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# Interpolation of suspension kinematics for the purpose of vehicle dynamics simulation

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Suspensions (for example, see Fig. 1 on the left) are usually 1 DOF systems. The expression of the position of the wheel support in terms of some parameter can be obtained by solving the constraints imposed by the joints present in the system. But this is not efficient as the same equations are solved several times. This paper explains another strategy consisting in solving the kinematic problem (at position and velocity level) for particular points and using these points to calculate all other positions by interpolation. The proposed interpolation is $C^1$ continuous in translation and rotation.

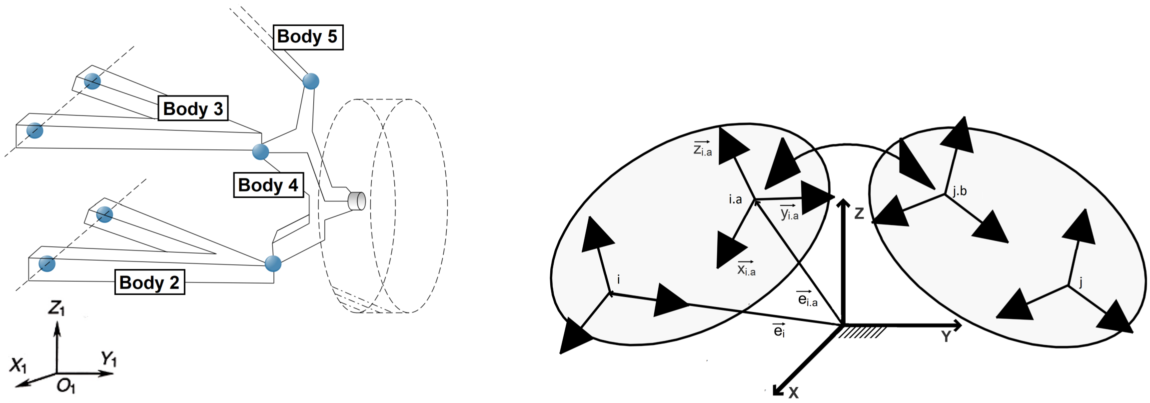


Figure 1. Double wishbone suspension (*left*) and main coordinate system, principal frames of the bodies and secondary frames of the bodies (*right*) [2]

The homogeneous transformation matrix giving the situation of the principal frame of body *i* can be written as

$$\mathbf{T}_{0,i} = \begin{pmatrix} & \mathbf{R}_{0,i} & & \{\mathbf{e}_i\}_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

where the columns of $\mathbf{R}_{0,i} = [\boldsymbol{x}_i \ \boldsymbol{y}_i \ \boldsymbol{z}_i]$ give the orientation of the axes of the principal frame of body *i* in the main coordinate system and $\{\mathbf{e}_i\}_0$ gives the position of the principal frame of the body $i$ in the main coordinate system $\{\}_0$ (see Fig. 1 on the right) .

The transformation matrix between the main coordinate system and the secondary coordinate system of body *i* is given by the dot product

$$\mathbf{T}_{0,i.a} = \mathbf{T}_{0,i} \cdot \mathbf{T}_{i,i.a} = \begin{pmatrix} \mathbf{x}_{i.a} & \mathbf{y}_{i.a} & \mathbf{z}_{i.a} & \mathbf{e}_{i.a} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

where $\mathbf{x}_{i.a}$, $\mathbf{y}_{i.a}$ and $\mathbf{z}_{i.a}$ are the unit vectors of the axes of secondary frame and $\mathbf{e}_{i.a}$ is the position of the secondary frame. All vectors are considered with respect to the main coordinate system.

Constraint equations allow to set joints between two arbitrary bodies (body *i* and body *j*) of the mechanical system. The joints are defined between the frames of two different bodies. The main frame of the body is usually placed into the mass center of the body so there is the need to define secondary frames on the bodies (frame *a* of body *i* = *i.a* and frame *b* of body *j* = *j.b*) between which are defined the joints.

Set of constraint equations ${}^1b$, ${}^2b$ ... ${}^6b$ is defined as

$$ {}^1b \equiv \mathbf{x}_{i.a} \cdot (\mathbf{e}_{i.a} - \mathbf{e}_{j.b}) = 0, \tag{3} $$
$$ {}^2b \equiv \mathbf{y}_{i.a} \cdot (\mathbf{e}_{i.a} - \mathbf{e}_{j.b}) = 0, \tag{4} $$
$$ {}^3b \equiv \mathbf{z}_{i.a} \cdot (\mathbf{e}_{i.a} - \mathbf{e}_{j.b}) = 0, \tag{5} $$
$$ {}^4b \equiv \mathbf{y}_{i.a} \cdot \mathbf{z}_{j.b} = 0, \tag{6} $$
$$ {}^5b \equiv \mathbf{z}_{i.a} \cdot \mathbf{x}_{j.b} = 0, \tag{7} $$
$$ {}^6b \equiv \mathbf{x}_{i.a} \cdot \mathbf{y}_{j.b} = 0. \tag{8} $$

The constraint equations relative to each classical joint can be presented as a subset of the six previous generic equations (e.g., spherical joint is represented by ${}^1b$, ${}^2b$ and ${}^3b$).

The system of the constraint (non-linear) equations could be expressed as

$$ \mathbf{F}(\mathbf{q}) = \left(b_1, b_2, \ldots, b_{n_C}\right)^T = \mathbf{0}, \tag{9} $$

where $n_C$ is the number of constraints (${}^jb \neq b_j$). The vector of unknown variables is defined as

$$ \mathbf{q} = \left(x_1,\ y_1,\ z_1,\ \phi_1,\ \ldots,\ \psi_{n_B}\right)^T = \left(q_1,\ q_2,\ q_3,\ q_4,\ \ldots,\ q_{6n_B}\right)^T, \tag{10} $$

where $n_B$ is number of all bodies.

All types of suspensions correspond to a one degree of freedom mechanism (assuming the rotation and the steering of the wheel are locked), so one variable in the system of equations has to be given (e.g., the vertical coordinate of a wheel support denoted as $u$). For a given value of $u$, the equations are solved in terms of $\mathbf{q}$ by the Newton-Raphson method.

For the interpolation it is important to define velocity $\mathbf{v}_i$ and angular velocity $\boldsymbol{\omega}_i$ of body *i*. Motion of body 1 depends on $q_1$ to $q_6$, so motion of body $i$ depends on $q_{6i-5}$ to $q_{6i}$. Then it is possible to consider

$$ \mathbf{T}_{0,i}(\mathbf{q}_i) = \begin{pmatrix} & \mathbf{R}(q_{6i-2}, q_{6i-1}, q_{6i}) & & \begin{matrix} q_{6i-5} \\ q_{6i-4} \\ q_{6i-3} \end{matrix} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11} $$

For the velocity of principal frame of body *i* is possible to write

$$ \{\mathbf{v}_i\}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \underbrace{0}_{\mathbf{d}_{i,6i-4}} & 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \dot{q}_{6i-5} \\ \dot{q}_{6i-4} \\ \dot{q}_{6i-3} \\ \dot{q}_{6i-2} \\ \dot{q}_{6i-1} \\ \dot{q}_{6i} \end{pmatrix}, \tag{12} $$

where $\mathbf{d}_{i,k}$ can be written as $\mathbf{d}_{i,k} = \frac{\partial \mathbf{v}_i}{\partial \dot{q}_k}$. It is also possible to write [2]

$$\boldsymbol{\omega}_i = \begin{pmatrix} 0 & 0 & 0 & \cos q_6 \cos q_5 & -\sin q_6 & 0 \\ 0 & 0 & 0 & \sin q_6 \cos q_5 & \cos q_6 & 0 \\ 0 & 0 & 0 & \underbrace{-\sin q_5}_{\boldsymbol{\delta}_{i,6i-2}} & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \dot{q}_{6i-5} \\ \dot{q}_{6i-4} \\ \dot{q}_{6i-3} \\ \dot{q}_{6i-2} \\ \dot{q}_{6i-1} \\ \dot{q}_{6i} \end{pmatrix}, \tag{13}$$

where $\boldsymbol{\delta}_{i,k}$ can be written as $\boldsymbol{\delta}_{i,k} = \frac{\partial \boldsymbol{\omega}_i}{\partial \dot{q}_k}$.

Kinematic quantities, which are calculated using presented approach, are important for the interpolation. The interpolation assumes that the evolution of a homogeneous transformation matrix $\mathbf{T}_{0,A}$ of an arbitrary body $A$ and derivatives $\mathbf{d}_{A,u}$ and $\boldsymbol{\delta}_{A,u}$ (obtained from the constraints at the velocity level) have been calculated for particular values of parameter $u$, where vectors $\mathbf{d}_{A,u}$ and $\boldsymbol{\delta}_{A,u}$ are defined as [2]

$$\mathbf{d}_{A,u} = \frac{\partial \mathbf{e}_A}{\partial u} = \frac{\partial \mathbf{v}_A}{\partial \dot{u}}, \qquad \boldsymbol{\delta}_{A,u} = \frac{\partial \boldsymbol{\omega}_A}{\partial \dot{u}}. \tag{14}$$

It is useful to store the data into the table in terms of a series of values $u$ ($u_0,\ u_1,\ u_2,\ \ldots,\ u_N$)

$$\begin{array}{cccc} u_0 & \mathbf{T}_{0,A}(u_0) & \{\mathbf{d}_{A,u}(u_0)\}_0 & \{\boldsymbol{\delta}_{A,u}(u_0)\}_0 \\ u_1 & \mathbf{T}_{0,A}(u_1) & \{\mathbf{d}_{A,u}(u_1)\}_0 & \{\boldsymbol{\delta}_{A,u}(u_1)\}_0 \\ u_2 & \mathbf{T}_{0,A}(u_2) & \{\mathbf{d}_{A,u}(u_2)\}_0 & \{\boldsymbol{\delta}_{A,u}(u_2)\}_0 \\ \vdots & \vdots & \vdots & \vdots \\ u_N & \mathbf{T}_{0,A}(u_N) & \{\mathbf{d}_{A,u}(u_N)\}_0 & \{\boldsymbol{\delta}_{A,u}(u_N)\}_0, \end{array}$$

where $\mathbf{T}_{0,A}$ consists of $\mathbf{R}_{0,A}$ and $\mathbf{e}_{0,A}$. The interpolation is now a matter of finding continuous functions between each row of previous table. For this Cubic Hermite splines are typically used. The functions are

$$h_{00}(\xi) = 2\xi^3 - 3\xi^2 + 1, \quad h_{01}(\xi) = -2\xi^3 + 3\xi^2, \tag{15}$$

$$h_{10}(\xi) = \xi^3 - 2\xi^2 + \xi, \quad h_{11}(\xi) = \xi^3 - \xi^2. \tag{16}$$

With respect to boundary conditions ($f(x_0) = f_0,\ f(x_1) = f_1,\ f'(x_0) = m_0,\ f'(x_1) = m_1$) is the interpolation of an arbirary function given by

$$f(x) = f_0 h_{00}\left(\frac{x - x_0}{x_1 - x_0}\right) + f_1 h_{01}\left(\frac{x - x_0}{x_1 - x_0}\right) + \\ + m_0 h_{10}\left(\frac{x - x_0}{x_1 - x_0}\right) \cdot (x_1 - x_0) + m_1 h_{11}\left(\frac{x - x_0}{x_1 - x_0}\right) \cdot (x_1 - x_0), \tag{17}$$

which can be used in a straightforward way to interpolate the position as

$$\mathbf{e}_A(u) = \mathbf{e}_A(u_i) h_{00}(\xi) + \mathbf{e}_A(u_{i+1}) h_{01}(\xi) + \mathbf{d}_{A,u}(u_i) h_{10}(\xi) \cdot (u_{i+1} - u_i) \\ + \mathbf{d}_{A,u}(u_{i+1}) h_{11}(\xi) \cdot (u_{i+1} - u_i), \tag{18}$$

where $\xi = \frac{u - u_i}{u_{i+1} - u_i}$ and $u_i$ ($i = 1, \ldots, N$) are chosen values of independent suspension parameter $u$ from the table. The velocity and the acceleration can be found in the same way. For the interpolation of rotation it is important to obtain relative rotation matrix $\mathbf{R}_{0,A}^{i,i+1}$ which is given by

$$\mathbf{R}_{0,A}^{i,i+1} = \mathbf{R}_{0,A}^{-1}(u_i) \cdot \mathbf{R}_{0,A}(u_{i+1}) = \mathbf{R}_{0,A}^{T}(u_i) \cdot \mathbf{R}_{0,A}(u_{i+1}). \tag{19}$$

It is possible to express every spatial rotation as the rotation of the angle $\theta$ around the unit vector $\mathbf{n}$ which define $\mathbf{R_{ra}}(\mathbf{n}, \theta)$ that is given by

$$\mathbf{R_{ra}}(\mathbf{n}, \theta) = \begin{pmatrix} n_x^2 V + C & n_x n_y V - n_z S & N_x n_z V + n_y S \\ n_x n_y V + n_z S & n_y^2 V + C & n_y n_z V - n_x S \\ n_x n_z V - n_y S & n_y n_z V + n_x S & n_z^2 V + C \end{pmatrix}, \tag{20}$$

where $C = \cos(\theta)$, $S = \sin(\theta)$ and $V = 1 - \cos(\theta)$. It is possible for any rotation matrix $\mathbf{R}_{0,A}^{i,i+1}$ to determine the axis and the angle to which it corresponds. The angle is given directly by

$$\theta^{i,i+1} = \arccos\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right), \tag{21}$$

where $r_{11},\ r_{12},\ r_{13},\ r_{21},\ r_{22},\ r_{23},\ r_{31},\ r_{32},\ r_{33}$ are elements of matrix $\mathbf{R}_{0,A}^{i,i+1}$ and it is assumed that $0 < \theta < \pi$.

The unit vector parallel to the axis can then be obtained by (if $\sin\theta \neq 0$)

$$\{\mathbf{n}^{i,i+1}\}_i = \frac{1}{2\sin\theta} \cdot \begin{Bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{Bmatrix}. \tag{22}$$

The full orientation matrix will then be calculated from

$$\mathbf{R}_{0,A}(u) = \mathbf{R}_{0,A}(u_i) \cdot \mathbf{R}_{0,A}^{int}(u) = \mathbf{R}_{0,A}(u_i) \cdot \mathbf{R}^1(u) \cdot \mathbf{R}^2(u) \cdot \mathbf{R}^3(u), \tag{23}$$

where matrices $\mathbf{R}^1(u)$, $\mathbf{R}^2(u)$ and $\mathbf{R}^3(u)$ correspond to

$$\mathbf{R}^1(u) = \mathbf{R_{ra}}\left(\frac{\{\boldsymbol{\delta}_{A,u}(u_i)\}_i}{\|\boldsymbol{\delta}_{A,u}(u_i)\|},\ \|\boldsymbol{\delta}_{A,u}(u_i)\| \cdot h_{10}(\xi(u)) \cdot (u_{i+1} - u_i)\right), \tag{24}$$

$$\mathbf{R}^2(u) = \mathbf{R_{ra}}(\{\mathbf{n}_{A,u}^{i,i+1}\}_i,\ \theta^{i,i+1} \cdot h_{01}(\xi(u))), \tag{25}$$

$$\mathbf{R}^3(u) = \mathbf{R_{ra}}\left(\frac{\{\boldsymbol{\delta}_{A,u}(u_{i+1})\}_{i+1}}{\|\boldsymbol{\delta}_{A,u}(u_{i+1})\|},\ \|\boldsymbol{\delta}_{A,u}(u_{i+1})\| \cdot h_{11}(\xi(u)) \cdot (u_{i+1} - u_i)\right). \tag{26}$$

The proposed methodology was implemented in MATLAB and tested on the double wishbone suspension (Fig. 1 on the left). The exact position and orientation (of a wheel or a sample for verifying the interpolation method) were compared with the calculated values obtained from the interpolation. It applies $\mathbf{R}_{i,e} = \mathbf{R}_i^T \cdot \mathbf{R}_e$, for the error in rotation, where $\mathbf{R}_{i,e}$ is the relative rotation matrix between interpolated and exact position, $\mathbf{R}_i$ and $\mathbf{R}_e$ are interpolated and exact rotation matrices. Error in rotation is then given by formula (21), applied to $\mathbf{R}_{i,e}$. The error in translation was evaluated as norm of the distance vector between the two versions (exact and interpolated). For the position the highest norm is smaller than $4 \times 10^{-3}$ mm and for the rotation the biggest difference is smaller than $4 \times 10^{-3}$ rad (0.03°).


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## References


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