Generalized Tellegen Principle Used for Energy Method for Systems Modeling

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Abstract – **This paper deals with dissipativity, stability, chaotic behavior and related structural properties of a relatively broad class of finite dimensional strictly causal systems. The class of nonlinear systems under consideration is described in the state-space representation form. System properties are investigated by a new approach based on a new abstract state energy concept, and on a proper generalization of the well known Tellegen's theorem as a form of the energy conservation principle. The resulting energy function is induced by the output signal power and determines both, the structure of a proper system representation as well as the corresponding system state space topology. The state minimality, as well as parameter minimality requirements plays a crucial role in the proposed method. Several examples are solved, and results of simulation are shown for illustration of fundamental ideas and basic attributes of the proposed method.**

Keywords - chaos; energy; nonlinear; state space; Tellegen)

I. INTRODUCTION

In many real-world situations some natural concepts, such as causality principle and different forms of conservation laws, have generally been recognized as system properties of crucial practical importance. For example, the Tellegen's theorem is well known in the field of electrical engineering [1, 2]. It is one of few general theoretical results that apply in non-linear and time-varying situations, too. In the paper a more general class of abstract strictly causal system representations is addressed. The proposed approach gets out from the hypothesis that any physically correct system representation must not be in contradiction not only with a set of measured data but also with a form of *an energy conservation principle*.

Thus, if a specific physical structure of the system under investigation would be explicitly known, then the concept of physical energy could serve as a fundamental tool for system analysis and synthesis.

II. TELLEGEN'S THEOREM USED FOR SYSTEMS

Tellegen's theorem is one of the most powerful theorems in network theory. Most of the energy distribution theorems and extremum principles in network theory can be derived from it. The Tellegen theorem provides a useful tool to analyze complex network systems including electrical circuits, biological and metabolic networks, pipeline transport networks, and chemical process networks. The classical Tellegen's principle can be seen as main reasons for using it as a starting point of proposed method. The main aim of the contribution is to develop a new approach based on a generalized form of the classical Tellegen's principle as an abstract formulation of the energy conservation law, and to investigate some possibilities of its systematic use to solution of basic problems of nonlinear system theory [3 - 6]. Some connections of dissipativity, state and parameter minimality, instability and chaos with are investigated from this point of view. In order to explain essential features of the theorem [2], consider an arbitrarily connected electrical network with n components and choose associated reference directions for branch voltages v_k and currents i_k. Let Kirchhoff's laws be given by the following equations:

$$Ai(t) = 0; \qquad Bv(t) = 0 \tag{1}$$

where A is a node incidence matrix, B is a loop incidence matrix and $i(t)$, $v(t)$ are defined as follows:

$$\begin{aligned} i(t) &= \left[i_1(t), i_2, ..., i_n(t)\right]^T \\ v(t) &= \left[v_1(t), v_2(t), ...\ v_n(t)\right]^T \end{aligned} \tag{2}$$

Let the vectors $i(t)$, $v(t)$ be the elements of an Euclidean space E_n and invoke the inner product:

$$\left\langle i(t),\, v(t)\right\rangle = \sum_{k=1}^{b} i_k(t)\, v_k(t) \tag{3}$$

Let I be the set of all the vectors $i(t)$ and V the set of all the vectors $v(t)$ satisfying the equations (1).

Theorem 1: (Tellegen's theorem) If $i(t)\in I$ and $v(t)\in V$ then it holds that:

$$\left\langle i(t), v(t)\right\rangle = 0 \tag{4}$$

It is worth noticing a close relation between physical correctness and Tellegen's theorem. It is also important to realize that the branch currents and voltages are chosen arbitrarily complied only with Kirchhoff's laws. It implies that different sets $\overline{I}$, $\overline{V}$ of the branch currents and voltages satisfying the laws can be selected and the relation:

$$\left\langle \overline{i}(t), \overline{v}(t)\right\rangle = 0,\ \ \overline{i}(t)\in\overline{I},\ \ \overline{v}(t)\in\overline{V} \tag{5}$$

still holds.

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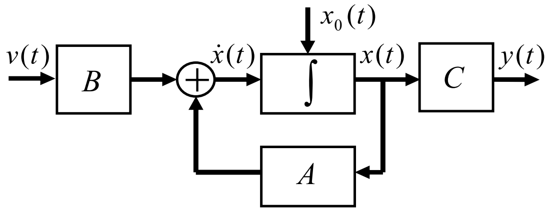

Figure 1. Block diagram of open loop system (without state or output feedback) with input *v(t)*, output *y(t)*, state *x(t)*, initial conditions $x_0(t)$ and matrices *A*, *B*, *C*

III. GENERALIZED TELLEGEN'S PRINCIPLE

Consider a class of state equivalent representations (see block diagram of Fig.1) described by equations (6) – (8) and structure [4] shown in Fig. 2.

$$\frac{dx(t)}{dt} = A\cdot x(t) + B\cdot v(t);\; y(t) = C\cdot x(t) \tag{6}$$

$$A=\begin{bmatrix} -\alpha_{11} & \alpha_2 & 0 & 0 & \cdots & 0 & 0 \\ -\alpha_2 & \alpha_{22} & \alpha_3 & 0 & \cdots & 0 & 0 \\ 0 & -\alpha_3 & \alpha_{33} & \alpha_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots -\alpha_{n-1} & \alpha_{n-1\,n-1} & \alpha_n \\ 0 & 0 & 0 & 0 & \cdots 0 & -\alpha_n & \alpha_{nn} \end{bmatrix} \tag{7}$$

$$B=\begin{bmatrix}\beta_1 & \beta_2 & \dots & \beta_n\end{bmatrix}^T;\; C=\begin{bmatrix}\gamma_1 & \gamma_2 & \dots & \gamma_n\end{bmatrix} \tag{8}$$

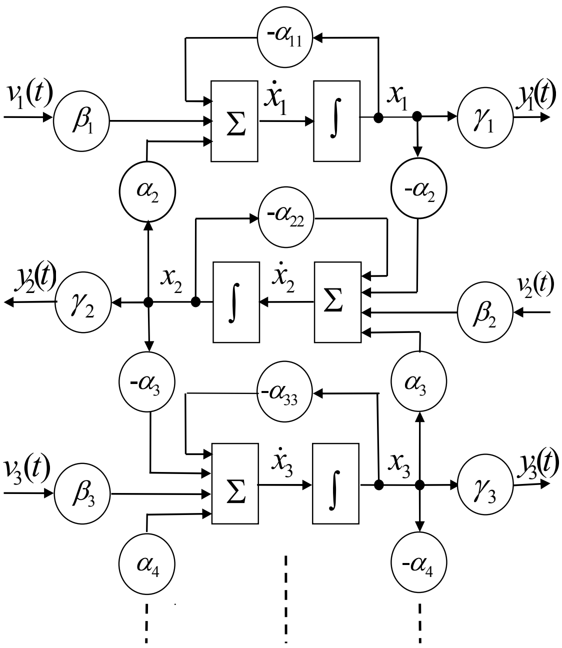

Figure 2. Physicaly correct structure for generalized Tellegen theorem (α_i should be function of x_i or time)

For the system described by previous equations and structure according Fig. 2 the generalized Tellegen theorem is given by

$$\left\langle x^T(t), \frac{dx(t)}{dt} \right\rangle = 0 \tag{9}$$

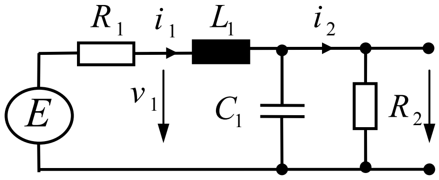

Figure 3. *RLC* circuit used as linear example

IV. LINEAR SYSTEM EXAMPLE

In this part the linear example is presented. Let us have a 2nd order *RLC* circuit shown in Fig. 3, which can be described by equations

$$\begin{aligned} R_1 i_1 + L\frac{di_1}{dt} + v_2 &= E \\ C\frac{dv_2}{dt} + \frac{v_2}{R_2} &= i_1 \end{aligned} \tag{10}$$

State space equations are

$$\begin{bmatrix} \frac{dv_2}{dt} \\ \frac{di_1}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_2 C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_1}{L} \end{bmatrix} \cdot \begin{bmatrix} v_2 \\ i_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \cdot E \tag{11}$$

or differential equation

$$\begin{aligned} &\frac{d^2 i_1}{dt^2} + \left(\frac{R_1}{L} + \frac{1}{CR_2}\right)\frac{di_1}{dt} + \left(\frac{1}{LC} + \frac{R_1}{LCR_2}\right) i_1 \\ &= \frac{1}{L}\frac{dE}{dt} + \frac{E}{LCR_2} \end{aligned} \tag{12}$$

Tellegen equation for power is

$$E\cdot i_1 - \left(v_{R1} i_1 + v_L i_L + v_C i_C + v_{R2} i_{R2}\right) = 0 \tag{13}$$

and after manipulation

$$E\cdot i_1 - \left(R_1 i_1^2 + L i_1 \frac{di_1}{dt} + C v_C \frac{dv_C}{dt} + \frac{v_C^2}{R_2}\right) = 0 \tag{14}$$

From (18) can be derived also equation for energy

$$\int_0^t E\cdot i_1 dt - \left(\int_0^t R_1 i_1^2 dt + \frac{1}{2} L i_1^2 + \frac{1}{2} C v_C^2 + \int_0^t \frac{v_C^2}{R_2} dt\right) = 0 \tag{15}$$

Suppose following values of circuit: R_1=0.5 [Ω]; *L*=2.5 [H]; *C*=0.1 [F]; R_2=20 [Ω]. Therefore eq. (11) is rewritten as

$$\begin{bmatrix} \frac{dv_2}{dt} \\ \frac{di_1}{dt} \end{bmatrix} = \begin{bmatrix} -0.5 & 10 \\ -0.4 & -0.2 \end{bmatrix} \cdot \begin{bmatrix} v_2 \\ i_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.4 \end{bmatrix} \cdot E \tag{16}$$

State transformation leads to system described by

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_2 C} & \frac{1}{\sqrt{LC}} \\ -\frac{1}{\sqrt{LC}} & -\frac{R_1}{L} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\sqrt{LC}} \end{bmatrix} \cdot v \tag{17}$$

$$\alpha_{11} = \frac{1}{R_2 C}; \alpha_2 = \frac{1}{\sqrt{LC}}; \alpha_{22} = -\frac{R_1}{L}; \qquad (18)$$
$$\beta_2 = \frac{1}{\sqrt{LC}}; \gamma_1 = 1$$

From generalized Tellegen equation (9) can be derived

$$\left\langle x^T(t), \frac{dx(t)}{dt} \right\rangle = \underbrace{-\alpha_{11} x_1^2(t) - \alpha_{22} x_2^2(t)}_{P_D} + \qquad (19)$$
$$\underbrace{\beta_2 x_2(t) v(t)}_{P_I} = P_D(t) + P_I(t) = 0$$

where P_D is dissipated power and P_I is input power. Proof is given in next equations (20) - (22).

$$x_1(t) = v_C(t)\sqrt{C}; \quad x_2(t) = i_1(t)\sqrt{L}; \quad v(t) = E\sqrt{C} \qquad (20)$$

$$P_I(t) = \beta_2 x_2(t) v(t) = \frac{1}{\sqrt{LC}} i_1(t)\sqrt{L} \cdot E(t)\sqrt{C} = \qquad (21)$$
$$E(t) \cdot i_1(t)$$

$$P_D(t) = -\alpha_{11} x_1^2 - \alpha_{22} x_2^2 = \qquad (22)$$
$$-\frac{1}{R_2 C}\left(v_C\sqrt{C}\right)^2 - \frac{R_1}{L}\left(i_1\sqrt{L}\right)^2 = -\frac{v_C^2}{R_2} - R_1 \cdot i_1^2$$

The generalized Tellegen principle described by eq. (9) can be used also for system supplied only by initial condition as well for nonlinear system. It will show that type of system can be easily described by energy or power function. It is important to note that state space energy can be derived from power by

$$V = \int_0^t \left\langle x^T, \dot{x} \right\rangle dt = \int_0^t \left(x_1 \dot{x}_1 + \ldots + x_n \dot{x}_n \right) dt = \sum_{i=1}^{n} x_i^2 \qquad (23)$$

Time evolution of energy for *RLC* circuit (Fig. 2), and time evolution of state space energy of abstract system described by eq. (19) are displayed in Fig. 4 - response on initial condition and unit step. Booth curves are exactly same (therefore only one curve is visible in Fig. 4).

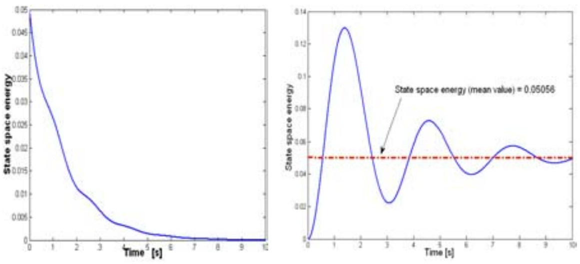

Figure 4. Left - Time evolution of energy in *RLC* circuit (Fig. 3) and time evolution of state space energy, eq. (21) as response on initial conditions $v_C(0)=1$ or $\boldsymbol{x}(0)=[x_1(0)\ x_2(0)]=[1\ 0]^T$. Booth curves are the same. Right - Time evolution of energy in *RLC* circuit (Fig. 3) and time evolution of state space energy, eq. (21) as response on unit step. Mean value of energy $E_C+E_L=0.05056$. Booth curves are the same.

V. NONLINEAR SYSTEM

In this part the example of nonlinear (but non-chaotic) system is presented [7 - 8]. System is described by eq. (24) or block diagram (Fig. 6). For nonlinear function $f(x_2)$, two functions were used: $(x_2)^2$ and $abs(x_2)$. Prescribed value is w and k is gain. The equivalent electronic system is shown in Fig. 8.

$$\dot{x}_1 = k\alpha_{11}\left(w - f(x_2)\right)x_1 + \alpha_2 x_2 \qquad (24)$$
$$\dot{x}_2 = -\alpha_2 x_1 - \alpha_{22} x_2$$

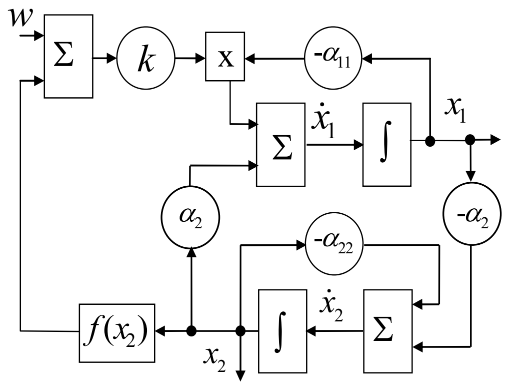

Figure 5. Block diagram of nonlinear system

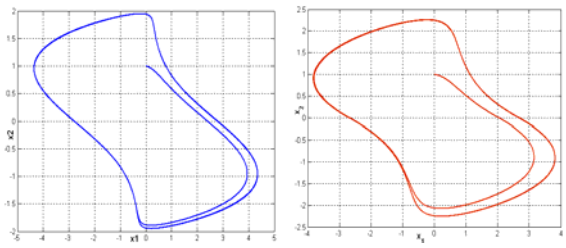

Figure 6. 2D phase portrait of nonlinear systems (x_2 versus x_1). Left: $f(x_2) = x_2^2$; w=1; k=10. Right: $f(x_2) = abs(x_2)$; w=1; k=10

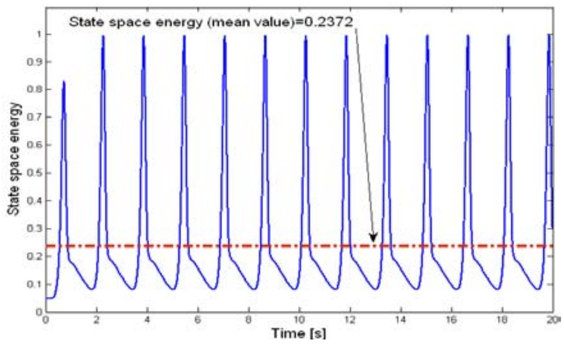

Figure 7. Time evolution of state space energy in nonlinear system for parameters: $f(x_2) = x_2^2$; w=1; k=10;

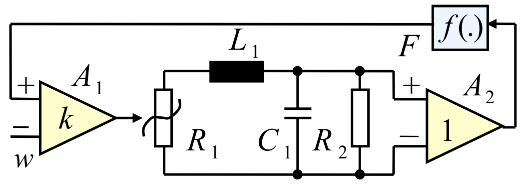

Figure 8. Equivalent nonlinear circuit. *A* - instrumentation amplifiers with gain *k* and 1; *F* - nonlinear function; R_1 - controlled resistor; L_1, C_1, R_2 - passive parts.

VI. CHAOTIC SYSTEM

The chaotic system is given by eq. (25) and structure see Fig. 9, where w is prescribed value and k_2

and k_3 are gains [9 - 10]. Simulation results for w=1; α_2=1; α_3=0.89; α_{33}=0.1; k_2=0.1; k_3=3 and initial condition [0 0.1 0] are shown in Fig, 10 - 12.

$$\begin{aligned} \dot{x}_1 &= \left(w - k_2 x_2^2 - k_3 x_3^2\right) x_1 - \alpha_2 x_2 \\ \dot{x}_2 &= \alpha_2 x_1 - \alpha_3 x_3 \\ \dot{x}_3 &= \alpha_3 x_2 - \alpha_{33} x_3 \end{aligned} \tag{25}$$

Chaotic behavior can be controlled by means of values k_2 and k_3.

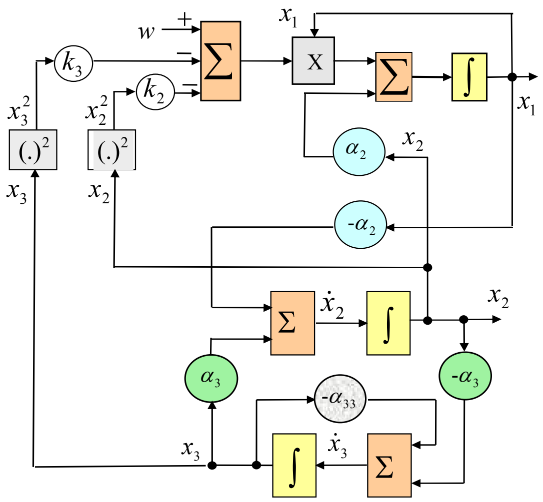

Figure 9. Structure of chaotic system

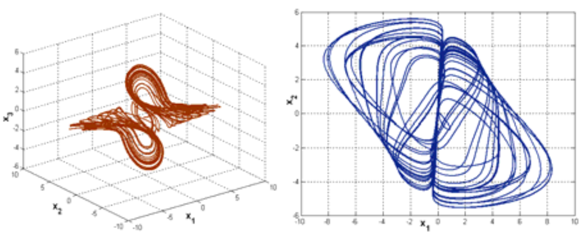

Figure 10. 3D phase portrait of chaotic system (left) and 2D phase portrait, x_2 versus x_1 (right)

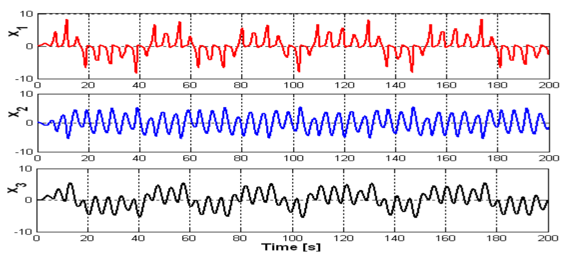

Figure 11. Time evolution of signals of chaotic system

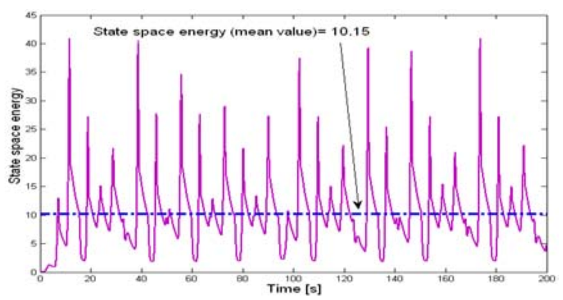

Figure 12. Time evolution of state space energy of chaotic system

VII. Conclusion

In this paper the generalized Tellegen principle was used for power or energy description of different types of physical systems. It was derived, that abstract state space energy based on Tellegen principle can be used for linear, nonlinear or chaotic system. Most important is, that time evolution of energy can be used for classification of systems - for nonlinear system is periodic around the mean value, for chaotic system is non-periodic around mean value.

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