Bounding the distance among longest paths in a connected graph

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Abstract

It is easy to see that in a connected graph any 2 longest paths have a vertex in common. For $k \geq 7$, Skupień in 1966 obtained a connected graph in which some k longest paths have no common vertex, but every $k-1$ longest paths have a common vertex. It is not known whether every 3 longest paths in a connected graph have a common vertex and similarly for $4, 5,$ and 6 longest path. Fujita et al. in 2015 give an upper bound on distance among 3 longest paths in a connected graph. In this paper we give a similar upper bound on distance between 4 longest paths and also for k longest paths, in general.

1 Introduction

In 1966 Gallai in [4] asked whether all longest paths in a connected graph have a vertex in common. Couple of years later, several counterexamples were found, see [9], [10], and [11]. In 1976 Thomassen in [8] showed that there exist infinitely many counterexamples to Gallai's question.

On the other hand, if we restrict to a special class of graphs, the answer to Gallai's question may become positive. For example in a tree, all longest paths must have a vertex in common. Klavžar and Petkovšek in [6] proved that it is also true for split graphs and cacti and Balister et al. in [2] proved it for the class of circular arc graphs.

Another approach to Gallai's question is to ask, what happens if we consider a fixed number of longest paths. It is easy to see that every 2 longest paths in a connected graph have a common vertex. For 3 longest paths, the question remains open. This has been originally asked by Zamfirescu in [12].

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Conjecture 1. [12] *For every connected graph, any* 3 *of its longest paths have a common vertex.*

There are few results dealing with this conjecture. Axenovich in [1] proved that it is true for connected outerplanar graphs and de Rezende et al. in [3] showed that Conjecture 1 is true for connected graphs in which all nontrivial blocks are hamiltonian.

For $k \geq 7$, Skupień in [7] obtained a connected graph in which some k longest paths have no common vertex, but every $k-1$ longest paths have a common vertex. Regarding this, it is still valid to ask wheter not only 3 but also 4, 5, and 6 longest path in a connected graph have a common vertex.

In [5] the authors introduced a parameter to measure the distance among the longest paths in a connected graph and proved an upper bound of this parameter for 3 longest paths. To state their result we give some definitions first.

Let G be a connected graph. Let $\ell(G)$ be the length of any longest path in G and $\mathcal{L}(G) = \{P \mid P \text{ is a path in } G \text{ with } |V(P)| = \ell(G)+1\}$ be a set of longest paths of G. For $x, y \in V(G)$, let $d_G(x, y)$ be the distance between x and y in G. For a vertex $x \in V(G)$ and a subset $U \subseteq V(G)$, let $d_G(x, U) = min\{d_G(x, y)| \ y \in U\}$. For $\mathcal{P} \subseteq \mathcal{L}(G)$ we call *path-distance-function* $f(G, \mathcal{P}) = min\{\sum_{P\in\mathcal{P}} d_G(v, V(P)) \mid v \in V(G)\}$.

For a class of graphs $\mathcal{G}$ and an integer k, we introduce *path-distance-ratio* $d_k(\mathcal{G}) = max\frac{f(G,\mathcal{P})}{|V(G)|}$, where the maximum is taken over all the graphs of $\mathcal{G}$ and their sets of longest paths $\mathcal{P} \subseteq \mathcal{L}(G)$ with $|\mathcal{P}| = k$.

Let $\mathcal{G}_c$ be a class of connected graphs. The question whether for every connected graph any 3 longest paths have a vertex in common translates into the question whether $d_3(\mathcal{G}_c) = 0$. On the other hand, Skupień in [7] constructed a graph on 17 vertices, in which there are 7 longest paths without a common vertex, this graph implies that $d_7(\mathcal{G}_c) \geq \frac{1}{17}$.

Now we can state the result by Fujita et al. from [5].

Theorem 2. [5] *Let $\mathcal{G}_c$ be a class of connected graphs. Then $d_3(\mathcal{G}_c) \leq \frac{1}{17}$.*

In this paper we prove similar results for 4 longest path and also for k longest paths, in general.

Theorem 3. *Let $\mathcal{G}_c$ be a class of connected graphs. Then $d_4(\mathcal{G}_c) \leq \frac{3}{16}$.*

By picking any vertex of a connected graph G, we see that $d_k(\mathcal{G}_c)$ can be bounded by k. We show that it can be improved as roughly $\frac{k}{6}$.

Theorem 4. *Let $\mathcal{G}_c$ be a class of connected graphs and let $k \geq 3$ be an integer. Then $d_k(\mathcal{G}_c) \leq \frac{k^3-4k^2+5k-2}{6k^2-8k}$.*

2 Proofs

In our proofs, we adapt ideas of [5]. We start by giving several technical definitions.

Let G be a connected graph. Let U and V be two sets of vertices of G, let P be a path in G and Q be a subpath of P. Let u and v be the end-vertices of Q, we say Q is a $U-V$ *path* on P if $u \in U$ and $v \in V$. A vertex of a path which is not its end-vertex is an *int-vertex* of the path. Let uPv denote the $\{u\} - \{v\}$ path on P. Futhermore, let $\check{u}Pv = uPv - u$, $uP\check{v} = uPv - v$ and $\check{u}P\check{u} = uPv - \{u, v\}$. For a set $\mathcal{P} = \{P, P_1, P_2, ..., P_{k-1}\} \subseteq \mathcal{L}(G)$ and $i \neq j \in \{1, 2, ..., k-1\}$, a $V(P_i) - V(P_j)$ path Q on P is *good* if $V(Q) \cap V(P_m) \neq \emptyset$ for every $m = 1, 2, ..., k-1$ and neither P_i nor P_j contain an int-vertex of Q. Let $t_{\mathcal{P}}(P)$ be the number of all good paths of P and $t'_{\mathcal{P}}(P)$ be the maximum number of all non-intersecting (no edge in common) good paths on P. By Proposition 3 in [5], every 2 longest paths intersect. Thus, we have that $t_{\mathcal{P}}(P) \geq t'_{\mathcal{P}}(P) \geq 1$ for every $P \in \mathcal{P}$. For a path $P \in \mathcal{P}$, let $X^i_{\mathcal{P}}(P)$ denote the set of all vertices of P which are exactly on i paths from $\mathcal{P}$. Let $n_i = |\bigcup_{P \in \mathcal{P}} X^i_{\mathcal{P}}(P)|$.

Lemma 5. *Let G be a connected graph of order n and $\mathcal{P} \subseteq \mathcal{L}(G)$ with $|\mathcal{P}| = k \geq 3$. If $f(G, \mathcal{P}) > 0$, then*

$$n \geq \frac{k \cdot \ell(G) + k + (k-2)n_1 + (k-3)n_2 + ... + n_{k-2}}{k-1}.$$

Proof. Clearly $n \geq n_1 + n_2 + ... + n_{k-1} + n_k$, where $n_k = 0$, and $n \geq k(\ell(G) + 1) - n_2 - 2n_3 - ... - (k-3)n_{k-2} - (k-2)n_{k-1}$. Hence $n \geq k \cdot \ell(G) + k - n_2 - 2n_3 - ... - (k-3)n_{k-2} - (k-2)(n - n_1 - n_2 - ... - n_{k-2})$ and the result follows. □

Lemma 6. *Let G be a connected graph and $\mathcal{P} \subseteq \mathcal{L}(G)$ with $|\mathcal{P}| = k$. If there exists a path $P \in \mathcal{P}$ with $t'_{\mathcal{P}}(P) = 1$, then $f(G, \mathcal{P}) = 0$.*

Proof. To the contrary, we suppose there is a path $P = v_1 v_2 ... v_{\ell(G)+1}$ with $t'_{\mathcal{P}}(P) = 1$ and $f(G, \mathcal{P}) > 0$. By $f(G, \mathcal{P}) > 0$, every good path on P contains an edge. We consider the 'left-most' good path Q on P; more formally, we consider the good path $Q = v_i v_{i+1} ... v_j$ such that there is no good path on P containing a vertex v_k with $k < i$. Let $\mathcal{P}_j$ denote the set of paths of $\mathcal{P}$ which contain v_j. By the choice of Q, some path of $\mathcal{P}_j$ contains no vertex v_k with $k < j$, and thus the length of $v_1 v_2 ... v_j$ is at most $\frac{1}{2}\ell(G)$. Similarly, we consider the 'right-most' good path $Q' = v_{i'} v_{i'+1} ... v_{j'}$ and we see that the length of $v_{i'} v_{i'+1} ... v_{\ell(G)+1}$ is at most $\frac{1}{2}\ell(G)$. By the assumption $t'_{\mathcal{P}}(P) = 1$, the paths Q and Q' have an edge in common, so $j > i'$, hence the length of P is shorter than $\ell(G)$, a contradiction. □

Lemma 7. *Let G be a connected graph and $\mathcal{P} \subseteq \mathcal{L}(G)$ with $|\mathcal{P}| = k \geq 3$. Let $P \in \mathcal{P}$ and let Q be a good path on $\mathcal{P}$. Then the following two statements hold:*

(i) $f(G, \mathcal{P}) \leq \frac{|V(Q)|-1}{2}(k-1)$;

(ii) $|X^1_{\mathcal{P}}(P) \cup X^2_{\mathcal{P}}(P) \cup ... \cup X^{k-2}_{\mathcal{P}}(P)| \geq t'_{\mathcal{P}}(P)(\frac{2}{k-1} f(G, \mathcal{P}) - 1)$.

Proof. Note that if $f(G, \mathcal{P}) = 0$, then the statement holds. Suppose $f(G, \mathcal{P}) \geq 1$. In particular, every good path on $\mathcal{P}$ contains at least two vertices. Let $x \in V(Q)$ such that

$\sum_{P' \in \mathcal{P}} d_G(x, P') \leq \sum_{P' \in \mathcal{P}} d_G(y, P')$ for every $y \in V(Q)$. Then

$$f(G, \mathcal{P}) \leq \sum_{P' \in \mathcal{P}} d_G(x, P') \leq \frac{|V(Q)| - 1}{2}(k-1).$$

For any path P of $\mathcal{P}$ and any good path Q' on P, no int-vertex of Q' is in $X^{k-1}_{\mathcal{P}}(P)$, therefore $|V(Q') \cap (|X^1_{\mathcal{P}}(P) \cup X^2_{\mathcal{P}}(P) \cup ... \cup X^{k-2}_{\mathcal{P}}(P))| \geq |V(Q')| - 2 \geq \frac{2}{k-1} f(G, \mathcal{P}) - 1$. Let $\mathcal{Q}$ be a maximum set of non-intersecting good paths on P. By the definition, $t'_{\mathcal{P}}(P) = |\mathcal{Q}|$, and we have

$$|X^1_{\mathcal{P}}(P) \cup X^2_{\mathcal{P}}(P) \cup ... \cup X^{k-2}_{\mathcal{P}}(P)| \geq |\cup_{Q \in \mathcal{Q}} (V(Q) \cap (X^1_{\mathcal{P}}(P) \cup X^2_{\mathcal{P}}(P) \cup ... \cup X^{k-2}_{\mathcal{P}}(P)))| \geq$$

$$\geq \sum_{Q \in \mathcal{Q}} (|V(Q)| - 2) \geq t'_{\mathcal{P}}(P) \left(\frac{2}{k-1} f(G, \mathcal{P}) - 1 \right).$$

□

Corollary 8. *Let G be a connected graph and $\mathcal{P} \subseteq \mathcal{L}(G)$ with $|\mathcal{P}| = 4$. Let $\mathcal{P} = \{P, P_1, P_2, P_3\}$ and let Q be a good path on $\mathcal{P}$. Then the following two statements hold:*

(i) $f(G, \mathcal{P}) \leq |V(Q)| - 1$*;*

(ii) $|X^1_{\mathcal{P}}(P) \cup X^2_{\mathcal{P}}(P)| \geq t'_{\mathcal{P}}(P)(f(G, \mathcal{P}) - 1)$.

Proof. The proof is the same as the proof of Lemma 7 with respect to the following. Let u, v be end-vertices of Q. Assume that Q is a $V(P_1) - V(P_2)$ path on P (otherwise we renumber the paths) and we consider a vertex $x \in V(Q) \cap V(P_3)$. Then

$$f(G, \mathcal{P}) \leq \sum_{P \in \mathcal{P}} d_G(x, P) = d_G(x, P_1) + d_G(x, P_2) \leq d_G(u, v) \leq |V(Q)| - 1.$$

Then we use Corollary 8(i) instead of Lemma 7(i) and the result follows. □

Proof of Theorem 4. Suppose that $f(G, \mathcal{P}) \geq 1$. Hence $t'_{\mathcal{P}}(P) \geq 2$ by Lemma 6. Let $P \in \mathcal{P}$ be a path minimizing $|X^1_{\mathcal{P}}(P) \cup X^2_{\mathcal{P}}(P) \cup ... \cup X^{k-2}_{\mathcal{P}}(P)|$. Let $\mathcal{P} - \{P\} = \{P_1, P_2, ..., P_{k-1}\}$ and u_i, v_i be the end-vertices of P_i for $i \in \{1, 2, ..., k-1\}$. Assume that Q is a good $V(P_1) - V(P_2)$ path on P with end-vertices u, v (otherwise we renumber paths $P_1, P_2, ..., P_{k-1}$). Let R be the shortest $\{u\} - V(P_2)$ path on P_1 and $x \in V(R) \cap V(P_2)$. We may assume that $|V(u_2 P_2 v)| \leq |V(u_2 P_2 x)|$ (see Figure 1).

We have $|V(R)| \geq 2$ from $f(G, \mathcal{P}) \geq 1$ and $|V(Q)| \geq \frac{2f(G,\mathcal{P})}{k-1} + 1$ from Lemma 7(i). Since $vQ\check{u}$ contains no vertex of $V(P_1)$, $vQuRx$ is a path in G. Futhermore, since $\check{v}QuP_1\check{x}$ contains no vertex of $V(P_2)$, $S_1 = v_2 P_2 vQuR\check{x}$, $S_2 = u_2 P_2 vQuRxP_2 v_2$, and $S_3 = u_2 P_2 xRuQ\check{v}$ are paths in G (see Figure 2).

By comparing the lengths of P_2 and S_1 and using Lemma 7(i) and $|V(R)| \geq 2$, we have

$$|V(u_2 P_2 v)| - 1 \geq |V(Q)| - 1 + |V(R)| - 2 \geq |V(Q)| - 1 \geq \frac{2f(G, \mathcal{P})}{k-1}.$$

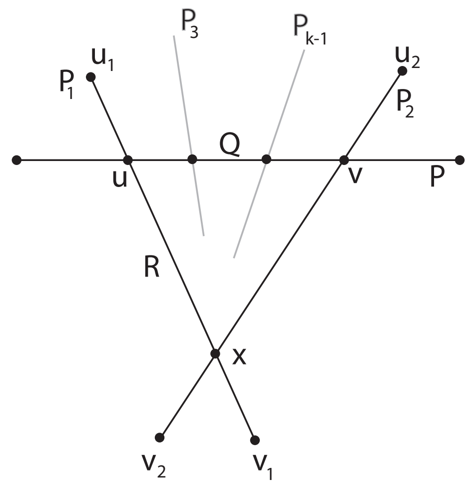

Figure 1: A good $V(P_1)-V(P_2)$ path Q and path R

Similarly for P_2 and S_2, we have

$$|V(vP_2x)|-1 \geq |V(Q)|-1+|V(R)|-1 \geq |V(Q)| \geq \frac{2f(G,\mathcal{P})}{k-1}+1.$$

Also for P_2 and S_3, we have

$$|V(xP_2v_2)|-1 \geq |V(Q)|-1+|V(R)|-2 \geq |V(Q)|-1 \geq \frac{2f(G,\mathcal{P})}{k-1}.$$

Therefore all together we have

$$\ell(G)=|V(P_2)|-1=|V(u_2P_2v)|-1+|V(vP_2x)|-1+|V(xP_2v_2)|-1 \geq$$
$$\geq \frac{2f(G,\mathcal{P})}{k-1}+\frac{2f(G,\mathcal{P})}{k-1}+1+\frac{2f(G,\mathcal{P})}{k-1}=\frac{6f(G,\mathcal{P})}{k-1}+1. \qquad (*)$$

Clearly $n_i=\frac{1}{i}\sum_{P'\in\mathcal{P}}X^i_{\mathcal{P}}(P')$. By the choice of P and $t'_{\mathcal{P}}(P')\geq 2$ for every $P'\in\mathcal{P}$ together with $(*)$, Lemma 5, and Lemma 7 we have

$$n \geq \frac{k\cdot\ell(G)+k+(k-2)\sum_{P'\in\mathcal{P}}X^1_{\mathcal{P}}(P')+\frac{k-3}{2}\sum_{P'\in\mathcal{P}}X^2_{\mathcal{P}}(P')+...+\frac{1}{k-2}\sum_{P'\in\mathcal{P}}X^{k-2}_{\mathcal{P}}(P')}{k-1} \geq$$

$$\geq \frac{k\cdot\ell(G)+k+\frac{1}{k-2}(\sum_{P'\in\mathcal{P}}X^1_{\mathcal{P}}(P')+\sum_{P'\in\mathcal{P}}X^2_{\mathcal{P}}(P')+...+\sum_{P'\in\mathcal{P}}X^{k-2}_{\mathcal{P}}(P'))}{k-1} \geq$$

$$\geq \frac{k\cdot\ell(G)+k+\frac{k}{k-2}(X^1_{\mathcal{P}}(P)+X^2_{\mathcal{P}}(P)+...+X^{k-2}_{\mathcal{P}}(P))}{k-1} \geq$$

$$\geq \frac{k(\frac{6f(G,\mathcal{P})}{k-1}+1)+k+\frac{2k}{k-2}(\frac{2}{k-1}f(G,\mathcal{P})-1)}{k-1}=\frac{(6k^2-8k)f(G,\mathcal{P})+2k^3-8k^2+6k}{(k-2)(k-1)^2},$$

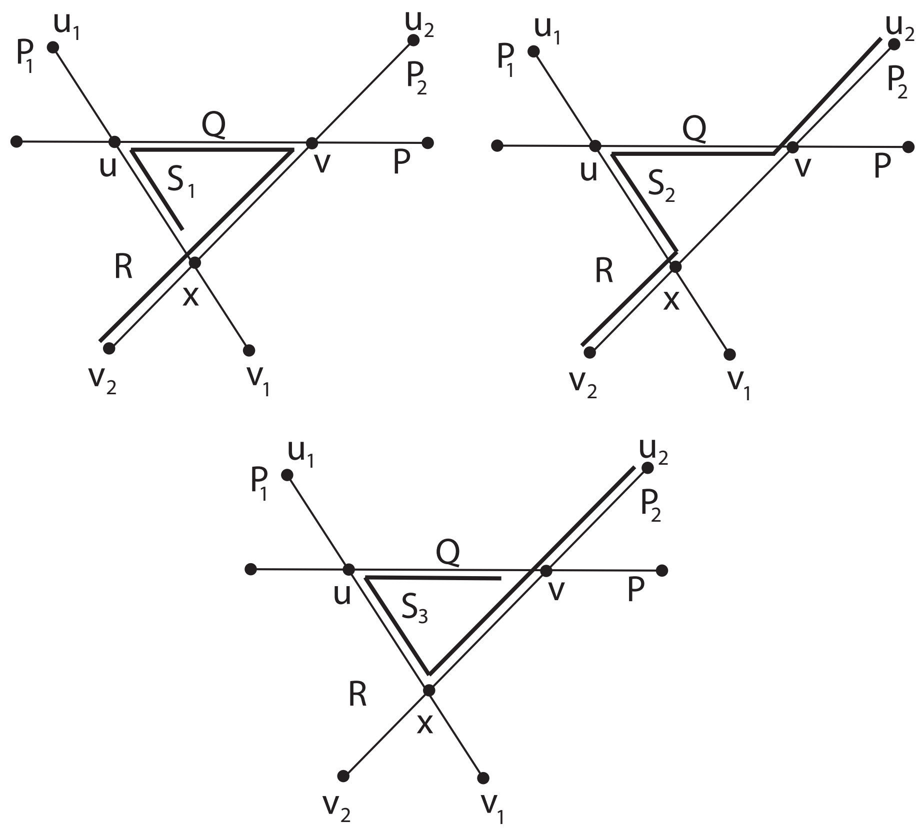

Figure 2: Paths S_1, S_2, and S_3

and hence $f(G,\mathcal{P}) \leq \frac{(k^3-4k^2+5k-2)n-2k^3+8k^2-6k}{6k^2-8k}$. This completes the proof of Theorem 4. □

Proof of Theorem 3. We proceed as in the proof of Theorem 4 and use Corollary 8(i) instead of Lemma 7(i).

By comparing the lengths of P_2 and S_1 and using Corollary 8(i) and $|V(R)| \geq 2$, we have

$$|V(u_2P_2v)| - 1 \geq |V(Q)| - 1 + |V(R)| - 2 \geq |V(Q)| - 1 \geq f(G,\mathcal{P}).$$

Similarly for S_2 and S_3, we have

$$|V(vP_2x)| - 1 \geq |V(Q)| - 1 + |V(R)| - 1 \geq |V(Q)| \geq f(G,\mathcal{P}) + 1,$$

$$|V(xP_2v_2)| - 1 \geq |V(Q)| - 1 + |V(R)| - 2 \geq |V(Q)| - 1 \geq f(G,\mathcal{P}).$$

Therefore all together we have

$$\ell(G) = |V(P_2)| - 1 = |V(u_2P_2v)| - 1 + |V(vP_2x)| - 1 + |V(xP_2v_2)| - 1 \geq$$

$$\geq f(G,\mathcal{P}) + f(G,\mathcal{P}) + 1 + f(G,\mathcal{P}) = 3f(G,\mathcal{P}) + 1. \qquad (**)$$

By the choice of P and $t'_{\mathcal{P}}(P') \geq 2$ for every $P' \in \mathcal{P}$ together with $(**)$, Lemma 7, and Lemma 6 we have

$$n \geq \frac{4\ell(G) + 4 + 2\sum_{P' \in \mathcal{P}} X^1_{\mathcal{P}}(P') + \frac{1}{2}\sum_{P' \in \mathcal{P}} X^2_{\mathcal{P}}(P')}{3} \geq$$

$$\geq \frac{4(3f(G,\mathcal{P})+1) + 4 + 4(f(G,\mathcal{P})-1)}{3} = \frac{16f(G,\mathcal{P})+4}{3},$$

and hence $f(G,\mathcal{P}) \leq \frac{3n-4}{16}$. This completes the proof of Theorem 3. □

3 Conclusion

As it was mentioned in Introduction, we extend Conjecture 1 to Conjecture 9.

Conjecture 9. *For every connected graph, any k of its longest paths have a common vertex for $3 \leq k \leq 6$.*

Conjecture 10 is an extension of a Conjecture stated in [5] for 3 longest paths. We prove that Conjecture 10 is equivalent with Conjecture 9.

Conjecture 10. *There exists a sublinear function g such that for every connected graph G of order n and every subset $\mathcal{P}$ of $\mathcal{L}(G)$ with $3 \leq |\mathcal{P}| \leq 6$, $f(G,\mathcal{P}) \leq g(n)$.*

Let $\mathcal{G}_n$ be a class of connected graphs of order at least n. In other words, using $d_k(\mathcal{G}_n)$ with $3 \leq k \leq 6$, Conjecture 10 translates into the following statement. The path distance ratio $d_k(\mathcal{G}_n)$ goes to 0 as n goes to infinity.

Theorem 11. *Conjecture 9 is true if and only if Conjecture 10 is true.*

Proof. Suppose Conjecture 9 holds. For every set $\mathcal{P}$ of k longest paths $(3 \leq k \leq 6)$ of every connected graph G, we have $f(G,\mathcal{P}) = 0$. Thus any non-negative sublinear function implies that Conjecture 10 holds.

Suppose Conjecture 10 holds. We prove the contrapositive statement, that is, if Conjecture 9 is not true, then neither is Conjecture 10. For $3 \leq k \leq 6$, we consider a connected graph G and a set $\mathcal{P}$ of its k longest paths so that they have no common vertex. We extend G by adding a pendant edge to every vertex, which is an end-vertex of a path of $\mathcal{P}$, and we note that each path of $\mathcal{P}$ prolonged with two of these new edges is a longest path in the extended graph. For a non-negative integer t, we subdivide every edge of the extended graph t times and we observe that the corresponding k paths, say $\mathcal{P}_t$, are longest paths in the resulting graph G_t. Let n be the number of vertices and m the number of edges of G. We see that G_t has at most $n + t(m+2k)$ vertices. By construction, $f(G_t,\mathcal{P}_t) \geq t$. We consider the sequence of graphs $(G_t)_{t=1}^{\infty}$ and we note that $f(G_t,\mathcal{P}_t)$ cannot be bounded from above by a sublinear function. □

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