k. A *partial k-tree* is a subgraph of a k-tree.

Zhu [62] proved that $\mathrm{col}_g(G) \leq 3k + 2$ for any partial k-tree. Wu and Zhu [60] showed that this is tight by presenting a family of partial k-trees that have coloring number $3k + 2$ and showing that $\mathrm{col}_g(G)$ is monotone on taking subgraphs.

5.2.4 Forests

Every forest has an ordering of its vertices such that each vertex has at most one backwards neighbor. If we apply the activation strategy on such ordering, then every uncolored vertex can have at most three colored neighbors after the play of Bob - its unique backward neighbor, and two forward neighbors. Thus, the game coloring number of forests is at most 4. This has been proved much earlier by Faigle et al. [18]. They used the notion ‘game chromatic number’ and stated the theorem for trees only, but the more general statement follows from their proof.

Theorem 5.6 (Faigle et al. [18])**.** $\chi_g(T) \leq 4$ *for every tree* T.

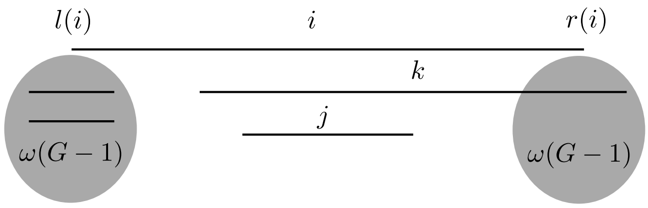


Figure 5.3: Bounding the number of colored intervals intersecting an uncolored interval i in the proof of Theorem 5.7.

Their strategy goes as follows: Alice starts with coloring any vertex r (the root) and defines T_0 to be the trivial tree consisting of the vertex r. This tree will grow as the game proceeds. Suppose that Bob colors a vertex v, and let u be the first vertex of T_0 on the path P from v to r. Alice first updates the tree $T_0 := T_0 \cup P$. If u is uncolored, she colors it. If u is colored, she colors any uncolored vertex of T_0. If there is no uncolored vertex in T_0, Alice colors any uncolored neighbor of T_0. Note that this strategy is essentially equivalent (up to some details about which vertex to color if u is colored) to the activation strategy since the vertices of T_0 correspond to the activated vertices.

Dunn et al. [16] characterized forests with game chromatic number 2 and provided a simple criterion to determine the game chromatic number of forests with no vertex of degree 3. The authors also showed that there exists a tree T with $\chi_g(T) = 4$ and $\Delta(G) = 3$. The problem of determining the game chromatic number of forests with a vertex of degree 3 is still open. It is not even known if there exists a polynomial time algorithm determining whether a forest with maximum degree 3 has game chromatic number 4.

5.2.5 Interval Graphs

An *interval graph* is a graphs that can be represented as the intersection graph of a family of intervals on the real line. It has one vertex for each interval in the family, and an edge between every pair of vertices corresponding to intervals that intersect.

Theorem 5.7 (Faigle et al. [18]). *For every interval graph G,*

$$col_g(G) \leq 3\omega(G) - 2.$$

Their proof uses the following strategy for Alice. After Bob colors an interval j, Alice colors the interval k that contains j and has the largest right endpoint. If there is no such interval, Alice colors the uncolored interval with the largest right endpoint among all uncolored intervals. Suppose that

Alice has just finished her move. Every uncolored interval i has at most $\omega(G)-1$ other intervals containing the left point $l(i)$ of i and at most $\omega(G)-1$ intersecting the right point $r(i)$ of i. It remains to bound the number of colored intervals inside i not containing the end points $l(i)$ and $r(i)$. The crucial observation is that for every such interval j, there must be an interval k containing $l(j)$ and $r(i)$. Indeed, first observe that j must have been colored by Bob, since Alice would have colored i instead (see Figure 5.3). Second, after Bob colored j, Alice colored an interval k which has larger right point than i. In addition, all such intervals k contain not only $r(i)$ but also the largest left point $l(j)$ among the intervals j. Therefore, the number of colored intervals inside i is at most $\omega(G)-2$. Together, the number of colored intervals intersecting i is at most $3\omega(G)-4$ before the move of Bob, at most $3\omega(G)-3$ before the move of Bob, and there is one color left for the move of Alice.

As for the lower bound, Faigle et al. [18], construct an interval graph G with game chromatic number $2\omega-1$, where $\omega=\omega(G)$. Let $G+H$ be the graph formed from the disjoint union of G and H by adding all possible edges between G and H, and let I_k be an empty graph with k vertices. Faigle et al. observe that the game chromatic number of two disjoint copies of $I_{2(\omega-1)}+K_{\omega-1}$ is at least $2(\omega-1)$. After Alice colors her first vertex in one copy, Bob only colors vertices in the second copy (so taking two disjoint copies is equivalent to letting Bob start the game on one copy). There, he will always color a vertex from the independent set $I_{2(\omega-1)}$ with a new color. He will use at least $\omega-1$ colors before Alice colors the last vertex of $K_{\omega-1}$, so the total number of colors needed is at least $^2(\omega-1)$. The authors further improve the bound to $2\omega-1$ by taking two copies of the above graph through a new vertex v to obtain $I_{2(\omega-1)}+K_{\omega-1}+\{v\}+K_{\omega-1}+I_{2(\omega-1)}$. Here, Bob starts in the untouched copy by coloring the vertex v and then applies the same strategy as above to the half whose $K_{\omega-1}$ has no colored vertices. Faigle et al. [18] claim without proof that interval graphs with game chromatic number $2\omega(G)$ can be constructed.

As far as we know, there is no other lower bound known on the game chromatic number or the game coloring number of interval graphs. In Section 5.3, we present a new construction of interval graphs G with $\mathrm{col}_g(G)\geq 2.5\omega(G)-3$.

5.3 New results

Theorem 5.8 (Kaiser-Petříčková). *For every $\omega\in\mathbb{N}$ there is an interval graph G with clique number ω such that*

$$col_g(G)\geq 2.5\omega-3.$$

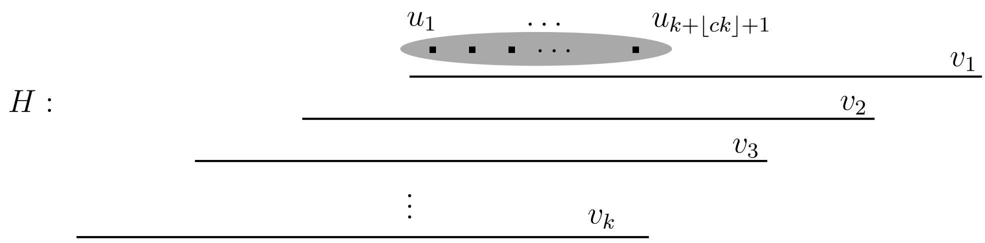


Figure 5.4: Graph H. The building block for the graph G in Theorem 5.8.

Proof. Without loss of generality we can assume that Bob starts the game since he can take two disjoint graphs defined below and discard the copy where Alice made her first move. Fix $\omega \in \mathbb{N}$. We show that there exists a graph G such that for any play of Alice, an uncolored vertex with at least $2k + \lfloor ck \rfloor$ (where $0 < c < 1$ will be specified later) colored neighbors appears at some point of the game. Let $H \simeq I_{k+\lfloor ck \rfloor+1} + K_k$, with the set of vertices in $I_{k+\lfloor ck \rfloor+1}$ called $X = \{u_1, \ldots, \ldots, u_{k+\lfloor ck \rfloor+1}\}$ and the set of vertices in K_k called $Y = \{v_1, \ldots, v_k\}$. See Figure 5.4 for an interval representation of H. Consider disjoint copies $H_1, \ldots, H_{2l+1}$, where $l \geq \frac{k-3}{2}$, of H, and denote by X_j and Y_j the sets of vertices in H_j that correspond to X and Y. We define G as the graph formed by connecting $H_1, \ldots, H_{2l+1}$ in series as in Figure 5.5. In particular, the copy of v_i in H_1 will be adjacent to the copies of $v_{i+1}, \ldots, v_k$ in H_2, and analogously for edges between H_j and H_{j+1} for $j = 2, \ldots, 2l$. Bob's strategy is to first color $k + \lfloor ck \rfloor$ vertices in each X_j with $j = 1, 3, 5, \ldots, 2l + 1$. We analyze two possible scenarios after the first $(k + \lfloor ck \rfloor)(l + 1)$ rounds:

1. There is an uncolored vertex in Y_j for some $j \in \{1, 3, \ldots, 2l + 1\}$. Then Bob colors the last vertex of X_j. Since Y_j induces a clique, there will eventually be a vertex in Y_j with at least $|X_j| + |Y_j| = k + \lfloor ck \rfloor + 1 + (k-1) = 2k + \lfloor ck \rfloor$ colored neighbors.

2. All vertices of $Y_1 \cup Y_3 \cup \cdots \cup Y_{2l+1}$ have been colored by Alice. This means that the number of colored vertices in $Y_2 \cup Y_4 \cup \cdots \cup Y_{2l}$ is at most $\lfloor ck \rfloor(l+1)$. It follows that there exists H_i with $i \in \{2, 4, \ldots, 2l\}$ that has at most $\frac{\lfloor ck \rfloor(l+1)}{l} = \lfloor ck \rfloor + \frac{\lfloor ck \rfloor}{l}$ colored vertices. Now, Bob will subsequently color the vertices of X_i. Bob will be able to color $k - (\lfloor ck \rfloor + \frac{\lfloor ck \rfloor}{l})$ vertices before Alice manages to color all vertices in Y_i. In addition to the colored neighbors in X_i, every vertex in Y_i has $k-1$ colored neighbors in $H_{i-1} \cup H_{i+1}$. So there will eventually be an uncolored vertex with at least $k - (\lfloor ck \rfloor + \frac{\lfloor ck \rfloor}{l}) + (k-1) + (k-1) = 3k - \lfloor ck \rfloor - \frac{\lfloor ck \rfloor}{l} - 2$ colored neighbors.

Choose c so that $3k - \lfloor ck \rfloor - \frac{\lfloor ck \rfloor}{l} - 2 = 2k + \lfloor ck \rfloor$. It follows that $\lfloor ck \rfloor =$

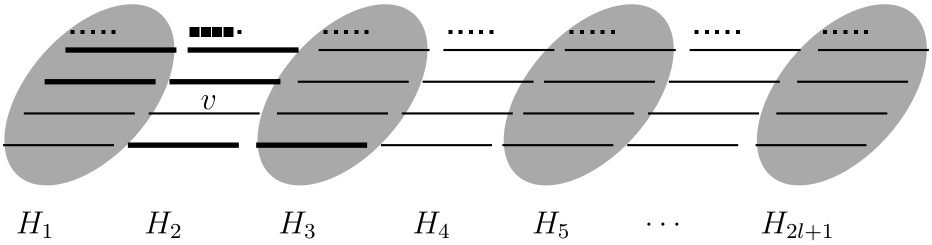
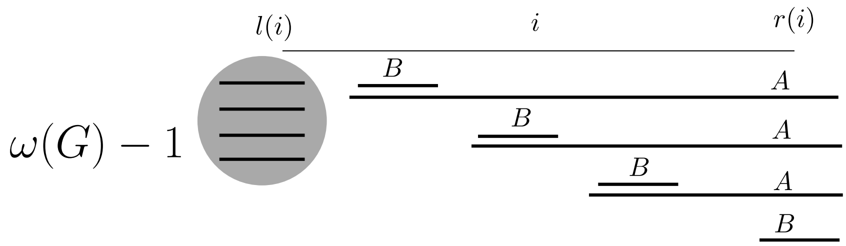


Figure 5.5: The graph G from Theorem 5.8. The bold intervals are the colored neighbors of the last uncolored vertex v of Y_i in the second scenario of the proof.

$\frac{k}{2}-1-\frac{k-2}{4l+2} \geq \frac{k}{2}-1-\frac{1}{2}$ for $l \geq \frac{k-3}{2}$. Finally, $2k+\lfloor ck \rfloor \geq 2.5k-1-\frac{1}{2} = 2.5\omega-4$. □

Observation 5.9. *We were unable to generalize the argument in the proof of Theorem 5.8 to bound the game chromatic number. The problem is that Alice can assign the same color to all vertices in the same level (i.e., for each $i \in [k]$, all copies of v_i in can receive the same color). Thus, in the second scenario, the number of colors in the neiborhood of the last uncolored vertex of Y_i is only $k-(\lfloor ck \rfloor + \frac{\lfloor ck \rfloor}{l}) + k - 1 < 2k$.*

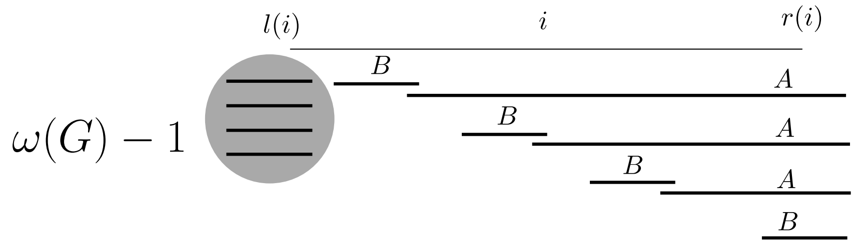
Beneath Theorem 5.7 we sketched the proof of Faigle et al.[18] that $\mathrm{col}_g \leq 3\omega(G)-2$ for every interval graph G. On the following picture we show an example of and interval graph for which this strategy of Bob forces Alice to use all $3\omega(G)-2$ "colors".



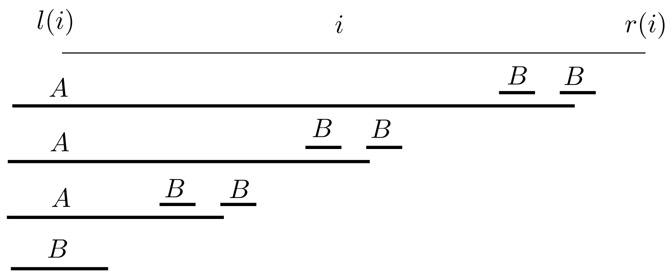
We show that the strategy of Alice given by Faigle et al.[18] may be slightly simplified.

Alternative strategy for Alice: After Bob colors an interval j, Alice colors the interval k that intersects j and has the largest right endpoint. If there is no such interval, Alice colors the uncolored interval with the largest right endpoint.

Let i be an uncolored interval at some point of the game after the move of Alice, and let $C(i)$ be the colored intervals intersecting i, $C_L(i)$ the colored intervals j containing the left endpoint $l(i)$ of i, and $C_R(i) := C(i) \setminus C_L(i)$. Since all intervals in $C_L(i)$ contain $l(i)$, we have $|C_L(i)| \leq \omega(G)-1$. We now count $|C_R(i)|$. The number of intervals containing $r(i)$ is again at most $\omega(G)-1$. Let j be a colored interval intersecting i but containing none of $l(i)$ and $r(i)$. Then j was colored by Bob since Alice would have colored i instead. Since i is not colored, there has to be some other interval k containing $r(i)$ that Alice colored after Bob colored j. In addition, the rightmost interval inside i intersects all of the intervals of "type k". Thus, the number of colored intervals inside i is at most $\omega(G)-2$. Together, for any $i \in U$, $|C(i)| \leq 3\omega(G)-4$. This number increases by at most 1 after the move of Bob, and so $\mathrm{col}_g(G) \leq 3\omega(G)-2$. An example of a graph in which Alice needs all $\mathrm{col}_g(G) \leq 3\omega(G)-2$ "colors" when following this strategy is shown on the following picture. Note that if we assume that i has $3\omega(G)-2$ colored neighbors, then we know that all intervals in $C_L(i)$ were colored by Alice. This observation might be useful for improving the upper bound on the game coloring number for interval graphs – Alice may be able to avoid marking too many such intervals.



Activation strategy for Alice: Suppose that Alice applies the activation strategy on the vertices ordered according to the position of the left points of the intervals. The activation strategy ensures that for every backward neighbor of an uncolored vertex/interval i there are at most two activated forward neighbors of i (i.e., at most two intervals in $C_R(i)$). The ordering ensures that the backward neighborhood of any vertex u together with u itself forms a clique. Hence for any uncolored interval i, the number of backwards neighbors is at most $\omega(G)-1$. The bound $3\omega(G)-2$ easily follows as in the previous two strategies. The following sharpness example however looks quite different:



5.4 Further research

In the previous section, we constructed three interval graphs that witness the upper bound $3\omega(G)-2$ on $\chi_g(G)$ for three different strategies. Their wide variety suggests that it might be possible to lower the best known upper bound $3\omega(G)-2$ for interval graphs closer to $2.5\omega(G)$.

There are many other open problems for coloring games or marking games. Improving the upper bound on the game chromatic number for planar graphs seems hard. The proof for the best known upper bound of 17 is long and uses a complicated dynamic ordering for the activation strategy. It seems that an additional idea on how to incorporate colors in Alice's strategy is needed.

One of the most intriguing question is whether more colors can hurt Alice:

Problem 5.10 (Zhu [62]). *Suppose $\chi_g(G) = k$. It is true that for any $k' > k$, Alice has a winning strategy for the coloring game played on G with k' colors?*

It is not even known whether there is a function f (where f depends only on k and is not the identity function) such that, if Alice wins the game on G with k colors, then Alice wins the game on G with $f(k)$?

The game coloring number cannot be bounded in terms of the chromatic number. Indeed, it is not hard to see that $\chi_g(K_{n,n}) = 3$ and $\mathrm{col}_g(K_{n,n}) = n+1$. Zhu [62] suggested that the reason for this might be that the game chromatic number is not a monotone parameter. (For example, the game chromatic number of $K_{n,n}$ is 3, but when we erase a perfect matching from $K_{n,n}$, the game chromatic number increases to n. The game coloring number, on the other hand, is a monotone parameter (Wu and Zhu [60]).) He therefore asked the following question:

Problem 5.11 (Zhu [62]). *Suppose that a hereditary class of graphs (i.e., a class of graphs closed under the operation of taking subgraphs) has bounded game chromatic number. Is it true that this class of graph also has bounded game coloring number?*

We end this section with one more open problem.

Problem 5.12 (Peterin [52]). *Is it true that $\chi_g(Q_n) = n+1$?*

Peterin [52] showed that it is true for $n \le 4$.

Chapter 6

Conclusion

This thesis is devoted to chromatic graph theory. At the beginning we presented several applications of graph coloring, defined basic terminology and notation, review fundamental results in the area, and introduced the four problems studied in the main part of the thesis.

First we investigated online Ramsey games, where Builder and Painter alternately color edges red or blue, and the goal of Builder is to ensure that a monochromatic copy of a target graph is eventually created, in which case we say that the target graph is unavoidable. We studied the game when the players were required to only add edges so that the resulting graph is planar. Grytczuk, Hałuszczak, and Kierstead [24] (2004) conjectured that the class of graphs unavoidable on planar graphs is exactly the class of outerplanar graphs. We however constructed an infinite family of non-outerplanar graphs for which Builder has a winning strategy when playing on planar graphs, showing that the conjecture does not hold. This extends the result of the Author's Master's Thesis ([53]), where it was proven that the graph $K_{2,3}$ is unavoidable on planar graphs. On the other hand, we generalized partial results of [53], and proved that all outerplanar graphs are unavoidable on planar graphs. The paper *Online Ramsey theory for planar graphs* is attached as it appeared in The Electronic Journal of Combinatorics.

Our second problem was to study the chromatic number of $G^{\frac{m}{n}}$, which is the graph formed from G by replacing edges with internally disjoint paths of length n and connecting vertices within distance at most m of an edge. Iradmusa [31], 2010) conjectured that if G is a connected graph with $\Delta(G) \geq 3$, $n, m \in \mathbb{N}$ and $1 < m < n$, then $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$. We showed that the conjecture does not hold in full generality by presenting a graph H for which $\chi(H^{\frac{3}{5}}) > \omega(H^{\frac{3}{5}})$. Next, we proved that the conjecture is true if m is even and $\Delta(G) \geq 4$. We also investigated the case when m is odd, where we obtained a general upper bound $\chi(G^{\frac{m}{n}}) \leq \omega(G^{\frac{m}{n}}) + 2$ for graphs with $\Delta(G) \geq 4$ and several other results. This is joint work with Stephen Hartke and Hong Liu. The paper *Coloring fractional powers of graphs* has been accepted for publication in the Journal of Graph Theory.

Our third topic was chromatic-choosability of powers of graphs. A graph G is said to be chromatic-choosable if its chromatic number $\chi(G)$ is equal to its list chromatic number $\chi_\ell(G)$. Disproving the List Square Coloring Conjecture, Kim and Park found an infinite family of graphs whose squares are not chromatic-choosable. Xuding Zhu asked whether there exists a k such that all k-th power graphs are chromatic-choosable. We answered this question in the negative: we showed that for any natural number k there is a family of graphs G such that $\chi_\ell(G^k) > \chi(G^k)$, where G^k denotes the k-th power of G. This is a joint work with Nicholas Kosar, Benjamin Reiniger, and Elyse Yeager. The paper *A note on list-coloring powers of graphs* is attached as it appeared in the journal Discrete Mathematics.

Our last topic was the graph coloring game of two players, Alice and Bob, who alternately color vertices of a given graph G. If at some point either player is not able to properly color an uncolored vertex, the game ends, and Bob wins. Otherwise, all vertices of G are properly colored at the end, and Alice wins. The *game chromatic number* $\chi_g(G)$ is the smallest number of colors needed such that Alice can always win. We reviewed the best known bounds on $\chi_g(G)$ for various classes of graphs. Then we constructed an infinity family of interval graphs with $\mathrm{col}_g(G) \geq 2.5\omega(G) - 3$, where $\mathrm{col}_g(G)$ is equal to $1 + k$ where k is the maximum back degree of a linear order produced by playing the game with both players using their optimal strategies. This is a joint (unpublished) result with Tomáš Kaiser.

Bibliography

[1] Noga Alon, *Choice numbers of graphs: a probabilistic approach*, Combin. Probab. Comput. 1(2) 1992, 107–114.

[2] E. Anand and R. Panneerselvam, *Literature Review of Open Shop Scheduling Problems,* Intelligent Information Management 7 (2015), 33–52.

[3] T. Bartnicki, J. Grytczuk, H. A. Kierstead, and X. Zhu, *The Map-Coloring Game*, American Mathematical Monthly 114(9) (2007).

[4] J. Beck, *Achievement games and the probabilistic method*, Combinatorics, Paul Erdős is Eighty, Bolyai Soc. Math. Stud. 1 (1993), 51–78.

[5] M. Behzad, *Graphs and their chromatic numbers*, Ph.D. Thesis, Michigan State University (1965).

[6] B. Bollabás and A. J. Harris, *List-colorings of graphs*, Graphs Combin. 1 (1985), 115–127.

[7] O. V. Borodin, A. V. Kostochka, D. R. Woodall, *List edge and list total colourings of multigraphs*, J. Combin. Theory Ser. B, 71 (1997), 184–204.

[8] J. Butterfield, T. Grauman, W. B. Kinnersley, K. G. Milans, C. Stocker, D. B. West, *On-line Ramsey Theory for Bounded Degree Graphs*, Electron. J. Comb. 18 (2011).

[9] E. Friedgut, Y. Kohayakawa, V. Rödl, A. Ruciński, P. Tetali, *Ramsey Games Against a One-Armed Bandit*, Comb. Probab. Comput. 12 (2003), 515–545.

[10] R. L. Brooks, *On colouring the nodes of a network*, Proc. Cambridge Philosophical Society, Math. Phys. Sci. 37 (1941), 194–197.

[11] G. J. Chaitin, *Register allocation & spilling via graph coloring*, Proceedings of the 1982 SIGPLAN symposium on Compiler construction, Boston, MA, (1982), 98–105.

[12] Charles J. Colbourn and Jeffrey H. Dinitz, editors. *The CRC handbook of combinatorial designs*, CRC Press Series on Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, (1996).

[13] D. Conlon, *On-line Ramsey numbers*, SIAM J. Discrete Math. 23 (2009), 1954–1963.

[14] R. Diestel, *Graph Theory*, Springer-Verlag, Heidelberg, New York, (2005).

[15] T. Dinski and X. Zhu, *Game chromatic number of graphs*, Discrete Math. 196 (1999), 109–115.

[16] C. Dunn, V. Larsen, K. Lindke, T. Retter, D. Toci, *The game chromatic number of trees and forest*, Discrete Mathematics and Theoretical Computer Science, Vol. 17(2) (2015), 31–48.

[17] A. Eisenblatter, M. Grotschel, A. M. C. A. Koster, *Frequency planning and ramifications of coloring*, Discussiones Mathematicae Graph Theory 22(1) (2002).

[18] U. Faigle, W. Kern, H. A. Kierstead, and W. T. Trotter, *On the game chromatic number of some classes of graphs*, Ars Combin. 35(17) (1993), 143–150.

[19] F. G., *Tinting maps*, The Athenaeum: 726 (1854).

[20] R. Gandhi, M. M. Halldorsson, G. Kortsarz, H. Shachnai, *Improved Results for Data Migration and Open Shop Scheduling*, ACM Transactions on Algorithms 2(1) (2006), 116–129.

[21] M. R. Garey, D. S. Johnson, L. Stockmeyer, *Some simplified NP-complete problems*, Proceedings of the Sixth Annual ACM Symposium on Theory of Computing (1974), 47–63.

[22] R. L. Graham, B. L. Rotschild, J. H. Spencer, *Ramsey Theory*, Wiley-Interscience Series in Discrete Mathematics, USA, (1980).

[23] H. Grötzsch, *Zur Theorie der diskreten Gebilde, VII: Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel*, Wiss. Z. Martin-Luther-U., Halle-Wittenberg, Math.-Nat. Reihe, 8 (1959), 109–120.

[24] J. A. Grytczuk, M. Hałuszczak, H. A. Kierstead, *On-line Ramsey Theory*, Electron. J. Combin. 11(1) (2004)..

[25] J. A. Grytczuk, H. A. Kierstead, P. Prałat *On-line Ramsey Numbers for Paths and Stars*, Discrete Mathematics and Theoretical Computer Science 10(3) (2008).

[26] D. J. Guan, X. Zhu, *Game chromatic number of outerplanar graphs*, J. Graph Theory 30 (1999), 67–70.

[27] A. Gyárfás, J. Lehel, *On-line and first fit colorings of graphs*, J. Graph Theory 12(2) (1988), 217–227.

[28] S. Hartke, H. Liu, Š. Petříčková, *On coloring of fractional powers of graphs*, accepted to JGT.

[29] W. He, X. Hou, K.-W. Lih, J. Shao, W. Wang, X. Zhu, *Edge-partitions of planar graphs and their game coloring numbers* J. Graph Theory 41 (2002), pp. 307–317.

[30] T. E. Huffman, *Total coloring of graphs.* Ph.D. dissertation, San Jose State University, (1989).

[31] M. N. Iradmusa, *On colorings of graph fractional powers*, Discrete Math. 310 (2010), 1551–1556.

[32] T. R. Jensen and B. Toft, *Graph coloring problems*, Wiley-Intersci. Ser. Discrete Math. Optim., John Wiley & Sons, (1995).

[33] S. Jahanbekam, J. Kim, S. O, and D. B. West, *On r-dynamic coloring of graphs*, Discrete Applied Mathematics 206 (2016), 65–72.

[34] H. A. Kierstead, G. Konjevod, *Coloring number and on-line Ramsey theory for graphs and hypergraphs*, Combinatorica 29(1) (2009), 49–64.

[35] H. A. Kierstead, *Coloring graphs on-line*, Lecture Notes in Computer Science 1442 (1998), 281–305.

[36] H. A. Kierstead, W. T. Trotter, *Planar graph coloring with an uncooperative partner*, J. Graph Theory 18 (1994), pp. 569–584.

[37] K. Kilakos, B. Reed, *Fractionally colouring total graphs*, Combinatorica 13(4) (1993), 435–440.

[38] S.-J. Kim, Y. S. Kwon, and B. Park, *Chromatic-choosability of the power of graphs*, Discrete Applied Mathematics 180(C) (2015), 120–125.

[39] S.-J. Kim and B. Park, *Counterexamples to the list square conjecture*, J. Graph Theory 78(4) (2015), 239–247.

[40] D. König, *Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre*, Math. Ann. 77 (1916), 453–465.

[41] N. Kosar, Š. Petříčková, B. Reiniger, E. Yeager, *A note on list-coloring powers of graphs*, Discrete Math. 332 (2014), 10–14.

[42] A. V. Kostochka and D. R. Woodall, *Choosability conjectures and multicircuits*, Discrete Math. 240(1–3) (2001), 123–143.

[43] A. Kurek, A. Ruciński, *Two variants of the size Ramsey number*, Discuss. Math. Graph Theory 25 (1–2) (2005), 141–149.

[44] F. T. Leighton, *A graph coloring algorithm for large scheduling problems*, Journal of Research of the Notional Bureau of Standards 84 (6) (1979), 489–506.

[45] M. Malkawi, M. A. Hassan, and O. A. Hassan, *A new exam scheduling algorithm using graph coloring*, Int. Arab J. Inf. Technol. 5 (1) (2008), 80–86.

[46] B. D. McKay, *A note on the history of the four-colour conjecture* (2012), arXiv:1201.2852.

[47] J. Mitchem, *On the history and solution of the four color map problem*, The College Math. Journal 12 (1981), 108–116.

[48] M. Molloy, B. Reed, *A Bound on the Total Chromatic Number*, Combinatorica, Vol. 18, No. 2 (2000), 241–280.

[49] J. A. Noel. *Choosability of graph powers*, (2013).
`http://www.openproblemgarden.org/op/choosability_of_graph_powers`

[50] Suil O, *Matchings, connectivity, and eigenvalues in regular graphs*, Dissertation, University of Illinois at Urbana-Champaign (2011).

[51] T. Park, C. Y. Lee, *Application of the graph coloring algorithm to the frequency assignment problem*, Journal of the Operations Research Society of Japan 39 (1996), 258–265.

[52] I. Peterin , *Game chromatic number of Cartesian product graphs*, Electronic Notes in Discrete Mathematics 29 (2007), 353–357.

[53] Š. Petříčková, *Combinatorial Game Theory*, Master's Thesis, University of West Bohemia, (2010).

[54] Š. Petříčková, *Online Ramsey Theory for Planar Graphs*, Electron. J. Combin. 12(1) (2014).

[55] S. Pirzada, A. Dharwadker, *Applications of Graph Theory*, Journal of the Korean society for industrial and applied mathematics 11 (4) (2007).

[56] F. P. Ramsey, *On a problem of formal logic*, Proceedings of the London Mathematical Society 30 (1930), 264–286.

[57] H. Tamura, K. Nakano, M. Sengoku, S. Shinoda, *On applications of graph/network theory to problems in communication systems*, ECTI Transactions on Computer and Information Technology 5(1) (2011), 8–14.

[58] T. Vetrík, *List coloring of complete multipartite graphs*, Discussiones Mathematicae Graph Theory 32 (2012), 31–37.

[59] V. G. Vizing, *Some unsolved problems in graph theory*, Uspekhi Mat. Nauk 23 (1968), 117–134.

[60] J. Wu, X. Zhu, *Lower bounds for the game colouring number of planar graphs and partial k-trees*, Discrete Math. 308 (2008) 2637–2642.

[61] X. Zhu, *The game coloring number of planar graphs*, J. Combin. Theory, Ser B, 75(2) (1999), 245–258.

[62] X. Zhu, *The game coloring number of pseudo partial k-trees*, Discrete Math., 215 (2000), 245–262.

[63] X. Zhu., personal communication.

[64] `http://world.mathigon.org/Graph_Theory`

Chapter 7

Appendix

Online Ramsey Theory for Planar Graphs

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Abstract

An online Ramsey game $(G, \mathcal{H})$ is a game between Builder and Painter, alternating in turns. During each turn, Builder draws an edge, and Painter colors it blue or red. Builder's goal is to force Painter to create a monochromatic copy of G, while Painter's goal is to prevent this. The only limitation for Builder is that after each of his moves, the resulting graph has to belong to the class of graphs $\mathcal{H}$. It was conjectured by Grytczuk, Hałuszczak, and Kierstead (2004) that if $\mathcal{H}$ is the class of planar graphs, then Builder can force a monochromatic copy of a planar graph G if and only if G is outerplanar. Here we show that the "only if" part does not hold while the "if" part does.

Keywords: Ramsey theory; Online Ramsey games; Planar graphs; Outerplanar graphs; Game theory; Builder and Painter

1 Introduction

For a fixed graph G and a class of graphs $\mathcal{H}$ such that $G \in \mathcal{H}$, an *online Ramsey game* $(G, \mathcal{H})$, defined by Grytczuk, Hałuszczak, and Kierstead [5], is a game between Builder and Painter with the following rules. The game starts with the empty graph on infinitely many vertices. On the i-th turn, Builder adds a new edge to the graph created in the first $i-1$ turns so that the resulting graph belongs to $\mathcal{H}$ (we say that Builder *plays on* $\mathcal{H}$), and Painter colors this edge blue or red. *Builder wins* if he can always force Painter to create a monochromatic copy of G (or *force* G for short). We then say that G is *unavoidable on* $\mathcal{H}$. A graph G is *unavoidable* if it is unavoidable on planar graphs. On the other hand, if Painter can ensure that a monochromatic copy of G is never created, then G is *avoidable on* $\mathcal{H}$. A class of graphs $\mathcal{H}$ is *self-unavoidable* if every graph of $\mathcal{H}$ is unavoidable on $\mathcal{H}$.

According to Ramsey's theorem, for every $t \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that every 2-coloring of the edges of K_n contains a monochromatic copy of K_t. Thus, without restricting to $\mathcal{H}$, Builder would always win the online Ramsey game by creating sufficiently large complete graph. The *size Ramsey number* $r(G)$ for a graph G is the minimum number of edges of a graph that contains a monochromatic copy of G in every 2-coloring of its edges. The *online size Ramsey number* $\tilde{r}(G)$ is the minimum m such that Builder can force G by playing on the class of graphs with at most m edges. Clearly, $\tilde{r}(G) \leqslant r(G)$ (Builder wins by presenting a graph of size $r(G)$ that contains a monochromatic copy of G for any 2-edge-coloring). However, Builder may be able to win using less than $r(G)$ edges since he can adapt his strategy to Painter's coloring. One can then ask whether or not $\tilde{r}(G) = o(r(G))$. The basic conjecture in the field, attributed to Rödl by Kurek and Ruciński [9], is that $\tilde{r}(K_t) = o(r(K_t))$. In 2009, Conlon [3] showed that $\tilde{r}(K_t) \leqslant 1.001^{-t}(r(K_t))$ for infinitely many t. On the other hand, if G is a path or a cycle, then both $\tilde{r}(G)$ and $r(G)$ are linear in $|V(G)|$ (see [1], [6], [7]).

Butterfield et al. [2] studied Online Ramsey games played on the class $\mathcal{S}_k$ of graphs with maximum degree at most k. The authors introduce an *online degree Ramsey number* $\tilde{r}_\triangle(G)$ as the least k for which G is unavoidable on $\mathcal{S}_k$.

Online Ramsey games played on various classes of graphs were studied by Grytczuk et al. [5]. They proved that the class of k-colorable graphs as well as the class of forests are self-unavoidable. (It was later shown by Kierstead and Konjevod [8] that the k-colorable graphs are self-unavoidable even if Painter uses more colors.) Various games played on planar graphs were investigated in [5]. It was shown, for example, that every cycle, as well as the graph $K_4 - e$, is unavoidable on planar graphs. They made the following conjecture:

Conjecture ([5])**.** The class of graphs unavoidable on planar graphs is exactly the class of outerplanar graphs.

Here we show that the conjecture is only partially true. In particular, it is true that the class of outerplanar graphs is a subclass of the class of graphs unavoidable on planar graphs.

Theorem 1. *Every outerplanar graph is unavoidable on planar graphs.*

However, we show that there exists an infinite family of planar but not outerplanar graphs which are unavoidable on planar graphs. Let $\theta_{i,j,k}$ denote the union of three internally disjoint paths of lengths i, j, k, respectively.

Theorem 2. *The graph $\theta_{2,j,k}$ is unavoidable for even j, k.*

The paper is organized as follows. In Section 2, we introduce notation. Section 3 gives a proof of Theorem 1, and Section 4 gives a proof of Theorem 2.

2 Notation

In this section, we first mention several notions that are particularly important for the next discussion. Besides these, we follow standard graph theory terminology (see Diestel [4]).

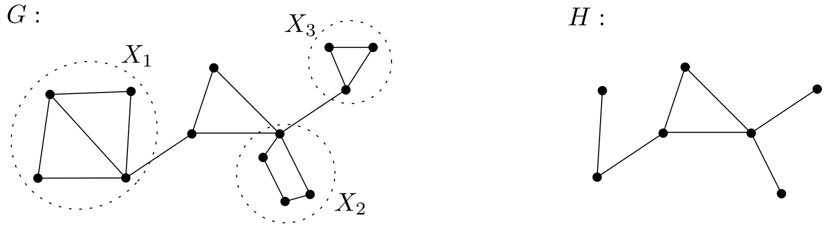


Figure 1: Graphs G and H such that $G \rhd H$.

All graphs considered here are simple and undirected. For a graph G, the set of vertices is denoted by $V(G)$ and the set of edges by $E(G)$. The length of a path is the number of its edges. If we replace an edge e of G with a path of length $k+1$ (i.e. we place k vertices of degree 2 on e), then we say that *e is subdivided* k-times. For a fixed graph G, a *copy* H of G is a graph isomorphic to G with $V(G) \cap V(H) = \emptyset$. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, their *union* is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. When we say that a graph is a *disjoint union* of G_1 and G_2, we are automatically assuming that $V_1 \cap V_2 = \emptyset$. A *planar graph* is a graph that can be drawn in the plane without edge crossings. An *outerplanar graph* is a planar graph that can be embedded so that all its vertices belong to the boundary of the outer face. A *red-blue graph* is a graph with its edges colored red or blue. A red-blue graph will often use the same name as its underlying (uncolored) graph.

Let G_1 and G_2 be two disjoint graphs containing cliques H_1, H_2 ($H_i \subseteq G_i$) of size $k \geqslant 0$. Let the vertices of H_i be labeled $v_1(G_i), \ldots, v_k(G_i)$. A *$k$-sum* $G_1 \oplus_k G_2$ of G_1 and G_2 is a graph formed from the disjoint union of G_1 and G_2 by identifying the vertex $v_j(G_1)$ with $v_j(G_2)$ for each $j = 1, \ldots, k$. To simply notation, we write $G_1 \oplus G_2$ if $k \leqslant 2$. Note that $G_1 \oplus G_2$ does not specify the appending cliques, and so it is not a well-defined operation. However, if $k = 1$, then we can make this notation precise and specify the appending vertex v by writing $G_1 \oplus_v G_2$ (which we will do often). For $k = 2$, we sometimes write $G_1 \oplus_e G_2$, where $e(G_i)$ is a non-oriented edge $v_1(G_i)v_2(G_i)$ (the resulting graph is again not always unique). Also, we abbreviate $((G_1 \oplus G_2) \oplus \ldots) \oplus G_n$ by $G_1 \oplus G_2 \oplus \cdots \oplus G_n$.

Let G be a graph, H a subgraph of G. If there exist planar graphs $X_1, \ldots, X_n$ such that $G = H \oplus X_1 \oplus \cdots \oplus X_n$, then we say that G is *reducible to* H, and we write $G \rhd H$. It is a well known fact that for $k \leqslant 2$, a k-sum of two planar graphs is planar, thus the following holds:

Remark 3. If H is a planar graph, and a graph G is reducible to H, then G is planar.

Informally, G is reducible to H if G can be formed from H by successively "appending" planar graphs on edges/vertices. So, Remark 3 says that if the starting graph H is planar, then so is G.

Consider an online Ramsey game on planar graph. A *strategy (for Builder)* $\mathbf{X}$ is a finite sequence of rules that tell Builder how to move on any given turn of the game, no matter how Painter plays. If a monochromatic copy of the target graph G arises, the game ends and Builder wins (provided that the final red-blue graph is planar). The *output graph* of strategy $\mathbf{X}$ is then the final red-blue graph with a fixed monochromatic copy of G, called a *winning copy* (of G by $\mathbf{X}$) and denoted simply by G if no confusion can arise. This winning copy adopts all notation from the target graph. For example, for a target graph G with vertices u, v and a cycle C, the two corresponding vertices and the corresponding cycle are again denoted by u, v, C in the chosen winning copy G. If Builder always wins when following strategy $\mathbf{X}$, then we say that G is *unavoidable by strategy* $\mathbf{X}$. The set of all output graphs of a strategy $\mathbf{X}$ is denoted $\mathcal{X}$ (the calligraphic version of the name of the strategy).

3 Outerplanar graphs

In this section we show that every outerplanar graph is unavoidable on the class of planar graphs. The idea behind our proof is based on the inductive proof of the self-unavoidability of forests presented by Grytczuk et al. [5]. Suppose that Builder's goal is to force a forest T. We can assume that T is a tree (since every forest is contained in some tree). Choose a leaf u of T, let v be the neighbor of u in T, and let $T' = T - u$. Builder forces $2|T| - 1$ monochromatic copies of T' (where the corresponding final graphs are pairwise disjoint), from which at least $|T|$ are of the same color, say blue. On those copies, Builder builds a new copy of T by adding edges between copies of v. If any one of the new added edges is blue, then that edge and a blue copy of T' appended to one of its endpoints form a blue copy of T. Otherwise, those edges form a red copy of T. We will call this strategy the *tree strategy*.

Since trees are planar, the tree strategy shows that forests are unavoidable (on planar graphs). Moreover, a generalized version of the tree strategy can be used for forcing a graph formed from a tree T by appending a copy of an unavoidable graph G to each vertex of T. Before presenting this strategy we need some notation.

Let T be a tree on vertices $v_1(T), \dots, v_n(T)$, and let G be a graph with an arbitrary vertex labeled by v. The ordered triple (T, G, v) denotes the graph $T \oplus_{v_1} G_1 \oplus_{v_2} \cdots \oplus_{v_n} G_n$, where for $i = 1, \dots, n$, G_i is a copy of G and $v_i(G_i) \in V(G_i)$ is the copy of v. We refer to the identified vertices $v_i(T) = v_i(G)$ in (T, G, v) by v_i. Next, let $\mathcal{S}$ be any set of red-blue graphs X such that each has a fixed monochromatic copy G_X of G. Let A be a red-blue graph with a fixed monochromatic subgraph $(T, G, v) = T \oplus_{v_1} G_1 \oplus_{v_2} \cdots \oplus_{v_n} G_n$. We say that A is $(T, \mathcal{S})$*-reducible* if $A \triangleright T \oplus_{v_1} X_1 \oplus_{v_2} \cdots \oplus_{v_n} X_n$, where for each $i \in [n]$ either X_i is a monochromatic copy of G, or $X_i \in \mathcal{S}$ such that $G_i = G_{X_i}$. In our proofs we take $\mathcal{S}$ to be the set of all final graphs of some strategy. For example, the set of all output graphs $\mathcal{X}$ of a strategy $\mathbf{X}$ is a set of red-blue graphs, each with a fixed monochromatic copy of G, and so, for any given tree T, we can talk about $(T, \mathcal{X})$-reducible graphs.

Suppose that G is unavoidable by strategy $\mathbf{X}$. We consider the following Builder's strategy for forcing a monochromatic copy of (T, G, v).

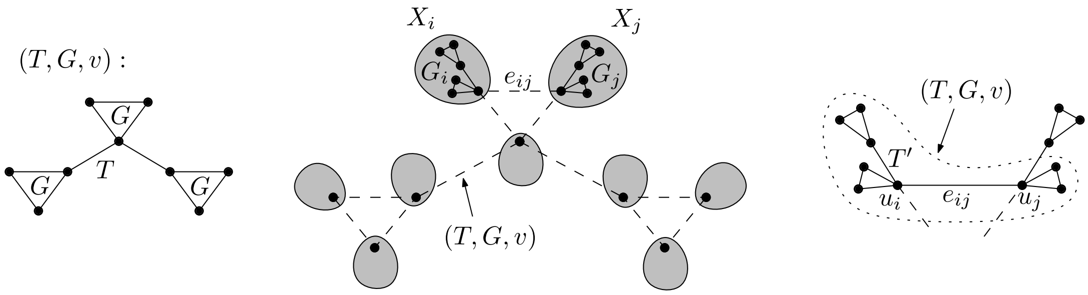


Figure 2: Forcing a monochromatic copy of (T, G, v), where T is a path of length 2, G a triangle, and v is an arbitrary vertex of $V(G)$.

strategy A $(T, G, v, \mathbf{X})$
Let $n = \lvert V(T)\rvert$, $k = \lvert V((T, G, v))\rvert$. 1. If $n = 1$, call strategy $\mathbf{X}$ and stop. 2. If $n > 1$, choose a leaf u of T, set $T' = T - u$, and call strategy $\mathbf{A}(T', G, v, \mathbf{X})$ $(2k - 1)$-times. Choose k copies $H_1, \ldots, H_k$ of (T', G, v) of the same color, and in ith of them label the vertex that corresponds to the neighbor of u in T by u_i. Add an edge $e_{ij} = u_i u_j$ if and only if $v_i v_j$ is an edge in (T, G, v).

To prove that (T, G, v) is unavoidable by strategy $\mathbf{A}(T, G, v, \mathbf{X})$, we have to ensure that no matter how Painter plays, a monochromatic copy of the target graph (T, G, v) eventually appears, and that the final graph is planar. Both parts are shown below using induction and reduction arguments that rely on Remark 3.

Lemma 4. *Let T be a tree, G a graph, and v a vertex of $V(G)$. If G is unavoidable by strategy $\mathbf{X}$, then (T, G, v) is unavoidable by strategy $\mathbf{A}(T, G, v, \mathbf{X})$, and every graph A of $\mathcal{A}(T, G, v, \mathbf{X})$ is $(T, \mathcal{X})$-reducible.*

Proof. We use all the notation introduced in strategy $\mathbf{A}$. The proof is by induction on the number n of vertices of T. If $n = 1$, then $(T, G, v) = G$, which is unavoidable by strategy $\mathbf{X}$ by the assumption. Since $\mathcal{A}(T, G, v, \mathbf{X}) = \mathcal{X}$, the graph A is $(T, \mathcal{X})$-reducible. Now let $n > 1$. The following two cases can arise.

Case 1: All edges e_{ij} are red. These edges form a red (T, G, v). Every final graph for forcing H_i is planar by the induction hypothesis. Observe that each such graph is appended to (T, G, v) by one vertex only. Thus, A is reducible directly to (T, G, v), and hence is $(T, \mathcal{X})$-reducible, which proves the planarity as well as the second part of the claim. See Figure 2.

Case 2: Some edge e_{ij} is blue. The graph H_i, the edge e_{ij}, and one copy of G contained in H_j form a blue (T, G, v). The graph A is planar by previous discussion, so the first part of the claim is complete. Let $A_i, A_j \in \mathcal{A}(T', G, v, \mathbf{X})$ be subgraphs of A that were used for forcing H_i and H_j, respectively. By the induction hypothesis, A_i is $(T', \mathcal{X})$-reducible.

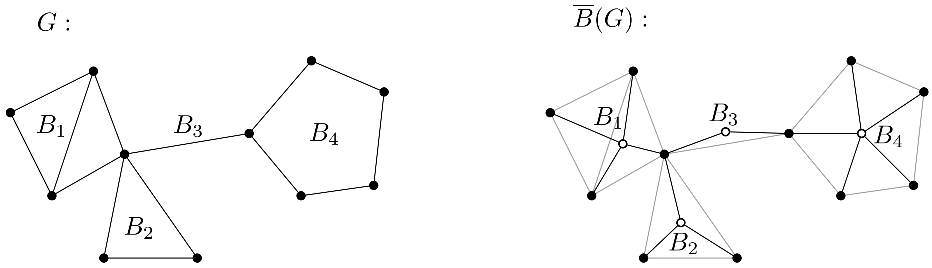


Figure 3: A graph G and its complete block graph $\overline{B}(G)$.

Similarly A_j is $(T', \mathcal{X})$-reducible, and therefore is $(u', \mathcal{X})$-reducible. Since the rest of A is reducible to e_{ij}, and e_{ij} shares with each of H_i and H_j only one vertex (the vertex u_i and u_j, respectively), we get the second part of the claim. □

A *block* is a maximal 2-connected subgraph. For a graph G with a vertex set $V = \{v_1, \ldots, v_k\}$ and blocks $B_1, \ldots, B_l$, the *complete block graph* $\overline{B}(G)$ is a graph on $V \cup \{B_1, \ldots, B_l\}$ formed by the edges $v_i B_j$ with $v_i \in V(B_j)$ (see Figure 3). Notice that $\overline{B}(G)$ can be obtained from the block graph $B(G)$ of G by adding edges with one endpoint of degree 1, and thus, $\overline{B}(G)$ is a tree for every connected graph G.

Remark 5. The union of an outerplanar graph G and its complete block graph $\overline{B}(G)$ is planar.

Let H be an outerplanar graph. The *weak dual* H^* of H is the graph obtained from the plane dual of H by removing the vertex that corresponds to the outer face of H. It is easy to see that H^* is a forest, which is a tree whenever H is 2-connected. If there exists a vertex $r \in V(H^*)$ such that H^* rooted in r (denoted by $H^*(r)$) is a full binary tree, then we call H a *full outerplanar graph.* The *height* $h(H)$ of a full outerplanar graph H is the number of levels in its full binary tree $H^*(r)$. The edge of a full outerplanar graph H incident to the face that corresponds to r, as well as to the the outer face, is the *central edge* e_H of H (see Figure 4, left). For the sake of convenience, a graph that consists of a single edge is also considered to be full outerplanar. Its height is then defined to be 0 and its central edge is the only edge of the graph.

Lemma 6. *For every outerplanar graph G there exists a full outerplanar graph H such that $G \subseteq H$.*

Proof. Let G_T be an almost triangulation of G, i.e. an outerplanar graph formed by triangulating the inner faces of G. The maximum degree of G_T^* is at most 3, and there exists a vertex $r \in V(G_T^*)$ of degree 1 or 2. Let $H^*(r)$ be a full binary tree of height $h(G_T^*(r))$ containing G_T^*. The graph H is then the desired full outerplanar graph. □

Recall that for a tree T with n vertices and m edges, a graph G, and a vertex $v \in V(G)$, we have $(T, G, v) = T \oplus_{v_1} G_1 \oplus_{v_2} \cdots \oplus_{v_n} G_n$. Let H be a full outerplanar graph, and

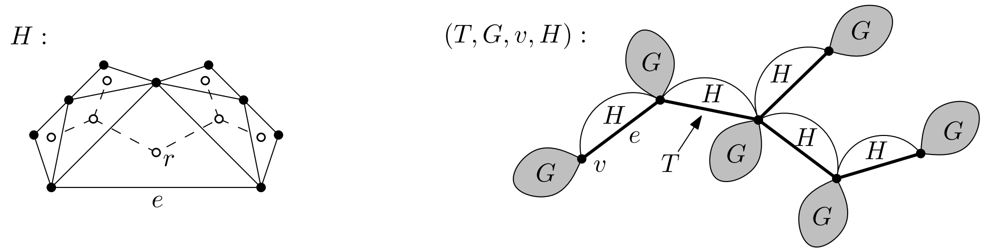


Figure 4: A full outerplanar graph H with its full binary tree $H^*(r)$ of height 3 (left). The structure of (T, G, v, h) (right).

for $i = 1, \ldots, m$, let H_i be a copy of H with the central edge $e_i(H_i)$. Then we define (T, G, v, H) as a graph $T \oplus_{v_1} G_1 \oplus_{v_2} \cdots \oplus_{v_n} G_n \oplus_{e_1} H_1 \oplus_{e_2} \cdots \oplus_{e_m} H_m$. So, (T, G, v, H) is simply the graph that arises from (T, G, v) if we "glue" a copy of H by its central edge to every edge of T (cf. Figure 4, right).

We now present a strategy $\mathbf{B}$ for forcing a monochromatic copy of (T, G, v, H), assuming that G is unavoidable by a strategy $\mathbf{X}$.

strategy B $(T, G, v, H, \mathbf{X})$

Let $t = |V(T)|$ and $h = h(H)$.

1. If $t = 1$, call strategy $\mathbf{X}$ and stop.
2. If $h = 0$, call strategy $\mathbf{A}(T, G, v, \mathbf{X})$ and stop.
3. Choose a leaf u of T and call its neighbor u'. Call strategy $\mathbf{B}(T', G', v', H', \mathbf{X}')$, where
    - $T' = \overline{B}(T, G, v, H)$,
    - $G' = (T - u, G, v, H)$,
    - v' is the vertex of $T - u \subseteq G'$ that corresponds to u',
    - H' is the full outerplanar graph of height $h - 1$,
    - $\mathbf{X}' = \mathbf{B}(T - u, G, v, H, \mathbf{X})$.

    Let $\{u_1, \ldots, u_k\}$ be the vertex set of (T, G, v, H), and thus also a subset of a vertex set of $\overline{B}(T, G, v, H) = T'$. Adopt this notation to the subgraph T' of the winning copy (T', G', v', H') found by strategy $\mathbf{B}(T', G', v', H', \mathbf{X}')$. Add an edge $e_{ij} = u_i u_j$ in (T', G', v', H') if and only if $u_i u_j$ is an edge in (T, G, v, H).

Let $\mathcal{S}$ be a set of red-blue graphs such that each $X \in \mathcal{S}$ contains a fixed monochromatic graph G. Then we set $\overline{\mathcal{S}} = \mathcal{S} \cup \{G \cup \overline{B}(G)\}$, where G is the fixed monochromatic graph.

Theorem 1. Every outerplanar graph is unavoidable (on planar graphs).

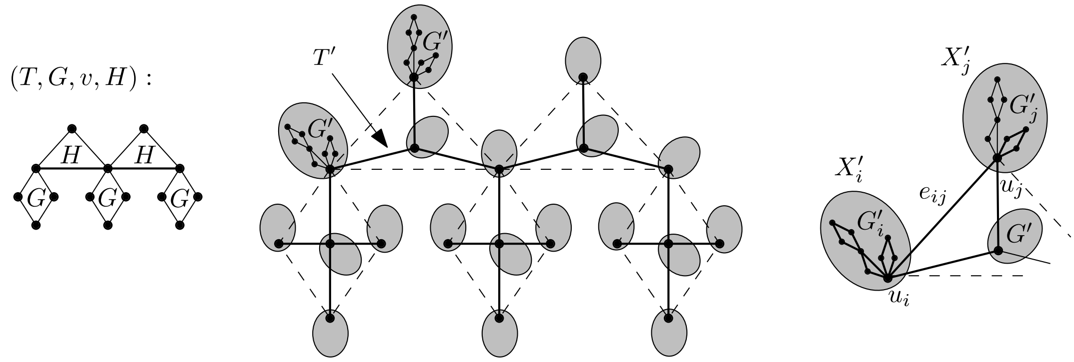


Figure 5: Forcing a monochromatic copy of (T, G, v, H), where T is a path of length 2, G is a cycle of length 4, v is any vertex of $V(G)$, and H is the full outerplanar graph of height 1.

Proof. We prove a stronger claim instead.

Claim. Let T be a tree, G an outerplanar graph, and $v \in V(G)$. If G is unavoidable by strategy $\mathbf{X}$, then (T, G, v, H) is unavoidable by strategy $\mathbf{B}(T, G, v, H, \mathbf{X})$, and every graph $B \in \mathcal{B}(T, G, v, H, \mathbf{X})$ is $(T, \overline{\mathcal{X}})$-reducible.

This statement implies Theorem 1 since every outerplanar graph G is contained in some full outerplanar graph H by Lemma 6, which can be written as $(e_H, (\{v\}, \emptyset), v, H)$, and is therefore unavoidable by the above claim.

We adopt all the notation used in strategy $\mathbf{B}$. Let S be the set of all 2-tuples $(h, t) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}$. On S, we define the lexicographic order $\trianglelefteq$, i.e. $(h_1, t_1) \trianglelefteq (h_2, t_2)$ exactly when $h_1 < h_2$, or $h_1 = h_2$ and $t_1 \leqslant t_2$ for all $h_1, h_2 \in \mathbb{N} \cup \{0\}$ and $t_1, t_2 \in \mathbb{N}$. The set S together with the relation $\trianglelefteq$ is linear, and we can apply induction.

We start with the basis. Suppose first that $h \geqslant 0$, and $t = 1$. Then $(T, G, v, H) = G$ and the claim is trivially satisfied. Let now $h = 0$, and $t \geqslant 1$. In this case we have $(T, G, v, H) = (T, G, v)$. By Lemma 4, (T, G, v) is unavoidable by $\mathbf{A}(T, G, v, \mathbf{X})$. So, every graph of $\mathcal{A}(T, G, v, \mathbf{X})$ is $(T, \mathcal{X})$-reducible, and thus $(T, \overline{\mathcal{X}})$-reducible since $\mathcal{X} \subseteq \overline{\mathcal{X}}$.

Suppose now that $h \geqslant 1$, $t \geqslant 2$. By the induction hypothesis ($(h, t-1) \trianglelefteq (h, t)$), $G' = (T - u, G, v, H)$ is unavoidable by strategy $\mathbf{X}' = \mathbf{B}(T - u, G, v, H, \mathbf{X})$, and every graph of $\mathcal{X}'$ is $(T - u, \overline{\mathcal{X}})$-reducible. Since G' is unavoidable by strategy $\mathbf{X}'$, it holds by the induction hypothesis ($(h-1, t) \trianglelefteq (h, t)$) that (T', G', v', H') is unavoidable by strategy $\mathbf{B}' = \mathbf{B}(T', G', v', H', \mathbf{X}')$, and every graph B' of $\mathcal{B}'$ is $(T', \overline{\mathcal{X}'})$-reducible. Say that the winning copy (T', G', v', H') in B' is blue. We distinguish the following two cases:

Case 1: All edges e_{ij} are red. These edges form a red copy of (T, G, v, H). The graph B' is $(T', \overline{\mathcal{X}'})$-reducible, and thus reducible to $T' = \overline{B}(T, G, v, H)$. Since B arose from B' by adding the edges forming (T, G, v, H), B is reducible to $(T, G \cup \overline{B}(G), v)$, and thus $(T, \overline{\mathcal{X}})$-reducible.

Case 2: At least one edge $e_{ij} = u_i u_j$ is blue. The endpoints of e_{ij} are connected by a path P in T' of length 2. There is a copy of H' appended along each of the edges of P. Those two copies of H' together with e_{ij} form a full outerplanar graph H of height h with central edge e_{ij} (see Figure 5). Let G'_i and G'_j be the blue copies of $G' = (T - u, G, v, H)$ appended to u_i and u_j, respectively. Then H, G'_i, and the copy of G in G'_j that is appended to u_j form a blue copy of (T, G, v, H). We can assume that Builder chooses this copy as the winning copy. We now prove the second part of the claim. Recall that B' is $(T', \overline{\mathcal{X}'})$-reducible. Let X'_i, X'_j be the graphs of $\overline{\mathcal{X}'}$ appended to u_i, u_j, respectively. So, X'_i is $(T - u, \overline{\mathcal{X}})$-reducible, and X_j is $(\{u_j\}, \overline{\mathcal{X}})$-reducible. Since the rest of the graph B is reducible to e_{ij}, we find that B is $(T, \overline{\mathcal{X}})$-reducible. See the diagram on the right in Figure 5. □

4 Non-outerplanar graphs

We now show that an infinite subclass of theta-graphs is unavoidable on planar graphs. Recall that a *theta-graph* (θ-graph) is the union of three internally disjoint paths that have the same two end vertices. We write $\theta_{i,j,k}$ for the theta-graph with paths of length i, j, k. For example, $K_{2,3}$ is the graph $\theta_{2,2,2}$.

Before stating the main theorem, we introduce a strategy for forcing even cycles. The unavoidability of cycles was proven in [5], but here we need the final graph to have a special type of plane embedding that we utilize in the proof of the main theorem.

Let C be a cycle of even length n that is unavoidable by strategy $\mathbf{X}$. If for every graph X of $\mathcal{X}$ there is a plane embedding of X such that

(G1) all vertices of $V(C)$ belong to the boundary of one common face, and

(G2) there exists a path $P \subset C$ of length $\frac{n}{2}$ such that all vertices of $V(P)$ lie the boundary of another face,

then we say that strategy $\mathbf{X}$ is a *good strategy.* The path P is then called a *good path* in C.

strategy C (C)
Let $n = \lvert V(C)\rvert$, $a = (n-1)$, and $b = \frac{n}{2} - 1$. 1. Force a monochromatic path $P = v_0 \dots v_{a^2+b}$ by the tree strategy. 2. In P, Connect the vertices v_0 and v_{a^2} by an edge e. 3. If e has the other color than P, add the path $P' = v_0 v_a v_{2a} \dots v_{a^2}$. Otherwise add the cycle $C' = v_b v_{(b+a)} v_{(b+2a)} \dots v_{(b+a^2)} v_b$.

Lemma 7. *Let C be an even cycle. Then strategy $\mathbf{C}(C)$ is a good strategy.*

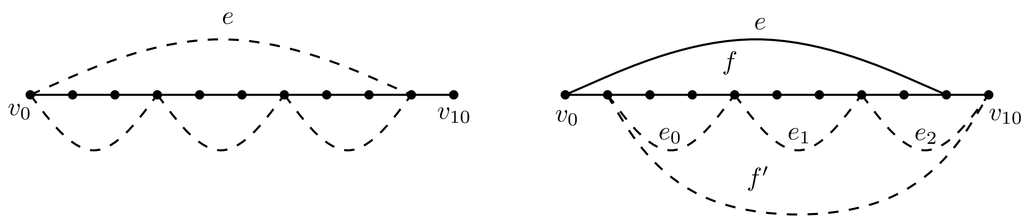


Figure 6: Forcing cycle of length 4 by strategy C.

Proof. We will follow the notation introduced in strategy **C**. We fix a planar embedding of an output graph of strategy $\mathbf{C}(C)$ as shown in Figure 6. By [5], every final graph of the tree strategy for forcing P is a forest, which is reducible to (the chosen monochromatic) P. Assume that P is blue. The following two cases can arise.

Case 1: The edge e is red. If Painter colors some edge of P' blue, a blue copy of C arises since there is a blue path of length $n-1$ between such two vertices. Otherwise, $P' \cup e$ is a red cycle C of length n. In both cases, all vertices of the monochromatic copy of C belong to two common faces. See Figure 6, left.

Case 2: The edge e is blue. Suppose that Painter colors some edge e' of $C' - v_b v_{(a^2+b)}$ blue. Since each such pair is connected by a blue path of length $a = n-1$, a blue cycle of length n arises. Condition (G1) is then satisfied by the face bounded by this cycle, and (G2) is satisfied by the face f bounded by the cycle $v_0 v_a v_{2a} \dots v_{a^2} v_0$, which contains a good path on $n/2+1$ vertices if $e' = v_{a(a-1)} v_{a^2}$ and on all n vertices in all the other cases. Suppose now that Painter colors the edge $v_b v_{(a^2+b)}$ blue. Then the blue copy of C is formed by this edge and the blue path starting at v_b, going through e, and ending at $v_{(a^2+b)}$. All of the vertices of the blue copy of C belong to the outer face, and there is a good path $v_b v_{b-1} \dots v_0 v_{a^2}$ of length $b+1 = \frac{n}{2}$ that belongs to f. Consider the last possibility when C' is red. Now, all of the vertices of $V(C)$ belong to the boundary of f', and all but the vertex $v_{(a^2+b)}$ of $V(C)$ belong to the boundary of f. See Figure 6, right. □

Theorem 2. The graph $\theta_{2,j,k}$ is unavoidable for even j, k.

Proof. For fixed j and k, let $j' = \frac{j}{2}$, $k' = \frac{k}{2}$. We consider disjoint cycles $C_1, \dots, C_{j'+k'+1}$ of length $k+2$. In ith of them, we label an arbitrary vertex by c_i and one of the two vertices in distance 2 from c_i by $v_0(C_i)$ if $i \leqslant j'+1$, and by $v_1(C_i)$ otherwise. Let $P_1, \dots, P_{j'+k'+2}$ be paths of length $j-1$, where in each P_i, one end is labeled by p_i, and another one by $v_0(P_i)$ if $i \leqslant j'+1$, and by $v_1(P_i)$ otherwise. Let

$$L := C_1 \oplus_{v_0} \cdots \oplus_{v_0} C_{j'+1} \oplus_{v_0} P_1 \oplus_{v_0} \cdots \oplus_{v_0} P_{j'+1}$$

and

$$R := C_{j'+2} \oplus_{v_1} \cdots \oplus_{v_1} C_{j'+k'+1} \oplus_{v_1} P_{j'+2} \oplus_{v_1} \cdots \oplus_{v_1} P_{j'+k'+2}.$$

Then we write H for a graph that is formed from the disjoint union of L and R by identifying p_1 with $p_{j'+k'+2}$, and $p_{j'+1}$ with $p_{j'+2}$ (see Figure 7, left). The cycle consisting of the paths P_1, $P_{j'+1}$, $P_{j'+2}$, and $P_{j'+k'+2}$ is denoted C_0.

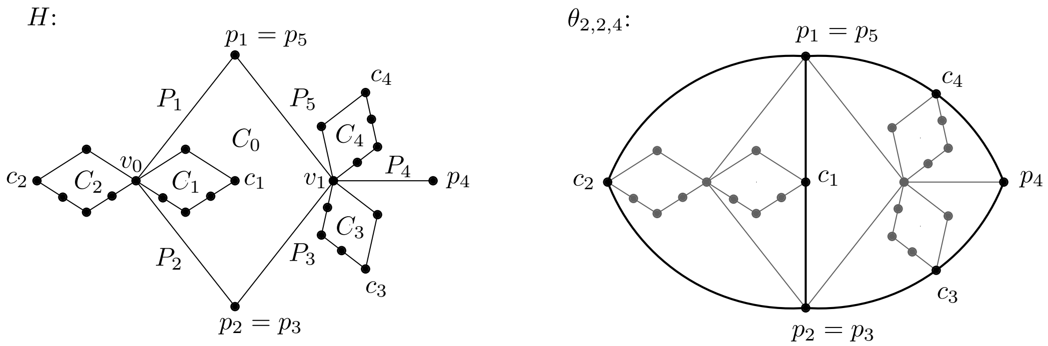


Figure 7: Graph H for $j = 2$, $k = 4$ (left) and forcing $\theta_{2,2,4}$ using H (right).

Observe first that having a monochromatic copy of a H, Builder could easily force $\theta_{2,j,k}$ (cf. Figure 7). The graph H is outerplanar, and hence unavoidable by Theorem 1. The problem is that by connecting the proper vertices of the monochromatic copy of H in the resulting graph, the planarity condition would be violated. Therefore, we have to change the strategy for forcing H.

For $n = 0, \dots, j' + k' + 1$, let $G_n := H[V(C_0 \cup \dots \cup C_n)]$. So, the graph $G_{j'+k'+1}$ is the graph H without the paths $P_2, \dots, P_{j'}, P_{j'+3}, \dots, P_{j'+k'+1}$. Let us refer to the blocks of G_n and the corresponding vertices of the complete block graph simply by $C_0, C_1, \dots, C_n$. Next, let V' be the set of vertices of G_n (and thus also of $\overline{B}(G_n)$) for which the distance from v_0 in G_n is even. For G_n, we define a *subdivided complete block graph* $\overline{B_S}(G_n)$ as a graph that arises from $\overline{B}(G_n)$ by subdividing each edge joining C_i $(i = 1, \dots, n)$ and a vertex of V' $(k-1)$-times. Observe that $\overline{B_S}(G)$ is a tree, and that $G \cup \overline{B_S}(G)$ is planar.

We now present strategy $\mathbf{D}$ for forcing G_n.

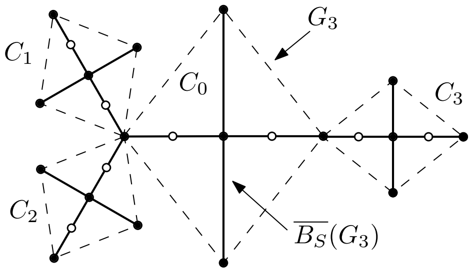
strategy D (G_n)
1. If $n = 0$, call strategy $\mathbf{C}(C_0)$. In C_0, find a good path P_0, denote the middle vertex of P_0 by v_0 and its opposite vertex in C_0 by v_1.
2. If $n \geqslant 1$, let $T' = \overline{B_S}(G_n)$, $G' = G_{n-1}$, $v' = v_0$ if $n \leqslant j' + 1$ and $v' = v_1$ otherwise, and $\mathbf{D}' = \mathbf{D}(G_{n-1})$. Call strategy $\mathbf{A}(T', G', v', \mathbf{D}')$.
3. In (T', G', v'), connect two vertices of $T' = \overline{B_S}(G_n)$ by an edge if and only if the corresponding vertices are connected by an edge in G_n.

We show by induction on n that G_n is unavoidable by strategy $\mathbf{D}(G_n)$, and that every graph D of $\mathcal{D}(G_n)$ can be embedded in the plane so that

(1) all vertices $v_0, v_1, p_1, p_{j'+1}, c_1, \dots, c_n$ belong to some face f_1, and

(2) (a) the vertices $v_0, p_1, p_{j'+1}$ belong to some face f_2, other than f_1, or

(b) there is a path $P = p_1 c_1 p_{j'+1}$ of the other color than G_n.



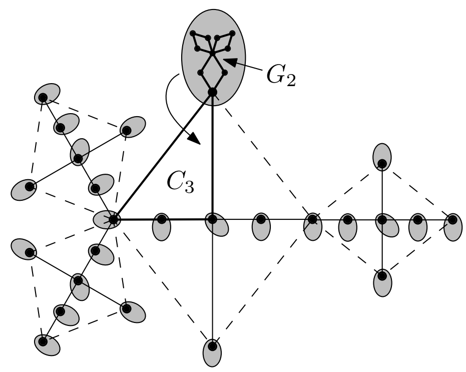


Figure 8: Left: The graphs G_3 (dashed) and $\overline{B_S}(G_3)$ (solid) for $j = 2$, $k = 2$. Black vertices represent the vertices of $\overline{B}(G_3)$ whereas white ones are the subdividing vertices. Right: Forcing G_3, Case 2 – one of the edges added to $(T', G', v') = (\overline{B_S}(G_3), G_2, v_2)$ is blue.

The base case is $n = 0$. By Lemma 7, strategy $\mathbf{C}(C_0)$ is a good strategy, i.e. every graph of $\mathcal{C}(C_0)$ can be embedded in such a way that all vertices of C_0 belong to one common face, and there is a path $P_0 \subset C_0$ of length $\frac{4(j-1)}{2} = 2(j-1)$ such that all vertices of $V(P_0)$ belong to the boundary of another face. The first part implies condition (1), and the second part implies condition (2)(a).

Now assume that $n \geqslant 1$. We first show that D is planar. Every graph of $\mathcal{D}(G')$ is planar by induction. Therefore, every graph A of $\mathcal{A}(T', G', v', \mathbf{D}')$ is reducible to T' by Lemma 4, and thus planar. Since $T' = \overline{B_s}(G_n)$, every graph that arises in Step 3 of strategy $\mathbf{D}(G_n)$ is reducible to $\overline{B_s}(G_n) \cup G_n$, which is planar.

Let us prove that a monochromatic copy of G_n arises when following strategy $\mathbf{D}(G_n)$, and that D fulfills Conditions (1) and (2). Suppose that the monochromatic copy of (T', G', v') in A is blue. Let us focus on the edges added in Step 3. The following two cases can arise.

Case 1: All these edges are red. Since all the edges form a copy of G_n, we get a red G_n. According to the discussion above, D is reducible to $\overline{B_s}(G_n) \cup G_n$. Every $\overline{B_s}(C_i)$ ($i = 0, \ldots, n$) can be drawn inside the cycle C_i of G_n. Then, all the vertices of G_n belong to the boundary of the outer face, which gives us Condition (1). Condition (2) is also satisfied as there is a blue path $p_1 C_0 p_{j'+1}$.

Case 2: At least one edge e is blue. The edge e joins the ends of a blue path P of length $k+1$ of T'. So, a cycle C_n of length $k+2$ is formed. Since there is a copy of $G' = G_{n-1}$ appended to each vertex of T' and $G_n = G_{n-1} \cup C_n$, $k+2$ blue copies of G_n are formed. Builder arbitrarily chooses one of them, called G_n. The graph A of $\mathcal{A}(T', G', v', \mathbf{D}')$ is $(T', \mathcal{D}')$-reducible by Lemma 4. Since $P \subseteq T'$, the graph A is $(P, \mathcal{D}')$-reducible. Let D' be the graph of $\mathcal{D}'$ that contains the copy of G_{n-1} in the chosen blue graph G_n. By the induction hypothesis, D' can be embedded so that Conditions (1) and (2) are satisfied (for G_{n-1} in D'). Therefore, the graph D can be embedded is such a way that all the vertices v_0, v_1, p_1, $p_{j'+1}$, $c_1, \ldots, c_{n-1}$ of G' lie in the boundary of a common

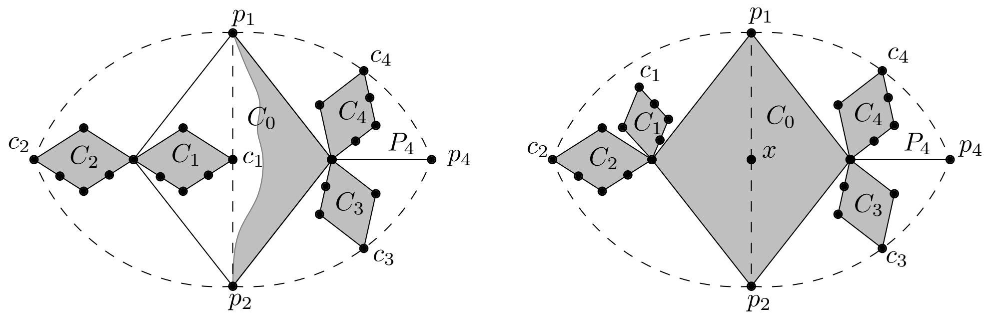


Figure 9: Adding edges of a copy of $\theta_{2,2,4}$ to H if the condition (2)a (left), resp. (2)b (right) is satisfied.

face and either $v_0, p_1, p_{j'+1}$ belong to another common face or there is red path of $p_1c_1p_{j'+1}$. Also, D can be embedded so that all vertices of C_n lie in the boundary of a common face. Now, G' can be drawn inside that face, which gives both Condition (1) and Condition (2).

Having a monochromatic copy of $G = G_{j'+k'+1}$, Builder can force $H = G \oplus_{v_0} T_0 \oplus_{v_1} T_1$, where $T_0 = P_2 \oplus_{v_0} \cdots \oplus_{v_0} P_{j'}$ and $T_1 = P_{j'+3} \oplus_{v_1} \cdots \oplus_{v_1} P_{j'+k'+1}$. Indeed, since G is unavoidable by strategy $\mathbf{D}(G_{j'+k'+1})$, the supergraph (T_0, G, v_0) of $G \oplus_{v_0} T_0$ is unavoidable by strategy $\mathbf{X} = \mathbf{A}(T_0, G, v_0, \mathbf{D}(G))$ by Lemma 4. Applying Lemma 4 again, we find that the supergraph $(T_1, (T_0, G, v_0), v_1)$ of $G \oplus_{v_0} T_0 \oplus_{v_1} T_1$ is unavoidable by strategy $\mathbf{A}(T_1, (T_0, G, v_0), v_1, \mathbf{X})$. In order to force $\theta_{2,j,k}$, it suffices to add the appropriate edges to $G \oplus_{v_0} T_0 \oplus_{v_1} T_1$. The whole process is summed up in strategy $\mathbf{E}$.

strategy E ($\theta_{2,j,k}$)
1. Call strategy $\mathbf{A}(T_1, (T_0, G, v_0), v_1, \mathbf{X})$, where $\mathbf{X} = \mathbf{A}(T_0, G, v_0, \mathbf{D}(G))$ and v_1 is an arbitrary vertex corresponding to v_1 of some copy of G in (T_0, G, v_0). Chose a monochromatic copy of H.
2. Add edges of the cycle $p_1c_2p_2c_3 \dots p_{j'+1}(= p_{j'+2})c_{j'+2}p_{j'+3} \dots c_{j'+k'+1}p_1$ to H. If there is not the path $p_1c_1p_{j'+1}$, also add the edges p_1c_1 and $c_1p_{j'+1}$.

As a consequence of Lemma 4, every graph of $\mathcal{A}(T_1, (T_0, G, v_0), v_1, \mathbf{X})$ can be embedded in such a way that Conditions (1) and (2) hold for G, and that all the vertices $p_2, \dots, p_{j'+k'+1}$ lie in the boundary of the face f_1. This means that adding the cycle in Step 3 of strategy $\mathbf{E}$ does not violate the planarity of the final graph. Finally, Condition (2) ensures that either there already is a path of length 2 connecting p_1 and $p_{j'+1}$ of the desired color, or Builder can add it by connecting p_1 to c_1 and c_1 to $p_{j'+1}$.

□

5 Further problems

The question of whether the class of planar graphs is self-unavoidable is still open. To disprove it, it suffices to find a single planar graph G such that Painter can ensure that a monochromatic copy of G never occurs when playing on planar graphs. The graph K_4 seems to be a good candidate.

Conjecture 8. *K_4 is avoidable on the class of planar graphs.*

Unfortunately, Painter's winning strategies seem to be much harder to find. So far, only one such strategy has been presented; namely a strategy showing that a triangle is avoidable on the class of outerplanar graphs given in [5].

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References

[1] J. Beck. On size Ramsey number of paths, trees and cycles I. *J. Graph Theory*, 7:115–130, 1983.

[2] J. Butterfield, T. Grauman, W. B. Kinnersley, K. G. Milans, C. Stocker, D. B. West. On-line Ramsey Theory for Bounded Degree Graphs. *Electron. J. Comb.*, 18(11):P136, 2011.

[3] D. Conlon. On-line Ramsey numbers. *SIAM J. Discrete Math.*, 23:1954–1963, 2009.

[4] R. Diestel. Graph Theory. *Springer-Verlag*, 2005.

[5] J. A. Grytczuk, M. Hałuszczak, H. A. Kierstead. On-line Ramsey Theory. *Electron. J. Combin.*, 11(1):R57, 2004.

[6] J. A. Grytczuk, H. A. Kierstead, P. Prałat. On-line Ramsey Numbers for Paths and Stars. *Discrete Math. Theor. Comp. Science*, 10(3):63–74, 2008.

[7] P. E. Haxell, Y. Kohayakawa, and T. Łuczak. The induced size-Ramsey number of cycles. *Combin. Probab. Comput.*, 4:217–239, 1995.

[8] H. A. Kierstead, G. Konjevod. Coloring number and on-line Ramsey theory for graphs and hypergraphs. *Combinatorica*, 29(1):49–64, 2009.

[9] A. Kurek, A. Ruciński, Two variants of the size Ramsey number, *Discuss. Math. Graph Theory*, 25(1–2):141–149, 2005.

Coloring fractional powers of graphs

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Abstract

For $m, n \in \mathbb{N}$, the fractional power $G^{\frac{m}{n}}$ of a graph G is the mth power of the n-subdivision of G, where the n-subdivision is obtained by replacing each edge in G with a path of length n. It was conjectured by Iradmusa that if G is a connected graph with $\Delta(G) \geq 3$ and $1 < m < n$, then $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$. Here we show that the conjecture does not hold in full generality by presenting a graph H for which $\chi(H^{\frac{3}{5}}) > \omega(H^{\frac{3}{5}})$. However, we prove that the conjecture is true if m is even and $\Delta(G) \geq 4$. We also study the case when m is odd, obtaining a general upper bound $\chi(G^{\frac{m}{n}}) \leq \omega(G^{\frac{m}{n}}) + 2$ for graphs with $\Delta(G) \geq 4$ and proving several sufficient conditions for when the conjecture holds.

Keywords: fractional power, chromatic number, regular graphs.
Math. Subj. Class.: 05C15

1 Introduction

Let G be a simple finite graph, and let m and n be positive integers. The *n-subdivision* of G, denoted by $G^{\frac{1}{n}}$, is the graph formed from G by replacing each edge with a path of length n. The *m-power* of G, denoted by G^m, is the graph constructed from G by joining every two distinct vertices with distance at most m in G. Iradmusa [2] defined the *fractional power* $G^{\frac{m}{n}}$ to be the m-power of the n-subdivision of G, that is, $G^{\frac{m}{n}} = (G^{\frac{1}{n}})^m$. Iradmusa investigated the relation of the chromatic number $\chi(G^{\frac{m}{n}})$ and the clique number $\omega(G^{\frac{m}{n}})$, and we present further results in this paper.

The study of colorings of fractional powers of graphs is motivated by the famous Total Coloring Conjecture. A *total coloring* of a graph G is a coloring of its vertices and edges such that no adjacent vertices, no adjacent edges, and no incident edge and vertex have the same color. The *total chromatic number* $\chi''(G)$ of G is the least number of colors in a total coloring. Behzad [1] and Vizing [6] independently formulated the Total Coloring Conjecture relating the total chromatic number to the maximum degree $\Delta(G)$, conjecturing that $\chi''(G) \leq \Delta(G) + 2$ for any simple graph G. Since $\chi''(G) = \chi(G^{\frac{2}{2}})$, and $\omega(G^{\frac{2}{2}}) = \Delta(G) + 1$ for every graph G with $\Delta(G) \geq 2$, we can reformulate the Total Coloring Conjecture as follows:

Total Coloring Conjecture ([1, 6]). *If G is a simple graph with $\Delta(G) \geq 2$, then $\chi(G^{\frac{2}{2}}) \leq \omega(G^{\frac{2}{2}}) + 1$.*

If $m < n$, the maximum cliques of $G^{\frac{m}{n}}$ are somewhat separated and fewer colors may be needed. Iradmusa [2] conjectured that this is always the case.

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Conjecture 1.1 ([2]). *If G is a connected simple graph with $\Delta(G) \geq 3$ and $1 < m < n$, then* $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$.

Iradmusa [2] showed that Conjecture 1.1 is true for $m = 2$ and gave several partial results for $m > 2$.

The purpose of this paper is to prove further results towards Conjecture 1.1. Our first result is that Conjecture 1.1 is true when m is even and $\Delta(G) \geq 4$.

Theorem 1.2. *If m is even and G is a connected graph with $\Delta(G) \geq 4$, then $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$.*

We also study the conjecture when m is odd. We show that the conjecture does not hold for all graphs. In particular, it is not true for the cartesian product $C_3 \Box K_2$ of C_3 and K_2 (triangular prism), when $m = 3$ and $n = 5$. However, we believe this is the only counterexample to Conjecture 1.1, and so we make the following conjecture.

Conjecture 1.3. *Conjecture 1.1 holds except for $G = C_3 \Box K_2$.*

Towards this conjecture we prove the general bound $\chi(G^{\frac{m}{n}}) \leq \omega(G^{\frac{m}{n}}) + 2$ when m is odd and $\Delta(G) \geq 4$, as well as prove the conjecture for complete graphs on at least 6 vertices.

Theorem 1.4. *If G be a connected non-complete graph with maximum degree Δ at least 4, then $\chi(G^{\frac{m}{n}}) \leq \omega(G^{\frac{m}{n}}) + 2$ for any odd m such that $3 \leq m < n$.*

Theorem 1.5. *Conjecture 1.1 holds for any complete graph on at least 5 vertices.*

Additionally, we give sufficient conditions for Conjecture 1.1 to hold when m is odd using r-dynamic colorings.

Corollary 1.6. *If G is a Δ-regular graph with $\Delta \geq 39$, $\chi(G) \leq \frac{\Delta}{4}$, and $1 < m < n$ with m odd, then* $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$.

This paper is organized as follows. In Section 2, we introduce notation and previous results. Our result for even m is given in Section 3 and results for odd m in Section 4. In Section 5 we investigate fractional powers of complete graphs and prove Theorem 1.5. Finally, Section 6 discusses the connection between r-dynamic colorings and Conjecture 1.1.

2 Notation and preliminaries

In this paper we only consider finite simple graphs. We also always assume that m and n are positive integers such that $m < n$. The graph $G^{\frac{m}{n}}$ is constructed from G in two steps. First, every edge uv is replaced by a path P_{uv} (called a *superedge*) of length n, forming $G^{\frac{1}{n}}$. Second, edges joining vertices of distance at most m in $G^{\frac{1}{n}}$ are added, forming $G^{\frac{m}{n}}$.

Let $(uv)_i$ $(i = 0, \ldots, n)$ be a vertex of $G^{\frac{m}{n}}$ that lies on P_{uv} and has distance i from u in $G^{\frac{1}{n}}$. If $i = 0$ or $i = n$, then $(uv)_i$ is a *branch vertex*, otherwise $(uv)_i$ is an *internal vertex*.

For an edge uv in G, the ordered $\lfloor \frac{m}{2} \rfloor$-tuple of vertices of $V(G^{\frac{m}{n}})$ defined by

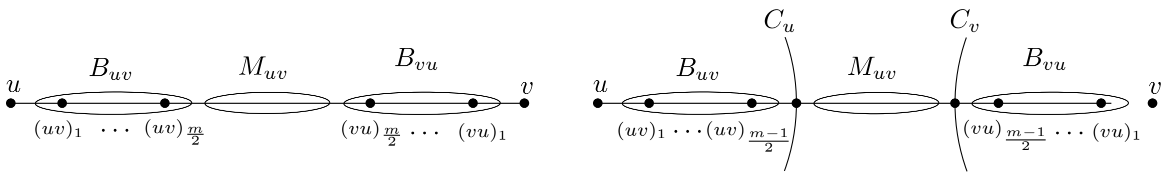
$$B_{uv} = ((uv)_1, \ldots, (uv)_{\lfloor \frac{m}{2} \rfloor})$$

is called a *bubble (at u)*. If m is odd, then we say that the set of vertices

$$C_u = \{(uv)_{(m+1)/2} \in V(G^{\frac{m}{n}}) : uv \in E(G)\}$$

is the *crust at u*. Lastly, M_{uv}, called a *middle part*, is the tuple of vertices between the two bubbles (or two crusts if m is odd) on the edge uv defined by

$$M_{uv} = ((uv)_{\lceil \frac{m}{2} \rceil + 1}, \ldots, (uv)_{n - (\lceil \frac{m}{2} \rceil + 1)}).$$



(a) m even. (b) m odd.

Figure 1: Bubbles, crusts, and middle parts.

See Figure 1 for a picture showing these vertices.

For a k-tuple $A = (a_1, \dots, a_k)$ we define $\overline{A}$ as the reverse $(a_k, \dots, a_1)$. Also, we write $A[i : j] = (a_i, \dots, a_j)$ and $A[i] = (a_i)$. So, we have $\overline{A}[i : j] = (a_{k-i+1}, \dots, a_{k-j+1})$ and $\overline{A}[i] = (a_{k-i+1})$. Note that in general $\overline{A}[i : j]$ does not equal $\overline{A[i : j]}$. If $B = (b_1, \dots, b_k)$ is another k-tuple, then $\varphi(A) = B$ means $\varphi(A[i]) = B[i]$ (i.e. $a_i = b_i$) for all $i \in [k]$ where φ is some function. To shorten notation, we often write $[k]$ instead of $\{1, \dots, k\}$. Lastly, we use the symbol $*$ to denote the concatenation of two tuples as follows: $(a_1, \dots, a_i) * (a_{i+1}, \dots, a_n) := (a_1, \dots, a_n)$.

We are ready to state results of Iradmusa that we will use later in this paper. The first theorem gives the exact values of $\omega(G^{\frac{m}{n}})$.

Theorem 2.1 ([2])**.** *If G is a graph and $n, m \in \mathbb{N}$ such that $m < n$, then*

$$\omega(G^{\frac{m}{n}}) = \begin{cases} m+1 & \text{if } \Delta(G) = 1, \\ \frac{m}{2}\Delta(G) + 1 & \text{if } \Delta(G) \geq 2 \text{ and } m \text{ is even}, \\ \frac{m-1}{2}\Delta(G) + 2 & \text{if } \Delta(G) \geq 2 \text{ and } m \text{ is odd}. \end{cases}$$

Since $\chi(H) \geq \omega(H)$ for any graph H, it suffices to show $\chi(G^{\frac{m}{n}}) \leq \omega(G^{\frac{m}{n}})$ to prove Conjecture 1.1. By Theorem 2.1, we thus need to construct a coloring of $G^{\frac{m}{n}}$ using $\frac{m}{2}\Delta(G) + 1$ colors if m is even, and $\frac{m-1}{2}\Delta(G) + 2$ colors if m is odd.

By [2, Lemma 1], if $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$, then $\chi(G^{\frac{m}{n+m+1}}) = \omega(G^{\frac{m}{n+m+1}})$. This result can be generalized as follows.

Lemma 2.2. *Let G be a graph, and $m, n \in \mathbb{N}$ such that $m < n$. If $\chi(G^{\frac{m}{n}}) \leq \omega(G^{\frac{m}{n}}) + c$ for some $c \geq 0$, then $\chi(G^{\frac{m}{n+m+1}}) \leq \omega(G^{\frac{m}{n+m+1}}) + c$.*

Proof. By Theorem 2.1, $\omega(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n+m+1}})$. By the proof of Lemma 1 in [2], it holds that $\chi(G^{\frac{m}{n+m+1}}) \leq \chi(G^{\frac{m}{n}})$. Therefore

$$\chi(G^{\frac{m}{n+m+1}}) \leq \chi(G^{\frac{m}{n}}) \leq \omega(G^{\frac{m}{n}}) + c = \omega(G^{\frac{m}{n+m+1}}) + c.$$

□

Theorem 2.3 ([2])**.** *If G is a connected graph with $\Delta(G) \geq 3$ and $m \in \mathbb{N}$, then $\chi(G^{\frac{m}{m+1}}) = \omega(G^{\frac{m}{m+1}})$.*

The next lemma follows by inductively applying Lemma 2.2 and Theorem 2.3. We will use it repeatedly.

Lemma 2.4. *If $\chi(G^{\frac{m}{n}}) \leq \omega(G^{\frac{m}{n}}) + c$ for all $n = m+2, \dots, 2m+1$ and some $c \in \mathbb{N} \cup \{0\}$, then $\chi(G^{\frac{m}{n}}) \leq \omega(G^{\frac{m}{n}}) + c$ for all n with $n > m$. In particular, if Conjecture 1.1 holds for $n = m+2, \dots, 2m+1$, then it holds for all n with $n > m$.*

Throughout this paper, we will also use the following well known result of König [4].

Lemma 2.5 ([4])**.** *For every graph G with maximum degree Δ there exists a Δ-regular graph containing G as an induced subgraph.*

Lemma 2.5 enables us to restrict our attention to regular graphs. Indeed, if G is not regular, we can find a Δ-regular graph H containing G, color H, and then use the coloring of H on G. Note that we only need G to be a subgraph of H, not necessarily induced.

The following result is proven in [2, Theorem 3] using a special type of vertex ordering and induction. Here we give another simple proof.

Lemma 2.6. *If G is a connected graph with maximum degree $\Delta \geq 3$, then there exists a proper coloring $h : V(G^{\frac{2}{3}}) \to \{0, \dots, \Delta\}$ using color 0 exactly on the branch vertices.*

Proof. First, we color all branch vertices with 0. It remains to color the internal vertices with colors $1, \dots, \Delta$. Let H be a graph that arises if we delete all branch vertices of $G^{\frac{2}{3}}$. Then $\Delta(H) = \Delta$, H is connected, and H is neither an odd cycle nor a complete graph. Thus, $\chi(H) \leq \Delta$ by Brooks's theorem. □

Observe that the graph H that arises if we delete all branch vertices of $G^{\frac{2}{3}}$ is isomorphic to the line graph of $G^{\frac{1}{2}}$. Therefore, we can think of coloring internal vertices of $G^{\frac{2}{3}}$ as coloring edges of $G^{\frac{1}{2}}$. So, Lemma 2.6 equivalently states that for a graph with degree $\Delta \geq 3$, there is a proper edge coloring $h : E(G^{\frac{1}{2}}) \to \{1, \dots, \Delta\}$. We refer to such a coloring as a *half-edge coloring of G*, and we refer to the 'halves' of edges of G that correspond to the edges in $G^{\frac{1}{2}}$ simply as *half-edges of G*. For an edge $e = uv$ in G, e_{uv} denotes the half-edge on e that is adjacent to u.

Lemma 2.7. *If G is a graph with maximum degree $\Delta \geq 3$, then there exists a (proper) half-edge coloring $h : E(G^{\frac{1}{2}}) \to [\Delta]$.*

3 Fractional powers with m even

In this section we show that Conjecture 1.1 is true when m is even and $\Delta(G) \geq 4$. We sketch here the basic idea of the technique used. Suppose that we have a half-edge coloring of G using $\Delta(G)$ colors. For each color a we introduce $\frac{m}{2}$ new colors $a_1, \dots, a_{\frac{m}{2}}$, and use them on each bubble whose corresponding half-edge has color a. Hence we use $\frac{m}{2}\Delta(G)$ colors to color the vertices of the bubbles. Next we show that it is possible to color the middle vertices from the same set of colors. By using an additional color 0 for branch vertices of $G^{\frac{m}{n}}$, we can conclude that $\chi(G^{\frac{m}{n}}) \leq \frac{m}{2}\Delta(G) + 1 = \omega(G^{\frac{m}{n}})$, as desired.

Theorem 3.1. *If m is even and G is a connected graph with $\Delta(G) \geq 4$, then $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$.*

Proof. By Lemma 2.5, we can assume that G is a Δ-regular graph, where $\Delta = \Delta(G)$. Also, by Lemma 2.4 it is sufficient to prove the claim for $m + 2 \leq n \leq 2m + 1$. By Theorem 2.1, $\omega(G^{\frac{m}{n}}) = \frac{m}{2}\Delta + 1$. Since trivially $\chi(G^{\frac{m}{n}}) \geq \omega(G^{\frac{m}{n}})$, we only need to construct a proper vertex coloring of $G^{\frac{m}{n}}$ that uses $\frac{m}{2}\Delta + 1$ colors.

Let $h : E(G^{\frac{1}{2}}) \to \{1, \dots, \Delta\}$, where $1 = (1_1, \dots, 1_{\frac{m}{2}}), \dots, \Delta = (\Delta_1, \dots, \Delta_{\frac{m}{2}})$, be a half-edge coloring of G whose existence is ensured by Lemma 2.7. Then we define a vertex coloring $\varphi : V(G^{\frac{m}{n}}) \to \{0, 1_1, \dots, 1_{\frac{m}{2}}, 2_1 \dots, 2_{\frac{m}{2}}, \dots, \Delta_1, \dots, \Delta_{\frac{m}{2}}\}$ as follows.

1. **branch vertices:** $\varphi(v) = 0$ for every branch vertex v.

2. **bubbles:** $\varphi(B_{uv}) = h(e_{uv})$. (Recall that this means $\varphi(B_{uv}[i]) = h(e_{uv})[i]\ \forall i \in [\frac{m}{2}]$.)

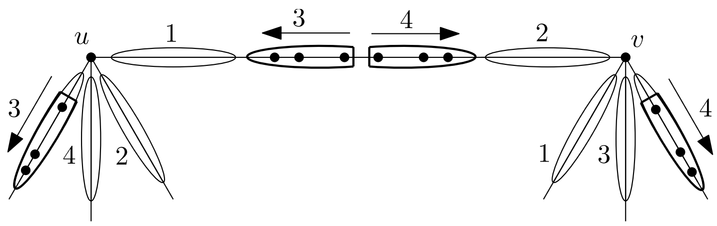


Figure 2: Coloring φ of $G^{\frac{m}{n}}$ for even m.

3. **middle parts:** Consider a superedge P_{uv}. Since $\Delta(G) \geq 4$, there exist two color-tuples a, $b \in [\Delta] \setminus \{h(e_{uv}), h(e_{vu})\}$. Let $k = \lceil \frac{1}{2}|M_{uv}| \rceil$ and $l = \lfloor \frac{1}{2}|M_{uv}| \rfloor$. Note that $k, l \leq \frac{m}{2}$ since $n \leq 2m+1$. Then we define $\varphi(M_{uv}[1:k]) = \overline{a}[1:k]$ and $\varphi(M_{vu}[1:l]) = \overline{b}[1:l]$. See Figure 2.

We show that the coloring φ is proper. We need to check that the distance between any two vertices x, y with the same color is greater than m in the subgraph $H := G^{\frac{1}{n}}$ of $G^{\frac{m}{n}}$. This is true if $\varphi(x) = \varphi(y) = 0$ since then x and y are branch vertices. We assume then that $\varphi(x) = \varphi(y) \neq 0$.

1. $x \in B_{uv}$ and $y \in B_{u'v'}$: First, $uv \neq u'v'$ since the colors of vertices within each bubble are pairwise distinct. Next, neither $u = u'$ nor $v = v'$ since h is proper and thus B_{uv} and $B_{u'v'}$ cannot be at the same vertex. Also, B_{uv} and $B_{u'v'}$ are not on the same superedge, so either $u \neq v'$ or $v \neq u'$. If $u \neq v'$ and $v \neq u'$, then uv and $u'v'$ are two vertex-disjoint edges in G and so $d_H(x,y) \geq n+2 > m$. If $u \neq v'$ and $v = u'$, then $d_H(x,y) = d_H(u,v) = n > m$ since $d_H(u,x) = d_H(u',y)$ by definition of φ. The case $u = v'$ and $v \neq u'$ is analogous.

2. $x \in M_{uv}$ and $y \in M_{u'v'}$: Since the colors of vertices within the middle part of a single superedge are pairwise distinct, $uv \neq u'v'$. But then $d_H(x,y) > m$ since any two vertices from two different middle parts are at distance at least $m+2$ in $G^{\frac{1}{n}}$.

3. $x \in B_{uv}$ and $y \in M_{u'v'}$: If uv and $u'v'$ are vertex-disjoint edges in G, then again $d_H(x,y) \geq n+2 > m$. So we may assume that $u = u'$. Then $v \neq v'$ since B_{uv} and $M_{u'v'}$ do not belong to the same superedge by definition of φ. Let i be such that $y = M_{uv}[i]$. Since $\varphi(x) = \varphi(y)$, we have by definition of φ that $x = B_{uv}[\frac{m}{2} - i + 1]$. It follows that $d_H(x,u) = \frac{m}{2} - i + 1$ and $d_H(u',y) = \frac{m}{2} + i$, which together gives $d_H(x,y) = \frac{m}{2} - i + 1 + \frac{m}{2} + i = m+1$.

□

We believe that Conjecture 1.1 also holds when m is even and $\Delta(G) = 3$ but have been unable to extend our methods to that case.

4 Fractional powers with m odd

We begin this section by proving that Conjecture 1.1 is not true for $G = C_3 \square P_2$ if $m = 3$ and $n = 5$. Before presenting the proof we need the following lemma.

Lemma 4.1. *Let m be odd. If φ is a proper vertex coloring of $G^{\frac{m}{n}}$ with $\omega(G^{\frac{m}{n}})$ colors, then all vertices belonging to the same crust have the same color.*

Proof. By Theorem 2.1, $\omega(G^{\frac{m}{n}}) = \frac{m-1}{2}\Delta + 2$. By the proof of Theorem 2.1 given in [2], maximal cliques in $G^{\frac{m}{n}}$ are induced by all vertices of distance at most $\frac{m-1}{2}$ and one vertex of distance $\frac{m+1}{2}$ from a branch vertex v of degree $\Delta(G)$ in G. In our terminology, a maximal clique consists of one

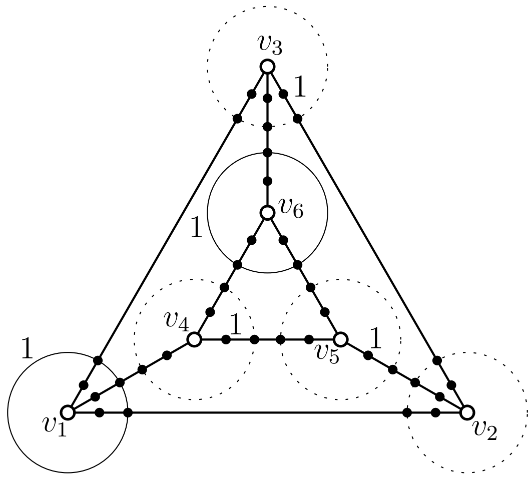


Figure 3: $(C_3\square K_2)^{\frac{3}{5}}$ is not 5-colorable.

branch vertex v with $d(v)=\Delta$, all bubbles at v and one vertex x of the crust C_v. The coloring φ uses $\frac{m-1}{2}\Delta(G)+1$ colors on v and the bubbles at v. Therefore, only one color remains for all the vertices of the crust C_v. □

Since all the vertices in the crust C_v have the same color in a coloring φ of $G^{\frac{m}{n}}$, we write $\varphi(C_v)$ for that common color. If $n \le 2m+1$, m is odd, and φ is a proper vertex coloring of $G^{\frac{m}{n}}$ with $\omega(G^{\frac{m}{n}})$ colors, then $\varphi(C_u) \ne \varphi(C_v)$ for every edge $uv \in E(G)$ since $|M_{uv}| \le m-1$. So a necessary condition for φ to be proper is that the vertex coloring f of G defined by $f(v) := \varphi(C_v)$ for every $v \in V(G)$ is proper.

Proposition 4.2. *If $G = C_3\square K_2$, then $\chi(G^{\frac{3}{5}}) > \omega(G^{\frac{3}{5}})$.*

Proof. By Lemma 2.1, $\omega(G^{\frac{m}{n}}) = \frac{m-1}{2}\Delta(G)+2$. Thus, $\omega(G^{\frac{3}{5}}) = 5$. By way of contradiction, suppose there exists a proper vertex coloring $\varphi : V(G^{\frac{3}{5}}) \to [5]$. Let $v_1,\dots,v_6$ be the vertices of $V(G)$ as on Figure 3. By Lemma 4.1, all vertices of the same crust have to receive the same color. Since there are six crusts and only five colors, at least two crusts have to receive the same color, say 1. Also, no two adjacent crusts can have the same color. So, we can assume that $\varphi(C_{v_1}) = \varphi(C_{v_6}) = 1$ (all other cases are symmetric).

Every maximal clique has to use all five colors $1,2,3,4,5$. We try to color some vertex of each clique with 1. The color 1 cannot be used for any other crust then on C_{v_1} and C_{v_6} and thus, it has to be used on some vertex in distance at most $\frac{m-1}{2} = 1$ from v_k for every $k \in \{2,3,4,5\}$.

For $k=3$, the only vertex that can receive color 1 is $(v_3v_2)_1$, because all of the others are too close to one of the two crusts C_{v_1}, C_{v_6}. Analogously for $k=4$, the vertex $(v_4v_5)_1$ has to receive color 1. Subsequently $(v_5v_2)_1$ has to receive the color 1, because $(v_5v_6)_1$ and v_5 are too close to the crust $C(v_6)$ and $(v_5v_4)_1$ is too close to $(v_4v_5)_1$. But then there is no vertex at distance at most $\frac{m-1}{2}$ from v_2 that can be colored with 1. Indeed, v_2 and $(v_2v_1)_1$ are too close to $C(v_1)$, $(v_2v_3)_1$ is too close to $(v_3v_2)_1$, and $(v_2v_5)_1$ is too close to $(v_5v_2)_1$. Hence we obtain a contradiction. □

We now present the two main theorems that support Conjecture 1.1 for odd m. We first find a general upper bound $\omega(G^{\frac{m}{n}})+2$ for the chromatic number of $G^{\frac{m}{n}}$ for any non-complete graph G with maximum degree $\Delta \ge 4$. Then we show that in about half of the possible choices for n, the chromatic number of $G^{\frac{m}{n}}$ is in fact equal to $\omega(G^{\frac{m}{n}})$.

For a graph G with $\Delta := \Delta(G)$, let $f : V(G) \to [\Delta]$ be a proper vertex coloring and let $h : E(G^{\frac{1}{2}}) \to [\Delta]$ be a half-edge coloring of G. A half-edge e_{uv} is called an *incompatible half-edge* if $h(e_{uv}) = f(v)$. If otherwise $h(e_{uv}) \neq f(v)$, then e_{uv} is called a *compatible half-edge*. If every vertex of $V(G)$ is adjacent to at most k incompatible half-edges, then we say that f and h are *k-incompatible*. If $k = 0$, then we say that that f and h are *compatible* rather than 0-incompatible.

Lemma 4.3. *For every connected non-complete graph G with maximum degree Δ and for every proper vertex coloring $f : V(G) \to [\Delta]$ there is a half-edge coloring $h : E(G^{\frac{1}{2}}) \to [\Delta]$ such that f and h are 2-incompatible.*

Proof. Let h be a half-edge coloring that has the minimal number of non-compatible half-edges. Let u be a vertex with three neighbors v_1, v_2, v_3 such that $h(e_{uv_i}) = f(v_i)$ for all $i = 1, 2, 3$. Let $a := h(e_{uv_1}) = f(v_1)$, $b := h(e_{uv_2}) = f(v_2)$, $c := h(e_{uv_3}) = f(v_i)$. Observe that either $h(e_{v_1u}) = b$ or $h(e_{v_2u}) = a$, otherwise we could switch colors on e_{uv_1} and e_{uv_2}, a contradiction with minimality. We can assume that $h(e_{v_1u}) = b$. It follows that $h(e_{v_3u}) = a$ (since otherwise we could switch colors on e_{uv_1} and e_{uv_3}), and next that $h(e_{v_2u}) = c$ (since otherwise we could switch colors on e_{uv_2} and e_{uv_3}). But now we can recolor e_{uv_1} with c, e_{uv_2} with a, e_{uv_3} with b, a contradiction with minimality. □

Theorem 4.4. *If m is odd and G is a connected non-complete graph with maximum degree $\Delta \geq 4$, then $\chi(G^{\frac{m}{n}}) \leq \omega(G^{\frac{m}{n}}) + 2$.*

Proof. Let $f : V(G) \to [\Delta]$ and $h : E(G^{\frac{1}{2}}) \to [\Delta]$, where $1 = (1_1, \ldots, 1_{\frac{m-1}{2}}), \ldots, \Delta = (\Delta_1, \ldots, \Delta_{\frac{m-1}{2}})$, be 2-incompatible colorings given by Lemma 4.3. Then we define an auxiliary (not necessarily proper) vertex coloring $\varphi' : V(G^{\frac{m}{n}}) \to \{0, \heartsuit, \diamondsuit, 1_1, \ldots, 1_{\frac{m-1}{2}}, \ldots, \Delta_1, \ldots, \Delta_{\frac{m-1}{2}}\}$ as follows.

1. **branch vertices**: $\varphi'(v) = 0$ for every branch vertex $v \in G^{\frac{m}{n}}$.

2. **crusts**: $\varphi'(C_v) = f(v)[\frac{m-1}{2}]$ for every branch vertex $v \in G^{\frac{m}{n}}$.

3. **bubbles**:

$$\varphi'(B_{uv}) = \begin{cases} (\heartsuit) * h(e_{uv})[1 : \frac{m-1}{2} - 1] & \text{if } h(e_{uv}) = f(u), \\ h(e_{uv})[1 : \frac{m-1}{2} - 1] * (\diamondsuit) & \text{if } h(e_{uv}) = f(v), \\ h(e_{uv}) & \text{otherwise.} \end{cases}$$

Observe that there can only be a conflict between vertices using color $\heartsuit$ and between vertices using color $\diamondsuit$. Vertices colored $\heartsuit$ are exactly vertices $(uv)_1$ for which $h(e_{uv}) = f(u)$, and we will call them $\heartsuit$-vertices. If $n = m + 2$, then for every superedge P_{uv}, the vertices $(uv)_1$ and $(vu)_1$ are of distance m in $G^{\frac{1}{2}}$. So, if both $(uv)_1$, $(vu)_1$ are $\heartsuit$-vertices, then we have a conflict. For each superedge containing two $\heartsuit$-vertices, we choose one of the two $\heartsuit$-vertices. We call the set of all the chosen $\heartsuit$-vertices as a $(\heartsuit \to 0)$-set, and the set of their neighboring branch vertices as a $(0 \to \heartsuit)$-set.

Vertices colored $\diamondsuit$ are vertices $(uv)_{(m-1)/2}$ for which $h(e_{uv}) = f(v)$, and we will call them $\diamondsuit$-vertices. Since f and h are 2-incompatible, we have at most two $\diamondsuit$-vertices around each branch vertex. The subgraph H of $G^{\frac{m}{n}}$ induced by all $\diamondsuit$-vertices then consists of disjoint union of paths and even cycles. So, we can properly color H using two colors $\diamondsuit_1$ and $\diamondsuit_2$. Let $\diamondsuit_1$-set and $\diamondsuit_2$-set denote the set of $\diamondsuit$-vertices colored $\diamondsuit_1$ and $\diamondsuit_2$, respectively.

We now define $\varphi : V(G^{\frac{m}{n}}) \to \{0, \heartsuit, \diamondsuit_1, \diamondsuit_2, 1_1, \ldots, 1_{\frac{m-1}{2}}, \ldots, \Delta_1, \ldots, \Delta_{\frac{m-1}{2}}\}$, for branch vertices, crusts, and bubbles.

$$\varphi(v) = \begin{cases} 0 & \text{if } v \in (\heartsuit \to 0)\text{-set}, \\ \heartsuit & \text{if } v \in (0 \to \heartsuit)\text{-set}, \\ \diamondsuit_1 & \text{if } v \in \diamondsuit_1\text{-set}, \\ \diamondsuit_2 & \text{if } v \in \diamondsuit_2\text{-set}, \\ \varphi'(v) & \text{otherwise (if } v \text{ is not from a middle part).} \end{cases}$$

4. **middle parts**: Let $k = \lceil \frac{|M_{uv}|}{2} \rceil$ and let $l = \lfloor \frac{|M_{uv}|}{2} \rfloor$. Let $a, b \in [\Delta]$ such that $\varphi(B_{uv}) = a$, $\varphi(B_{vu}) = b$. Then we have the following three (up to symmetry) cases.

   (a) $\varphi(C_u), \varphi(C_v) \in \{a_{\frac{m-1}{2}}, b_{\frac{m-1}{2}}\}$. Then we fix $c, d \in [\Delta] \setminus \{a, b\}$ ($c \neq d$ exist since $\Delta \geq 4$) and define $\varphi(M_{uv}[1:k]) = \overline{c}[1:k]$ and $\varphi(M_{vu}[1:l]) = \overline{d}[1:l]$.

   (b) $\varphi(C_u) = c_{\frac{m-1}{2}}$, $\varphi(C_v) = a_{\frac{m-1}{2}}$, where $c \in [\Delta] \setminus \{a, b\}$. Then we fix $d \in [\Delta] \setminus \{a, b, c\}$ and define $\varphi(M_{uv}[1:k]) = \overline{c}[2:k] * (\heartsuit)$ and $\varphi(M_{vu}[1:l]) = \overline{d}[1:l]$.

   (c) $\varphi(C_u) = c_{\frac{m-1}{2}}$, $\varphi(C_v) = d_{\frac{m-1}{2}}$, where $c, d \in [\Delta] \setminus \{a, b\}$. Then we define $\varphi(M_{uv}[1:k]) = \overline{c}[2:k] * (\heartsuit)$ and $\varphi(M_{vu}[1:l]) = \overline{d}[2:l] * (\diamondsuit_1)$.

□

Corollary 4.5 (of the proof of Theorem 4.4). *If m is odd and G is a non-complete graph with maximum degree $\Delta \geq 5$ and such that there are compatible proper colorings $f : V(G) \to [\Delta]$ and $h : E(G^{\frac{1}{2}}) \to [\Delta]$, then $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$.*

Proof. If f and h are compatible, then there are no $\diamondsuit$-vertices in $G^{\frac{m}{n}}$, so we do not need to use colors $\diamondsuit_1$ and $\diamondsuit_2$. The only other place where we use one of these colors is the middle part, in Case (c), where we use the color $\diamondsuit_1$. But if $\Delta \geq 5$, then there exists a fifth color $e \in [\Delta] \setminus \{a, b, c, d\}$. So, we can define $\varphi(M_{uv}[1:k]) = \overline{c}[2:k] * (\heartsuit)$ and $\varphi(M_{vu}[1:k]) = \overline{e}[1:k]$. □

When each middle part has at least $\frac{m+1}{2}$ and at most $m-1$ vertices, the situation gets easier, as the following theorem shows. (Note that $n = 2|B_{uv}| + 2 + |M_{uv}| + 1$.)

Theorem 4.6. *Let m be odd and G be a non-complete graph with $\Delta(G) \geq 3$. If $\frac{3m+5}{2} \leq n \leq 2m$, then $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$. If $n = 2m+1$, then $\chi(G^{\frac{m}{n}}) \leq \omega(G^{\frac{m}{n}}) + 1$.*

Proof. Let $\Delta = \Delta(G)$. By Lemma 2.5, we can assume that G is Δ-regular. We start with the first part of the claim, so we suppose $\frac{3m+5}{2} \leq n \leq 2m$. Since $\omega(G^{\frac{m}{n}}) = \frac{m-1}{2}\Delta + 2$ and $\omega(G^{\frac{m}{n}}) \leq \chi(G^{\frac{m}{n}})$, we only need to show a coloring of $G^{\frac{m}{n}}$ that uses $\frac{m-1}{2}\Delta + 2$ colors.

Let $f : V(G) \to [\Delta]$ be a proper vertex coloring of G and let $h : E(G^{\frac{1}{2}}) \to [\Delta]$ be a proper half-edge coloring of G. Let uv be an edge of G. If $f(u) = h(e_{uv})$, then the vertex $(u, v)_1$ is called a $\heartsuit$-vertex. For each superedge containing two $\heartsuit$-vertices, we choose one and put it in the (originally empty) set called ($\heartsuit \to 0$)-set. The set of branch vertices that have a neighbor in the ($\heartsuit \to 0$)-set is called ($0 \to \heartsuit$)-set. We define a coloring $\varphi : V(G^{\frac{m}{n}}) \to \{0, \heartsuit, 1_1, \ldots, 1_{\frac{m-1}{2}}, \ldots, \Delta_1, \ldots, \Delta_{\frac{m-1}{2}}, \}$ as follows.

1. **branch vertices:** $\varphi(v) = \heartsuit$ if $v \in (0 \to \heartsuit)$-set, $\varphi(v) = 0$ otherwise.

2. **$\heartsuit$-vertices:** $\varphi(v) = 0$ if $v \in (\heartsuit \to 0)$-set, $\varphi(v) = \heartsuit$ otherwise.

3. **crusts:** $\varphi(C_v) = f(v)[1]$.

4. **bubbles:** If there is no $\heartsuit$-vertex in B_{uv}, then $\varphi(B_{uv}) = h(e_{uv})$. Otherwise $\varphi(B_{uv}[2 : \frac{m-1}{2}]) = h(e_{uv})[2 : \frac{m-1}{2}]$.

5. **middle parts:** Let $k = \left\lceil \frac{1}{2}|M_{uv}| \right\rceil$ and let $l = \left\lfloor \frac{1}{2}|M_{uv}| \right\rfloor$. If $k = \frac{m-1}{2}$ and $h(e_{uv}), h(e_{vu}), f(u), f(v)$ are pairwise distinct (so there is no $\heartsuit$-vertex on P_{uv}), then we define $\varphi(M_{uv}[1:k]) = \overline{f(v)}[1:k-1] * (\heartsuit)$. In all other cases we define $f(M_{uv}[1:k]) = \overline{f(v)}[1:k]$. Lastly, we define $f(M_{vu}[1:l]) = \overline{f(u)}[1:k]$.

The second part of the theorem follows immediately by inserting one more vertex $(uv)_*$ on every superedge of $G^{\frac{m}{2m}}$ and coloring it with a new color. In particular, to obtain $G^{\frac{m}{2m+1}}$ from $G^{\frac{m}{2m}}$, we replace the edge $M_{uv}[k]M_{vu}[l]$ by a path $M_{uv}[k](uv)_*M_{vu}[l]$ for every edge $uv \in E(G)$, and color all the new vertices $(uv)_*$ by a new color. □

5 Fractional powers for complete graphs

Lemma 5.1. *$\chi(G^{\frac{3}{5}}) = \omega(G^{\frac{3}{5}})$ for any complete graph G on at least 5 vertices.*

Proof. We show that we can color $G^{\frac{3}{5}}$ with $\omega(G^{\frac{3}{5}}) = \Delta(G)+2$. Let $v_1,\dots,v_r$ be vertices of G. Let $\Delta := \Delta(G) \geq 4$. We construct a coloring $\varphi : V(G^{\frac{3}{5}}) \to \{0,\dots,\Delta+1\}$ as follows. First, we color all branch vertices of $G^{\frac{3}{5}}$ with 0. Second, we let $\varphi(v) = i$ for every vertex of the same crusts C_{v_i}. Finally, we need to color neighbors (in $G^{\frac{1}{5}}$) of $v_1,\dots,v_r$.

Observe that all the vertices of distance 3 (in $G^{\frac{1}{5}}$) from a branch vertex v_i have mutually different colors $1,\dots,i-1,i+2,\dots,\Delta+1$, which have to be used for the neighbors of v_i. Suppose that we colored all the neighbors of $v_1,\dots,v_{i-1}$ and we want to color v_i. We construct an auxiliary bipartite graph (A_i,B_i) with parts $A_i := \{(v_iv_j)_1 : j \in [\Delta+1]\setminus\{i\}\}$ and $B_i := [\Delta+1]\setminus\{i\}$. There is an edge between $(v_iv_j)_1 \in A_i$ and $b \in B_i$ if and only if b is not forbidden for $(v_iv_j)_1$, i.e. if b can be used on $(v_iv_j)_1$ such that the resulting coloring remains proper. Each vertex $(v_iv_j)_1$ has one or two forbidden colors: the color $\varphi((v_iv_j)_3) = j$, and in case $j < i$ also the color $\varphi((v_iv_j)_4)$. We show that either (A_i,B_i) has a perfect matching or that we can switch colors on some vertices and redefine (A_i,B_i) so that the new (A_i,B_i) has a perfect matching.

By Hall's Theorem (see e.g. [7]), (A_i,B_i) has a perfect matching if and only if $|S| \leq |N(S)|$ for all $S \subseteq A_i$. Since every vertex of A_i has degree at least $\Delta-2$ in (A_i,B_i), $|N(S)| \geq \Delta-2$, we S can only violate Hall's condition if $|S| = \Delta-1$ or $|S| = \Delta$.

Suppose first that $|S| = \Delta-1$. Since $\Delta \geq 4$, there are three vertices $(v_iv_j)_1$, $(v_iv_k)_1$, and $(v_iv_l)_1$ in S. If $|N(S)| = \Delta-2$, then each of these vertices has the same two forbidden colors $b_1,b_2 \in B_i$. But this is not possible since the forbidden colors j,k,l for $(v_iv_j)_1$, $(v_iv_k)_1$, $(v_iv_l)_1$, respectively, are pairwise distinct.

Suppose next that $|S| = \Delta$, so $S = A_i$. We claim that if $i \in [r-2]$, then $N(S) = B_i$. Indeed, since $r-1$ and r are the only forbidden colors for $(v_iv_{r-1})_1$ and $(v_iv_r)_1$, respectively, and these are distinct, we conclude that $N(\{(v_iv_{r-1})_1,(v_iv_r)_1\}) = B_i$, and thus $N(S) = B_i$.

Suppose that $i = r-1$. If a perfect matching does not exist in (A_i,B_i), then $|N(A_i)| = \Delta-1$. Since the only forbidden color for $(v_iv_r)_1$ is r, we have $\varphi((v_iv_j)_4) = r$, or equivalently $\varphi((v_jv_i)_1) = r$, for all $j \in [\Delta]\setminus\{i\}$. Let $k \in [\Delta]\setminus\{i,r-1,r\}$ such that $\varphi((v_1v_k)_1) \neq r-1$. Then we switch colors on vertices $(v_1v_i)_1$ and $(v_1v_k)_1$ (see Figure 4a). For the redefined bipartite graph (A_i,B_i) we then have $|N(A_i)| = |A_i|$, which means that there is a perfect matching in (A_i,B_i).

Finally, let $i = r$, and suppose $N(A_i) = \Delta-1$. Let $b \in B_i \setminus N(A_i)$, so b is forbidden for every vertex in A_i. Exactly one vertex $(v_iv_b)_3$ of $\{(v_iv_j)_3 : j = [\Delta]\setminus\{i\}\}$ has color b. So, all vertices of $\{(v_iv_j)_4 : j = [\Delta]\setminus\{i,b\}\}$ have color b. Let $l \in [\Delta]\setminus\{i,b,r\}$ such that $\varphi((v_1v_l)_1) \neq r$. Then we switch colors on vertices $(v_1v_i)_1$ and $(v_1v_l)_1$ to obtain a new graph (A_i,B_i) with $|N(A_i)| = |A_i|$ as needed; see Figure 4b. □

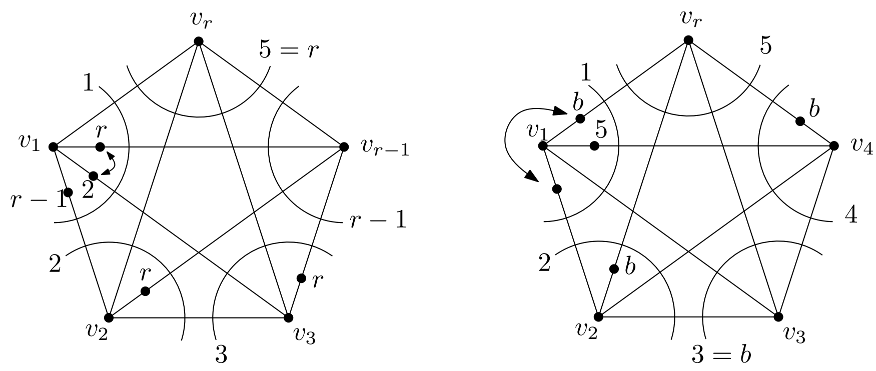
Theorem 5.2. *If G is a complete graph on at least 5 vertices, then $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$.*

Proof. We only need to consider the case when m is odd since Theorem 3.1 can be applied to complete graphs.

We first show that for all $m \geq 3$, there exists a proper coloring of $\varphi : V(G^{\frac{m}{m+2}}) \to \{0,\dots,\frac{m-1}{2}\Delta+1\}$ such that the vertices colored 0 are exactly the branch vertices. We use induction on m. The base case is exactly Lemma 5.1.

Let $m \geq 5$. By the induction hypothesis, there exists a proper vertex coloring $\varphi' : V(G^{\frac{m-2}{m}}) \to \{0,\dots,\frac{m-3}{2}\Delta+1\}$. Let $f : V(G^{\frac{2}{3}}) \to \{0,\frac{m-3}{2}\Delta+2,\dots,\frac{m-1}{2}\Delta+1\}$ be a proper coloring such that 0 is used exactly on branch vertices (existing by Lemma 2.6). Then we define

$$\varphi((uv)_i) = \begin{cases} f((uv)_i) & \text{in } G^{\frac{2}{3}} \quad \text{if } i \in \{0,1\}, \\ \varphi'((uv)_{i-1}) & \text{in } G^{\frac{m-2}{m}} \quad \text{otherwise.} \end{cases}$$



(a) $i = r - 1$ (b) $i = r$

Figure 4

It is now easy to complete the proof. Suppose that we seek a coloring of $G^{\frac{m}{n}}$ for some $n > m+2$ that uses $\frac{m-1}{2}\Delta + 2$ colors. Then we adopt the above coloring φ for all $(uv)_i$ with $i = 0, \ldots, \frac{m+1}{2}$; thus φ provides a coloring of bubbles and crusts in $G^{\frac{m}{n}}$. All that remains is to color the middle parts. By Lemma 2.4, we can assume that $m + 2 \leq n \leq 2m + 1$, and so $0 \leq |M_{uv}| \leq m - 1$. For $i = 1, \ldots, \lceil \frac{1}{2}|M_{uv}| \rceil$, let S_i be the set of colors used on vertices of distance i from u or from v in $G^{\frac{1}{n}}$. By construction of φ, we have $S_i \cap S_j = \emptyset$ if $i \neq j$. Therefore, for each P_{uv} there exist at least 2 colors c_1, c_2 in S_i that are not used on P_{uv}. Then we define $\varphi(M_{uv}[i]) = c_1$ and $\varphi(M_{vu}[i]) = c_2$ (if $M_{vu}[i]$ exists). □

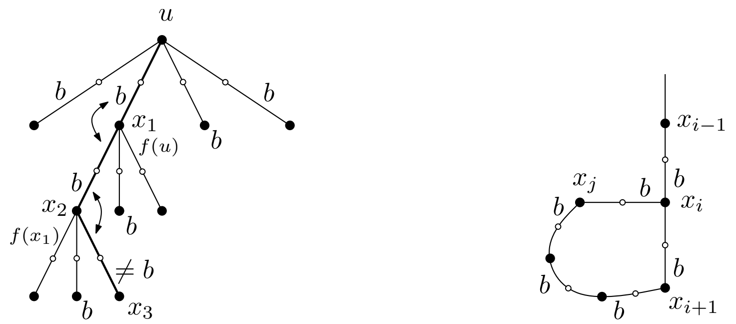
6 Dynamic coloring

In this section we assume that G is a graph with maximum degree $\Delta \geq 5$ and that m is odd. By Remark 6.2, if there exist compatible colorings $f : V(G) \to [\Delta]$ and $h : E(G^{\frac{1}{2}}) \to [\Delta]$ and $\Delta(G) \geq 6$ (or $\Delta(G) \geq 5$ and an additional assumption), then $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$. The purpose of this section is to find a sufficient condition for existence of compatible colorings f and h for G. An *r-dynamic proper k-coloring* of a graph G is a proper coloring $g : V(G) \to [k]$ such that for every vertex $v \in V(G)$ the number of colors used on $N(v)$ is at least $\min\{r, d(v)\}$.

Lemma 6.1. *Let G be a graph with maximum degree $\Delta \geq 5$. For every 4-dynamic proper Δ-coloring f of G there exists a coloring $h : E(G^{\frac{1}{2}}) \to [\Delta]$ such that f and h are compatible.*

Proof. We subsequently color half-edges around each vertex, in any fixed order of the vertices of G. Suppose that we want to color the half-edges around u. We construct an auxiliary bipartite graph (A, B) with parts $A := \{e_{uv} : v \in N(u)\}$ and $B := [\Delta]$. There is an edge between $e_{uv} \in A$ and $b \in B$ if and only if e_{uv} can be colored with b, i.e. exactly when $b \neq f(v)$ and $b \neq h(e_{vu})$ (in case e_{vu} is already colored). We show that either (A, B) has a perfect matching or that we can switch colors on some half-edges and redefine (A, B) so that the new (A, B) has a perfect matching. If $|N(S)| \geq |S|$ for all $S \subseteq A$, then the existence of a perfect matching is ensured by Hall's Theorem. So suppose that there exists $S \subseteq A$ such that $|N(S)| < |S|$. Every vertex of A has degree at least $\Delta - 2$, where $\Delta \geq 5$. So $|N(S)| \geq \Delta - 2$ and therefore $|S| \geq \Delta - 1$.

First, suppose that $|S| = \Delta - 1$. Let $e_{uv'} \in A \setminus S$. Since f is 4-dynamic, there are at least four colors on $N_G(v)$. This implies that there are three vertices v_1, v_2, v_3 in $N_G(v) \setminus \{v'\}$ with pairwise different colors. If $|N(S)| \leq \Delta - 2$, then there are $b_1, b_2 \in B \setminus N(S)$. But then one of the vertices v_1, v_2, v_3 is colored neither b_1 nor b_2. Therefore one of b_1, b_2 is not forbidden for that vertex, a contradiction.



(a) Switching along the path $ux_1x_2 \dots x_k$.

(b) There is no loop.

Figure 5: Coloring half-edges around u.

Second, suppose that $|S| = \Delta$, i.e. $|A| > |N(A)|$. Since then $|N(A)| = \Delta - 1$, there is a color $b \in B$ that is forbidden for every vertex of A. So, for every $e_{uv} \in A$, either $f(v) = b$ or $h(e_{vu}) = b$ (if defined). Since f is 4-dynamic, there exists $x_1 \in N_G(v)$ such that $f(x_1) \neq b$. Then necessarily $h(e_{x_1u}) = b$. Let now $x_2 \in N_G(x_1) \setminus \{u\}$ such that $f(x_2) \neq b$ and $h(e_{x_1x_2}) \neq f(u)$. Such a vertex exists since f is 4-dynamic. If $h(e_{x_2x_1}) \neq b$, then we can switch the colors on e_{x_1u} and $e_{x_1x_2}$. Then the new graph (A, B) has a perfect matching and the so far defined coloring h is proper and compatible with f. If otherwise $h(e_{x_2x_1}) = b$, then we search for $x_3 \in N_G(x_2) \setminus \{x_1\}$ such that $f(x_3) \neq b$ and $h(e_{x_2x_3}) \neq f(x_1)$, and check whether or not $h(e_{x_3x_2}) \neq b$. We repeat this process until we find a vertex x_k with $h(e_{x_kx_{k-1}}) \neq b$. Then we switch colors on $e_{x_ix_{i-1}}$ and $e_{x_ix_{i+1}}$ for all $i = 1, \dots, k-1$, where $x_0 = u$ (see Figure 5a). The process has finitely many steps since $|V(G)|$ is finite and we cannot have a loop. Indeed, suppose that we found a sequence $x_1, x_2, \dots, x_i, x_{i+1}, \dots, x_j, x_i, \dots$. Then $h(e_{x_{i-1}x_i}) = b$ and $h(e_{x_jx_i}) = b$ since otherwise the sequence would end at x_{i-1} and x_j, respectively. But $e_{x_{i-1}x_i}$ and $e_{x_jx_i}$ are adjacent and cannot have the same color, a contradiction (see Figure 5b). □

Remark 6.2. *Lemma 6.1 holds under the weaker assumption that f is 3-dynamic and such that for every vertex v with exactly three colors used on $N(v)$, at least two colors are used twice on $N(v)$.*

The following theorem by Jahanbekam, Kim, O, and West [3], [5, Corollary 5.2.6] provides sufficient conditions for a graph to be r-dynamic proper k-colorable.

Theorem 6.3. *If G is a Δ-regular graph with $\Delta \geq 7r \ln r$, then there exists an r-dynamic proper $(r\chi(G))$-coloring of G.*

We seek a 4-dynamic proper Δ-coloring of G. For that we need $\Delta \geq \lceil 28 \ln 4 \rceil = 39$ and $\chi(G) \leq \frac{\Delta}{4}$. The following corollary is immediate.

Corollary 6.4. *If G is a Δ-regular graph with $\Delta \geq 39$, $\chi(G) \leq \frac{\Delta}{4}$, and $1 < m < n$ with m odd, then $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$.*

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References

[1] M. Behzad, *Graphs and their chromatic numbers*, Ph.D. Thesis, Michigan State University (1965).

[2] M. N. Iradmusa, *On colorings of graph fractional powers*, Discrete Math. 310 (2010), 1551–1556.

[3] S. Jahanbekam, J. Kim, S. O, and D. B. West, *On r-dynamic coloring of graphs*, Discrete Applied Mathematics 206 (2016), 65–72.

[4] D. König, *Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre*, Math. Ann. 77 (1916), 453–465.

[5] S. O, *Matchings, connectivity, and eigenvalues in regular graphs*, Ph.D. Thesis, University of Illinois at Urbana-Champaign (2011).

[6] V. G. Vizing, *Some unsolved problems in graph theory*, Uspekhi Mat. Nauk 23 (1968) 117-134.

[7] D. B. West, *Introduction to graph theory*, Prentice Hall, Inc. (1996).

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Note

A note on list-coloring powers of graphs

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ABSTRACT

Recently, Kim and Park have found an infinite family of graphs whose squares are not chromatic-choosable. Xuding Zhu asked whether there is some k such that all kth power graphs are chromatic-choosable. We answer this question in the negative: we show that there is a positive constant c such that for any k there is a family of graphs G with $\chi(G^k)$ unbounded and $\chi_\ell(G^k) \geq c\chi(G^k)\log\chi(G^k)$. We also provide an upper bound, $\chi_\ell(G^k) < \chi(G^k)^3$ for $k > 1$.

1. Introduction

The *list-chromatic number* of a graph G, denoted by $\chi_\ell(G)$, is the least k such that for any assignment of lists of size k to the vertices of G, there is a proper coloring of $V(G)$ where the color at each vertex is in that vertex's list. A graph is said to be *chromatic-choosable* if $\chi_\ell(G) = \chi(G)$. The *kth power* of a graph G, denoted by G^k, is the graph on the same vertex set as G such that uv is an edge if and only if the distance from u to v in G is at most k.

The List Total Coloring Conjecture (LTCC) asserts that $\chi_\ell(T(G)) = \chi(T(G))$ for every graph G, where $T(G)$ is the total graph of G. The List Square Coloring Conjecture (LSCC) was introduced in [5], as it would imply the LTCC. The LSCC asserts that squares of graphs are chromatic-choosable. However, the LSCC was recently disproved by Kim and Park [4], who constructed a family of graphs G with $\chi(G^2)$ unbounded and $\chi_\ell(G^2) \geq c\chi(G^2)\log\chi(G^2)$. Xuding Zhu asked whether there is any k such that all kth powers are chromatic-choosable [7]. In this note we give a negative answer to Zhu's question, with a bound on $\chi_\ell(G^k)$ that matches that of Kim and Park for $k = 2$.

Theorem 3.4. *There is a positive constant c such that for every $k \in \mathbb{N}$, there is an infinite family of graphs G with $\chi(G^k)$ unbounded such that*

$$\chi_\ell(G^k) \geq c\chi(G^k)\log\chi(G^k).$$

While preparing this note, it has come to our attention that Kim, Kwon, and Park have arrived at a similar result [3]. They have found, for each k, an infinite family of graphs G whose kth powers satisfy $\chi_\ell(G^k) \geq \frac{10}{9}\chi(G^k) - 1$.

Let $f_k(m) = \max\{\chi_\ell(G^k) : \chi(G^k) = m\}$. Then Theorem 3.4 says that $f_k(m) \geq cm\log m$. Kwon (see [6]) observed that $f_2(m) < m^2$. We extend this observation to larger k in Section 4.

Theorem 4.1. *Let $k > 1$. If k is even, then $f_k(m) < m^2$. If k is odd, then $f_k(m) < m^3$.*

Question 1.1. *What is the correct order of magnitude of $f_k(m)$? Does it depend on k?*

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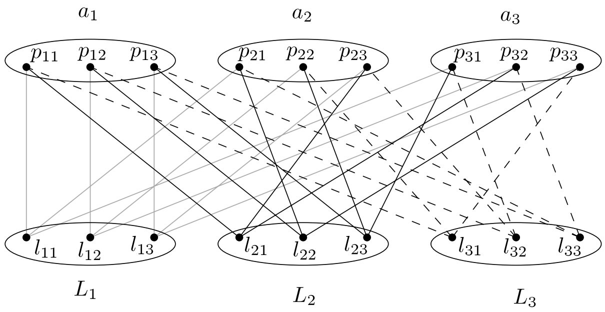

Fig. 1. The graph H, here with $n = 3$.

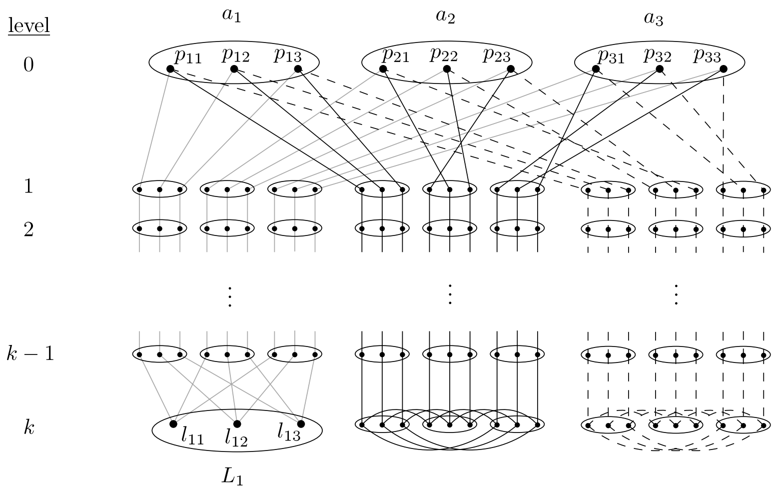

Fig. 2. The graph G when $n = 3$.

2. Construction

The example of Kim and Park [4] for $k = 2$ is based on complete sets of mutually orthogonal latin squares. We will use this structure to find examples for all k, but we find the language of affine planes to be more convenient.

Take an affine plane $(\mathcal{P}, \mathcal{L})$ on n^2 points. Let $\{L_0, L_1, \ldots, L_n\}$ be the decomposition of $\mathcal{L}$ into parallel classes. Recall that we call the elements of $\mathcal{P}$ the *points* and the elements of $\mathcal{L}$ the *lines* of the plane, and that we have the following properties (see for instance [2]):

- Each line is a set of n points.
- For each pair of points, there is a unique line containing them.
- Two lines in the same parallel class do not intersect.
- Two lines in different parallel classes intersect in exactly one point.
- Such a plane exists whenever n is a (positive) power of a prime.

Form the bipartite graph H with parts $\mathcal{P}$ and $B = \mathcal{L} - L_0$, with $p\ell \in E(H)$ if and only if $p \in \ell$. Let $a_1, \ldots, a_n$ denote the lines of L_0. Consider the refinement $\mathcal{V}'$ of the bipartition of H obtained by partitioning $\mathcal{P}$ into $a_1, \ldots, a_n$ and B into $L_1, \ldots, L_n$. Note that $H[a_i, L_j]$ is a matching for each i and j. In Fig. 1, the graph H is shown with $n = 3$. Edges are drawn differently according to which parallel class their line-endpoint belongs to, and the parts of $\mathcal{V}'$ are indicated.

Let $k \geq 2$. Subdivide the edges of H into paths of different lengths: edges incident to L_1 are subdivided into paths of length k, while edges not incident to L_1 are subdivided into paths of length $k+1$. For an edge $p\ell \in E(H)$, denote the vertices along the subdivision path as $p = (p\ell)_0, (p\ell)_1, (p\ell)_2, \ldots$. If $\ell \in L_1$, then $(p\ell)_k = \ell$, and if $\ell \notin L_1$, then $(p\ell)_{k+1} = \ell$. For a vertex $(p\ell)_i$, say its *level* is i, its *point* is p, and its *line* is ℓ (levels are well-defined, and points and lines of vertices of degree 2 are well-defined). Form the graph G by, for each $\ell \in \bigcup_{2 \leq i \leq n} L_i$, adding edges to make the neighborhood of ℓ a clique and then deleting ℓ. For each $i, j \in [n]$ and $m \in \{0, \ldots, k\}$, let $V_{i,j,m} = \{(p\ell)_m : p\ell \in E(H), p \in a_i, \ell \in L_j\}$; then $\{V_{i,j,m} : i, j \in [n], m \in \{0, \ldots, k\}\}$ is a partition of $V(G)$ into sets of size n, which we call $\mathcal{V}$. In Fig. 2, the graph G is shown. Again we use $n = 3$, and here the parts of $\mathcal{V}$ are indicated.

3. Proof of Theorem 3.4

Lemma 3.1. *G^{4k} is multipartite with partition $\mathcal{V}$.*

Proof. Let p and q be two points in some a_i. Any path from p to q must start by increasing levels, arriving at $(p\ell)_k$. If $\ell \notin L_1$, then the path must move from $(p\ell)_k$ to $(p'\ell)_k$ for some p' not on a_i. Continuing along the path to level 0, we arrive at p'. Since p' is not on a_i, p' and q are on a common line $\ell' \in \bigcup_{i=1}^n L_i$. If $\ell' \in L_1$, the shortest path from p' to q is to increase levels to ℓ' and decrease levels to q. If $\ell' \in \bigcup_{i=2}^n L_i$, the shortest path from p' to q is to increase levels to $(p'\ell')_k$, move over to $(q\ell')_k$, and then decrease levels to q. Notice, if p and p' are on a common line in L_1, p' and q cannot be on a common line in L_1 because then p and q would be on a common line in L_1. Thus, the path uses at least 3 vertices in level k, and so has length at least $4k + 1$.

Let $\ell_1, \ell_2 \in L_1$. Any path would have to have both ends decrease to level 0. If both ℓ_1 and ℓ_2 connect to points in some a_i, then since these vertices are a distance at least $4k + 1$ apart, the path between ℓ_1 and ℓ_2 would have length at least $4k + 1$. Otherwise, the paths from ℓ_1 and ℓ_2 arrive at points on different lines in L_0, say p and q, respectively. These two points are on a common line not in L_0 or L_1, say ℓ. The shortest path between p and q is to go from p to $(p\ell)_k$, over to $(q\ell)_k$, and finally to q. However, this results in a path between ℓ_1 and ℓ_2 of length at least $4k + 1$.

Let $(p\ell_1)_k$, $(q\ell_2)_k$ be two vertices in the same part other than L_1; that is, p, q are both on some a_i and ℓ_1, ℓ_2 are two lines in the same parallel line class. If a path joining them starts by decreasing levels from both ends to level 0, that is connects $(p\ell_1)_k$ to p and $(q\ell_2)_k$ to q, then since p and q are a distance at least $4k + 1$ apart, the path between $(p\ell_1)_k$ and $(q\ell_2)_k$ would have length at least $4k + 1$. Otherwise, at least one of $(p\ell_1)_k$ or $(q\ell_2)_k$ must first go to $(p'\ell_1)_k$ or $(q'\ell_2)_k$. Without loss of generality connect $(p\ell_1)_k$ to $(p'\ell_1)_k$. Now, any path must connect $(p'\ell_1)_k$ to p' and $(q\ell_2)_k$ to q. These are on a common line not in L_0, however, increasing levels from each of p' and q to level k results in a total of at least $4k + 1$ steps.

Now consider two degree-two vertices in the same part. Any path joining them has ends that either increase or decrease levels from the endpoint. If the path increases levels from both ends or decreases levels from both ends, then we arrive at different vertices in the same level 0 or level k part. Since the rest of the path must have length at least $4k+1$, the total path must have length at least $4k + 1$. Otherwise, one end increases levels and the other decreases levels. The resulting point, p, is not on the resulting line, ℓ. The path must next increase levels from p to a line. If this line is in the same parallel line class as ℓ, then the resultant path has length over $4k + 1$. Otherwise, since this line is not in the same class as ℓ, these two lines share a common point. The shortest completion of the path is through this point. However, since at least one of these lines is not in L_1, the path must contain at least 3 vertices in level k. Thus, the path has length at least $4k + 1$. □

Lemma 3.2. *The subgraph of G^{4k} induced by the vertices in levels 0 through $k - 1$ is complete multipartite with partition $\mathcal{V}$ restricted to those levels.*

Proof. Consider two points p, q on different lines in L_0. They are on a common line $\ell \in \bigcup_{i=1}^n L_i$. If $\ell \in L_1$, connect p to ℓ then ℓ to q. If $\ell \notin L_1$, connect p to $(p\ell)_k$ to $(q\ell)_k$ to q. In each case the path has length at most $2k + 1 < 4k$.

Consider two vertices in different parts at level i, $1 \le i \le k - 1$. Either their points are on different lines in L_0 or their lines are from different parallel classes. If their points are from different lines in L_0, go to these points. These points share a common line not in L_0. Connect via the path between this line. This takes at most $2i+2k+1 \le 4k-1$ steps. If their lines are from different parallel classes, increase levels to level k. These two lines share a common point. By, if necessary, first changing vertices at level k, connecting through this point, we get a path of length at most $2(k - i) + 2 + 2k = 4k - 2i + 2 \le 4k$.

Finally, consider two vertices in levels i and j, $0 \le i < j < k$. Start a path joining them by decreasing levels from the lower-level vertex, and increasing levels from the larger-level vertex. Let the point we arrive at from decreasing the lower-level vertex be p. If the increasing from the larger-level vertex takes us to a line in L_1, we can connect from this line to a point on a different line of L_0 than p, say q. Now p and q are on a common line not in L_0. Connecting through this gives us a path of length at most $k - 1 + k + 2k + 1 = 4k$. If instead the increasing from the larger-level vertex takes us to a vertex of the form $(q\ell)_k$, $\ell \notin L_1$, then let ℓ' be the line through p in L_1. Now ℓ and ℓ' intersect at a point, say q'. We can complete the path by going from $(q\ell)_k$ to $(q'\ell)_k$ to q' to ℓ' to p. This takes a total of at most $k - 1 + 1 + 3k = 4k$ steps. □

We will use the following result of Alon.

Lemma 3.3 ([1]). *Let K_{r*s} denote the complete r-partite graph with each part of size s. There are two constants, d_1 and d_2, such that*

$$d_1 r \log s \le \chi_\ell(K_{r*s}) \le d_2 r \log s.$$

Everything is now in place to complete the proof.

Theorem 3.4. *There is a positive constant c such that for every $k \in \mathbb{N}$, there is an infinite family of graphs G with $\chi(G^k)$ unbounded such that*

$$\chi_\ell(G^k) \ge c\,\chi(G^k) \log \chi(G^k).$$

Proof. Since G^{4k} is multipartite on $kn^2 + 1$ parts, $\chi(G^{4k}) \le kn^2 + 1$, and so $n \ge \sqrt{(\chi(G^{4k}) - 1)/k}$.

Since G^{4k} contains a complete multipartite subgraph with $(k-1)n^2$ parts of size n, we have from Lemma 3.3 that

$$\begin{aligned}\chi_\ell(G^{4k}) &\ge d_1(k-1)n^2\log n\\ &\ge d_1\frac{k-1}{k}\left(\chi(G^{4k})-1\right)\log\sqrt{\frac{\chi(G^{4k})-1}{k}}\\ &= \frac{d_1}{2}\frac{k-1}{k}\left(\chi(G^{4k})-1\right)\left(\log(\chi(G^{4k})-1)-\log k\right)\\ &\ge \frac{d_1}{4}\left(\chi(G^{4k})-1\right)\left(\log(\chi(G^{4k})-1)-\log k\right).\end{aligned}$$

Taking n large enough makes $\chi(G^{4k})$ as large as we like, and so by taking a constant c just smaller than $d_1/4$ and taking n sufficiently large, we obtain

$$\chi_\ell(G^{4k}) \ge c\,\chi(G^{4k})\log\chi(G^{4k}).$$

The family $\{G^4\}$ is an infinite family of graphs whose kth powers have the desired properties. □

4. Upper bound

We now provide an upper bound on $\chi_\ell(G^k)$ in terms of $\chi(G^k)$. Recall that $f_k(m) = \max\{\chi_\ell(G^k) : \chi(G^k) = m\}$.

Theorem 4.1. *Let $k > 1$. If k is even, then $f_k(m) < m^2$. If k is odd, then $f_k(m) < m^3$.*

When k is even, this follows from Kwon's observation (see [6]) that it holds for $k = 2$. When k is odd, we generalize the argument and prove the following.

Theorem 4.2. *Let $k \ge 3$, k odd. Then for any G, $\chi_\ell(G^k) \le \Delta(G)\,\chi(G^k)^2$.*

Theorem 4.1 follows by noting that $\Delta(G) < \omega(G^k) \le \chi(G^k)$ when $k > 1$.

Proof of Theorem 4.2. Let x be a vertex with maximum degree in G^k. Let A be the set of vertices at distance $\lceil k/2\rceil$ from x in G. Let $B(v, r)$ denote the ball of radius r centered at v in G. Note that $\Delta(G^k) = \max\{|B(v, k)| - 1 : v \in V(G)\}$ and $\omega(G^k) \ge \max\{|B(v, \lfloor k/2\rfloor)| : v \in V(G)\}$.

Since k is odd (and bigger than 1), we have

$$B(x, k) \setminus B(x, \lfloor k/2\rfloor) \subseteq \bigcup_{y\in A} B(y, \lfloor k/2\rfloor). \tag{1}$$

Let S be the set of vertices at distance $\lfloor k/2\rfloor$ from x in G. Then S is a clique in G^k, so $|S| \le \omega(G^k)$. Also, A is contained in the neighborhood of S, and each vertex in S also has at least one neighbor outside of A (closer to x). Hence $|A| \le (\Delta(G)-1)|S| \le (\Delta(G)-1)\omega(G^k)$. So

$$\begin{aligned}\chi_\ell(G^k) &\le 1 + \Delta(G^k) && \text{(degeneracy)}\\ &= |B(x, k)|\\ &\le |B(x, \lfloor k/2\rfloor)| + \sum_{y\in A}|B(y, \lfloor k/2\rfloor)| && \text{(equation (1))}\\ &\le (1+|A|)\max_{v\in V(G)}|B(v, \lfloor k/2\rfloor)| && \text{(bounding terms in sum)}\\ &\le \left(1+(\Delta(G)-1)\omega(G^k)\right)\omega(G^k)\\ &\le \Delta(G)\,\omega(G^k)^2\\ &\le \Delta(G)\,\chi(G^k)^2. \end{aligned}$$

□

5. Remarks

Using constructions similar to that of Section 2, we have found infinite families of graphs G whose kth powers are complete multipartite on roughly $kn^2/4$ parts each of size n, but only when $k \not\equiv 0 \bmod 4$. The construction presented here is messier and does not yield complete multipartite powers, but it proves the theorem for all values of k simultaneously.

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References

[1] Noga Alon, Choice numbers of graphs: a probabilistic approach, Combin. Probab. Comput. 1 (2) (1992) 107–114.
[2] Charles J. Colbourn, Jeffrey H. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs, in: CRC Press Series on Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, 1996.
[3] Seog-Jin Kim, Young Soo Kwon, Boram Park, Chromatic-choosability of the power of graphs. arXiv:1309.0888.
[4] Seog-Jin Kim, Boram Park, Counterexamples to the list square conjecture, J. Graph Theory, in press. arXiv:1305.2566.
[5] Alexandr V. Kostochka, Douglas R. Woodall, Choosability conjectures and multicircuits, Discrete Math. 240 (1–3) (2001) 123–143.
[6] Jonathan A. Noel, Choosability of graph powers, August 2013. http://www.openproblemgarden.org/op/choosability_of_graph_powers.
[7] Xuding Zhu, personal communication.

Publications

- *Typical Structure of maximal triangle-free graphs*,
  J. Balogh, H. Liu, Š. Petříčková, M. Sharifzadeh,
  Forum of Mathematics, Sigma, 3(20), 2015.
- *The number of the maximal triangle-free graphs*,
  J. Balogh, Š. Petříčková,
  Bull. London Math. Soc., 46(5), 2014.
- *Coloring fractional powers of graphs*,
  S. Hartke, H. Liu, Š. Petříčková,
  accepted to JGT, arXiv:1212.3898.
- *A note on list-coloring powers of graphs*,
  N. Kosar, Š. Petříčková, B. Reiniger, E. Yeager,
  Discrete Mathematics, 332, 2014.
- *Online Ramsey Theory for Planar Graphs*,
  Š. Petříčková,
  Electron. J. Combin., 12(1), 2014.
- *On the periodic motion of a generalized oscillator in two dimensions*,
  Š. Petříčková, P. Nečesal,
  Journal of Interdisciplinary Mathematics, 12(2), 2009.