FRACTIONAL COVERS AND MATCHINGS IN FAMILIES OF WEIGHTED d-INTERVALS

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Abstract. A *d-interval* is a union of at most d disjoint closed intervals on a fixed line. Tardos [9] and the second author [6] used topological tools to bound the transversal number τ of a family H of d-intervals in terms of d and the matching number ν of H. We investigate the weighted and fractional versions of this problem and prove upper bounds that are tight up to constant factors. We apply both the topological method and an approach of Alon [1]. For the use of the latter, we prove a weighted version of Turán's theorem. We also provide a proof of the upper bound of [6] that is more direct than the original proof.

1. Introduction

A *d-interval* is the union of at most d disjoint closed intervals on a fixed line. The j-th component of a d-interval h, counted from the left to right, will be denoted by h^j.

We call a d-interval h *separated* if its intersection with each interval $(i, i+1)$, where $0 \leq i < d$, is either empty or coincides with one component of h. For our purposes, we can equivalently picture a separated d-interval as the union of d possibly empty intervals, one on each of d fixed parallel lines (which is the definition used in [10]). We shall also consider *discrete* d-intervals where the lines are replaced by finite linearly ordered sets. All results and conjectures below are easily seen to be equivalent to their discrete versions, and sometimes we shall switch to the discrete versions without further argument. We shall assume that our hypergraphs are finite.

We will be interested in properties of hypergraphs whose vertex sets are the points of the lines considered above, and whose edges are d-intervals. We call them *hypergraphs of d-intervals*.

A *matching* in a hypergraph $H = (V, E)$ with vertex set V and edge set E is a set of disjoint edges. A *cover* is a subset of V meeting all edges. The *matching number* $\nu(H)$ is the maximal size of a matching, and the *covering number*, or *transversal number* $\tau(H)$ is the minimal size of a cover.

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The fractional relaxations of ν and τ are denoted, as usual, by ν^* and τ^*, respectively. By Linear Programming duality, $\nu^* = \tau^*$.

An old result of Gallai is that if H is a hypergraph of (1-)intervals, then $\tau(H) = \nu(H)$. In [4, 5] the authors raised the problem of the best bound on the ratio $\frac{\tau}{\nu}$ in hypergraphs of 2-intervals, with the natural conjecture being that the answer is 2. This was proved by Tardos [9], using algebraic topology. A simpler proof was then found by the second author [6], who also extended the theorem to hypergraphs consisting of d-intervals:

Theorem 1.1. *For a hypergraph H of d-intervals,*

$$\frac{\tau(H)}{\nu(H)} \leq d^2 - d + 1.$$

In the separated case, the upper bound improves to $d^2 - d$.

Matoušek [8] showed that this bound is not far from the truth: there are examples of intersecting hypergraphs of d-intervals in which $\tau = \frac{\tau}{\nu} = \Omega(\frac{d^2}{\log d})$. The proof of Theorem 1.1 in [6] relies on reducing the problem to a discrete problem on d-uniform hypergraphs and the fact that in such hypergraphs $\tau^* \leq d\nu$.

Another approach was taken by Alon [1] who proved a slightly weaker upper bound for the non-separated case, namely $\frac{\tau}{\nu} \leq 2d^2$. Like in [6], there is a central role in this proof to fractional versions. Alon breaks his bound into the following two results:

Theorem 1.2. $\tau^* \leq 2d\nu$.

Theorem 1.3. $\tau \leq d\tau^*$.

Theorem 1.3 follows by a relatively simple sampling argument. (As will be shown below, it also follows from the arguments in [6].) The bound in Theorem 1.2 is proved using Turán's theorem.

As noted, both approaches use fractional parameters. This makes finding the right relations between them and the integral parameters of interest. In particular, the following conjecture is appealing:

Conjecture 1.4. *In a hypergraph of separated d-intervals, $\tau^* \leq d\nu$.*

The conjecture is not true in the non-separated case, as shown by an intersecting family of 2-intervals with $\tau^* \geq \frac{11}{4}$ constructed by Gyárfás [3, p. 45]. An open question in [3] is whether $\tau^* < 3$ for intersecting families of (non-separated) 2-intervals.

Besides the fractional parameters, we shall study also the weighted versions of the matching and covering numbers, as defined below. For a real valued function f on a set T, we let $f[T]$ denote the sum $\sum_{t \in T} f(t)$.

Definition 1.5. Given a hypergraph H, a *weight system on $E(H)$* is a function $w : E(H) \to \mathbb{N}$. A function $g : V(H) \to \mathbb{N}$ is called a *w-cover* if $\sum_{v \in e} g(v) \geq w(e)$ for all $e \in E(H)$. Let $\nu_w(H)$ be the maximum of $w[F]$

over all matchings F in H, and let $\tau_w(H)$ be the minimum of $g[V]$ over all w-covers g.

It is likely that the bounds (conjectured as well as proved) that are true in the non-weighted case are true also in the weighted case.

Conjecture 1.6. *Let H be a hypergraph of separated d-intervals and w a weight system on $E(H)$. Then $\tau^*_w(H) \le d\nu_w(H)$.*

In Section 3 we shall prove a weighted and directed version of Turán's theorem, and in Section 4 we shall use it to prove:

Theorem 1.7. *If H is a hypergraph of d-intervals and w a weight system on $E(H)$, then $\tau^*_w(H) \le 2d\nu_w(H)$.*

A straightforward generalization of the proof of Theorem 1.3 will yield:

Theorem 1.8. *If H is a hypergraph of d-intervals and w a weight system on $E(H)$, then $\tau_w(H) \le d\tau^*_w(H)$.*

Combining the two results, we have:

Corollary 1.9. *If H is a hypergraph of d-intervals and w a weight system on $E(H)$, then $\tau_w(H) \le 2d^2\nu_w(H)$.*

By Theorem 1.8, Conjecture 1.6, if true, would imply the separated case of the following:

Conjecture 1.10. *In a hypergraph of d-intervals $\tau_w \le d^2\nu_w$.*

This conjecture is true for intersecting hypergraphs, since by Theorem 1.1, there exists a cover of size at most d^2-d+1, and putting weight $\max_{h\in H} w(h) = \nu_w$ on each of its points constitutes a weighted cover.

In Section 5 we shall show that the case of the weight of each d-interval being its length has a special stature, namely that in order to prove an upper bound of αd^2 on $\frac{\tau_w}{\nu_w}$, it is enough to prove that in every hypergraph H of d-intervals

$$\frac{\tau^*_\ell}{\nu_\ell} \le \alpha d,$$

where $\ell(h)$ is the length of $h \in H$. The proof uses the KKMS (Knaster-Kuratowski-Mazurkewitz-Shapley) theorem. In Section 6 we shall use the same theorem, and a generalization of it due to Komiya, to provide a simplified proof of Theorem 1.1. In Section 7 we shall use the KKMS theorem to prove an upper bound of 4d on $\frac{\tau^*}{\nu}$. This result is weaker than Theorem 1.2, but we think it is still worth presenting, since an improved topological approach may well be the right tool for finding the right bound.

2. Examples for sharpness

If true, then Conjectures 1.4 and 1.6 are sharp for all d. In the next example $\nu = 1$ and $\tau^* = d$:

Example 2.1. It is known that the edge set of K_{2d} is decomposable into d Hamiltonian paths $P_1, \dots, P_d$ (see [12]). Let $V(K_{2d}) = [2d]$. Each path P_j can be viewed as a permutation π_j on $[2d]$. Let $H = \{e_1, \dots, e_{2d}\}$, where e_i is the separated interval whose j-th component is $e_i^j = [\pi_j^{-1}(i), \pi_j^{-1}(i) + 1]$. For every $k, \ell \in [2d]$ the edge (k, ℓ) of K_{2d} belongs to some path P_j, namely $k = \pi_j(i),\ \ell = \pi_j(i+1)$ or $\ell = \pi_j(i),\ k = \pi_j(i+1)$ for some i. This implies that e_k and e_ℓ meet in the j-th line. Thus $\nu(H) = 1$. Putting weight $\frac{1}{2}$ on each e_i yields a fractional matching of size d. The set of points $\{2k \mid 1 \le k \le d\}$ on the first line constitutes a cover of H of size d. This proves that $\tau(H) = \tau^*(H) = d$.

The bound given in Theorem 1.3 is also sharp, at least asymptotically, even for separated d-intervals. This is shown by the following example. Let n be an integer, let $[0,1]^{\cup d}$ denote the union of d disjoint copies of the unit interval, and take H to be the set of d-intervals $e \subseteq [0,1]^{\cup d}$ satisfying $|e| > \frac{1}{n}$, where $|f|$ denotes the total length of a d-interval f, namely the sum of the lengths of its components.

Assertion 2.2. $\tau(H) \ge nd^2 - d$.

Proof. Let C be a cover for H. Let C_i be obtained from the intersection of C with the i-th line by the addition of 0 and 1, and let ℓ_i be the longest distance between two consecutive points of C_i. Then, clearly, $\sum_{i \le d} \ell_i \le \frac{1}{n}$, and the intersection of C with the i-th line contains at least $\frac{1}{\ell_i} - 1$ points from C. The latter means that $|C| \ge \sum_{i \le d} \frac{1}{\ell_i} - d$. By the harmonic-arithmetic average inequality, $\sum_{i \le d} \frac{1}{\ell_i} \ge \frac{d^2}{\sum_{i \le d} \ell_i} \ge nd^2$, and hence $|C| \ge nd^2 - d$. □

Assertion 2.3. $\nu^*(H) \le nd$.

Proof. Let α be a fractional matching in H. Denoting the value of α on an edge e by α_e, and the characteristic function of e by I_e, we have

$$d = |[0,1]^{\cup d}| \ge \int \sum \alpha_e I_e = \sum \alpha_e \int I_e = \sum \alpha_e |e|,$$

where the integration is over $[0,1]^{\cup d}$. Since $|e| > \frac{1}{n}$ for all $e \in H$ it follows that $\sum \alpha_e < nd$. Alternatively, the constant function n constitutes a 'continuous fractional cover' of H of size nd, and it can be approximated by discrete fractional covers as well as we please. □

By the above, for every n there exists a d-interval hypergraph with $\frac{\tau}{\nu^*} \ge d - \frac{1}{n}$.

3. A weighted version of Turán's theorem

In this section we prove a weighted version of Turán's theorem. Let $G = (V, E)$ be a graph. Recalling notation defined above, for a set X of vertices $w[X]$ is $\sum_{x \in X} w(x)$. For a set F of edges let $\tilde{w}[F] = \sum_{uv \in F}(w(u) + w(v))$. If G is directed, given disjoint subsets A and B of V, we write $E(A, B) = \{xy \mid$

$x \in A, y \in B\}$, where xy is an edge directed from x to y. If $v \in V \setminus A$ we write $E(v, A)$ and $E(A, v)$ for $E(\{v\}, A)$ and $E(A, \{v\})$, respectively. The set of neighbors of a vertex v is $N(v) = \{u \mid uv \in E \textit{ or } vu \in E\}$. We let $\alpha_w(G)$ denote the maximal sum of weights on an independent set in V.

Theorem 3.1. *Let $G = (V, E)$ be a graph and let $w : V \to \mathbb{N}^+$ be a weight function on its vertices. Let $W = w[V]$ and $K = \alpha_w(G)$. Then*

$$\tilde{w}[E] \geq \frac{W^2}{K} - W. \tag{1}$$

Note that if all weights are 1, this is just Turán's theorem. We will actually need a result about directed graphs that implies Theorem 3.1 as an easy corollary, upon replacing every edge in G by two oppositely directed edges:

Theorem 3.2. *Let D be a directed graph in which every pair of adjacent vertices is connected by at least two directed edges, not necessarily in the same direction. Let w be a weight function on $V = V(D)$, and write $W = w[V]$ and $K = \alpha_w(D)$. Then*

$$\sum_{xy \in E(D)} w(x) \geq \frac{W^2}{K} - W.$$

Proof. Choose an independent set A of total weight K. Write $B = V \setminus A$. By the induction hypothesis

$$\sum_{xy \in E(D[B])} w(x) \geq \frac{w[B]^2}{K} - w[B]. \tag{2}$$

Let v be a vertex not in A. By the maximality property of A, we have

$$\sum \{w(u) \mid u \in A \cap N(v)\} \geq w(v), \tag{3}$$

for otherwise the sum of weights in $A \setminus N(v) + v$ would be larger than in A. In particular, note that $|A \cap N(v)| \geq 1$.

We claim that

$$\sum_{uv \in E(A,v)} w(u) + \sum_{vu \in E(v,A)} w(v) \geq 2w(v). \tag{4}$$

Indeed, if all $A - v$ edges are directed from A to v, this follows from (3). If there exist two (possibly parallel) edges directed from v to A then (4) is obvious. So, there remains the case that there is precisely one $A - v$ edge, say va, directed from v to A. Then

$$\sum_{uv \in E(A,v)} w(u) + \sum_{vu \in E(v,A)} w(v) = 2 \sum \{w(u) \mid u \in A \cap N(v)\} - w(a) + w(v).$$

Therefore if $w(a) \le w(v)$ then (4) is proved by (3). Else we have

$$\sum_{uv \in E(A,v)} w(u) + \sum_{vu \in E(v,A)} w(v) \ge w(v) + w(a) > 2w(v),$$

as claimed.

Hence

$$\sum_{uv \in E(A,B)} w(u) + \sum_{vu \in E(B,A)} w(v) \ge 2w[B]. \tag{5}$$

Using (2) and (5), we get:

$$\sum_{xy \in E(D)} w(x) \ge \frac{w[B]^2}{K} - w[B] + 2w[B] = \frac{w[B]^2}{K} + w[B]. \tag{6}$$

Since $W = w[A] + w[B]$ and $w[A] = K$, we have

$$\begin{aligned} \frac{W^2}{K} - W &= \frac{w[A]^2 + 2w[A]w[B] + w[B]^2}{K} - w[A] - w[B] \\ &= \frac{w[B]^2}{K} + w[B]. \end{aligned}$$

Together with (6) this proves the desired inequality.

□

4. Weighted matchings

In this section we prove Theorems 1.7 and 1.8. But before that, here is some motivation to the study of the weighted case: a (well known) connection to edge-colorings. The *edge chromatic number* $\chi_e(H)$ of a hypergraph H is the minimal number of matchings needed to cover all edges of the hypergraph. A *fractional edge coloring* is a non-negative function f on the set $\mathcal{M}$ of matchings in H, satisfying the condition that $\sum_{e \in M \in \mathcal{M}} f(M) \ge 1$ for all $e \in E(H)$. The fractional edge chromatic number $\chi_e^*(H)$ is the minimum, over all fractional edge colorings f of H, of $f[\mathcal{M}]$. (It is easy to see that this minimum exists.)

Lemma 4.1. *For any hypergraph $H = (V, E)$, if $\tau_w^*(H) \le \alpha \nu_w(H)$ for every weight system w on $E(H)$ then $\chi_e^*(H) \le \alpha \Delta(H)$.*

Proof. Write q for $\chi_e^*(H)$. By LP duality, $q = \max w[E]$, where the maximum is over all weight functions w on E satisfying the condition that $w[M] \le 1$ for every matching M in H. Taking w for which this maximum is attained, we have $\nu_w(H) = 1$, and hence by the assumption of the lemma, $\tau_w^*(H) \le \alpha$. But since every vertex participates in at most $\Delta(H)$ edges in a fractional w-covering of the edges, $\tau_w^* \ge \frac{w[E]}{\Delta(H)} = \frac{q}{\Delta(H)}$. Combining these inequalities yields $q \le \alpha \Delta(H)$, as desired. □

In [5] the conjecture was raised that in a hypergraph H of 2-intervals $\chi_e(H)$ does not exceed 2 times the maximal size of an intersecting subhypergraph. The following conjecture strengthens this, and generalizes it to all d:

Conjecture 4.2. *If H is a hypergraph of d-intervals, and the maximum degree of a point on any line is Δ, then $\chi_e(H) \leq d\Delta$.*

If true, then this is sharp by Example 2.1. There $\Delta = 2$, and since $\nu = 1$ we have $\chi_e = |E| = 2d$.

As noted already in [5], relaxing the problem by a factor of 2 brings Conjecture 4.2 within easy reach:

Theorem 4.3. *Under the assumptions of Conjecture 4.2, $\chi_e(H) \leq 2d(\Delta - 1)$.*

Proof. Let D be a digraph whose vertex set is $E(H)$, and in which e sends an arrow to f if an endpoint of some interval component of e belongs to f. Then the outdegree of every vertex of D (=edge of H) is at most $2d(\Delta-1)$, and hence the number of directed edges is at most $2d(\Delta-1)|V|$. As is easy to realize, for every pair of vertices of D, if there is an edge between them then there are two, hence the number of edges in the underlying undirected graph G is at most $d(\Delta-1)|V|$, so the average degree in G is at most $2d(\Delta-1)$. Since the same is true for any induced subgraph of G, the theorem follows by successively removing vertices of minimum degree and coloring them greedily in the reverse order. $\square$

By Lemma 4.1, Conjecture 1.6 would imply the fractional version of Conjecture 4.2, namely $\chi_e^*(H) \leq d\Delta(H)$, for separated d-intervals.

The proof of Theorem 1.8 is almost identical to the proof in the non-weighted case:

Proof of Theorem 1.8. Let g be a rational valued fractional w-cover of H of minimal size, and let $|g| = \frac{p}{q}$. By duplicating points we may assume that g has value $\frac{1}{q}$ on each point belonging to a set P of p points, and that $d|q$. Write $m = \frac{q}{d}$. Let Q be a set obtained by taking every m-th point in P, in the left to right order on the line. Every $e \in H$ satisfies $|P \cap e| \geq qw(e)$, and hence has a component containing $\frac{qw(e)}{d}$ points from P. This component contains then at least $\frac{qw(e)}{dm} = w(e)$ points from Q. Thus Q is a w-cover, and its size is $\frac{p}{m} = \frac{pd}{q} = d\tau^*_w$. $\square$

Proof of Theorem 1.7. Write $K = \nu_w(H)$. We need to show that $\nu^*_w(H) \leq 2dK$, meaning that for every fractional matching f we have $\sum_{e \in H} w(e)f(e) \leq 2dK$. Multiplying by a common denominator, removing edges on which $f = 0$ and duplicating edges if necessary, we can assume that $f(e) = \frac{1}{q}$

for all $e \in H$, for some natural number q. We write W for $w[H]$. In this terminology, we have to show that

$$\frac{W}{q} \leq 2dK. \tag{7}$$

Let D be a digraph obtained from the line graph $L = L(H)$ by duplicating each edge in $E(L)$, and directing the two copies of each edge according to the following rule. Suppose that two edges e_1 and e_2 in H meet, namely $e_1e_2 \in E(L)$. Choose an interval component c_1 of e_1 that meets an interval component c_2 of e_2. It is now possible to choose two pairs (x, c_i), where x is an endpoint of c_{3-i}, and belongs to c_i. For each choice of such a pair direct a copy of the edge e_1e_2 from e_{3-i} to e_i (namely, from the piercing edge to the pierced one).

Let $\tilde{D}$ be D with all loops ee added. By Theorem 3.2, $\sum_{e\in H} w(e)deg_D^+(e) \geq W(W-K)/K$ (here deg^+ denotes the outdegree). Dividing both sides by W gives that the weighted average of $deg_D^+(e)$ (weighted by $w(e)$) is at least $\frac{W}{K} - 1$, and hence the weighted average of $deg_{\tilde{D}}^+(e)$ is $\frac{W}{K}$. Hence there is an edge $a \in H$ with $deg_D^+(a) \geq \frac{W}{K} - 1$, and considering a as adjacent also to itself, its outdegree in $\tilde{D}$ is at least $\frac{W}{K}$. That is, it pierces by its endpoints at least this many edges. Since a has at most $2d$ endpoints, it has an endpoint x meeting at least $\frac{W}{2dK}$ edges. Since $\sum_{e\in H,\ x\in e} f(e) \leq 1$, it follows that $W/(2dKq) \leq 1$, proving (7). $\square$

5. KKMS and the special role of lengths

A natural weight on a d-interval is its total length. In this section we show that in some sense, to be specified below, this is the general case. The tool showing this is the KKMS theorem (to be given below).

For a hypergraph H define a weight function $\ell = |h|$ for every $h \in H$. Call 1.6L the special case of Conjecture 1.6 in which $w = \ell$. Call a hypergraph $H = (V, E)$ *balanced* if it has a perfect fractional matching, namely a weight system on E such that for each vertex v, the weights of the edges containing v sum up to 1.

Lemma 5.1. *If a hypergraph H on a vertex set $[k]$ is balanced and $\tau^*_\ell(H) \leq \alpha\nu_\ell(H)$ then H contains a matching M such that $\sum_{m\in M} |m| \geq \frac{k}{\alpha}$.*

Proof. Let f be a perfect fractional matching on H, then

$$\sum_{h\in H} f(h)|h| = k.$$

Therefore,

$$\sum_{h\in H} f(h)\ell(h) = k,$$

which implies that $\tau^*_\ell = \nu^*_\ell(H) \geq k$. Thus we have $\nu_\ell(H) \geq \frac{k}{\alpha}$, which proves the lemma. $\square$

Therefore, by Theorem 1.7 we have the following:

Corollary 5.2. *In a balanced discrete hypergraph of d-intervals on $[k]$ there exists a matching of total size $\frac{k}{2d}$. If the d-intervals are separated and Conjecture 1.6L is true, then there exists a matching of total size $\frac{k}{d}$.*

In the continuous case, for a d-intervals hypergraph to be balanced it has to consist of intervals that are half closed (say at the left) and half open (at the right). Assuming this, here is an appealing special case of Conjecture 1.6L: A balanced hypergraph of separated d-intervals in $[0,1]^{\cup d}$ has a matching of total length 1. By the first part of the observation, there exists in such a hypergraph a matching of total length $\frac{1}{2}$.

Theorem 5.3. *Let α be such that for every hypergraph of d-intervals $\tau_\ell^* \leq \alpha d \nu_\ell$. Then for every hypergraph of d-intervals and for every weight function w, $\tau_w \leq \alpha d^2 \nu_w$.*

The simplex Δ_k is the set of all points $\vec{x} = (x_1, x_2, \dots, x_{k+1}) \in \mathbb{R}_+^{k+1}$ satisfying $\sum_{i \leq k+1} x_i = 1$. For $S \subseteq [k+1]$ let $F(S)$ be the face of Δ_k consisting of the points $\vec{x}$ satisfying $x_i = 0$ for all $i \notin S$.

Theorem 5.4 (KKMS)**.** *Suppose that with every subset T of $[k+1]$ there is associated a subset B_T of Δ_k, so that all B_T are closed or all of them are open, and $F(R) \subseteq \bigcup_{T \subseteq R} B_T$ for every $R \subseteq [k+1]$. Then there exists a balanced set $\mathcal{T}$ of subsets of $[k+1]$, satisfying $\bigcap_{T \in \mathcal{T}} B_T \neq \emptyset$.*

This is a generalization, by Shapley [11], of the famous KKM (Knaster-Kuratowski-Mazurkiewicz) theorem, which is the case where the only nonempty B_T's are $B_{\{i\}}$ for singletons $\{i\} \subset [k+1]$.

Proof of Theorem 5.3. We may assume that all edges of H are contained in $(0,1)$. Assume that $\tau_w(H) > k$. The theorem will follow if we show that $\nu_w(H) > \frac{k}{\alpha d^2}$. The assumption that $\tau_w(H) > k$ implies that for every point $\vec{x} = (x_1, x_2, \dots, x_{k+1})$ in $P = \Delta_k$, the points $p_{\vec{x}}(m) = \sum_{i \leq m} x_i$, $1 \leq m \leq k$ do not constitute a weighted cover for H. Note that the points $p_{\vec{x}}(m)$ are taken with multiplicity, namely if such a point repeats q times, it is given weight q. The above can be stated as:

Assertion 5.5. *For each $\vec{x} \in \Delta_k$ there exists $h \in H$ that contains fewer than $w(h)$ points $p_{\vec{x}}(m)$ (counted with multiplicity).*

Given $\vec{x} \in P$, let $p_{\vec{x}}(0) = 0$ and $p_{\vec{x}}(k+1) = 1$. For $T \subseteq [k+1]$, define $L(T, \vec{x}) = \bigcup_{t \in T} [p_{\vec{x}}(t-1), p_{\vec{x}}(t)]$. Let $c(T, \vec{x})$ be the number of connected components of $L(T, \vec{x})$. For $X \subseteq [0,1]$ let $np(X, \vec{x})$ be the number of points $p_{\vec{x}}(m)$ (counted with multiplicity) in X. Note that $np(int(L(T, \vec{x})), \vec{x}) \geq |T| - c(T, \vec{x})$ ($int(X)$ is the interior of X).

We want now to define sets B_T, towards an application of the KKMS theorem. A definition that almost works is this: $\vec{x} \in B_T$ if there exists an edge

$h \in H$ that is contained in $int(L(T,\vec{x}))$ and $w(h) \geq np(int(L(T,\vec{x})),\vec{x})+1$. The problem is that with this definition B_T is not necessarily open (and obviously also not necessarily closed). For example, let $T=\{1,3\}$, and let $\vec{x}$ be such that $x_2=0$. It is possible that there exists $h \in H$ properly contained in $L(T,\vec{x})) = [x'_0,x'_1] \cup [x'_1+x'_2, x'_1+x'_2+x'_3] = [x_0, x_1+x_2+x_3]$, so $\vec{x} \in B_T$ by this definition, but for points $\vec{x}'$ arbitrarily close to $\vec{x}$, in which $x'_2>0$, there is no $h \in H$ contained in $L(T,\vec{x}') = [x'_0,x'_1] \cup [x'_1+x'_2, x'_1+x'_2+x'_3]$, and so $\vec{x}' \notin B_T$. For this reason, the definition of B_T will be a bit more involved.

Let $A_T(>\varepsilon)$ (resp. $A_T(\geq \varepsilon)$) be the set of those points $\vec{x}$ for which the following hold:

- $c(T,\vec{x}) \leq d$,
- there exists an edge $h \in H$ contained in $int(L(T,\vec{x}))$, satisfying $w(h) \geq np(int(L(T,\vec{x})),\vec{x})+1$, and
- $dist(h, \partial(L(T,\vec{x})) > \varepsilon$ (resp. $dist(h,\partial(L(T,\vec{x})) \geq \varepsilon$.

(Here $\partial(X)$ denotes the boundary of X, and $dist$ stands for "distance".)

Assertion 5.6. *$F(S) \subseteq \bigcup_{\varepsilon>0}\bigcup_{T\subseteq S} A_T(>\varepsilon)$ for every subset S of $[k+1]$.*

Proof. If $\vec{x} \in F(S)$ then by Assertion 5.5 there exists $h \in H$ that is covered by fewer than $w(h)$ points $p_{\vec{x}}(m)$. Since h has at most d interval components, this means that there exists some $R \subseteq [k+1]$ such that $h \subset int(L(R,\vec{x})$, $w(h) > np(int(L(R,\vec{x})),\vec{x})$, and $c(R,\vec{x}) \leq d$. Let $T = R \cap S$. Since $x_i = 0$ for $i \notin S$ we have $int(L(R,\vec{x})) = int(L(T,\vec{x}))$. Taking small enough ε, we have then $\vec{x} \in A_T(>\varepsilon)$. This proves the assertion. □

Since $F(S)$ is compact there exists $\varepsilon(S)>0$ such that $F(S) \subseteq \bigcup_{T\subseteq S} A_T(>\varepsilon(S))$. Let $\delta = \frac{1}{2}\min\{\varepsilon(S) \mid S \subseteq [k+1]\}$ and for $T \subseteq [k+1]$ define $B_T = A_T(\geq \delta)$. Then $F(S) \subseteq \bigcup_{T\subseteq S} B_T$ for all $S \subseteq [k+1]$.

Assertion 5.7. *The sets B_T are closed.*

Proof. Let $\vec{x}$ be a limit point of the sequence $\vec{x}^n \in B_T$. For every n let h^n be an edge witnessing the fact that $\vec{x}^n \in B_T$. Since H is finite, there is an edge $h \in H$ such that $h^n = h$ for infinitely many values of n. Then $h \subseteq L(T,\vec{x})$, and its distance from the boundary of $L(T,\vec{x})$ is at least δ, meaning that $\vec{x} \in B_T$. □

We have shown that the sets B_T satisfy the conditions of the KKMS theorem, so by this theorem there exists a balanced collection $\mathcal{T}$ of discrete d-intervals in $[k+1]$ such that $\bigcap_{T\in\mathcal{T}} B_T \neq \emptyset$. Then $\tau^*_\ell(\mathcal{T}) \leq \alpha d \nu_\ell(\mathcal{T})$, and by Lemma 5.1 there is a matching $M \in \mathcal{T}$ with $\sum_{m\in M}|m| \geq \frac{k+1}{\alpha d}$. Therefore,

$$\sum_{m\in M}(|m| - c(M,\vec{x})+1) \geq \sum_{m\in M}\frac{|m|}{c(M,\vec{x})} \geq \frac{1}{d}\sum_{m\in M}|m| \geq \frac{k+1}{\alpha d^2}. \tag{8}$$

Choose a point $\vec{x} \in \bigcap_{T\in\mathcal{T}} B_T$. For every $T \in \mathcal{T}$, the fact that $\vec{x} \in B_T$ means that there exists $h(T) \in H$ such that $h(T) \subset int(L(T,\vec{x}))$ and

$w(h(T)) \geq np(int(L(T, \vec{x})), \vec{x}) + 1 \geq |T| - c(T, \vec{x}) + 1$. Therefore by (8) the proof of the theorem will be complete if we show that for disjoint $T_1, T_2 \in \mathcal{T}$ the edges $h(T_1)$ and $h(T_2)$ are disjoint. If they do meet, then any point $z \in h(T_1) \cap h(T_2)$ lies in the interior of a connected component of $L(T_1, \vec{x})$ and in the interior of a connected component of $L(T_2, \vec{x})$. There exists then t such that $x_t \neq 0$, and $z \in [p_{\vec{x}}(t-1), p_{\vec{x}}(t)]$. Then $t \in T_1 \cap T_2$, a contradiction. This concludes the proof of the theorem.

6. A proof of Theorem 1.1 using KKMS

In this section we use the KKMS theorem to give a short proof of Theorem 1.1. In [6], a certain version of Borsuk's theorem was used. The KKMS theorem and its generalizations get to the point more directly.

We shall also use (as is done in [6]) a theorem of Füredi [2]:

Theorem 6.1. *If all edges of a hypergraph H are of size at most d, then $\nu(H) \geq \frac{\nu^*(H)}{d-1+\frac{1}{d}}$. If, in addition, $d > 2$ and H does not contain a copy of the d-uniform projective plane then $\nu(H) \geq \frac{\nu^*(H)}{d-1}$.*

The first half of the following theorem just restates the non-separated part of Theorem 1.1:

Theorem 6.2. [6] *If H is a hypergraph of d-intervals then:*

(1) $\tau(H) \leq (d^2 - d + 1)\nu(H)$, *and*
(2) $\tau(H) \leq d\nu^*(H)$.

Proof. Like before, we assume that all edges of H are contained in $(0, 1)$. Our aim is to show that if $\tau(H) > k$ then $\nu(H) > \frac{k}{d^2-d+1}$. Every point $\vec{x} = (x_1, x_2, \ldots, x_{k+1})$ in $P = \Delta_k$ corresponds to a distribution of k points $p_{\vec{x}}(m) = \sum_{i \leq m} x_i$, $1 \leq m \leq k$ on L. Set $p_{\vec{x}}(0) = 0$ and $p_{\vec{x}}(k+1) = 1$.

For a subset I of $[k+1]$ let B_I consist of all vectors $\vec{x} \in \Delta_k$ for which there exists an edge $h \in H$ satisfying: (a) h does not contain any point $p_{\vec{x}}(m)$, and (b) for each $i \in I$ there exists at least one $j \leq d$ such that $h^j \subseteq (p_{\vec{x}}(i-1), p_{\vec{x}}(i))$. Note that if $B_I \neq \emptyset$ then $|I| \leq d$.

Clearly, the sets B_I are open. By the assumption that $\tau > k$, for every $\vec{x} \in P$ the points $p_{\vec{x}}(m)$, $1 \leq m \leq k$, do not cover H, meaning that there exists $h \in H$ not containing any $p_{\vec{x}}(m)$. This, in turn, means that $\vec{x} \in B_I$ for some $I \subseteq [k+1]$. We have thus shown that $P = \bigcup B_I$.

Let $F = F(J)$ be a face of Δ_k. If $\vec{x} \in F(J)$ then $(p_{\vec{x}}(i-1), p_{\vec{x}}(i)) = \emptyset$ for $i \notin J$, and hence it it impossible to have $h^j \subseteq (p_{\vec{x}}(i-1), p_{\vec{x}}(i))$. Thus $\vec{x} \in B_I$ for some $I \subseteq J$. This proves that $F \subseteq \bigcup_{I \subseteq J} B_I$.

By Theorem 5.4 there exists a balanced set $\mathcal{I}$ of subsets of $[k+1]$, satifying:

(1) $\bigcap_{I \in \mathcal{I}} B_I \neq \emptyset$, and
(2) $|I| \leq d$ for all $I \in \mathcal{I}$.

Since $\mathcal{I}$ is balanced, (2) implies that $\nu^*(\mathcal{I}) \geq \frac{k+1}{d}$. By Theorem 6.1, $\nu(\mathcal{I}) \geq \frac{\nu^*(\mathcal{I})}{d-1+\frac{1}{d}} \geq \frac{k+1}{d(d-1+\frac{1}{d})} > \frac{k}{d^2-d+1}$.

Let M be a matching in $\mathcal{I}$ of size at least $m = \lceil \frac{k}{d^2-d+1} \rceil$.

Let $\vec{x}$ be a point in $\bigcap_{I \in \mathcal{I}} B_I$. For every $I \in \mathcal{I}$ let $h(I)$ be the edge of H witnessing the fact that $\vec{x} \in B_I$. Then the edges $h(I),\ I \in M$ form a matching of size m in H, proving the lower bound on ν.

To prove (2), let $f : \mathcal{I} \to \mathbb{R}^+$ be the fractional matching of size at least $\frac{k+1}{d}$ whose existence we have just proved. Then the function $\tilde{f} : H \to \mathbb{R}^+$ defined by $\tilde{f}(h) = f(I)$ if $h = h(I)$ and $\tilde{f}(h) = 0$ otherwise, is a fractional matching of size at least $\frac{k+1}{d}$ in H. □

For the convenience of the reader, we restate also the separated case of Theorem 1.1:

Theorem 6.3. *In a hypergraph H of separated d-intervals, $\tau(H) \le (d^2 - d)\nu(H)$.*

To prove this we use the following extension of the KKMS theorem proved by Komiya [7]:

Theorem 6.4. *Let P be a polytope, and let a point $q(F) \in F$ be chosen for every face F of P. Let also B_F be a an assignment of an open subset of P to every face F, satisfying the condition that every face G is contained in $\bigcup_{F \subseteq G} B_F$. Then there exists a collection $\mathcal{F}$ of faces such that $q(P) \in conv\{q(F) \mid F \in \mathcal{F}\}$ and $\bigcap_{F \in \mathcal{F}} B_F \neq \emptyset$.*

The theorem is true also if all B_F are closed. The KKMS theorem is the case in which $P = \Delta_k$ and each $q(F)$ is the center of the face F.

Proof of Theorem 6.3. It clearly suffices to prove that if H is a hypergraph of d-intervals satisfying $\tau(H) > kd$ then $\nu(H) \ge \frac{k+1}{d-1}$. We may assume that the ground set of H is the d-fold product $(0,1) \times \ldots \times (0,1)$. We apply Komiya's theorem to $P = \Delta_k \times \Delta_k \times \ldots \times \Delta_k$, the d-fold product of the k-dimensional simplex Δ_k by itself. A point in P has the form:

$$\vec{x} = ((x_1^1, x_2^1, \ldots, x_{k+1}^1), (x_1^2, x_2^2, \ldots, x_{k+1}^2), \ldots, (x_1^d, x_2^d, \ldots, x_{k+1}^d))$$

where $x_i^j \ge 0$ and $\sum_{i=1}^{k+1} x_i^j = 1$ for every j. For $\vec{x} \in P$ let $p_{\vec{x}}(i,j) = \sum_{a \le i} x_a^j$.

Let V be the set of all pairs $\{(i,j) \mid 1 \le i \le k+1,\ 1 \le j \le d\}$. A vertex $\vec{t}$ of P corresponds to a d-tuple $(i^1, 1), (i^2, 2), \ldots, (i^d, d)$ of vertices in V, by the rule $t_b^a = 1$ if $b = i^a$ and $t_b^a = 0$ otherwise (here we are using with respect to $\vec{t}$ the same notation we used for points in P denoted by $\vec{x}$, namely t_b^a is the b coordinate of $\vec{t}$ in the a-th copy of $(0,1)$). To any such vertex we can assign an edge $e_{\vec{t}} = \{(i^1, 1), (i^2, 2), \ldots, (i^d, d)\}$ in the complete d-partite hypergraph with vertex set V and sides $V^j = \{(i,j) \mid 1 \le i \le k+1\}$. Let $B_{\vec{t}}$ be the set of all points $\vec{x} \in P$ for which there exists $h \in H$ satisfying $h^j \subseteq (p_{\vec{x}}(i^j - 1, j), p_{\vec{x}}(i^j, j))$ for all j. Let also $q(\vec{t}) = \vec{t}$ (the only possible choice). For all other faces F of P let $B_F = \emptyset$. Let $q(P)$ be the uniformly all $\frac{1}{k+1}$ vector. The points $q(F)$ for all other faces F do not come into play, so we do not define them.

By our assumption, for no $\vec{x} \in P$ is the set of all points $p_{\vec{x}}(i,j)$ ($1 \le i \le k$, $1 \le j \le d$) a cover for H. Hence $\bigcup B_{\vec{t}} = P$, where the union is over all vertices $\vec{t}$ of P. As in the previous proof, it is also easy to see that for every face $F = conv(T)$ (where T is a set of vertices) $F \subseteq \bigcup\{B_{\vec{t}} \mid \vec{t} \in T\}$. By Theorem 6.4, there exists a set Q of vertices, such that $q(P) \in conv(q(\vec{t}) \mid \vec{t} \in Q)$ and $\bigcap_{\vec{t} \in Q} B_{\vec{t}} \neq \emptyset$.

Let $E = \{e_{\vec{t}} \mid \vec{t} \in Q\}$. Then the hypergraph $D = (V,E)$ is d-partite, and the fact that $q(F) \in conv(q(\vec{t}) \mid \vec{t} \in Q)$ means that D is balanced. This in turn implies that $\nu^*(D)$ is $k+1$ (the size of one side of D). Since D is d-partite, by Theorem 6.1 $\nu(D) \ge \frac{\nu^*(D)}{d-1} \ge \frac{k+1}{d-1}$ (for $d=2$ we are using here König's theorem, rather than Füredi's theorem). Let M be a matching in D with $|M| \ge \frac{k+1}{d-1}$. Let $\vec{x}$ be a point in $\bigcap_{\vec{t} \in Q} B_{\vec{t}}$. By the definition of the sets $B_{\vec{t}}$, for every edge $e = \{(i^1,1),(i^2,2),\ldots,(i^d,d)\} \in M$ there exists an edge $h = h(e) \in H$ with $h^j \subseteq (p_{\vec{x}}(i^j-1,j), p_{\vec{x}}(i^j,j))$. Clearly, the edges $h(e)$, $e \in M$ are disjoint, proving that $\nu(H) \ge \frac{k+1}{d-1}$, as desired. □

7. Bounding the ratio $\frac{\tau^*}{\nu}$ using topology

As already mentioned, it is likely that in order to find the right upper bound on $\frac{\tau^*}{\nu}$ a topological method will be needed. In this section we describe an approach that yields the bound $4d$.

Theorem 7.1. *If H is a hypergraph of d-intervals then $\tau^*(H) \le (4d-6+\frac{3}{d})\nu(H)$.*

Proof. As before, we assume that all edges are contained in $(0,1)$. We will show that if $\tau^*(H) > \alpha$ then $\nu(H) > \frac{\alpha}{4d-6+\frac{3}{d}}$. Let $k = \lfloor \alpha d \rfloor$. The assumption that $\tau^*(H) > \alpha$ implies that for every point $\vec{x} = (x_1, x_2, \ldots, x_{k+1})$ in $P = \Delta_k$, putting weights $\frac{1}{d}$ on each of the points $p_{\vec{x}}(m) = \sum_{i \le m} x_i$, $1 \le m \le k$, does not constitute a fractional cover for H. Note that the points $p_{\vec{x}}(m)$ are taken with multiplicity, namely if such a point repeats q times, it is given weight $\frac{q}{d}$. Therefore, for each $\vec{x} \in \Delta_k$ there exists $h \in H$ that contains fewer than d points $p_{\vec{x}}(m)$ (counted with multiplicity). This implies the following:

Assertion 7.2. *For each $\vec{x} \in \Delta_k$ there exists $h \in H$ that meets at most $2d-1$ intervals of the form $[p_{\vec{x}}(m), p_{\vec{x}}(m+1)]$.*

Proof. Let $h \in H$ be a d-interval that contains fewer than d points $p_{\vec{x}}(m)$. For each interval component h^j of h, the number of intervals $[p_{\vec{x}}(i), p_{\vec{x}}(i+1)]$ that h^j meets is equal to the number of points $p_{\vec{x}}(i)$ it contains, plus 1. Since h has at most d interval components, it follows that the total number of intervals $[p_{\vec{x}}(i), p_{\vec{x}}(i+1)]$ that h meets is at most $d-1+d = 2d-1$. □

Given $\vec{x} \in P$, let $p_{\vec{x}}(0) = 0$ and $p_{\vec{x}}(k+1) = 1$. For $T \subseteq [k+1]$ define $L(T,\vec{x}) = \bigcup_{t \in T}[p_{\vec{x}}(t-1), p_{\vec{x}}(t)]$. We want now to define sets B_T, towards an application of the KKMS theorem. Similarly to the case in the proof of

Theorem 5.3, in order to make sure that the sets are closed we will use a compactness argument.

If $|T| \geq 2d$ we let B_T be the empty set.

Suppose that $|T| \leq 2d-1$. Let $A_T(>\varepsilon)$ (resp. $A_T(\geq \varepsilon)$) be the set of those points $\vec{x}$ for which there exists an edge $h \in H$ contained in $int(L(T,\vec{x}))$ and satisfying $dist(h, \partial(L(T,\vec{x}))) > \varepsilon$ (resp. $dist(h, \partial(L(T,\vec{x}))) \geq \varepsilon$.

Assertion 7.3. *$F(S) \subseteq \bigcup_{\varepsilon>0} \bigcup_{T\subseteq S} A_T(>\varepsilon)$ for every subset S of $[k+1]$.*

Proof. Let $\vec{x}$ be a point in $F(S)$. By Assertion 7.2 there exist $h \in H$ and $R \subseteq [k+1]$ such that $h \subseteq int(L(R,\vec{x}))$ and $|R| \leq 2d-1$. Let $T = R \cap S$. Since $x_i = 0$ for $i \notin S$ we have $int(L(R,\vec{x})) = int(L(T,\vec{x}))$. Taking small enough ε, we have then $\vec{x} \in A_T(>\varepsilon)$. This proves the assertion. □

As in the proof of Theorem 5.3, define $B_T = A_T(\geq \delta)$ for small enough δ. Then $F(S) \subseteq \bigcup_{T\subseteq S} B_T$ for all $S \subseteq [k+1]$.

The sets B_T are closed and satisfy the conditions of the KKMS theorem. Hence there exists a balanced collection $\mathcal{T}$ of subsets of $[k+1]$ such that $\bigcap_{T\in\mathcal{T}} B_T \neq \emptyset$. Let $f : \mathcal{T} \to \mathbb{R}^+$ be a perfect fractional matching. Since $|T| \leq 2d-1$ for all $T \in \mathcal{T}$, it follows that $\nu^*(\mathcal{T}) \geq \frac{|V(\mathcal{T})|}{2d-1} = \frac{k+1}{2d-1}$. Therefore by Theorem 6.1 $\nu(\mathcal{T}) \geq \frac{k+1}{(2d-1)(2d-2+\frac{1}{2d-1})} > \frac{\alpha}{4d-6+\frac{3}{d}}$, namely there exists a matching M of size larger than $\frac{\alpha}{4d-6+\frac{3}{d}}$ in $\mathcal{T}$.

Choose a point $\vec{x} \in \bigcap_{T\in\mathcal{T}} B_T$. For every $T \in \mathcal{T}$ let $h(T)$ be an edge of H witnessing $\vec{x} \in B_T$, a fact entailing $h(T) \subseteq int(L(T,\vec{x}))$. Using the same argument as in the proof of Theorem 5.3 we have that for disjoint $T_1, T_2 \in \mathcal{T}$ the edges $h(T_1)$ and $h(T_2)$ are disjoint, which completes the proof of the theorem.

□

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References

[1] N. Alon, Piercing d-intervals, Disc. Comput. Geom. 19 (1998), 333–334.
[2] Z. Füredi, Maximum degree and fractional matchings in uniform hypergraphs, Combinatorica 1, 2 (1981), 155–162.
[3] A. Gyárfás, The combinatorics of intervals, Lecture notes, 2009.
[4] A. Gyárfás and J. Lehel, A Helly-type problem in trees, in: P. Erdős, A. Rényi and V.T. Sós, eds., *Combinatorial Theory and its Applications* (North-Holland, Amsterdam, 1970), 571–584.
[5] A. Gyárfás and J. Lehel, Covering and coloring problems for relatives of intervals, Discrete Mathematics 55 (1985), 167–180.
[6] T. Kaiser, Transversals of d-intervals, Disc. Comput. Geom. 18, 2, (1997), 195–203.
[7] H. Komiya, A simple proof of K-K-M-S theorem, Econ. Theory 4 (1994), 463–466.
[8] J. Matoušek, Lower bounds on the transversal numbers of d-intervals, Disc. Comput. Geom. 26 (3) (2001), 283–287.

[9] G. Tardos, Transversals of 2-intervals, a topological approach, Combinatorica 15, 1 (1995), 123–134.

[10] G. Tardos, Transversals of d-intervals — comparing three approaches, in: European Congress of Mathematics, Vol. II (Budapest, 1996), Progress in Mathematics 169 (Birkhäuser, Basel, 1998), 234–243.

[11] L. S. Shapley, On balanced games without side payments, in: T. C. Hu and S. M. Robinson (eds.), Mathematical Programming, Math. Res. Center Publ. 30 (Academic Press, New York, 1973), 261–290.

[12] R. G. Stanton, D. D. Cowan and L. O. James, Some results on path numbers, Proc. Louisiana Conf. on Combinatorics, Graph Theory and Computing (1970), 112–135.

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