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Optimal control method for solution of inverse problem

J. Radová[a], J. Machalová[a]

[a] Faculty of Science, Palacký Univerzity Olomouc, 17. listopadu 1192/12, 771 46 Olomouc, Czech Republic

In this contribution we deal with using of the optimal control method for solution of an inverse problem for nonlinear Gao beam which was published by Gao in [1]. The inverse problem will be formulated as the minimization of a cost functional which depends on a state problem. The state problem is represented by the Gao beam equation

$$\begin{cases} \sigma I u^{IV} - \sigma\alpha(u')^2 u'' = f & \text{in } (0,L), \\ u(0) = u'(0) = 0, \\ u(L) = u'(L) = 0, \end{cases} \tag{1}$$

where u is an unknown deflection of the beam, σ is the Young's elastic modulus, $I = \frac{2}{3}h^3$ is the area moment of inertia, h is the thickness of the beam and L is the length of the beam. The remaining symbols indicate the following

$$\alpha = 3h(1-\nu^2) \qquad \text{and} \qquad f = (1-\nu^2)q,$$

where ν denotes Poisson ratio and q is the given vertical load. The clamped beam is considered which corresponds to the prescribed boundary conditions.

It remains to specify the cost functional. Let $||\cdot||$ be L^2-norm and let $z \in L^2((0,L))$ be a given function. Then we define the cost functional $J(\sigma, u(\sigma)) : U_{ad} \times H_0^2((0,L)) \to \mathbb{R}^1$ by

$$J(\sigma, u(\sigma)) = \frac{1}{2}||u(\sigma) - z||^2,$$

where

$$U_{ad} = \{\sigma \in L^\infty((0,L)) \,|\, \sigma_{\min} \le \sigma \le \sigma_{max} \text{ in } (0,L), \sigma|_{K_i} \in P_0(K_i), i = 1,\dots,n\}$$

is the set of admissible parameters with given constants $0 < \sigma_{\min} < \sigma_{\max}$ and $K_i = [x_{i-1}, x_i]$, $i = 1,\dots,n$, where $x_0 = 0 < x_1 < x_2 < \dots < x_n = L$. It means that $U_{ad} \subset \mathbb{R}^n$ is the set of piecewise constant functions over the partition of $(0,L)$.

Now we can define the inverse problem as a following minimization problem

$$\begin{cases} \text{Find function } \sigma^* \in U_{ad} \text{ such that} \\ J(\sigma^*, u(\sigma^*)) = \min\limits_{\sigma \in U_{ad}} J(\sigma, u(\sigma)), \\ \text{where } u(\sigma) \text{ solves the state problem } (1). \end{cases} \tag{2}$$

The numerical realization of this problem is based on using finite element method and consists of two parts. The first part is the discretization of the state problem which does not make any troubles and it is described for example in [2, 3]. The second part concerns the minimization of a cost function $\mathbf{I}$ which arises from the discretization of the cost functional J and can be written in the form

$$\mathbf{I}(\boldsymbol{\sigma}) := J(\boldsymbol{\sigma}, \mathbf{u}(\boldsymbol{\sigma})).$$

The minimization is based on iterative process which generates a sequence $\{\boldsymbol{\sigma}^k\}$ with the given initial approximation $\boldsymbol{\sigma}^0$ such that

$$\lim_{k\to\infty} \mathbf{I}(\boldsymbol{\sigma}^k) = \mathbf{I}(\boldsymbol{\sigma}^*), \qquad \text{where} \quad \mathbf{I}(\boldsymbol{\sigma}^*) = \min_{\boldsymbol{\sigma}\in\mathbf{U}} \mathbf{I}(\boldsymbol{\sigma}).$$

For the given $\boldsymbol{\sigma}^k$ we compute $\mathbf{u}(\boldsymbol{\sigma}^k)$ as a solution of the discretized state problem and the next iteration $\boldsymbol{\sigma}^{k+1}$ is found in the form $\boldsymbol{\sigma}^{k+1} = \boldsymbol{\sigma}^k + \alpha\mathbf{d}^k$, where $\alpha > 0$ and $\mathbf{d}^k$ is a descent direction. This direction is chosen in such a way that $\mathbf{I}(\boldsymbol{\sigma}^k + \alpha\mathbf{d}^k) < \mathbf{I}(\boldsymbol{\sigma}^k)$ for all $\alpha \in (0, \overline{\alpha})$, $\overline{\alpha} > 0$. The important step for this algorithm is the choice of the step length α which is obtained by using line search techniques. The cost function can be written in the form

$$\mathbf{I}(\boldsymbol{\sigma}) = \frac{1}{2}\left(\mathbf{S}\mathbf{u}(\boldsymbol{\sigma}) - \mathbf{z}, \mathbf{S}\mathbf{u}(\boldsymbol{\sigma}) - \mathbf{z}\right)_p,$$

where $\mathbf{z} \in \mathbb{R}^p$ is a given vector, $(.,.)_p$ denotes the inner product in $\mathbb{R}^p$ and $\mathbf{S} \in \mathbb{R}^{p\times n}$ is a matrix representing the restriction mapping from $\mathbb{R}^n$ onto $\mathbb{R}^p$. For computation of the descent direction we need to derive the expression for the gradient of the function $\mathbf{I}$. It is obvious that

$$\begin{aligned} \mathbf{I}'(\boldsymbol{\sigma}) &= \frac{1}{2}\left(\mathbf{S}\mathbf{u}'(\boldsymbol{\sigma}), \mathbf{S}\mathbf{u}(\boldsymbol{\sigma}) - \mathbf{z}\right)_p + \frac{1}{2}\left(\mathbf{S}\mathbf{u}(\boldsymbol{\sigma}) - \mathbf{z}, \mathbf{S}\mathbf{u}'(\boldsymbol{\sigma})\right)_p = \\ &= \left(\mathbf{S}\mathbf{u}(\boldsymbol{\sigma}) - \mathbf{z}, \mathbf{S}\mathbf{u}'(\boldsymbol{\sigma})\right)_p = \left(\mathbf{S}^T\left(\mathbf{S}\mathbf{u}(\boldsymbol{\sigma}) - \mathbf{z}\right), \mathbf{u}'(\boldsymbol{\sigma})\right)_p. \end{aligned}$$

The problematic part $\mathbf{u}'(\boldsymbol{\sigma})$ can be eliminated by using adjoint state problem, for more details see in [5].

Numerical results for the nonlinear Gao beam are compared with results for the classical linear Euler-Bernoulli beam [4] and numerical computations are realized by using MATLAB.

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