SIAM J. DISCRETE MATH.
Vol. 31, No. 2, pp. 890–907

ON DOMINATING EVEN SUBGRAPHS IN CUBIC GRAPHS*

ROMAN ČADA[†], SHUYA CHIBA[‡], KENTA OZEKI[§], AND KIYOSHI YOSHIMOTO[¶]

Abstract. It is known that a 3-edge-connected graph has a spanning even subgraph in which every component contains at least five vertices, and the lower bound is best possible. A natural question arises of whether we can improve the lower bound by changing the spanning property with the dominating property. In this paper, we show that a 3-edge-connected cubic graph has a dominating even subgraph in which every component contains at least six vertices.

Key words. even subgraph, dominating subgraph, 2-factor, cubic graph

AMS subject classifications. 05C38, 05C69

DOI. 10.1137/16M1066622

1. Introduction.

In this paper, we consider finite graphs without loops. An *even* graph is a graph in which every vertex has a positive even degree and a subgraph H of a graph G is said to be *dominating* if $G - V(H)$ is edgeless. In this paper, a *cycle* is a connected 2-regular graph and a cycle with l vertices is called an l-*cycle*. A 2-*factor* is a spanning 2-regular subgraph of a graph. An *edge-cut* is a minimal set of edges whose removal increases the number of components of the graph. We call an edge-cut with l edges an l-*cut*. An edge-cut is said to be *essential* if both of the two new components after deleting it have at least one edge.

For a vertex subset $X \subset V(G)$, the set of edges joining X and $V(G)-X$ is denoted by $\partial(X)$ or simply ∂X. If X consists of one vertex u, then we denote it simply by $\partial(u)$. For a subgraph H of G, we use ∂H instead of $\partial(V(H))$. For terminology and notation not defined in this paper, we refer the readers to [5].

In this paper we consider cubic graphs, i.e., 3-regular graphs. A classical result by Petersen [17] says that a bridgeless cubic graph has a 2-factor. This well-known result was generalized by Fleischner [10] as follows: a bridgeless graph with minimum degree at least three has a spanning even subgraph in which every component has at least three vertices. If we restrict ourselves to simple graphs, then the lower bound on the order of components is improved to four in [13]. Jackson and Yoshimoto considered 3-edge-connected graphs and showed the following.

THEOREM A (Jackson and Yoshimoto [14]). *A 3-edge-connected graph with n vertices has a spanning even subgraph in which each component contains at least $\min\{5, n\}$ vertices.*

---

*Received by the editors March 18, 2016; accepted for publication (in revised form) January 9, 2017; published electronically May 18, 2017.
http://www.siam.org/journals/sidma/31-2/M106662.html
Funding: The first author's research was partly supported by the project P202/12/G061 of the Czech Science Foundation and by the project LO1506 of the Czech Ministry of Education, Youth and Sports. The second author's work was supported by JSPS KAKENHI grants 26400187 and 26800083. The third author's work was in part supported by JSPS KAKENHI grant 25871053, by Grant for Basic Science Research Projects from the Sumitomo Foundation and by JST ERATO grant JPMJER1305, Japan. The fourth author's work was supported by JSPS KAKENHI grant 26400190.
[†]Department of Mathematics, University of West Bohemia, Univerzitni 8, 306 14 Pilsen, Czech Republic, Centre of Excellence IT, Institute for Theoretical Computer Science, Charles University, 116 36 Prague 1, Czech Republic, and European Centre of Excellence NTIS, New Technologies for the Information Society, Technicka 8, 301 00 Pilsen, Czech Republic (cadar@kma.zcu.cz).
[‡]Kumamoto University, Kumamoto 860-8555, Japan (schiba@kumamoto-u.ac.jp).
[§]Yokohama National University, Yokohama 240-8501, Japan (ozeki-kenta-xr@ynu.ac.jp).
[¶]Nihon University, Tokyo 101-8308, Japan (yoshimoto@math.cst.nihon-u.ac.jp).

They also gave an infinite family of 3-edge-connected cubic graphs in which every 2-factor contains 5-cycles. Thus the lower bound in the theorem is best possible in some sense. Kaiser and Škrekovski gave an interesting result, which also generalizes the Petersen's theorem.

THEOREM B (Kaiser and Škrekovski [15]). *Every graph has an even subgraph which intersects all* 3*-cuts and* 4*-cuts.*

If a given graph is bridgeless and cubic, then for any $u \in V(G)$, $\partial(u)$ is always a 3-cut of the graph, and so the above theorem implies the following.

COROLLARY C. *A bridgeless cubic graph has a* 2*-factor which intersects all* 3*-cuts and* 4*-cuts.*

If a given cubic graph is 3-edge-connected, then for an l-cycle C where $l \in \{3, 4\}$, ∂C is an l-cut of the graph, and so Corollary C implies that a 3-edge-connected cubic graph has a 2-factor in which every component contains at least five vertices.

Matthews and Sumner [16] conjectured that 4-connected claw-free graphs are Hamiltonian and Ryjáček [18] showed the Matthews–Sumner conjecture is equivalent to the conjecture by Thomassen and others [2, 4, 19] that 4-connected line graphs are Hamiltonian. Fleischner and Jackson [11] showed that the conjecture on line graphs, and so the Matthews–Sumner conjecture, is equivalent to the conjecture by Ash and Jackson [3] that an essentially 4-edge-connected cubic graph has a dominating cycle. Thus it is interesting and important to study the behavior of dominating subgraphs in cubic graphs. In this paper, we prove the following using Corollary C.

THEOREM 1.1. *A* 3*-edge-connected cubic graph has a Hamilton cycle or a dominating even subgraph F such that every component in F contains at least six vertices and F intersects all essential* 3*-cuts.*

In section 2, we give several preparations for the proof of Theorem 1.1, and in section 3, the proof will be given. Furthermore, we will give remarks on even subgraphs of 3-edge-connected cubic graphs and the traveling salesman problem in section 4.

We conjecture that Theorem 1.1 can be generalized as in Theorem A.

CONJECTURE 1.2. *A* 3*-edge-connected graph with n vertices has a dominating even subgraph in which each component has at least* $\min\{6, n\}$ *vertices.*

Also it is a natural question to ask about the lower bound "6" in Theorem 1.1 and Conjecture 1.2.

PROBLEM 1.3. *What is the maximum integer k such that any* 3*-edge-connected graph has a dominating even subgraph in which each component has at least* $\min\{k, n\}$ *vertices?*

The following example implies that the upper bound must be at most nine.

FACT 1.4. *There is an infinite family of* 3*-edge-connected cubic graphs in which every dominating even subgraph has a cycle of order at most nine.*

Proof. We construct such a cubic graph. Let S be the graph as in Figure 1, where S has 34 vertices, 49 edges, and 4 "half-edges" whose one end is in S. Later we define the other ends of the half-edges.

Let m and l be positive integers with $3m = 4l$. Let B be l copies of S and A be m mutually disjoint triangles with three half-edges incident to each vertex of the triangle. See Figure 2. Since A and B have $3m$ and $4l$ half-edges, respectively, and $3m = 4l$, we can pair up half-edges in A with half-edges in B. It is easy to pair them so that the obtained graph G is 3-edge-connected.

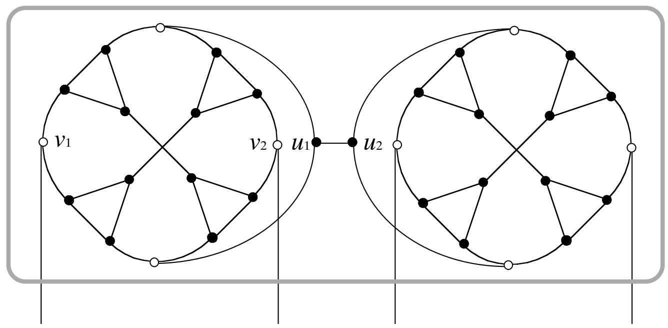

FIG. 1.

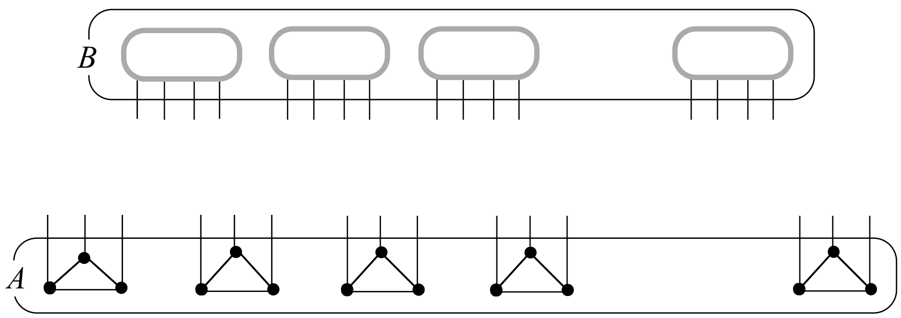

FIG. 2.

We show that any dominating even subgraph F in G has a cycle of order at most nine. If there is a triangle T in A such that $\partial T \cap F = \emptyset$, then obviously F contains the 3-cycle T as a component. Suppose $\partial T \cap F \neq \emptyset$ for all triangles T in A. Since F is an even subgraph, $|\partial T \cap F| = 2$ for all triangles T in A, and so F contains $2m$ edges joining A and B. Since $2m = 8l/3$, there is a component S in B such that $|\partial S \cap F| = 4$.

Let u_1u_2 be the edge in the middle of S; see Figure 1. Since F is a dominating subgraph in G, at least one of the vertices u_1 and u_2 is contained in F, say, u_1. Let L_S be the left component of $S - u_1u_2$. Since F is an even subgraph, we have $|\partial L_S \cap F| = 2$, and hence $u_1u_2 \notin F$ and $\partial(u_1) - \{u_1u_2\} \subset F$. Note that both v_1 and v_2 are contained in F. Then by the structure of L_S, it is an easy observation that v_1 and v_2 belong to the same component of F which is different from the one containing u_1, and furthermore the component containing u_1 has at most nine vertices. This completes the proof of Fact 1.4. ∎

The following question is also natural.

PROBLEM 1.5. *Does a* 3*-edge-connected cubic graph have a dominating even subgraph* F *such that every component in* F *contains at least six vertices and* F *intersects all essential* 3*-cuts and* 4*-cuts?*

2. Preparations. First, we give some additional notation. The set of all the neighbors of a vertex $x \in V(G)$ is denoted by $N_G(x)$ or simply $N(x)$ and its cardinality by $d_G(x)$ or $d(x)$. For a subgraph H of G, we denote $N_G(x) \cap V(H)$ by $N_H(x)$ and its cardinality by $d_H(x)$. For simplicity, we denote $|V(H)|$ by $|H|$ and "$u_i \in V(H)$" by "$u_i \in H$." Similarly $G - V(H)$ is denoted by $G - H$.

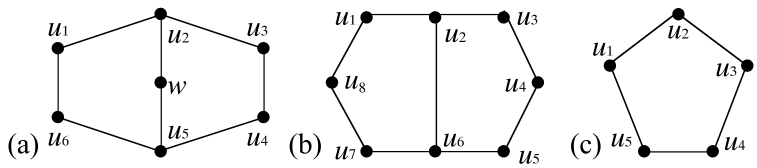

Fig. 3.

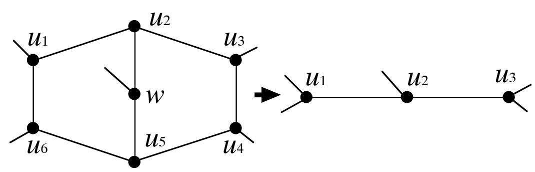

Fig. 4. *Reduction of* 2*-cell.*

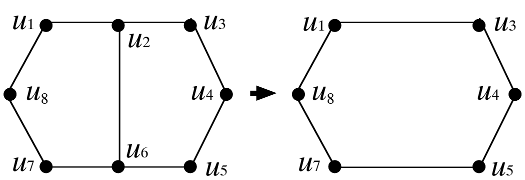

Fig. 5. *Reduction of* 1*-cell.*

Recall that an edge-cut is said to be *essential* if both of the two new components after deleting it have at least one edge. This definition directly implies the following fact, which will be implicitly used in our proofs.

Fact 2.1. *For a* 2*-edge-connected cubic graph* G *and for a* k*-cut* T*, both of the following hold:*

- *If* $k = 2$*, then* T *is always an essential cut.*
- *If* $k = 3$ *and* $T = \partial S$ *for some* $S \subseteq V(G)$ *with* $|S| \geq 2$ *and* $|V(G) - S| \geq 2$*, then* T *is an essential cut.*

An *i-cell* is the union of two 5-cycles in a cubic graph which have i common edges. See Figure 3(a), (b). We call a 5-cycle a 0*-cell*. In the proof of Theorem 1.1, we will construct a dominating even subgraph from a 2-factor of a cubic graph which is obtained by reducing those cells. Hence we define reductions for those cells first.

Let D be a 2-cell in G, and let $u_1u_2 \cdots u_6u_1$ be the 6-cycle and w the remaining vertex in D. See Figure 4. Let G' be the graph obtained from G by contracting all of the paths u_1u_6, u_2wu_5, u_3u_4 and removing the edges u_6u_5 and u_5u_4. We denote this reduction by $G' = G|D$.

Let D be a 1-cell in G and $u_1u_2 \cdots u_8u_1$ the 8-cycle of D. See Figure 5. Let G' be the graph obtained from G by removing the edge u_2u_6 and contracting both of the edges u_1u_2 and u_7u_6. We denote by $G' = G|D$ this reduction.

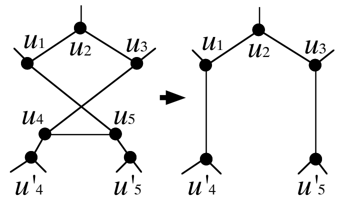

FIG. 6. *Reduction of* 5*-cycle.*

Let $D = u_1 \cdots u_5 u_1$ be a 5-cycle without chord. Let $u_i' \in V(G - C)$ which is adjacent to u_i for $1 \le i \le 5$. See Figure 6. Let G' be the graph obtained from G by removing the edges u_1u_5, u_5u_4, u_4u_3 and identifying u_1, u_4 and u_3, u_5, respectively. We denote by $G' = G|_{u_2}D$ this reduction.

We say that a 5-cycle C is *good in* G if there is an essential 3-cut T in G such that $|T \cap \partial C| \ge 2$. If C has exactly one chord and $|G| \ge 8$, then C is always good because ∂C is an essential 3-cut. If a 2- or 1-cell contains a good 5-cycle, then the cell is also called *good*. A cell which is not good is called *bad*. Notice that in a bad cell, every 5-cycle is bad.

We need the following fact in the proof of Theorem 1.1.

FACT 2.2. *Let $i \in \{2, 1, 0\}$. If a 3-edge-connected cubic graph G has a bad i-cell D, then $G|D$ or $G|_{u_2}D$ is 3-edge-connected.*

This fact is obtained from the following two lemmas.

LEMMA 2.3. *Let $D = u_1u_2u_3u_4u_5u_1$ be a 5-cycle of a 3-edge-connected cubic graph. If there is an essential 3-cut T such that $T \cap E(D) \ne \emptyset$, then D is good.*

Proof. Suppose D is bad and there is an essential 3-cut T such that $T \cap E(D) \ne \emptyset$. Since D is bad and G is 3-edge-connected, D has no chord. Let $u_i' \in N_{G-C}(u_i)$ for $1 \le i \le 5$. Since T is an essential edge-cut of a cubic graph and G is 3-edge-connected, no pair of edges in T is adjacent, and so $T \cap E(D)$ contains two independent edges, say, u_1u_2, u_4u_5. Then $(T - \{u_1u_2, u_4u_5\}) \cup \{u_1u_1', u_5u_5'\}$ is an essential 3-cut containing two edges in ∂D, a contradiction. ∎

LEMMA 2.4. *For $k \in \{2, 3\}$ and a k-edge-connected cubic graph G, the following hold:*

1. *Let D be a 2-cell and $u_1u_2 \cdots u_6u_1$ be the 6-cycle in D. See Figure 4. If $G|D$ is not k-edge-connected, then G has an essential k-cut containing $\{u_1u_2, u_5u_6\}$ or $\{u_2u_3, u_4u_5\}$.*
2. *Let D be a 1-cell and $u_1u_2 \cdots u_8u_1$ be the 8-cycle in D. See Figure 5. If $G|D$ is not k-edge-connected, then G has an essential k-cut containing $\{u_1u_8, u_4u_5, u_2u_6\}$ or $\{u_3u_4, u_7u_8, u_2u_6\}$.*
3. *Let $D = u_1u_2 \cdots u_5u_1$ be a 5-cycle and u_j' be the vertex in $G - D$ which is adjacent to u_j for $1 \le j \le 5$. See Figure 6. If $G|_{u_2}D$ is not k-edge-connected, then G has an essential k-cut containing $\{u_1u_1', u_4u_4'\}$ or $\{u_3u_3', u_5u_5'\}$.*

Proof. Let $G' = G|D$ or $G' = G|_{u_2}D$, respectively, T be a minimum edge-cut of G', and D' be the subgraph in G' corresponding to D. Let $S \subset V(G')$ such that $\partial S = T$ and $u_1 \in S$. Suppose $|T| \le k - 1$. Since G is k-edge-connected, T is not an edge-cut of G, and so $T = \partial S$ divides D'. For a vertex $u \in V(D)$, we denote a vertex in $G - D$ adjacent to u by u' if it exists.

1. Since $|T \cap D'| = 1$, by symmetry, we may suppose $T \cap D' = \{u_1u_2\}$. Since T is a minimum cut, no pair of edges in T is adjacent, and so $\{u'_1, u'_6\} \subset S$. Thus $\partial(S \cup \{u_6\})$ is an essential k-cut containing $\{u_1u_2, u_5u_6\}$ of G.
2. Since T divides $V(D')$, $|T \cap D'| = 2$. By symmetry, we have four cases.
   If $T \cap D' = \{u_1u_3, u_5u_7\}$, then ∂S is also a $(k-1)$-cut of G, a contradiction.
   If $T \cap D' = \{u_1u_3, u_4u_5\}$, then $\partial(S \cup \{u_2, u_6\})$ is a $(k-1)$-cut of G, a contradiction.
   If $T \cap D' = \{u_1u_8, u_3u_4\}$, then $\partial(S - \{u_1, u_3\})$ is a $(k-1)$-cut of G, a contradiction.
   If $T \cap D' = \{u_1u_8, u_4u_5\}$, then since $\{u_1, u_3\} \subset S$, $\partial(S \cup \{u_2\})$ is an essential k-cut containing $\{u_1u_8, u_4u_5, u_2u_6\}$ of G.
3. Since $T = \partial S$ divides D', $|T \cap D'| = 1$. By symmetry, we may suppose $T \cap D' = \{u_1u_2\}$. Since $\{u'_1, u'_4\} \subset S$ and G is k-edge-connected, $\partial(S - u_1)$ is an essential k-cut containing $\{u_1u'_1, u_4u'_4\}$ of G. □

Proof of Fact 2.2. If D is a 5-cycle and $G|_{u_2}D$ is not 3-edge-connected, then by Lemma 2.4, D is good. If D is a 2- or 1-cell and $G|D$ is not 3-edge-connected, then there exist a 5-cycle C in D and an essential 3-cut T of G such that $|D \cap T| \geq 2$ by Lemma 2.4. Thus by Lemma 2.3, D is good. □

3. Proof of Theorem 1.1. Let G be a 3-edge-connected cubic graph. We may assume that G is not Hamiltonian; otherwise we are done. First we define a sequence of bad cells in G which will be reduced.

Let

$$\mathcal{D}_1 = \{D_1, D_2, \ldots, D_p\}$$

be a maximal set of mutually disjoint 2-cells in G such that D_{i+1} is bad in G_i for each $0 \leq i \leq p-1$, where $G_0 = G$ and $G_i = G_{i-1}|D_i$ for $1 \leq i \leq p$. If there is no bad 2-cell in G, then we define $\mathcal{D}_1 = \emptyset$ and $p = 0$. We denote the subgraph in G_i corresponding to D_i by D'_i. See Figure 7. Notice that $G - \bigcup_{l \leq i} D_l = G_i - \bigcup_{l \leq i} D'_l$ and, by Fact 2.2, each G_i is 3-edge-connected for every $0 \leq i \leq p$. By the maximality of $\mathcal{D}_1$, obviously the following claim holds.

CLAIM 3.1. *There is no* 2*-cell in* $G - \bigcup_{l \leq p} D_l = G_p - \bigcup_{l \leq p} D'_l$ *which is bad in* G_p.

Let

$$\mathcal{D}_2 = \{D_{p+1}, D_{p+2}, \ldots, D_{p+q}\}$$

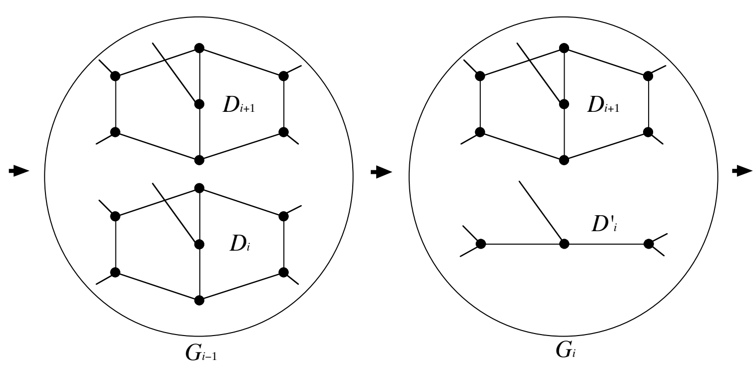

FIG. 7.

be a maximal set of mutually disjoint 1-cells in $G - \bigcup_{1\le l\le p} D_l$ such that D_{i+1} is bad in G_i for each $p \le i \le p+q-1$, where $G_i = G_{i-1}|D_i$ for $p+1 \le i \le p+q$. If there is no bad 1-cell in $G - \bigcup_{1\le l\le p} D_l$, then we define $\mathcal{D}_2 = \emptyset$ and $q = 0$. The subgraph in G_i corresponding to D_i is denoted by D'_i. In this case also, $G - \bigcup_{l\le i} D_l = G_i - \bigcup_{l\le i} D'_l$ and, by Fact 2.2, each G_i is 3-edge-connected for any $0 \le i \le p+q$.

CLAIM 3.2. *There is no 1-cell in $G - \bigcup_{l\le p+q} D_l = G_{p+q} - \bigcup_{l\le p+q} D'_l$ which is bad in G_{p+q} and there is no 2-cell C in $G_{p+j} - \bigcup_{i\le p+j} D'_i$ which is bad in G_{p+j} for any $0 \le j \le q$.*

Proof. By the maximality of $\mathcal{D}_2$, we have the first statement. If there is $1 \le j \le q$ such that $G_{p+j} - \bigcup_{i\le p+j} D'_i$ contains a 2-cell C which is bad in G_{p+j}, then obviously C is bad in G_{p+j-1} also, and so C is bad in G_p. This contradicts Claim 3.1. ◻

Let $\mathcal{D}_0$ be a maximal set of mutually disjoint bad 5-cycles in $G - \bigcup_{i\le p+q} D_i$. For $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_0$, we define a vertex subset R^* of G, whose vertices may not be contained in a dominating even subgraph of G which is constructed later.

First, for each cell D_i in $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_0$, we define pairs of vertices in D_i.

1. Let D_i be a 2-cell in $\mathcal{D}_1$ and $u_1u_2u_3u_4u_5u_6u_1$ be the 6-cycle in D_i. See Figure 3(a). The pairs of D_i are $\{u_1, u_3\}$ and $\{u_4, u_6\}$.
2. Let D_i be a 1-cell in $\mathcal{D}_2$ and $u_1u_2 \cdots u_8u_1$ be the 8-cycle in D_i. See Figure 3(b). We define the pair of D_i by $\{u_8, u_4\}$.
3. For a 5-cycle $D_i = u_1u_2 \cdots u_5u_1$ in $\mathcal{D}_0$, the pair is defined by arbitrary two adjacent vertices in D_i, e.g., $\{u_1, u_2\}$. See Figure 3(c).

Let $\mathcal{P}_0$ be the set of all the pairs for all $D_l \in \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_0$. For each pair $\{u_i, u_j\} \in \mathcal{P}_0$, let $E_{u_i,u_j} = \partial(\{u_i, u_j\}) \cap \partial D_l$, where $\{u_i, u_j\} \subset D_l \in \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_0$. Obviously $0 \le |E_{u_i,u_j}| \le 2$. Let

$$\mathcal{P} = \{\{u_i, u_j\} \in \mathcal{P}_0 : |E_{u_i,u_j}| = 2\} \text{ and } \mathcal{Q} = \bigcup_{\{u_i,u_j\}\in\mathcal{P}} E_{u_i,u_j}.$$

We define a bipartite graph H on the partite sets $\mathcal{P}$ and $\mathcal{Q}$ by defining the adjacency relation so that $\{u_i, u_j\} \in \mathcal{P}$ and $e \in \mathcal{Q}$ are adjacent if and only if $e \in E_{u_i,u_j}$. Since each element in $\mathcal{Q}$ is adjacent to at most two pairs in $\mathcal{P}$, for any $\mathcal{S} \subset \mathcal{P}$,

$$2|\mathcal{S}| = |E_H(\mathcal{S}, N(\mathcal{S}))| \le |E_H(N(\mathcal{S}), \mathcal{P})| \le 2|N(\mathcal{S})|.$$

Thus by Hall's theorem, there is a matching M in H covering $\mathcal{P}$. Let $\varphi : \mathcal{P} \to \mathcal{Q}$ be the injection defined by M, i.e., for each $\{u_i, u_j\} \in \mathcal{P}$, the pair is adjacent to $\varphi(\{u_i, u_j\}) \in \mathcal{Q}$ by M. Let

$$R^* = \{u_k : u_k \in \{u_i, u_j\} \text{ is the end of the edge } \varphi(\{u_i, u_j\}) \text{ for some } \{u_i, u_j\} \in \mathcal{P}\}.$$

Notice that there are no edges connecting two vertices in R^*, except for those connecting u_1 and u_6 or connecting u_3 and u_4 for some 2-cell in $\mathcal{D}_1$.

Let

$$\mathcal{D}_3 = \{D_{p+q+1}, D_{p+q+2}, \ldots, D_{p+q+r}\} \subset \mathcal{D}_0$$

be a maximal subset of $\mathcal{D}_0$ such that for $p+q \le i \le p+q+r-1$,

1. D_{i+1} is bad in G_i, where $u_i \in R^* \cap D_i$, and $G_i = G_{i-1}|_{u_i}D_i$, and
2. D_{i+1} intersects neither the 3-cycle nor the 4-cycle in G_i.

If there is no such 5-cycle, then we define $\mathcal{D}_3 = \emptyset$ and $r = 0$. We denote by D'_i the subgraph in G_i corresponding to $D_i \in \mathcal{D}_3$. Notice that each D_i has no chord since D_i is bad in G_{i-1} and each G_i is 3-edge-connected for any $p+q \le i \le p+q+r$ by Fact 2.2.

Claim 3.3. *If there exist $0 \le j \le r-1$ and $i \in \{2,1,0\}$ such that $G_{p+q+j} - \bigcup_{l \le p+q+j} D'_l - D_{p+q+j+1}$ contains an i-cell C which is good in G_{p+q+j}, then C is good in $G_{p+q+j+1}$ also.*

Proof. Since C is good in G_{p+q+j}, there exist an essential 3-cut T of G_{p+q+j} and a 5-cycle C_1 in C such that $|T \cap \partial C_1| \ge 2$. Let H_1 and H_2 be the two components in $G_{p+q+j} - T$. Since $C \subset G_{p+q+j} - \bigcup_{i \le p+q+j} D'_i - D_{p+q+j+1}$, the 5-cycle $D_{p+q+j+1}$ is contained in $H_1 - C$ or $H_2 - C$, and so both T and C exist in $G_{p+q+r+1}$ also. Thus C is good in $G_{p+q+r+1}$. ◻

Claim 3.4.

1. *If $G_{p+q+j} - \bigcup_{i \le p+q+j} D'_i$ has a bad 5-cycle, then the 5-cycle is bad in G_{p+q} also.*
2. *There is neither a 2- nor a 1-cell C in $G_{p+q+j} - \bigcup_{i \le p+q+j} D'_i$ which is bad in G_{p+q+j} for any $0 \le j \le r$.*

Proof. Claim 3.3 implies the first statement immediately. By Claim 3.3, if there is $0 \le j \le r$ such that $G_{p+q+j} - \bigcup_{i \le p+q+j} D'_i$ contains a 2- or 1-cell C which is bad in G_{p+q+j}, then C is bad in $G_{p+q+j-1}$, and so C is bad in G_{p+q}. This contradicts Claim 3.2. ◻

Let

$$S_0 = \emptyset \text{ and } S_i = \bigcup_{1 \le l \le i} D'_l$$

for $1 \le i \le p+q+r$. We call a vertex in S_i a *yellow* vertex.

We extend R^*. Let $D_i \in \mathcal{D}_1$ and $u_1u_2u_3u_4u_5u_6u_1$ be the 6-cycle. See Figure 3(a). We define $R(D_i) = \{u_2, u_5\}$. Let $D_i \in \mathcal{D}_2$ and $u_1u_2u_3u_4u_5u_6u_7u_8u_1$ be the 8-cycle. See Figure 3(b). We define $R(D_i) = \{u_2, u_6\}$. Let

$$R_0 = R^* \cup \bigcup_{1 \le l \le p+q} R(D_l) \ (\subset V(G))$$

and

$$R_i = R_0 - \bigcup_{1 \le l \le i} V(D_l) \ (\subset V(G_i))$$

for $1 \le i \le p+q+r$. We call a vertex in R_i a *red* vertex. By the definition of S_i and R_i, there is no vertex in G_i which is both red and yellow. Notice that

$$(1) \qquad R_i = \begin{cases} R_{i+1} \cup (R^* \cap D_{i+1}) \cup R(D_{i+1}) & \text{for } 0 \le i \le p+q-1, \\ R_{i+1} \cup (R^* \cap D_{i+1}) & \text{for } p+q \le i \le p+q+r-1. \end{cases}$$

For Theorem 1.1, it is enough to show that for all $0 \le i \le p+q+r$, G_i has an even subgraph F_i such that the following hold:

1. F_i intersects all essential 3-cuts in G_i.
2. Every component of F_i contains at least five vertices. Especially if a 5-cycle C is a component of F_i, then C contains a yellow vertex, i.e., $C \cap S_i \ne \emptyset$.
3. Every vertex in $G_i - F_i$ is red, i.e., $G_i - F_i \subset R_i$, and $|D_l - F_i| \le 1$ for $i+1 \le l \le p+q+r$.

Indeed by the second condition, any component in F_0 contains at least six vertices as $S_0 = \emptyset$. Although $(R^* \cap V(D_i)) \cup R(D_i) \subset R_0$ $(1 \le i \le p)$ and $R(D_i) \subset R_0$ $(p+1 \le i \le p+q)$ may not be independent, since $|D_l - F_0| \le 1$ for all $1 \le l \le p+q+r$, we have $G_0 - F_0(\subset R_0)$ is independent, i.e., F_0 is dominating $G = G_0$. Therefore F_0 is a desired even subgraph.

We construct F_i inductively. First we show the existence of F_{p+q+r}. Notice that G_i is 3-edge-connected for every $0 \leq i \leq p+q+r$ by Fact 2.2 as we reduced bad cells.

Claim 3.5. *There is a 2-factor F_{p+q+r} in G_{p+q+r} such that*

1. *F_{p+q+r} intersects all 3-cuts and 4-cuts in G_{p+q+r} and*
2. *each 5-cycle C in F_{p+q+r} contains a yellow vertex, i.e., $C \cap S_{p+q+r} \neq \emptyset$.*

Proof. Since G_{p+q+r} is a 3-edge-connected cubic graph, by Corollary C, G_{p+q+r} has a 2-factor F_{p+q+r} which intersects all 3-cuts and 4-cuts. We choose F_{p+q+r} such that the number of components is as small as possible.

Suppose F_{p+q+r} contains a 5-cycle C without a yellow vertex. If C is good in G_{p+q+r}, then there is an essential 3-cut T such that $|T \cap \partial C| \geq 2$, and F_{p+q+r} does not intersect the 3-cut T, a contradiction. Therefore C is bad in G_{p+q+r}. Since C has no yellow vertex, $C \subset G_{p+q+r} - S_{p+q+r}$, and so the 5-cycle C exists in G_{p+q} and, by Claim 3.4, C is bad in G_{p+q} also.

Suppose $C \notin \mathcal{D}_0$. By the maximality of $\mathcal{D}_0$, there is a bad 5-cycle $D \in \mathcal{D}_0$ intersecting C. If $|E(C \cap D)| \leq 2$, then $C \cup D$ is a 2- or 1-cell in G_{p+q}. Since both C and D are bad in G_{p+q}, $C \cup D$ is bad in G_{p+q}. This contradicts Claim 3.2.

If $|E(C \cap D)| = 3$, then D is a 5-cycle in G_{p+q+r} also. However, F_{p+q+r} does not contain the vertex in $D - C$ as C is a component of F_{p+q+r}. This is a contradiction.

Therefore $C \in \mathcal{D}_0$. Since C is bad in G_{p+q+r} and $C \notin \mathcal{D}_3$, C intersects a 3- or a 4-cycle C_1 in G_{p+q+r}. Since C is bad, C has no chord, and so $C_1 - C \neq \emptyset$. If $C_1 - C$ is a vertex w, then w is not contained in F_{p+q+r} as C is a component of the 2-factor F_{p+q+r}. This is a contradiction.

If $C_1 - C$ contains an edge ww', then there is a component C_2 in F_{p+q+r} containing the edge ww'. Since the symmetric difference $\widetilde{C} = C \triangle C_1 \triangle C_2$ is a cycle, the subgraph $(F_{p+q+r} - C \cup C_2) \cup \widetilde{C}$ is a 2-factor of G_{p+q+r} in which the number of components is less than F_{p+q+r}. This contradicts the choice of F_{p+q+r}. Thus C contains a yellow vertex. ∎

Suppose G_{i+1} has a desired even subgraph F_{i+1} for $1 \leq i+1 \leq p+q+r$. Since G_{i+1} has no vertex which is both yellow and red and F_{i+1} contains every vertex which is not red in G_{i+1}, we have $\bigcup_{l \leq i+1} D'_l \subset F_{i+1}$.

Claim 3.6. *If F_i is an even subgraph of G_i obtained from F_{i+1} by replacing edges in D'_{i+1} with edges in D_{i+1}, i.e.,*

$$E(F_{i+1}) - E(D'_{i+1}) = E(F_i) - E(D_{i+1}),$$

then the following holds:

1. *F_i intersects all essential 3-cuts in G_i.*
2. *Every component C of F_i intersecting no edge of D_{i+1} contains at least five vertices. Especially if C is a 5-cycle, then C contains a yellow vertex, i.e., $C \cap S_i \neq \emptyset$.*
3. *Every vertex in $G_i - F_i - D_{i+1}$ is red, i.e., $G_i - F_i - D_{i+1} \subset R_i$, and $|D_j - F_i| \leq 1$ for $j \geq i+2$.*

Proof.

1. Let T be any essential 3-cut of G_i. Since D_{i+1} is bad in G_i, $T \cap E(D_{i+1}) = \emptyset$ by Lemma 2.3. This implies T is an essential 3-cut of G_{i+1} and $T \cap E(D'_{i+1}) = \emptyset$. Thus $F_{i+1} - E(D'_{i+1}) = F_i - E(D_{i+1})$ intersects T.

2. Obviously C is a component of F_{i+1} also. Thus $|C| \geq 5$ and C contains a yellow vertex in $S_{i+1} - V(D'_{i+1}) = S_i$ if $|C| = 5$.
3. Since $G_i - D_{i+1} = G_{i+1} - D'_{i+1}$, we have $G_i - D_{i+1} - F_i = G_{i+1} - D'_{i+1} - F_{i+1} \subset R_{i+1} \subset R_i$ by (1). For $j \geq i+2$,

$$D_j - F_i = D_j - F_{i+1},$$

and so we have $|D_j - F_i| = |D_j - F_{i+1}| \leq 1$. □

In the remaining part of this paper, we will construct a desired even subgraph F_i of G_i from F_{i+1} by replacing edges in D'_{i+1} with edges in D_{i+1}. From the above claim, it is enough to show that F_i satisfies the following:

A1. Every component C containing an edge of D_{i+1} in F_i contains at least five vertices. Especially if C is a 5-cycle, then C contains a yellow vertex, i.e., $C \cap S_i \neq \emptyset$.

A2. A vertex in $D_{i+1} - F_i$ is red, i.e., $D_{i+1} - F_i \subset R_i$, and $|D_{i+1} - F_i| \leq 1$.

We divide our argument into the following three cases:
1. $0 \leq i \leq p-1$.
2. $p+q \leq i \leq p+q+r-1$.
3. $p \leq i \leq p+q-1$.

The first case is easier than the other cases. If there is a vertex in $G_i - D_{i+1}$ which is adjacent to $u \in D_{i+1}$, then we denote the vertex by u'.

1. $0 \leq i \leq p-1$, i.e., $D_{i+1} \in \mathcal{D}_1$.

Since $V(D'_{i+1}) \subset F_{i+1}$ and F_{i+1} is an even subgraph, $|F_{i+1} \cap \partial D'_{i+1}|$ is 2 or 4. If the subgraph induced by $V(D_{i+1})$ contains an edge that is not in $E(D_{i+1})$, then $|\partial D_{i+1}| = 3$. This implies D_{i+1} contains a good 5-cycle, i.e., D_{i+1} is good in G_i. This contradicts our assumption. Therefore, both $\{u_1, u_3\}$ and $\{u_4, u_6\}$ contain a red vertex.

Case 1. $|F_{i+1} \cap \partial D'_{i+1}| = 4$.

Since $D'_{i+1} \subset F_{i+1}$, by symmetry, we may suppose

$$F_{i+1} \cap \partial D'_{i+1} = \{u_1u'_6, u_1u'_1, u_2w', u_3u'_3\},$$

and then F_{i+1} contains the edge u_2u_3. See Figure 8. Let F_i be the even subgraph obtained from F_{i+1} by replacing

$$u'_1u_1u'_6 \text{ and } w'u_2u_3u'_3 \text{ by } u'_1u_1u_6u'_6 \text{ and } w'wu_5u_4u_3u'_3.$$

Since every component C containing an edge of D_{i+1} in F_i contains at least six vertices, A1 holds. Since $u_2 \in R(D_{i+1}) \subset R_i$, A2 holds.

Case 2. $|F_{i+1} \cap \partial D'_{i+1}| = 2$.

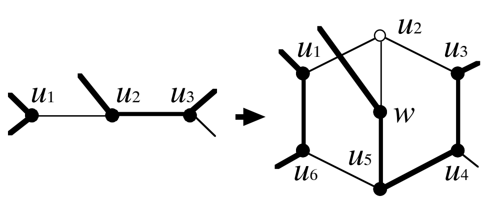

Fig. 8.

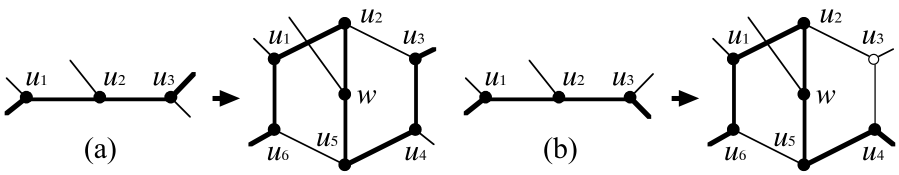

FIG. 9.

Since $D'_{i+1} \subset F_{i+1}$, $F_{i+1} \cap \partial D'_{i+1}$ does not contain the edge u_2w'. Thus by symmetry, we may suppose

$$F_{i+1} \cap \partial D'_{i+1} \text{ is } \{u_1u'_6, u_3u'_3\} \text{ or } \{u_1u'_6, u_3u'_4\}.$$

See Figure 9. If the intersection is $\{u_1u'_6, u_3u'_3\}$, then the even subgraph F_i obtained from F_{i+1} by replacing

$$u_1u_2u_3 \text{ by } u_6u_1u_2wu_5u_4u_3$$

is a desired even subgraph because both A1 and A2 hold as $V(D_{i+1}) \subset F_i$.

Suppose $F_{i+1} \cap \partial D'_{i+1} = \{u_1u'_6, u_3u'_4\}$. For the pair $\{u_1, u_3\}$, if u_3 is red, i.e., $u_3 \in R_i$, then the even subgraph F_i obtained from F_{i+1} by replacing

$$u'_6u_1u_2u_3u'_3 \text{ by } u'_6u_6u_1u_2wu_5u_4u'_4$$

is a desired even subgraph because both A1 and A2 hold. Similarly if $u_1 \in R_i$, then the even subgraph F_i obtained from F_{i+1} by replacing

$$u'_6u_1u_2u_3u'_4 \text{ by } u'_6u_6u_5wu_2u_3u_4u'_4$$

is a desired even subgraph.

2. $p + q \le i \le p + q + r - 1$, i.e., $D_{i+1} \in \mathcal{D}_3$.

In this case, $D_{i+1} = u_1u_2 \cdots u_5u_1$ is a 5-cycle. By symmetry, we may suppose u_2 is red, i.e., $u_2 \in R_i$. Since $V(D'_{i+1}) \subset F_{i+1}$, $|F_{i+1} \cap \partial D'_{i+1}|$ is 2 or 4.

Case 1. $|F_{i+1} \cap \partial D'_{i+1}| = 2$.

Notice that $F_{i+1} \cap \partial D'_{i+1}$ does not contain $u_2u'_2$ because $V(D'_{i+1}) \subset F_{i+1}$. Hence by symmetry, we have the following three cases:

$$F_{i+1} \cap \partial D'_{i+1} \text{ is } \{u_1u'_4, u_3u'_5\}, \{u_1u'_4, u_3u'_3\} \text{ or } \{u_1u'_1, u_3u'_3\}.$$

(i) Suppose $F_{i+1} \cap \partial D'_{i+1} = \{u_1u'_4, u_3u'_5\}$. See Figure 10(a). As $V(D'_{i+1}) \subset F_{i+1}$, F_{i+1} contains the path $u'_4u_1u_2u_3u'_5$. Let F_i be the even subgraph in G_i which is obtained from F_{i+1} by replacing

$$u'_4u_1u_2u_3u'_5 \text{ by } u'_4u_4u_3u_2u_1u_5u'_5.$$

Obviously A1 holds. Since $V(D_{i+1}) \subset F_i$, A2 holds.

(ii) Suppose $F_{i+1} \cap \partial D'_{i+1} = \{u_1u'_4, u_3u'_3\}$. See Figure 10(b). Then F_{i+1} contains the path $u'_4u_1u_2u_3u'_3$ as $V(D'_{i+1}) \subset F_{i+1}$. Let F_i be the even subgraph in G_i which is obtained from F_{i+1} by replacing

$$u'_4u_1u_2u_3u'_3 \text{ by } u'_4u_4u_5u_1u_2u_3u'_3.$$

Obviously both A1 and A2 hold.

(iii) Suppose $F_{i+1} \cap \partial D'_{i+1} = \{u_1u'_1, u_3u'_3\}$. See Figure 10(c). Let F_i be the even subgraph in G_i which is obtained from F_{i+1} by replacing

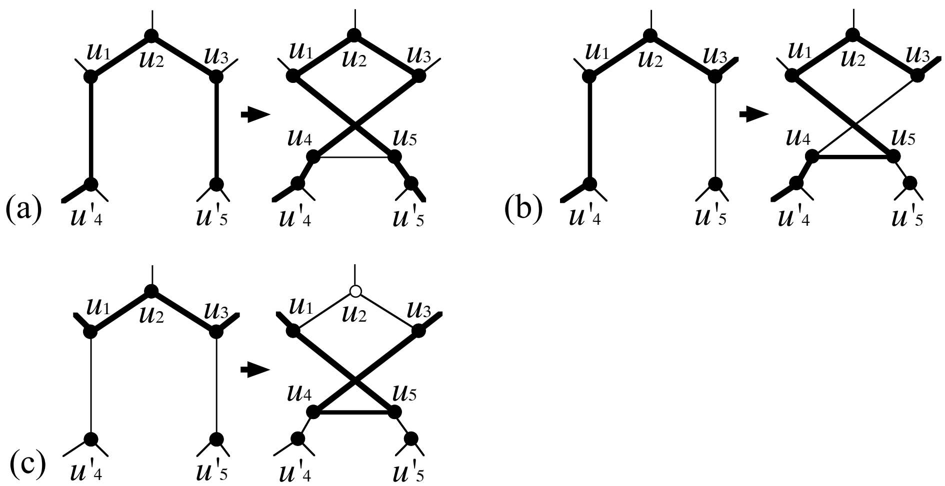

FIG. 10.

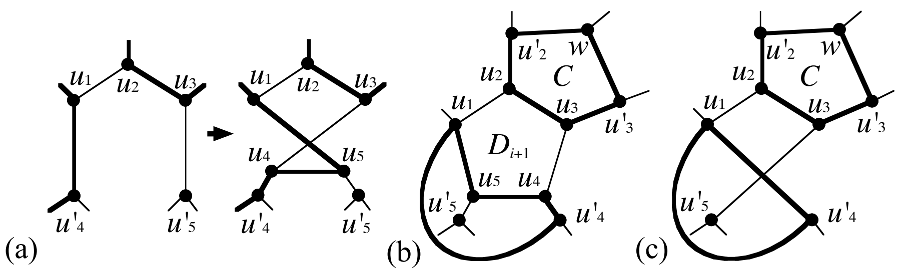

FIG. 11.

$$u_1'u_1u_2u_3u_3' \text{ by } u_1'u_1u_5u_4u_3u_3'.$$

Since the component in F_i containing an edge in D_{i+1} contains at least six vertices, A1 holds. As $D_{i+1} - F_i = \{u_2\} \subset R_i$, A2 holds.

Case 2. $|F_{i+1} \cap \partial D'_{i+1}| = 4$.

As $V(D'_{i+1}) \subset F_{i+1}$, $F_{i+1} \cap \partial D'_{i+1}$ is not $\{u_1u_4', u_1u_1', u_3u_3', u_3u_5'\}$. Thus by symmetry, we have two cases.

(i) Suppose

$$F_{i+1} \cap \partial D'_{i+1} = \{u_1u_4', u_1u_1', u_2u_2', u_3u_3'\},$$

and then $u_2u_3 \in F_{i+1}$. See Figure 11(a). Let F_i be the even subgraph in G_i which is obtained from F_{i+1} by replacing

$$u_4'u_1u_1' \text{ by } u_4'u_4u_5u_1u_1'.$$

Since $V(D_{i+1}) \subset F_i$, A2 holds.

The component continuing $u_4'u_4u_5u_1u_1'$ of F_i contains at least six vertices. Suppose $C = u_2u_3u_3'wu_2'u_2$ is a 5-cycle and $C \cap S_i = \emptyset$. See Figure 11(b). Then $\widetilde{C} = C \cup D_{i+1}$ is a 1-cell in $G_i - \bigcup_{j \le i} D'_j$. Since $D_{i+1} \in \mathcal{D}_3$, D_{i+1} is bad in G_i. Suppose that C is good and let $\overline{T}$ be an essential 3-cut such that

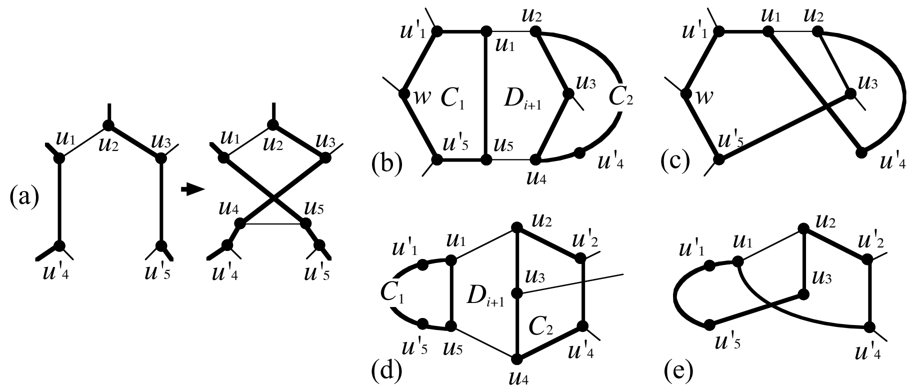

FIG. 12.

$|T \cap \partial C| \geq 2$. Since D_{i+1} is bad, $T \cap \partial C \subset \partial C - \{u_2u_1, u_3u_4\}$ by Lemma 2.3. Since T is an essential 3-cut of G_{i+1} also and F_{i+1} contains C as a component, F_{i+1} does not intersect T, a contradiction. See Figure 11(c). Thus both D_{i+1} and C are bad, and so $\widetilde{C}$ is a bad 1-cell in G_i. This contradicts Claim 3.4.

(ii) Suppose

$$F_{i+1} \cap \partial D'_{i+1} = \{u_1u'_4, u_1u'_1, u_2u'_2, u_3u'_5\},$$

and then $u_2u_3 \in F_{i+1}$. See Figure 12(a).

Let F_i be the even subgraph in G_i which is obtained from F_{i+1} by replacing

$$u'_4u_1u'_1 \text{ and } u'_2u_2u_3u'_5 \text{ by } u'_4u_4u_3u_2u'_2 \text{ and } u'_1u_1u_5u'_5.$$

Since $V(D_{i+1}) \subset F_i$, A2 holds.

Let C_1 and C_2 be the components in F_i containing u_1u_5 and $u_2u_3u_4$, respectively. Suppose C_1 or C_2 contains at most five vertices, and then $C_1 \neq C_2$. Since D_{i+1} intersects neither the 3-cycle nor the 4-cycle, C_1 or C_2 is a 5-cycle.

Suppose $C_1 = u_1u_5u'_5wu'_1u_1$ is a 5-cycle and $C_1 \cap S_i = \emptyset$. See Figure 12(b). Then $\widetilde{C_1} = C_1 \cup D_{i+1}$ is a 1-cell in G_i. By Claim 3.4, $\widetilde{C_1}$ is good. Since D_{i+1} is bad, C_1 is good, and so there is an essential 3-cut T in G_i such that $|T \cap \partial C_1| \geq 2$, and by Lemma 2.3 $T \cap \partial C_1 \subset \partial C_1 - \{u_1u_2, u_5u_4\}$. Thus T is an essential 3-cut of G_{i+1}. Since F_{i+1} contains the path $u_1u'_1wu'_5u_3$, F_{i+1} does not intersect T, a contradiction. See Figure 12(c).

Suppose $C_2 = u_2u_3u_4u'_4u'_2u_2$ is a 5-cycle and $C_2 \cap S_i = \emptyset$. See Figure 12(d). Then $\widetilde{C_2} = C_2 \cup D_{i+1}$ is a 2-cell in G_i. By Claim 3.4, $\widetilde{C_2}$ is good, and so, as in the above case, G_i has an essential 3-cut T such that $|T \cap \partial C_2| \geq 2$, and $T \cap \partial C_2 \subset \partial C_2 - \{u_1u_2, u_4u_5\}$. Hence T is an essential 3-cut of G_{i+1}. Since F_{i+1} contains the path $u'_5u_3u_2u'_2u'_4u_1$, F_{i+1} does not intersect T, a contradiction. See Figure 12(e).

3. $p \leq i \leq p + q - 1$, i.e., $D_{i+1} \in \mathcal{D}_2$.

Since $V(D'_{i+1}) \subset F_{i+1}$, $|F_{i+1} \cap \partial D'_{i+1}|$ is 0, 2, 4, or 6.

Case 1. $|F_{i+1} \cap \partial D'_{i+1}| = 0$.

In this case, the 6-cycle $u_1u_3u_4u_5u_7u_8$ is contained in F_{i+1}, and replacing it with the 8-cycle $u_1u_2u_3u_4u_5u_6u_7u_8$, we obtain the even subgraph F_i in G_i. Obviously, both A1 and A2 hold.

Case 2. $|F_{i+1} \cap \partial D'_{i+1}| = 2$.

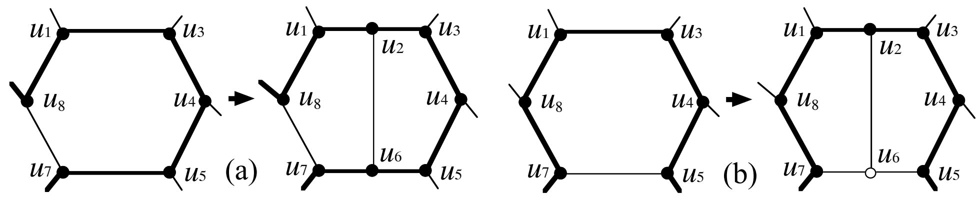

FIG. 13.

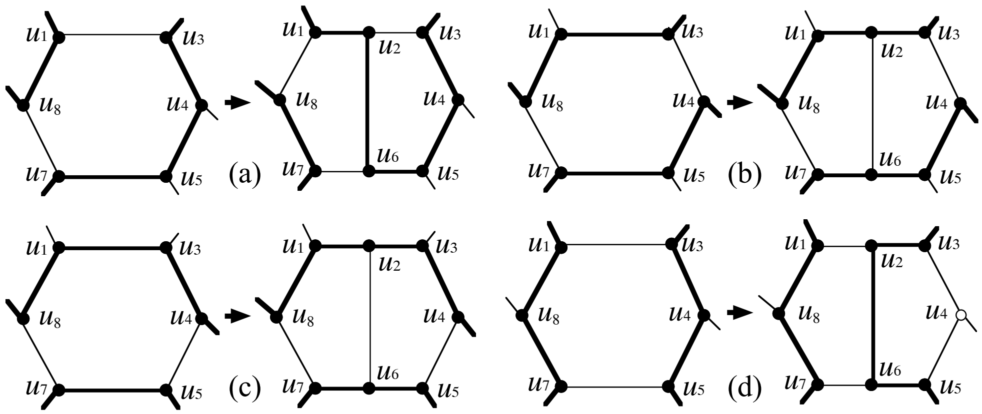

FIG. 14.

Since $V(D'_{i+1}) \subset F_{i+1}$ and F_{i+1} is an even subgraph, by symmetry we have two cases. If $F_{i+1} \cap \partial D'_{i+1} = \{u_7u'_7, u_8u'_8\}$, then F_{i+1} contains the path $u_8u_1u_3u_4u_5u_7$. See Figure 13(a). Then the even subgraph F_i obtained from F_{i+1} by replacing

$$u'_8u_8u_1u_3u_4u_5u_7u'_7 \text{ by } u'_8u_8u_1u_2u_3u_4u_5u_6u_7u'_7$$

is a desired even subgraph because both A1 and A2 hold. Similarly, we can show the case of $F_{i+1} \cap \partial D'_{i+1} = \{u_7u'_7, u_5u'_5\}$ since $u_6 \in R(D_{i+1}) \subset R_i$. See Figure 13(b).

Case 3. $|F_{i+1} \cap \partial D'_{i+1}| = 4$.

By symmetry, we have four cases:

(i) $F_{i+1} \cap \partial D'_{i+1} = \{u_7u'_7, u_8u'_8, u_1u'_1, u_3u'_3\}$.
Since $V(D'_{i+1}) \subset F_{i+1}$, F_{i+1} contains the paths u_8u_1 and $u_3u_4u_5u_7$. See Figure 14(a). Let P_1 and P_2 be the two paths obtained from the cycles in F_{i+1} intersecting D'_{i+1} by removing edges in $E(D'_{i+1})$ and isolated vertices. By symmetry, we may suppose $u_3 \in P_1$. If $u_7 \in P_1$ or $u_8 \in P_1$, then the even subgraph F_i obtained from F_{i+1} by replacing

$$u'_1u_1u_8u'_8 \text{ and } u'_3u_3u_4u_5u_7u'_7 \text{ by } u'_7u_7u_8u'_8 \text{ and } u'_3u_3u_4u_5u_6u_2u_1u'_1$$

is a desired even subgraph because both A1 and A2 hold. See Figure 14(a). In the case of $u_1 \in P_1$, let F_i be the even subgraph obtained from F_{i+1} by replacing

$$u'_3u_3u_4u_5u_7u'_7 \text{ by } u'_3u_3u_4u_5u_6u_7u'_7.$$

Obviously A1 holds. Since $u_2 \in R(D_{i+1}) \subset R_i$, A2 holds.

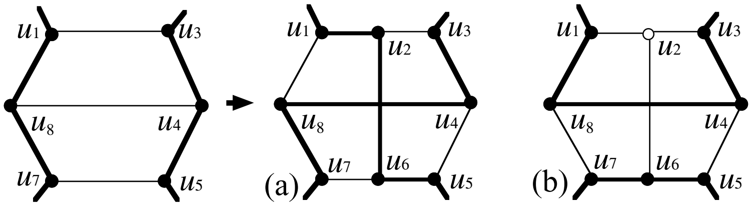

FIG. 15.

(ii) $F_{i+1} \cap \partial D'_{i+1} = \{u_7u'_7, u_8u'_8, u_3u'_3, u_4u'_4\}$.
Then F_{i+1} contains the paths $u_8u_1u_3$ and $u_4u_5u_7$. See Figure 14(b). Then the even subgraph F_i obtained from F_{i+1} by replacing

$$u'_8u_8u_1u_3u'_3 \text{ and } u'_4u_4u_5u_7u'_7 \text{ by } u'_8u_8u_1u_2u_3u'_3 \text{ and } u'_4u_4u_5u_6u_7u'_7$$

is a desired even subgraph because both A1 and A2 hold.

(iii) Similarly we can show the case that $F_{i+1} \cap \partial D'_{i+1} = \{u_7u'_7, u_8u'_8, u_4u'_4, u_5u'_5\}$. See Figure 14(c).

(iv) $F_{i+1} \cap \partial D'_{i+1} = \{u_7u'_7, u_1u'_1, u_3u'_3, u_5u'_5\}$.
Suppose first that $u_4u_8 \in E(G)$. Then, F_{i+1} contains either the paths $u_7u_8u_1$ and $u_3u_4u_5$, or the paths $u_1u_8u_4u_3$ and u_5u_7, or u_1u_3 and $u_5u_4u_8u_7$. See Figure 15 for the case of the paths $u_7u_8u_1$ and $u_3u_4u_5$. Let P_1 and P_2 be the two paths obtained from the cycles in F_{i+1} intersecting D'_{i+1} by removing edges in $E(D'_{i+1})$ and isolated vertices. By symmetry, we may suppose $u_3 \in P_1$. If $u_1 \in P_1$ or $u_5 \in P_1$, then the even subgraph F_i obtained from F_{i+1} by replacing the paths inside D'_{i+1}

$$\text{by } u_7u_8u_4u_3 \text{ and } u_1u_2u_6u_5$$

is a desired even subgraph because both A1 and A2 hold. See Figure 15(a).

In the case of $u_7 \in P_1$, let F_i be the even subgraph obtained from F_{i+1} by replacing the paths inside D'_{i+1}

$$\text{by } u_1u_8u_4u_3 \text{ and } u_7u_6u_5.$$

See Figure 15(b). Obviously A1 holds. Since $u_2 \in R(D_{i+1}) \subset R_i$, A2 holds.

Therefore, we may assume that $u_4u_8 \notin E(G)$. Since there are components in F_{i+1} containing $u_7u_8u_1$ and $u_3u_4u_5$, both u'_4 and u'_8 exist. Thus $|E_{u_4,u_8}| = 2$, and so one of u_4 and u_8 is in R_i. By symmetry, we may suppose $u_4 \in R_i$. Let F_i be the even subgraph obtained from F_{i+1} by replacing

$$u'_3u_3u_4u_5u'_5 \text{ by } u'_3u_3u_2u_6u_5u'_5.$$

See Figure 14(d). Obviously A2 holds.

Suppose the component C_1 of F_i containing $u_7u_8u_1$ is a 5-cycle and $C_1 \cap S_i = \emptyset$. Let $C_1 = u_1u_8u_7u'_7u'_1u_1$ and $C_2 = u_1u_2u_6u_7u_8u_1$. See Figure 16(a). Then $C = C_1 \cup C_2$ is a 2-cell in G_i. By Claim 3.2, C is good in G_i. Since D_{i+1} is bad, C_2 is bad, and so C_1 is good. Thus there is an essential 3-cut T such that $|T \cap \partial C_1| \geq 2$. Since C_2 is bad, $T \cap \partial C_1 \subset \partial C_1 - \{u_1u_2, u_7u_6\}$. Hence T is an essential 3-cut of G_{i+1}. Since F_{i+1} contains C_1 as a component, F_{i+1} does not intersect T, a contradiction. See Figure 16(b).

Case 4. $|F_{i+1} \cap \partial D'_{i+1}| = 6$.

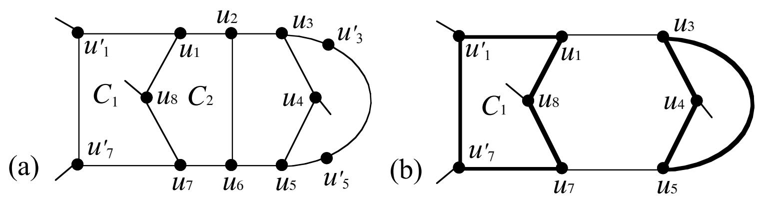

FIG. 16.

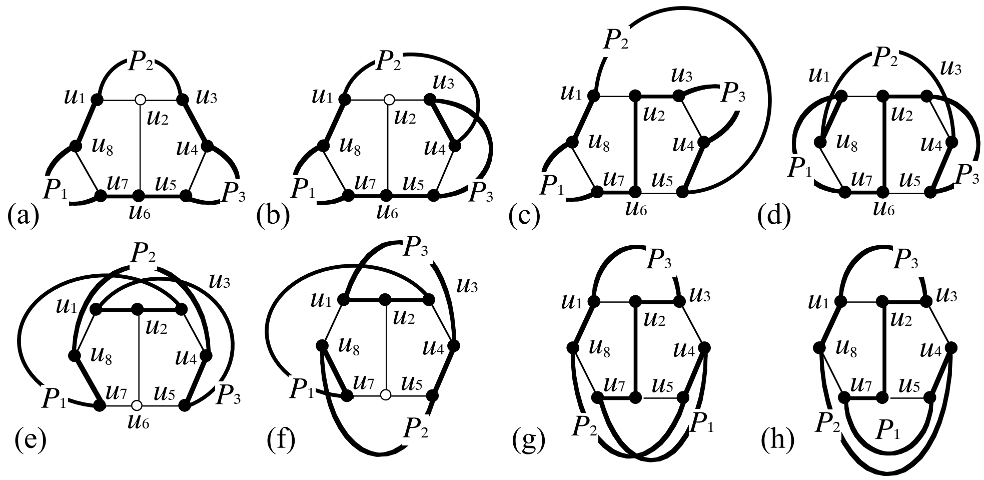

FIG. 17.

In this case, F_{i+1} contains all the edges in $\partial D'_{i+1}$. Let P_1, P_2, and P_3 be the three paths obtained from the cycles in F_{i+1} intersecting D'_{i+1} by removing edges in $E(D'_{i+1})$. By symmetry, we may suppose $u_7 \in P_1$. It is easy to confirm that for all the following cases, both A1 and A2 hold.

(i) The ends of P_1 are u_7 and u_8.

(a) If the ends of P_2 are u_1 and u_3, then let F_i be the even subgraph obtained from F_{i+1} by replacing the cycles in F_{i+1} intersecting D'_{i+1} with the cycle

$$u_7P_1u_8u_1P_2u_3u_4P_3u_5u_6u_7.$$

See Figure 17(a).

(b) If the ends of P_2 are u_1 and u_4, then let F_i be the even subgraph obtained from F_{i+1} by replacing the cycles in F_{i+1} intersecting D'_{i+1} with the cycle

$$u_7P_1u_8u_1P_2u_4u_3P_3u_5u_6u_7.$$

See Figure 17(b).

(c) If the ends of P_2 are u_1 and u_5, then let F_i be the even subgraph obtained from F_{i+1} by replacing the cycles in F_{i+1} intersecting D'_{i+1} with the cycle

$$u_7P_1u_8u_1P_2u_5u_4P_3u_3u_2u_6u_7.$$

See Figure 17(c).

Notice that by symmetry, we finished showing all the cases where there is a path joining u_i and u_{i+1} for any i by the case (i).

(ii) The ends of P_1 are u_7 and u_1.
In this case, the ends of P_2 are u_8 and u_4; otherwise there is a path joining u_i and u_{i+1} for some i. Let F_i be the even subgraph obtained from F_{i+1} by replacing the cycles in F_{i+1} intersecting D'_{i+1} with the cycle

$$u_7P_1u_1u_8P_2u_4u_5P_3u_3u_2u_6u_7.$$

See Figure 17(d).

(iii) The ends of P_1 are u_7 and u_3.
If the ends of P_2 are u_8 and u_4, then let F_i be the even subgraph obtained from F_{i+1} by replacing the cycles in F_{i+1} intersecting D'_{i+1} with the cycle

$$u_7P_1u_3u_2u_1P_3u_5u_4P_2u_8u_7.$$

See Figure 17(e).
If the ends of P_2 are u_8 and u_5, then let F_i be the even subgraph obtained from F_{i+1} by replacing the cycles in F_{i+1} intersecting D'_{i+1} with the cycle

$$u_7P_1u_3u_2u_1P_3u_4u_5P_2u_8u_7.$$

See Figure 17(f).

(iv) The ends of P_1 are u_7 and u_4.
If the ends of P_2 are u_8 and u_5, then let F_i be the even subgraph obtained from F_{i+1} by replacing the cycles in F_{i+1} intersecting D'_{i+1} with the cycle

$$u_7P_1u_4u_5P_2u_8u_1P_3u_3u_2u_6u_7.$$

See Figure 17(g). The case that the ends of P_2 are u_8 and u_3 is the same as case (iii). See Figure 17(f).

(v) The ends of P_1 are u_7 and u_5.
If the ends of P_2 are u_8 and u_4, then let F_i be the even subgraph obtained from F_{i+1} by replacing the cycles in F_{i+1} intersecting D'_{i+1} with the cycle

$$u_7P_1u_5u_4P_2u_8u_1P_3u_3u_2u_6u_7.$$

See Figure 17(h). The case that the ends of P_2 are u_8 and u_3 is the same as case (iv). See Figure 17(g).

Now we completed the proof.

4. Closing remarks. The *traveling salesman problem (TSP)* is used to find a spanning closed walk of short length in a given graph. The typical method for TSP on 3-edge-connected cubic graphs is as follows. First, we find a 2-factor F in a given 3-edge-connected cubic graph G and take a certain connected graph T (e.g., a spanning tree) in the graph G/F obtained from G by contracting all components in F, and then we obtain a connected subgraph $F \cup T$ of G. By modifying it suitably, we can get a spanning closed walk whose length is a certain function on $|E(T)|$. Since T must be a connected subgraph of G/F, $|E(T)|$ is at least the number of components of F minus one, and so the lower number of components in F gives the better bounds.

Aggarwal, Garg, and Gupta [1] used Theorem A to begin with a 2-factor having at most $n/5$ components and showed the existence of a spanning closed walk of length at most $4n/3$ in a 3-edge-connected cubic graph of order n. This result was further improved to 2-edge connected or connected cubic graphs, graphs of maximum degree at most 3, or better bounds than $4n/3$; see [6, 8, 9].

Because of the above reasons, several researchers have been interested in a 2-factor in cubic graphs such that the number of 5-cycles is small; see [7]. Instead of using a 2-factor, we can use an even subgraph satisfying certain conditions on the order of each component. In fact, such structures have appeared in [8, 9] as intermediate products, which is called an *R-factor* in [8]. For those intermediate products, it is not necessarily dominating, but the dominating property may help us to obtain good bounds, i.e., we expect that Theorem 1.1 has a potential application to the TSP.

Acknowledgments. The authors would like to thank the referees for valuable suggestions and comments.

REFERENCES

[1] N. Aggarwal, N. Garg, and S. Gupta, *A 4/3-Approximation for TSP on Cubic 3-Edge-Connected Graphs*, arXiv:1101.5586v1, 2011.
[2] B. R. Alspach and C. D. Godsil, eds., *Cycles in Graphs,* Ann. Discrete Math. 27, North-Holland, Amsterdam, 1985.
[3] P. Ash and B. Jackson, *Dominating cycles in bipartite graphs,* in Progress in Graph Theory, A. Bondy and U. S. R. Murty, eds., Academic Press, New York, 1984, pp. 81–87.
[4] J. C. Bermond and C. Thomassen, *Cycles in digraphs—a survey*, J. Graph Theory, 5 (1981), pp. 1–43.
[5] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Grad. Texts in Math. 244, Springer, New York, 2008.
[6] S. Boyd, R. Sitters, S. van der Ster, and L. Stougie, *The traveling salesman problem on cubic and subcubic graphs*, Math. Program., 144 (2014), pp. 227–245.
[7] B. Candráková and R. Lukot'ka, *Avoiding 5-circuits in 2-factors of cubic graphs*, SIAM J. Discrete Math., 29 (2015), pp. 1387–1405.
[8] B. Candráková and R. Lukot'ka, *Cubic TSP—a 1.3-Approximation*, arXiv:1506.06369v1, 2015.
[9] J. Correa, O. Larré, and A. Soto, *TSP tours in cubic graphs: Beyond 4/3*, SIAM J. Discrete Math., 29 (2015), pp. 915–939.
[10] H. Fleischner, *Spanning Eulerian subgraphs, the splitting lemma, and Petersen's theorem*, Discrete Math., 101 (1992), pp. 33–37.
[11] H. Fleischner and B. Jackson, *A note concerning some conjectures on cyclically 4-edge-connected 3-regular graphs,* in Graph Theory in Memory of G. A. Dirac, L. D. Andersen, I. T. Jakobsen, C. Thomassen, B. Toft, P. D. Vestergaard, eds., Ann. Discrete Math. 41, North-Holland, Amsterdam, 1988, pp. 171–177.
[12] D. Gamarnik, M. Lewenstein, and M. Sviridenko, *An improved upper bound for the TSP in cubic 3-edge-connected graphs*, Oper. Res. Lett., 33 (2005), pp. 467–474.
[13] B. Jackson and K. Yoshimoto, *Even subgraphs of bridgeless graphs and 2-factors of line graphs*, Discrete Math., 307 (2007), pp. 2775–2785.
[14] B. Jackson and K. Yoshimoto, *Spanning even subgraphs of 3-edge-connected graphs*, J. Graph Theory, 62 (2009), pp. 37–47.
[15] T. Kaiser and R. Škrekovski, *Cycles intersecting edge-cuts of prescribed sizes*, SIAM J. Discrete Math., 22 (2008), pp. 861–874.
[16] M. M. Matthews and D. P. Sumner, *Hamiltonian results in $K_{1,3}$-free graphs*, J. Graph Theory, 8 (1984), pp. 139–146.
[17] J. Petersen, *Die Theorie der regulären Graphs*, Acta Math., 15 (1891), pp. 193–220.
[18] Z. Ryjáček, *On a closure concept in claw-free graphs*, J. Combin. Theory. Ser. B, 70 (1997), pp. 217–224.
[19] C. Thomassen, *Reflections on graph theory*, J. Graph Theory, 10 (1986), pp. 309–324.