Flexible Moment Invariant Bases for 2D Scalar and Vector Fields

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ABSTRACT

Complex moments have been successfully applied to pattern detection tasks in two-dimensional real, complex, and vector valued functions.

In this paper, we review the different bases of rotational moment invariants based on the generator approach with complex monomials. We analyze their properties with respect to independence, completeness, and existence and present superior bases that are optimal with respect to all three criteria for both scalar and vector fields.

Keywords

Pattern detection, moment invariants, scalar fields, vector fields, flow fields, generator, basis, complex, monomial

1 INTRODUCTION

Pattern detection is an important tool for the generation of expressive scientific visualizations. Scientific datasets are ever increasing in size, yet the bandwidth of the human visual channel remains constant. Pattern detection algorithms allow us to reduce this abundance of information to simply features in which the scientist is interested.

One of the challanges in pattern detection is that physical phenomena expressed in coordinates usually come with some degrees of freedom that make the search more complex and time-consuming than inherently necessary. The underlying feature is present no matter how it is oriented. Likewise, the exact position or the scale in which a pattern occurs should not change whether or not it is detected. Using pattern detection algorithms that are independent with respect to these coordinate transformations can therefore significantly accelerate the process.

A common and successful class of such algorithms is based on moment invariants. These are characteristic descriptors of functions that do not change under certain transformations. They can be constructed from moments in two different ways: the generator approach and normalization. Moments are the projections of a function onto a function space basis.

During normalization, certain moments are put into a predefined standard position. The remaining moments are then automatically invariant with respect to this transformation. In contrast, the generator approach uses algebraic relations to explicitly define a set of moment invariants that are constructed from the moments through addition, multiplication, or other arithmetic operations.

Each of these approaches comes with its own advantages and disadvantages. Depending on the application, one may be superior to the other. In this paper, we will concentrate on the generator approach. We begin with a review of generators currently in the literature for two-dimensional scalar and vector fields, demonstrating their differences and dicussing shortcomings; we present a flexible basis able to overcome them.

A set of moment invariants should have the following three important qualities:

Completeness: The set is complete if any arbitrary moment invariant can be constructed from it.

Independence.: The set is independent if none of its elements can be constructed from its other elements.

Existence: The set is existent, in other words flexible, if it is generally defined[1] without requiring any specific moments[2] to be non-zero.

[1] We use the arithmetic meaning of defined. For example, the operation $1/x$ is defined for $x \neq 0$ and undefined if $x = 0$.

[2] As a counter example, the so far suggested basis for real valued functions requires at least one moment to be non-zero that suffices $p_0 - q_0 = 1$.

Completeness ensures that the set has the power to discriminate two objects that differ by something other than only a rotation. Independence accelerates feature detection by preventing comparison of redundant values. Finally, existence guarantees that the set can detect any pattern and does not have restrictions to its specific form, such as having a non-vanishing linear component.

In the real-valued case, a complete and independent set of moment invariants was proposed by Flusser in [1]. We build upon his results to construct a basis that generally exists. Since our basis is flexible, it can be adapted, making it robust even if all moments that correspond to rotational non-symmetric complex monomials are close to zero. Further, it is automatically suitable for the detection of symmetric patterns without prior knowledge of the specific symmetry.

Schlemmer et al. [2] were pioneers in the field, being the first to extend the concept of moment invariants to vector fields. Their suggested generator falls short of being a bona fide basis, according to their own definition, as it does not meet the requirements of completeness and independence. A proof can be found in Section 5.1. Later, Flusser et al. [3] proposed the first complete and independent basis of moment invariants for flow fields. In this paper, we build upon these efforts and introduce a novel basis that meets the full set of standards for a basis. As in the real-valued case, our suggested basis is independent, complete, solves the inverse problem, and additionally is generally existent.

2 RELATED WORK

In 1962, moment invariants were introduced to the image processing society by Hu [4]. He used a set of seven rotation invariants.

Teague [5] and Mostafa and Psaltis citeAMP84 advocated for the use of complex moments. This particularly simplifies the construction of rotation invariants as rotations take the simple form of products with complex exponentials.

In 2000, Flusser [1] presented a calculation rule to compute a complete and independent basis of moment invariants of arbitrary order for 2D scalar functions. He also showed that the invariants by Hu [4] are not independent and that his basis solves the inverse problem [6].

Building on Flusser's work, Schlemmer et al. [2] were the first to derive moment invariants for vector fields. In their pioneering work in 2007, they provided a set of five invariants. Later, in his thesis, Schlemmer also presented a general rule for moments of arbitrary order [7].

Apart from the use of complex numbers, moment tensors are the other common framework for the construction of moment invariants. They were suggested by Dirilten and Newman in 1977 [8]. The principal idea is that tensor contractions to zeroth order are naturally invariant with respect to rotation. It is more difficult to answer questions of completeness or independence in the tensor setting [9], but in contrast to the complex appproach, it generalizes more easily to three-dimensional functions. Pinjo et al. [10], for example, estimated 3D orientations from the contractions to first order, which behave like vectors. Another path that has been successfully taken uses spherical harmonics [11, 12, 13, 14] and their irreducible representation of the rotation group. A generalization of the tensor approach to vector fields was suggetsed by Langbein and Hagen [15].

In contrast to the derivation of explicit calculation rules that generate invariants, normalization can be used. A description of normalization for scalar fields can be found in [3]. Bujack et al. followed the normalization approach to construct moment invariants for two-dimensional [16] and three-dimensional [17] vector fields. Additionally, while Liu and Ribeiro [18] do not call it moment normalization, they follow a very similar approach.

The interested reader can find a detailed introduction to the theory of moment invariants in [3] and an overview of feature-based flow visualization in [19].

3 REAL-VALUED FUNCTIONS

Two-dimensional real valued functions $\mathbb{R}^2 \to \mathbb{R}$ are often embedded into the complex plane $\mathbb{C} \sim \mathbb{R}^2 \to \mathbb{R} \subset \mathbb{C}$ to make use of the easy representation of rotations in the setting of complex numbers. We briefly revisit the foundation of moment invariant bases of complex monomials. A more detailed introduction can be found in [3].

For a function $f : \mathbb{C} \to \mathbb{C}$ and $p,q \in \mathbb{N}$, the complex moments $c_{p,q}$ are defined by

$$c_{p,q} = \int_{\mathbb{C}} z^p \bar{z}^q f(z)\,\mathrm{d}z. \tag{3.1}$$

Let $f'(z) : \mathbb{C} \to \mathbb{C}$ differ from f by an inner rotation by the angle $\alpha \in (-\pi,\pi]$

$$f'(z) = f(e^{-i\alpha} z), \tag{3.2}$$

then, the moments $c'_{p,q}$ of f' satisfy

$$c'_{p,q} = e^{i\alpha(p-q)} c_{p,q}. \tag{3.3}$$

Starting with (3.3), Flusser [1] shows that a rotational invariant can be constructed by choosing $n \in \mathbb{N}$ and for $i = 1,..,n$ integers $k_i, p_i, q_i \in \mathbb{N}_0$. If they satisfy

$$\sum_{i=1}^{n} k_i(p_i - q_i) = 0, \tag{3.4}$$

then, the expression

$$I = \prod_{i=1}^{n} c_{p_i,q_i}^{k_i} \tag{3.5}$$

is invariant with respect to rotation. From this formula, infinitely many rotation invariants can be generated, but most of them are redundant. In order to minimize redundancy, Flusser constructs a basis of independent invariants. The following definitions and the theorem stem from [1].

Definition 3.1. An invariant J of the shape (3.5) is considered to be dependent on a set $I_1, ..., I_k$ if there is a function F containing the operations multiplication, involution with an integer exponent and complex conjugation, such that $J = F(I_1, ..., I_k)$.

Definition 3.2. A basis of a set of rotation invariants is an independent subset such that any other element depends on this subset.

3.1 Flusser's Basis

The following basis was suggested by Flusser in [1], where the proof of the theorem can be found.

Theorem 3.3. *Cited from [1]. Let M be a set of complex moments of a real-valued function, $\bar{M}$ the set of their complex conjugates and $c_{p_0,q_0} \in M \cup \bar{M}$ such that $p_0 - q_0 = 1$ and $c_{p_0,q_0} \neq 0$. Let $\mathscr{I}$ be the set of all rotation invariants created from the moments of $M \cup \bar{M}$ according to (3.5) and $\mathscr{B}$ be constructed by*

$$\forall p,q, p \geq q \wedge c_{p,q} \in M \cup \bar{M} : \phi(p,q) := c_{p,q} c_{q_0,p_0}^{p-q} \in \mathscr{B}, \tag{3.6}$$

then $\mathscr{B}$ is a basis of $\mathscr{I}$.

This basis satisfies another important property as it solves the inverse problem, meaning up to the one degree of freedom stemming from the rotational invariance, the original moments can be unambiguously reconstructed from the basis [6].

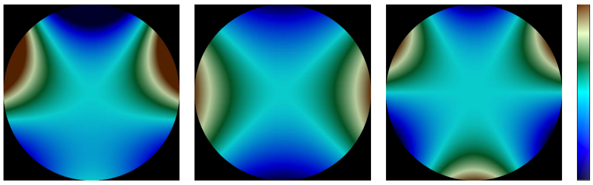
In certain situations, it may occur that no non-zero moment with $p_0 - q_0 = 1$, required for Theorem 3.3, can be found. In this case, Flusser's basis is undefined. However, it is sufficient for c_{q_0,p_0} to have a value close to zero to make the produced invariants unstable and therefore unusable.

Example 3.4. The function

$$f(x,y) = (-y^3 + 3x^2y + x^2 - y^2)\chi(x^2 + y^2 \leq 1) \tag{3.7}$$

with χ corresponding to the characteristic function, has the complex moments $c_{2,0} = \pi/6$, $c_{0,2} = \pi/6$, $c_{3,0} = i\pi/8$, $c_{0,3} = -i\pi/8$, $c_{3,1} = \pi/8$, $c_{1,3} = \pi/8$.

All other moments up to fourth order are zero. There is no $p_0 - q_0 = 1$ with $c_{p_0,q_0} \neq 0$. Therefore, the basis from Theorem 3.3 does not exist. Still, it would be possible to construct moment invariants for f, for example, $c_{3,1}c_{0,2} = \pi^2/48$.



Function (3.7) without rotational symmetry. | Its quadratic part with two-fold symmetry. | Its cubic part with three-fold rotational symmetry.

Figure 1: The function (3.7) from Example 3.4 and its components visualized using the height colormap.

It should be noted that the situation of vanishing moments always occurs with symmetric functions. In this case, Flusser et al. [20] provide a different basis, tailored toward the specific n-fold rotational symmetry, which needs to be known in advance. However, as can be seen in Example 3.4, all moments with $p_0 - q_0 = 1$ can be zero for non-symmetric functions, too.

3.2 Flexible Basis

Motivated by Example 3.4, we propose the following basis. Since it is adaptive, it exists for any pattern.

Theorem 3.5. *Let $M = \{c_{p,q}, p+q \leq o\}$ be the set of complex moments of an arbitrary real-valued function $f : \mathbb{R}^2 \to \mathbb{R}$ up to a given order $o \in \mathbb{N}$. If there is a $0 \neq c_{p_0,q_0} \in M$ with $p_0 - q_0 < 0$, we define the set $\mathscr{B}$ by $\mathscr{B} := \{\phi(p,q), p+q \leq o, p \geq q\}$ with*

$$\phi(p,q) := c_{p,q} c_{p_0,q_0}^{-\frac{p-q}{p_0-q_0}}, \tag{3.8}$$

and otherwise by $\mathscr{B} := \{c_{p,p}, p+p \leq o\}$. Then $\mathscr{B}$ is a basis of all rotation invariants of M, which is generally existent independent of f.

Before embarking on the proof of this theorem, we would like to provide useful context towards a better understanding of the proof.

We start by noting that this basis is tailored toward a given function. Different functions may result in different bases and a basis that exists for one function may not exist for another function. In order to maximize stability, we suggest choosing the lowest order moment, c_{p_0,q_0}, with a magnitude above the average:

$$|c_{p_0,q_0}| \geq \frac{\sum_{p+q<o} |c_{p,q}|}{\sum_{p+q<o}}. \tag{3.9}$$

The fraction in the exponent of (3.8) corresponds to a root of a complex number, which has $|p_0 - q_0|$ solutions. It is not necessary to store the invariants for all complex roots, but only for a single arbitrary but consistent one. However, during the comparison step with the pattern, we need to take this ambiguity into account and compare the arbitrary root of the function to each of the multiple roots of the pattern. We do not need to store the multiple roots of the pattern either as we can compute the missing ones if we know just one invariant $\phi(p,q)$ and the chosen p_0,q_0 from (3.9) using the rule

$$\phi(p,q)e^{\frac{2i\pi k}{p_0-q_0}} \tag{3.10}$$

for $k = 1,...,p_0-q_0$. Please note though it is crucial that all elements $\phi(p,q)$ of the set of stored invariants were generated using the same complex root. We show in detail why it is necessary to work with this ambiguity in Subsection 3.3.

Proof. This proof consists of four parts.

Invariance. We can see from (3.5) and (3.4) that the elements $\phi(p,q)$ are rotation invariant, because of $1(p-q)+(p_0-q_0)(-(p-q)/(p_0-q_0)) = 0$. The elements $c_{p,p}$ are naturally invariant with respect to arbitrary rotations, because of (3.2).

Completeness. We will solve the inverse problem. The assertion then follows from the fundamental theorem of moment invariants [21]. Analogous to [6], we can pick one orientation to remove the degree of freedom that comes from the rotation invariance. We assume $c_{p_0,q_0} \in \mathbb{R}^+$. Firstly, since $c_{p_0,q_0} \in \mathbb{R}^+$, it coincides with its absolute value, which can be constructed from $\phi(q_0,p_0)$ via

$$\begin{aligned} c_{p_0,q_0} &= |c_{p_0,q_0}| = \sqrt{\overline{c_{p_0,q_0}}c_{p_0,q_0}} = \sqrt{c_{q_0,p_0}c_{p_0,q_0}} \\ &= \sqrt{c_{q_0,p_0}c_{p_0,q_0}^{-\frac{q_0-p_0}{p_0-q_0}}} = \sqrt{\phi(q_0,p_0)} \end{aligned} \tag{3.11}$$

because real valued functions suffice

$$c_{p,q} = \overline{c_{q,p}}. \tag{3.12}$$

Please note that the invariant $\phi(q_0,p_0)$ is part of the basis, because from the restriction on the normalizer $p_0-q_0 < 0$ follows the restriction for the elements of the basis $p > q$ with $p = q_0, q = p_0$. Secondly, for all $p > q$, the original moment $c_{p,q}$ can be reconstructed from any of the possibly multiple $\phi(p,q)$ using the calculation rule

$$c_{p,q} = \phi(p,q)c_{p_0,q_0}^{\frac{p-q}{p_0-q_0}}. \tag{3.13}$$

Then, for all $p < q$, the original moments can afterwards be reconstructed from $c_{q,p}$ using the relation (3.12). Finally, for $p = q$, the moments are already part of the basis.

Existence. If all moments with $p_0-q_0 \neq 0$ are zero, the basis reduces to $\{c_{p,p}, p+p \leq o\}$. It is known from [20] that this is a basis for circular symmetric functions[3]. For all other functions, a non-zero non-symmetric moment c_{p_0,q_0} with $p_0-q_0 \neq 0$ can be chosen. If it should suffice $p_0-q_0 > 0$, then we automatically know from (3.12), that $c_{q_0,p_0} \neq 0$, too. It satisfies the constraint $q_0-p_0 < 0$ and the basis exists as defined.

Independence. We use the polar representation $c_{p_0,q_0} = re^{i\phi}$ of the normalizer of a function f to construct the new function

$$f'(z) := r^{\frac{1}{p_0-q_0}} f(e^{\frac{i\phi}{p_0-q_0}}z). \tag{3.14}$$

Using (3.2), we see that moments of f' suffice $c'_{p,q} = c_{p,q}c_{p_0,q_0}^{-(p-q)/(p_0-q_0)}$ and therefore coincide with the basis elements $\phi(p,q)$ of f. Since the moments of f' are independent, so is the basis. If no normalizer c_{p_0,q_0} can be found, the basis consists solely of moments and is therefore independent, too. □

Example 3.6. The flexible basis exists for the function (3.7) from Example 3.4 and Figure 1. In agreement with (3.9) among the moments up to fourth order, we pick $p_0 = 0, q_0 = 2$. Then, the non-zero elements of the basis are

$$\begin{aligned} \phi(2,0) &= c_{2,0}c_{0,2} = \frac{\pi^2}{36}, \\ \phi(3,0) &= c_{3,0}c_{0,2}^{\frac{3}{2}} = \pm\frac{i\pi\sqrt{\pi}^3}{8\sqrt{6}^3}, \\ \phi(3,1) &= c_{3,1}c_{0,2} = \frac{\pi^2}{48}. \end{aligned} \tag{3.15}$$

Pleas note that during the pattern recognition task, the flexible basis that is tailored toward the pattern will be evaluated on the field where the chosen normalizer c_{p_0,q_0} may vanish. The moment invariants always become unstable if the moment c_{p_0,q_0} is close to zero, which leads to very high values in the invariants. But because of 3.9 these areas must be very different from the pattern. So this kind of instability does not influence the result of the pattern matching.

3.3 Multiple Complex Roots

In this subsection, we will show why the proposed treatment of the multiple complex roots is necessary in order to guarantee independence, invariance, completeness, and existence. It may be skipped on first reading.

Invariance. If we restrict the basis from Theorem 3.5 to one representative of the possibly multiple complex roots, the resulting set is no longer invariant with respect to rotation. Without loss of generality, let us choose the root with the lowest non-negative angle to

[3] We call a function circular symmetric or completely rotationally symmetric if its rotated version coincides with the original function independent from the rotation angle α, meaning it suffices $\forall\alpha \in [0,2\pi) : f(z) = R_\alpha f(z)$. One could say, it is n-fold symmetric with $n = \infty$.

the positive real axis. Then, using function f from (3.7) as in Example 3.6, we would pick $\sqrt{\pi/6}$ as the representative complex root of $c_{0,2} = \pi/6$. The generated set would have the form $\phi(2,0) = c_{2,0}c_{0,2} = \pi^2/36$, $\phi(3,0) = c_{3,0}c_{0,2}^{3/2} = i\pi\sqrt{\pi}^3/8\sqrt{6}^3$, $\phi(3,1) = c_{3,1}c_{0,2} = \pi^2/48$. Let f' be f if we rotate it by π, then the moments of

$$f'(x,y) = (y^3 - 3x^2y + x^2 - y^2)\chi(x^2+y^2 \le 1) \quad (3.16)$$

are the same as in Example (3.4) except that the ones of odd order in the middle row change their sign. As a result, the chosen representative root of $c_{0,2}$ is still $\sqrt{\pi/6}$, and the new generated set differs from the previous, because $\phi(3,0) = c_{3,0}c_{0,2}^{3/2} = -i\pi\sqrt{\pi}^3/8\sqrt{6}^3$ has the opposite sign.

Completeness. In many applications, the full discriminative power of a complete basis is not necessarily required. In these cases, we can replace $\phi(p,q)$ from Theorem 3.5 by the simpler formula

$$\phi'(p,q) := c_{p,q}^{p_0-q_0} c_{p_0,q_0}^{-(p-q)}. \quad (3.17)$$

The resulting generator $\mathscr{B}$ can be used instead of the basis from Theorem 3.5. It has only one unique element for each p,q because it does not contain complex roots. But note that this set is not generally complete. To prove that, we revisit the function from Example 3.6 with moments calculated up to fourth order. If we use the basis from (3.8), the invariant $c_{3,1}c_{0,2} = \pi^2/48$ is part of the basis and can therefore be constructed from the basis trivially.

However, if we use $\phi'(p,q)$ from (3.17), we get $\phi'(2,0) = c_{2,0}^2c_{0,2}^2 = \pi^4/6^4$, $\phi'(3,0) = c_{3,0}^2c_{0,2}^3 = -\pi^5/8^26^3$, $\phi'(3,1 = c_{3,1}^2c_{0,2}^2 = \pi^4/8^26^2$, from which $c_{3,1}c_{0,2}$ cannot be constructed. We can only use $\phi'(3,1) = (c_{3,1}c_{0,2})^2$, which does not contain the more detailed information that $c_{3,1}c_{0,2} = \pi^2/48$ was actually positive. As an example, the function

$$\begin{aligned} g(x,y) =& (31(x^2-y^2) - 40(x^4-y^4) - y^3 + 3x^2y) \\ & \chi(x^2+y^2 \le 1) \end{aligned} \quad (3.18)$$

shown in Figure 2 has the moments $c_{2,0} = \pi/6$, $c_{0,2} = \pi/6$, $c_{3,0} = -i\pi/8$, $c_{0,3} = i\pi/8$, $c_{3,1} = -\pi/8$, $c_{1,3} = -\pi/8$. The basis from Theorem 3.5 shows the difference between g and f, because here $\phi_g(3,1) = c_{3,1}c_{0,2} = -\pi^2/48$ has opposite sign than $\phi_f(3,1) = \pi^2/48$ in (3.15). In contrast to that, the generator defined in (3.17) assumes the exact same values $\phi'_g(3,1) = c_{3,1}^2c_{0,2}^2 = \pi^4/8^26^2 = \phi'_f(3,1)$ for g as for f.

Existence. If we restrict ourselves to moments that have no symmetry with respect to rotation whatsoever, i.e. $p_0 - q_0 = 1$, then we have no complex roots and get one unique solution for each p,q. In this case, the basis reduces to the one suggested by Flusser and it may not exist even for non-symmetric functions as was already seen in Example 3.4.

Independence. Considering the multiplicity of the complex roots does not violate the independence if we interpret them in the following way. The multiple roots of an invariant are not independent invariants themselves, but merely manifestations of the same invariant. We do not have to store them separately, because we can construct all roots from one representative using formula (3.10).

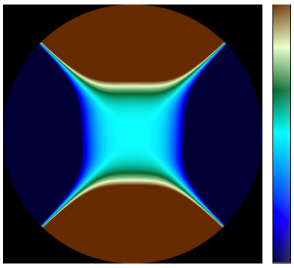


Figure 2: The function $g(x,y)$ from (3.18) visualized using the height color map. The generator (3.17) produces the same invariants as for $f(x,y)$ from Figure 1, even though they are clearly different.

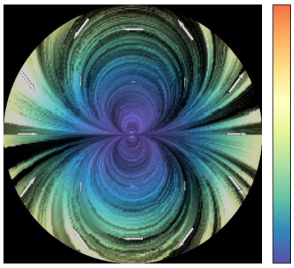


Figure 3: Arrow glyphs and line integral convolution (LIC) [22] of the function (5.10) from Example 5.2. Color and size of the arrows represent the speed. The generator (5.5) does not exist for this pattern.

4 COMPLEX FUNCTIONS

The bases from the previous section were tailored towards real valued functions. Since they satisfy $c_{p,q} = \overline{c_{q,p}}$, it was sufficient to only include $\phi(p,q)$ for $p > q$. Analogous to Theorem 3.5, a flexible basis for arbitrary complex functions that behave under rotations as given in (3.3) can be constructed using the following theorem.

Theorem 4.1. *Let $M = \{c_{p,q}, p+q \le o\}$ be the set of complex moments of a complex function up to a given order $o \in \mathbb{N}$. If there is a $0 \ne c_{p_0,q_0} \in M$ with $p_0 - q_0 \ne 0$, we define the set $\mathscr{B}$ by $\mathscr{B} := \{\phi(p,q), p+q \le o\} \setminus \{\phi(p_0,q_0)\} \cup \{|c_{p_0,q_0}|\}$ with*

$$\phi(p,q) := c_{p,q} c_{p_0,q_0}^{-\frac{p-q}{p_0-q_0}}, \quad (4.1)$$

and otherwise by $\mathscr{B} := \{c_{p,p}, p+p \le o\}$. Then $\mathscr{B}$ is a basis of all rotation invariants of M that exists for any arbitrary complex function.

Proof. The proof works analogously to the proof of Theorem 3.5. □

5 FLOW FIELDS

We can interpret a complex function $f : \mathbb{C} \to \mathbb{C}$ as a two-dimensional vector field by means of the isomorphism between the complex and the Euclidean plane. Analogously to scalar functions, we can make use of the complex moments $c_{p,q}$ as defined in (3.1).

In contrast to the scalar case, flow fields transform by a total rotation. Therefore, we assume that $f'(z) : \mathbb{C} \to \mathbb{C}$ suffices

$$f'(z) = e^{i\alpha} f(e^{-i\alpha} z). \tag{5.1}$$

In this case, the moments $c'_{p,q}$ of f' are related to the moments of f by

$$c'_{p,q} = e^{i\alpha(p-q+1)} c_{p,q}. \tag{5.2}$$

A proof can, for example, be found in [16].

Schlemmer and Heringer [2] showed that analogously to (3.5), any expression of the shape

$$I = \prod_{i=1}^{n} c_{p_i,q_i}^{k_i} \tag{5.3}$$

with $n \in \mathbb{N}$ and for $i = 1, ..., n : k_i, p_i, q_i \in \mathbb{N}_0$ is invariant to total rotation, if

$$\sum_{i=1}^{n} k_i(p_i - q_i + 1) = 0, \tag{5.4}$$

because of (5.2).

5.1 Schlemmer's Generator

The first moment invariants for vector fields were suggested by Schlemmer et al. in 2007 [2]. In that paper, instead of presenting a rule for the generation of moment invariants of arbitrary order, a set of five invariants was explicitly stated. Two years later, in his thesis [7], Schlemmer provided the general formula with which invariants of arbitrary order can be produced. The five moments from [2] are exactly the invariants that are produced from this formula if the maximal order of the moments is restricted to two. We therefore assume that Schlemmer at al. used this formula in their 2007 paper [2], although not explicitly stated.

Theorem 5.1. *Cited from [7]. Let M be the set or a subset of all complex moments $c_{p,q}$ of order $(p+q) \in \{0, ..., o\}$, $o \geq 2$. Let $\mathscr{I}$ be the set of all moment invariants being constructed according to (3.5) from the elements of M. Let $c_{\dot{p},\dot{q}}$ and $c_{\ddot{p},\ddot{q}} \in M$, with $\dot{p} - \dot{q} = \ddot{q} - \ddot{p} = 2$ and $c_{\dot{p},\dot{q}}$ as well as $c_{\ddot{p},\ddot{q}} \neq 0$ If the set $\mathscr{B}$ is constructed as follows:*

$$\mathscr{B} = \{\phi(p,q) := c_{p,q} c_{\dot{p},\dot{q}}^{a_{p-q}} c_{\ddot{p},\ddot{q}}^{b_{p-q}}, c_{p,q} \in M\}, \tag{5.5}$$

with

$$a_m = \begin{cases} 0, & \text{if } m \geq -1 \\ (|m|+1) \operatorname{div} 3, & \text{if } m \leq -2 \end{cases} \tag{5.6}$$

and

$$b_m = \begin{cases} m+1, & \text{if } m \geq -1 \\ (m+1) \bmod 3, & \text{if } m \leq -2 \end{cases} \tag{5.7}$$

then $\mathscr{B}$ is a basis of $\mathscr{I}$.

This theorem in fact happens to be slightly incorrect. Schlemmer's generator is neither independent nor complete and therefore no basis in the sense of Definition 3.2. We prove why in the two following paragraphs and give two explicit examples. In our opinion, this minor inaccurateness does not lessen the impact of their contribution to the pattern detection and flow visualization communities.

Independence. This generator is not independent, because the invariant $\phi(\dot{p},\dot{q})$ and $\phi(\ddot{p},\ddot{q})$ are identical. We can see that from $\dot{p} - \dot{q} = 2, \ddot{p} - \ddot{q} = -2$, and

$$\begin{aligned} \phi(\dot{p},\dot{q}) &\overset{(5.5)}{=} c_{\dot{p},\dot{q}} c_{\dot{p},\dot{q}}^{a_2} c_{\ddot{p},\ddot{q}}^{b_2} \overset{(5.6),(5.7)}{=} c_{\dot{p},\dot{q}} c_{\dot{p},\dot{q}}^{0} c_{\ddot{p},\ddot{q}}^{3} = c_{\dot{p},\dot{q}} c_{\ddot{p},\ddot{q}}^{3}, \\ \phi(\ddot{p},\ddot{q}) &\overset{(5.5)}{=} c_{\ddot{p},\ddot{q}} c_{\dot{p},\dot{q}}^{a_{-2}} c_{\ddot{p},\ddot{q}}^{b_{-2}} \overset{(5.6),(5.7)}{=} c_{\ddot{p},\ddot{q}} c_{\dot{p},\dot{q}}^{1} c_{\ddot{p},\ddot{q}}^{2} = c_{\dot{p},\dot{q}} c_{\ddot{p},\ddot{q}}^{3}. \end{aligned} \tag{5.8}$$

Completeness. This generator is not complete, because the magnitudes $|c_{\dot{p},\dot{q}}|$ and $|c_{\ddot{p},\ddot{q}}|$ cannot be reconstructed from its elements. That follows from the fact that given the moments $c_{\dot{p},\dot{q}}$ and $c_{\ddot{p},\ddot{q}}$ of a function f, any function f' with $c'_{\dot{p},\dot{q}} = s^3 c_{\dot{p},\dot{q}}$ and $c'_{\ddot{p},\ddot{q}} = c_{\ddot{p},\ddot{q}}/s$ with arbitrary $s \in R^+$ will produce the same $\phi(\dot{p},\dot{q}) = \phi(\ddot{p},\ddot{q})$, because of

$$\phi'(\dot{p},\dot{q}) \overset{(5.8)}{=} c'_{\dot{p},\dot{q}} c'^{3}_{\ddot{p},\ddot{q}} = s^3 c_{\dot{p},\dot{q}} (\frac{1}{s} c_{\ddot{p},\ddot{q}})^3 = \phi(\dot{p},\dot{q}). \tag{5.9}$$

The generator can be transformed into a basis via $\mathscr{B} \setminus \{\phi(\dot{p},\dot{q})\} \cup \{|c_{\dot{p},\dot{q}}|\}$. But even with this correction, the basis is not well-chosen. For one, it is unnecessarily complicated, because it requires evaluation of the two auxiliary functions (5.6) and (5.7) and each element can consist of up to thee factors. Further, it does not exist for functions that do not have non-zero $c_{\dot{p},\dot{q}} \neq 0$ as well as $c_{\ddot{p},\ddot{q}} \neq 0$ with $\dot{p} - \dot{q} = \ddot{q} - \ddot{p} = 2$. This situation is similar to the one in Subsection 3.1. But in this case, even two non-vanishing moments of specific orders need to be present, which increases the number of cases in which the generator does not exist.

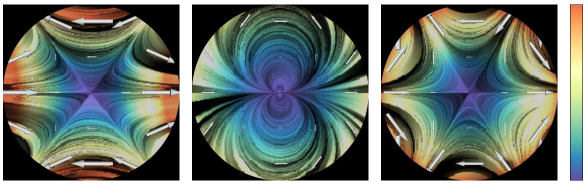
Example 5.2. The vector field given by the function

$$f(z) = z^2 \chi(|z| \leq 1) \tag{5.10}$$

has only one non-zero moment up to third order $c_{0,2} = \pi/3$. It is visualized in Figure 3. Even though it is not symmetric, Schlemmer's generator does not exist, because $c_{\dot{p},\dot{q}} \neq 0$ cannot be found to suffice $\dot{p} - \dot{q} = 2$.

Example 5.3. The vector field given by the function

$$f(z) = (z^2 + 2\bar{z}^2)\chi(|z| \leq 1), \tag{5.11}$$



The function has no symmetry. Its two-fold symmetric part. Its three-fold symmetric part.

Figure 4: Arrow glyphs and LIC of the function (5.11) from Example 5.3 and its components. The color and the size of the arrows represent the speed of the flow.

with χ being the characteristic function, is visualized in Figure 4. It has two non-zero moments up to third order

$$c_{0,2} = \frac{\pi}{3}, \quad c_{2,0} = \frac{2\pi}{3}. \tag{5.12}$$

Here, Schlemmer's generator does exist, because we can choose $c_{\dot{p},\dot{q}} = c_{2,0}$ and $c_{\ddot{p},\ddot{q}} = c_{0,2}$, but it contains only the redundant information

$$\begin{aligned}\phi(0,2) &= c_{0,2}c_{2,0}^{a_{-2}}c_{0,2}^{b_{-2}} = c_{0,2}c_{2,0}^{1}c_{0,2}^{2} = 2(\frac{\pi}{3})^4,\\ \phi(2,0) &= c_{2,0}c_{2,0}^{a_2}c_{0,2}^{b_2} = c_{2,0}c_{2,0}^{0}c_{0,2}^{3} = 2(\frac{\pi}{3})^4,\end{aligned} \tag{5.13}$$

from which we cannot reconstruct the magnitudes of the moments.

5.2 Flusser et al.'s Basis

A straight forward approach to generate a basis of moment invariants for vector fields was suggested by Flusser et al. in [3].

Theorem 5.4. *Let M be the set of moments up to the order $o \in \mathbb{N}$ and $c_{p_0,q_0} \neq 0$ satisfying $p_0 - q_0 = -2$. Further let $\mathscr{I}$ be the set of all rotation invariants created from the moments of M according to (5.3) and $\mathscr{B}$ be constructed by*

$$\forall p,q,p+q \leq o : \phi(p,q) := c_{p,q}c_{p_0,q_0}^{(p-q+1)} \in \mathscr{B}, \tag{5.14}$$

then $\mathscr{B} \setminus \{\phi(p_0,q_0)\} \cup \{|\phi(p_0,q_0)|\}$ is a basis of $\mathscr{I}$.

This produces not only an independent and complete set, but is also more flexible than Schlemmer's generator as it only needs a single specific non-zero moment, not two. Further, it is simpler and more intuitive because it does not need any additional series such as (5.6) and (5.7).

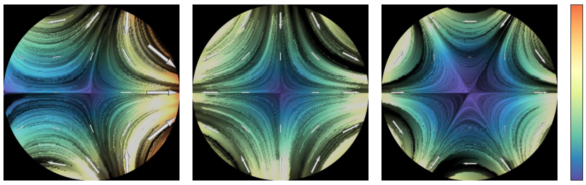
Example 5.5. Flusser's basis exists for the vector field given by the function (5.10) from Example 5.2 and Figure 3. It has one non-zero element $|c_{0,2}| = 2\pi/3$.

Example 5.6. Flusser's basis exists for the vector field given by the function (5.11) from Example 5.3, visualized in Figure 4, and, up to one degree of freedom, the moments can be reconstructed from the basis

$$|c_{0,2}| = \frac{2\pi}{3}, \quad \phi(2,0) = c_{2,0}c_{0,2}^{3} = 8(\frac{\pi}{3})^4. \tag{5.15}$$

To show that, we fix the rotational degree of freedom by setting $c_{0,2} \in \mathbb{R}^+$ and get

$$c_{0,2} = |c_{0,2}| = 2\frac{\pi}{3}, \quad c_{2,0} = \phi(2,0)c_{0,2}^{-3} = \frac{\pi}{3}. \tag{5.16}$$



The function has no rotational symmetry. Its linear part with two-fold symmetry. Its quadratic part with three-fold symmetry.

Figure 5: Arrow glyphs and LIC of the function (5.17) from Example 5.7. The color and the size of the arrows represent the speed of the flow.

Example 5.7. The vector field given by the function

$$f(z) = (\bar{z} + \bar{z}^2)\chi(|z| \leq 1) \tag{5.17}$$

has three non-zero moments up to third order

$$c_{1,0} = \frac{\pi}{2}, \quad c_{2,0} = \frac{\pi}{3}, \quad c_{2,1} = \frac{\pi}{4} \tag{5.18}$$

and is visualized in Figure 5. Here, Flusser's basis does not exist because we cannot find any $c_{p_0,q_0} \neq 0$ with $p_0 - q_0 = -2$, even though the function is not symmetric.

5.3 Flexible Basis

Analogous to the scalar case, we can derive a robust basis even for patterns that do not have a numerically significant moment of one-fold symmetry.

Theorem 5.8. *Let $M = \{c_{p,q}, p+q \leq o\}$ be the set of complex moments of a vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$ up to a given order $o \in \mathbb{N}$. If there is a $0 \neq c_{p_0,q_0} \in M$ with $p_0 - q_0 + 1 \neq 0$, we define the set $\mathscr{B}$ by $\mathscr{B} := \{\phi(p,q), p+q \leq o\} \setminus \{\phi(p_0,q_0)\} \cup \{|c_{p_0,q_0}|\}$ with*

$$\phi(p,q) := \phi(p,q) := c_{p,q}c_{p_0,q_0}^{-\frac{p-q+1}{p_0-q_0+1}}, \tag{5.19}$$

and otherwise by $\mathscr{B} := \{c_{p,p+1}, p+p+1 \leq o\}$. Then $\mathscr{B}$ is a basis of all rotation invariants of M, which generally exists independent of f.

Proof. The proof works analogously to the proof of Theorem 3.5. □

Remark 5.9. This last basis of invariants is equivalent to the normalization approach proposed by Bujack et al. [23].

Algorithm 1 Pattern Detection with Flexible Basis.

Input: $N_x \times N_y$ scalar field: f, $B_r(0)$ pattern: g, scales: $\{s_1,..,s_{N_s}\}$, maximum moment order: n,

for $p+q \leq n$ **do**
  moments of pattern: $c^g_{p,q} \overset{(3.1)}{=} \int_{B_r(0)} z^p \bar{z}^q g(z)\,\mathrm{d}z$,
end for
for $o = 0,..,n, p = 0,..,o, q = p,..,o-p,$ **do**
  if $|c_{p,q}| \geq \frac{\sum_{p+q<o} |c_{p,q}|}{\sum_{p+q<o}}$ **then**
    choose normalizer (3.9) $p_0 = p, q_0 = q$
    break
  end if
end for
for $p+q \leq n$ **do**
  basis for pattern: $\phi^g(p,q) \overset{(3.8)}{=} c^g_{p,q}(c^g_{p,q})^{-\frac{p-q}{p_0-q_0}}$,
end for
for $x \in N_x \times N_y, s = s_1,..,s_{N_s}$ **do**
  for $p+q \leq n$ **do**
    field mom.: $c^f_{p,q}(x,s) = \int_{B_s(x)} z^p \bar{z}^q f(z)\,\mathrm{d}z$,
  end for
  for $p+q \leq n$ **do**
    basis: $\phi^f(p,q)(x,s) \overset{(3.8)}{=} c^f_{p,q}(x,s)(c^f_{p,q})^{-\frac{p-q}{p_0-q_0}}$,
  end for
  Euclidean distance over $|p_0-q_0|$ roots (3.10): $D(x,s) = \min_{k=1,..,|p_0-q_0|} (\sum_{p+q\leq n} (\phi^f(p,q)(x,s) - \phi^g(p,q) e^{\frac{2i\pi k}{p_0-q_0}})^2)^{\frac{1}{2}}$,
end for

Output: similarity of the pattern p to the field f at position x and scale s: $S(x,s) = D(x,s)^{-1}$.

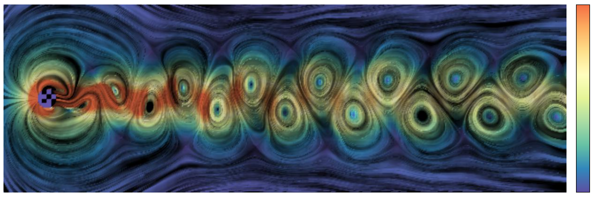
Example 5.10. The flexible basis exists for the vector field (5.17) from Example 5.7, visualized in Figure 5. Any of the three non-zero moments up to third order (5.18) can be chosen as normalizer c_{p_0,q_0}. In order to maximize stability, the proposed algorithm would choose $c_{p_0,q_0} = c_{1,0}$, resulting in two solutions of the complex square root $c_{1,0}^{-1/2} = \pm\sqrt{\frac{\pi}{2}}$ and the basis

$$|c_{1,0}| = \frac{\pi}{2}, \quad \phi(2,0) = \pm\frac{\sqrt{2\pi}}{3}, \quad \phi(2,1) = \frac{1}{2}. \tag{5.20}$$

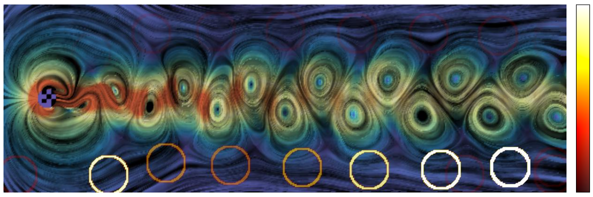
The algorithmic description of the pattern detection for the scalar case can be found in Algorithm 1.

6 EXPERIMENT

We apply the different vector field bases to a pattern detection task in a vector field. The dataset is a computational fluid dynamics simulation of the flow behind a cylinder. The characteristic pattern of the fluid is called the von Kármán vortex street. A visualization of the vortices with removed average flow can be found in Figure 6a. The direction of the flow is visualizaed using line integral convolution [22] and the speed is color coded using the colormap from Figure 7.



The non-flexible bases do not exist for moments up to first order. The algorithm does not produce any output.



The flexible basis does exist with normalizer $c_{1,0}$. The pattern from Figure 7 and its repetitions are correctly detected.

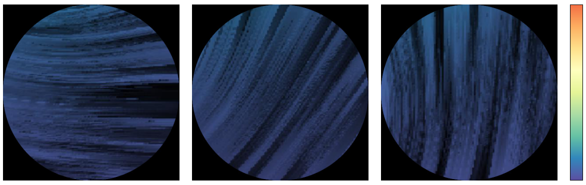
Figure 6: Result of the pattern detetction task using only moments up to first order. The speed of the flow is encoded using the colorbar on the top, the similarity of the field to the pattern using the colorbar on the bottom.

In our experiments, we consider moments up to first order in Figure 6 and moments up to second order in Figure 8. Please note that the basis suggested by Schlemmer [7] from Theorem 5.1 and the one suggested by Flusser [3] from Theorem 3.3 do not exist for moments calculated only up to first order, because a moment c_{p_0,q_0} with $p_0 - q_0 = -2$ cannot be found using only $c_{0,0}, c_{1,0}$, and $c_{0,1}$. For moments up to second order, there is only one potential moment $c_{p_0,q_0} = c_{0,2}$ satisfying $p_0 - q_0 = -2$, which is why there is only one basis configuration for these two approaches. They coincide for the moments up to second order, except for the magnitude of the normalizer $|c_{0,2}|$. The remaining moment invariants are

$$c_{0,0}c_{0,2}, \quad c_{0,1}, \quad c_{1,0}c_{0,2}^2, \quad c_{1,1}c_{0,2}, \quad c_{2,0}c_{0,2}^3, \tag{6.1}$$

as already presented in [2].

Then, as long as the normalizer $c_{0,2}$ is numerically nonzero, all three bases will produce stable and identical results up to minor numerical differences. To show the difference between the flexible and non-flexible bases, we therefore use the pattern from Figure 7a, which satisfies $|c_{0,2}| < 0.01$. This pattern was extracted from the dataset itself. Its position in the von Kármán vortex street can be found in the lower, rightmost circle of Figure 6b. Since the only element which differs in the two non-flexible bases is close to zero, the results of the two are almost identical. The differences are numerically small and cannot be perceived by the human eye. To save space, we plot only the instance that corresponds to Schlemmer's basis. The other is identical.

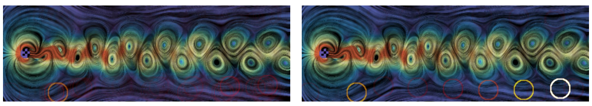


(a) Pattern cut out from the dataset. (b) The pattern rotated by $\pi/3$. (c) The pattern rotated by $\pi/2$.

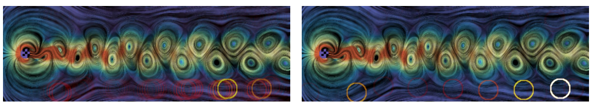
Figure 7: The pattern in different orientations. It was cut out from the dataset at the position of the lower right-most white circle in Figure 6b.

The output of our pattern detection algorithm are circles that indicate the position, the size, and the similarity of the matches. Similarity is encoded in the colormap in the bottom row of Figure 6. The higher the similarity, the brighter the color of the corresponding circle. The color white applies to all matches that have a Euclidean distance of all the moment invariants of less than 0.02. A more detailed description of the algorithm and the visualization can be found in [16].

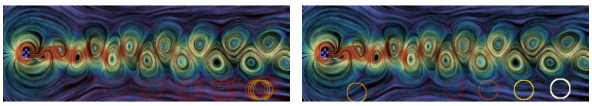
In Figure 6b, we can see that the flexible basis exists even for this pattern and that it correctly finds the pattern's original position. It further detects similar occurrences as it repeats itself in the periodic von Kármán street. As expected, the further we move towards the obstacle, the similarity in each repetition decreases, as indicated by the decreasing brightness of the circles.



non-flexible bases for the pattern oriented as in Figure 7a. Flexible bases for the pattern oriented as in Figure 7a.



non-flexible bases for the pattern oriented as in Figure 7b. Flexible bases for the pattern oriented as in Figure 7b.



non-flexible bases for the pattern oriented as in Figure 7c. Flexible bases for the pattern oriented as in Figure 7c.

Figure 8: Result of the pattern detetction task using moments up to second order. The result of the algorithm using the non-flexible bases is unstable (left). It depends on the orientation of input pattern. In contrast to that, the flexible basis produces consistent results (right).

Figure 8 compares the output of the algorithm using the flexible basis from Theorem 5.8 and the two non-flexible bases for moments up to second order. To show the instability of the non-flexible bases, we used three different instances of the pattern. They differ solely by their orientation. Theoretically, the invariants of all three bases should be invariant with respect to this degree of freedom and produce the same results for all three instances. But as can be seen in the left column of Figure 8, this is not true for the non-flexible bases. Depending on the orientation of the pattern, the similarity of the exact location of the pattern in the field is rather low. Sometimes its position is not the match with the highest similarity, or multiple fuzzy matches occur. On the right side, we can see that the flexible basis produces coherent, stable, and correct results independent from the orientation of the pattern.

7 DISCUSSION

We have reviewed the different bases of moment invariants built from complex monomials using the generator approach and compared their behavior with respect to three important qualities such a basis should suffice: independence, completeness and general existence.

For scalar fields, the basis suggested by Flusser [1] is complete and independent, but it only exists if the pattern has a non-zero moment that is not rotationally symmetric. We have extended his basis to one that always exists, no matter how the values of the moments of a function are distributed.

For vector fields, the first generator approach was suggested by Schlemmer [7]. We show that his set of moment invariants is neither complete nor independent and therefore does not satisfy the properties of a basis. As a result, Flusser et al. [3] were the first to provide a basis of moment invariants for vector fields using the generator approach. As in the scalar case, their basis is complete and independent, but requires a non-zero moment that has no rotational symmetry. We have derived an extension that exists for arbitrary vector fields and found it to coincide with the normalization approach by Bujack et al. [16].

One of the most interesting observations in this work is the equivalence of the optimal generator approach with the optimal normalization approach. Assuming that this fact should also be true for three-dimensional fields, it might be used for the study of 3D moment invariants. The 3D situation is much more complex and neither the generator nor the normalization approach have so far resulted in a set of moment invariants that is complete, independent, and generally existing. Assuming equivalence might guide future research to improve both methods.

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