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System response with random imperfections in coefficients on the space of realizations

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Parameters of dynamic systems are usually disturbed by a random noise, which emerges due to an imperfect function of the system, due to external influences, etc. These noises are random functions of time. The task assignment, however, may require determination of response statistics when the parameters of the system have indefinite values due to scattering of production, aging, wear or degradation of the system, etc. It turns out that the nature of the formulated task where the statistical set consists of individual realizations of the system is quite different from the case when parameters are influenced by random variables in time.

Therefore, a different assignment of the task is presented. Parameters of the system response statistics are looked for, when items of an initial statistical set are individual cases of the system itself. In other words, the coefficients have two sources of perturbation. Random noises, usually introduced as the Gaussian white noises, which are functions of time, and the random imperfection, which represents statistics in the set of implementations of the system described by a certain probability density.

From this point of view, it is crucial if, for instance, the material density is limited on both sides of the phase variables. If it is not limited (it is driven, e.g., by the normal distribution), the system will lose stochastic stability immediately or after a certain period of time. Therefore, the width of the band containing admissible cases should be strictly limited, see, e.g., [6]. Furthermore, many more papers can be cited investigating specific attributes of stochastic stability phenomena related with the topic dealt in this study, see, e.g. [1, 5] or [7]. The adequate limitations strongly depend in each parameter on the intensity of the parametric noise applied, correlation with an additive excitation noise and type of probability density of structural parameters.

Let us consider a simple system the function of which can be described by a stochastic differential equation of the first order

$$\dot{u}(t) = -(C + p + w(t))u(t) \; + \; f(t) \; + \; \varphi(t) \,, \tag{1}$$

where:

C – constant, nominal value of a system parameter;

p – deviation (imperfections) of the parameter from its nominal value; discrete set of values in realisations with zero mathematical mean value and known probability density;

$f(t)$ – useful signal (deterministic part of excitation);

$w(t)$, $\varphi(t)$ – time variable parametric perturbation or additive perturbation of excitation; Gaussian white noises of constant intensities $s_w, s_\varphi, s_{w\varphi}$.

The initial condition of the response $u(0) = u_\circ$ is a random quantity of probability density $h_\circ(u_\circ)$. The imperfection p can be considered as a constant. Within every individual realization of the system or time period of its service, the parameter is not subjected to differentiation or integration in time.

For every value of p we can consider in Eq. (1) the processes $w(t), \varphi(t), u(t)$ as Markov processes in time. With respect to Eq. (1), the Fokker-Planck equation can be deduced. Using the Ito white noise definition, see [2, 3, 8] and other monographs, the relevant equation can be written as

$$\begin{aligned}\frac{\partial h(u,t)}{\partial t} =& \frac{\partial}{\partial u}\left[\left(\left(C + p - \frac{1}{2}s_{ww}\right)u(t) + \frac{1}{2}s_{w\varphi} - f(t)\right)h(u,t)\right] \\ &+ \frac{1}{2}\frac{\partial^2}{\partial u^2}\left[\left(s_{ww}u^2(t) - 2s_{w\varphi}u(t) + s_{\varphi\varphi}\right)h(u,t)\right], \end{aligned} \tag{2}$$

where:

$h = h(u,t)$ – probability density function of the system response;
s_{ww}, $s_{w\varphi}$, $s_{\varphi\varphi}$ – intensity or cross-intensity of parametric and additive noises $w(t)$, $\varphi(t)$.

Using Eq. (2) for the construction of the equations for the first and second unknown stochastic moments of the response, it can be obtained

$$\dot{u}_s^i(t) = -(C + p^i - \frac{1}{2}s_w)u_s^i(t) \;-\; \frac{1}{2}s_{w\varphi} \;+\; f(t); \quad u_s^i(0) = u_{s\circ}^i\,, \tag{3}$$

$$\dot{D}_u^i(t) = -2(C + p^i - s_w)D_u^i(t) \;-\; 2s_{w\varphi}u_s^i(t) \;+\; s_\varphi; \quad D_u^i(0) = D_{u\circ}^i\,, \tag{4}$$

where:

$u_s^i(t)$ – mathematical mean value of the response for the i-th realization of the parameter imperfection;
$D_u^i(t)$ – variance of the response for the i-th realization of the parameter imperfection;
$u_{s\circ}^i$, $D_{u\circ}^i$ – random initial conditions (we will introduce the assumption of the statistical independence of initial conditions and parameter imperfections).

The solution of these equations can be expressed by means of Green functions in the form of

$$u_s^i(t) = u_{sg}^i(t,0,p^i)\cdot u_{s\circ}^i \;+\; \int_0^t u_{sg}^i(t,\tau,p^i)(f(\tau) - \frac{1}{2}s_{w\varphi})\mathrm{d}\tau\,, \tag{5}$$

$$D_u^i(t) = D_{ug}^i(t,0,p^i)\cdot D_{u\circ}^i \;+\; \int_0^t D_{ug}^i(t,\tau,p^i)(s_\varphi - 2s_{w\varphi}u_s^i(\tau))\mathrm{d}\tau\,, \tag{6}$$

where $u_{sg}^i(t,\tau,p^i)$ and $D_{ug}^i(t,\tau,p^i)$ are Green functions arising from Eqs. (3) and (4) for annulled right-hand sides and initial conditions of $u_{sg}^i(\tau,\tau,p^i) = 1$, $D_{ug}^i(\tau,\tau,p^i) = 0$ or $u_{sg}^i(\tau,\tau,p^i) = 0$, $D_{ug}^i(\tau,\tau,p^i) = 1$, respectively,

$$u_{sg}^i(t,\tau,p^i) = \exp\left[-\left(C + p^i - \frac{1}{2}s_w\right)(t-\tau)\right], \tag{7}$$

$$D_{ug}^i(t,\tau,p^i) = \exp\left[-2\left(C + p^i - s_w\right)(t-\tau)\right]. \tag{8}$$

To determine the mathematical mean value $u_s(t)$ and variance of the response $D_u(t)$ on the set of realizations, we will apply the mathematical mean value operator $\mathbf{E}\{\cdot\}$ to Eqs. (5) and

(6), making use of the fact that in the given case the operators of mathematical mean value and integration are mutually commutable, and of the statistical independence of initial conditions from imperfections. That means

$$u_s(t) = u_{sg}(t,0)\cdot u_{s\circ} \; + \int\limits_0^t u_{sg}(t,\tau)(f(\tau) - \frac{1}{2}s_{w\varphi})\mathrm{d}\tau\,, \tag{9}$$

$$D_u(t) = D_{ug}(t,0)\cdot D_{u\circ} \; + \int\limits_0^t D_{ug}(t,\tau)(s_\varphi - 2s_{w\varphi}u_s(\tau))\mathrm{d}\tau\,, \tag{10}$$

$$u_{sg}(t,\tau) = \mathbf{E}\{u^i_{sg}(t,\tau,p^i)\}\;; \quad D_{ug}(t,\tau) = \mathbf{E}\{D^i_{ug}(t,\tau,p^i)\}\,. \tag{11}$$

The kernels of integrals in Eqs. (9) and (10) implicitly depend, in the meaning of Eq. (11), on the probability density of the imperfection p, while the influence of particularly the additive noise $\varphi(t)$ is expressed in Eqs. (9) and (10) relatively distinctly.

Let us pay attention to some special cases of imperfection probability density distribution:

(i) Normal distribution of imperfections. The uncertainty of the quality of the individual parts of the system is commonly characterized by the normal distribution or its mean value and by variance D_p, see [4] and many others,

$$h(p) = \frac{1}{\sqrt{2\pi D_p}}\cdot\exp\left(-\frac{p^2}{2D_p}\right). \tag{12}$$

Let us assume that the useful signal $f(t) = \text{const}$. If $C > s_w/2$, then $0 < t < t_m = 2(C - \frac{1}{2}s_w)/D_p$, i.e., within a finite time interval, the system is stochastically stable in probability. Provided that $C \le s_w/2$, the system is unstable from the very beginning, i.e., for all $t > 0$. These conclusions would not change even if $f(t) \ne \text{const}$. The initial or "deferred" loss of stability in both cases results from the dispersal of imperfections D_p different from zero.

(ii) Uniform distribution of imperfections. The parameter imperfection should be obviously limited on both sides within a finite interval $-\Delta \le p \le \Delta$. The simplest distribution of probability complying with this requirement is the uniform distribution, see [2, 3] and others,

$$h(p) = \begin{cases} \frac{1}{2d} & ; \quad -\Delta < p < \Delta, \\ 0 & ; \quad p < -\Delta\;;\; p > \Delta. \end{cases} \tag{13}$$

It can be shown that for $t \to \infty$ the mathematical mean value equals

$$\lim_{t\to\infty} u_s(t) = u_{sn} = \frac{f_\circ - \frac{1}{2}s_{w\varphi}}{2\Delta}\cdot lg\,\frac{C - \frac{1}{2}s_w + \Delta}{C - \frac{1}{2}s_w - \Delta}. \tag{14}$$

This limit exists, if the imperfection interval is limited by $C \; > \; \frac{1}{2}s_w \; + \; \Delta$.

(iii) Truncated normal distribution of imperfections. The probability density of imperfections p is described by the truncated Gaussian distribution for $|p| \le \Delta$, see [4] and a number of additional papers cited herein,

$$h(p) \;=\; \begin{cases} 0 & ;\; p \le C - \Delta, \\ \frac{\mu}{\sqrt{2\pi D_p}}\cdot\exp\left(-\frac{(p-C)^2}{2D_p}\right) & ;\; C - \Delta < p < C + \Delta, \\ 0 & ;\; p \ge C + \Delta, \end{cases} \tag{15}$$

$$\mu^{-1} \;=\; 2\Phi\left(\frac{\Delta}{\sqrt{2D_p}}\right) \;=\; \frac{2}{\sqrt{2\pi}} \int\limits_{0}^{\Delta/\sqrt{D_p}} e^{-\xi^2/2}\, d\xi. \tag{16}$$

If Δ is finite and $C > s_w/2 + \Delta$, the influence of initial conditions successively disappears with growing t and the system reveals to be stable in the mathematical mean value. The permissible width Δ of the zone of imperfections is determined by the white noise intensity s_w. It is decreasing with the increasing noise intensity and vice versa. The permissible zone is wider for the truncated normal distribution than for the uniform distribution.

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