An influence of unilateral sources and sinks in reaction-diffusion systems exhibiting Turing's instability on bifurcation and pattern formation

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Abstract

We consider a general reaction-diffusion system exhibiting Turing's diffusion-driven instability. In the first part of the paper, we supplement the activator equation by unilateral integral sources and sinks of the type $\left(\int_K \frac{u(x)}{|K|}\,dK\right)^-$ and $\left(\int_K \frac{u(x)}{|K|}\,dK\right)^+$. These terms measure an average of the concentration over the set K and are active only when this average decreases bellow or increases above the value of the reference spatially homogeneous steady state, which is shifted to the origin. We show that the set of diffusion parameters in which spatially heterogeneous stationary solutions can bifurcate from the reference state is smaller than in the classical case without any unilateral integral terms. This problem is studied for the case of mixed, pure Neumann and periodic boundary conditions. In the second part of the paper, we investigate the effect of both unilateral terms of the type u^-, u^+ and unilateral integral terms on the pattern formation using numerical experiments on the system with well-known Schnakenberg kinetics.

Keywords: reaction-diffusion systems, pattern formation, Turing's instability, unilateral integral terms, numerical experiments
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1. Introduction

In 1952 Alan M. Turing wrote his paper *"The Chemical Basis of Morphogenesis"* [1] on biological pattern formation, in which he described that a reaction of two chemicals and diffusion in the space can lead to the production of a spatially heterogeneous structure. That means, the spatially homogeneous steady state of the system, that is stable in the absence of diffusion, could be destabilized by diffusion and produce a spatially heterogeneous state, in other words a "pattern". This behaviour is quite counter-intuitive since diffusion is usually perceived as a stabilization effect. Twenty years later, Gierer and Meinhardt published their paper [2], in which they analysed this problem in more detail and introduced so called "short range activation-long range inhibition" mechanism. Their research was completely independent, because they did not know about Turing's paper. A lot of work has been done in this area in past few decades. This effect is usually called Turing's instability, Turing's effect or diffusion-driven instability (diffusion "drives" stable homogeneous state unstable). The theory is very thoroughly summarized e.g., in the second instalment of J. D. Murray's *"Mathematical biology"* [3].

A pattern formation is not a process exclusive to reaction-diffusion equations, but they are one of the most common models for pattern formation problems. We consider a classical system of

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two reaction-diffusion equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \Delta u + f(u,v), \\ \frac{\partial v}{\partial t} &= d_2 \Delta v + g(u,v), \end{aligned} \tag{1}$$

with functions $u = u(x,t), v = v(x,t)$ describing concentrations of some chemicals and f, g being reaction kinetics as our starting point.

The idea of Turing's instability is not perfect and one of problems it suffers is the strict requirement that diffusion coefficients must be different. Actually, for the most usual reaction kinetics this difference must be significant. The fact that one chemical diffuses with very different intensity than the other is not very realistic in many problems. There were efforts to modify or generalize the idea of Turing's instability. For example Rovinsky and Menzinger presented so called differential-flow-induced chemical instability [4] in the system, in which one of the chemicals is immobilized and the other diffuses and flows. This idea leads to a similar destabilization as by Turing's DDI. This mechanism was experimentally verified on Belousov-Zhabotinsky reaction (see [5]). Their work later inspired Klika et al. [6] to add advection to the complete reaction-diffusion system (no immobilized chemical). They were able to show that, even if diffusion coefficients are equal, the homogeneous steady state can be destabilized utilizing relatively small amount of advection.

Several authors focused on the analysis of the bifurcation during the destabilization of the spatially homogeneous steady state and bifurcating patterns. The set of couples of diffusion parameters, such that the reference steady state is stable or unstable, is usually denoted D_S or D_U, respectively, in following papers. Kučera et al. studied unilateral conditions related to the inhibitor equation (the equation for the chemical that is inhibiting its production) and some of these results were first mentioned in [7]. The abstract result for variational inequalities concerning destabilization of the reference spatially homogeneous steady state was showed in [8] and the influence of unilateral conditions on bifurcation of patterns was revealed in [9]. There were also other papers concerning similar problems e.g., [10], [11], [12], [13]. It is known that in the classical case the bifurcation from the reference steady state is excluded in D_S. However, if unilateral conditions are present in boundary conditions for the inhibitor equation, this bifurcation exists under some conditions. Hence, the set of couples of diffusion parameters, for which the bifurcation occurs and a bifurcating pattern exists, is larger than in the classical case without unilateral conditions. It was natural to ask, what effect will have unilateral conditions related to the activator equation (the equation for the chemical that is activating its production). The problem was explored in [14] and it was shown that the effect is exactly opposite, i.e., the set of couples of diffusion parameters, for which the bifurcation occurs and a bifurcating pattern exists, is smaller than in the classical case without unilateral conditions.

Later, the idea of unilateral conditions and variational inequalities was replaced by the idea of adding unilateral sources and sinks to the activator equation (see [15]) or to the inhibitor equation (see [16]), respectively. The unilateral source or sink is based on the negative part $\psi^- = \max\{-\psi, 0\}$ or the positive part $-\psi^+ = -\max\{\psi, 0\}$ of the function ψ, respectively (ψ represents the variable of an activator or an inhibitor). These terms act in the interior of the domain and they seem to be more natural than unilateral conditions given by variational inequalities. Unilateral terms cannot be linearized and the study of the stability is difficult and so far an open problem. Therefore only bifurcations were studied in these papers. It was again showed that if there are unilateral terms in the inhibitor equation, then the set of couples of diffusion parameters, for which the bifurcation occurs and a bifurcating pattern exists, is larger than in the classical case without unilateral terms. The case with unilateral terms in the activator equation again brings opposite results. There were performed many numerical experiments in [17] and [18] for the case of the unilateral source τv^- (and its modifications) in the inhibitor equation for the specific reaction kinetics.

In our paper we will investigate the impact of unilateral terms involving the integral average over some subset of the domain Ω, where we will work. To put it in the context with the previous

work mentioned above, for example we had the unilateral source $s_-(x)\frac{\psi^-}{1+\psi^-}$ in our paper [15]. This term measures exactly the value of ψ in each point x of Ω and if the value drops under certain threshold (in this case the zero), then the production of ψ is switched on locally on the support of $s_-(x)$. One simple example of unilateral terms we introduce in this paper is

$$\chi^K(x)\left(\int_K \frac{\psi(x)}{|K|}\,dK\right)^-,$$

where K is a subset of the domain Ω and $\chi^K(x)$ is the characteristic function of the set K. We denote by $|K|$ the Lebesgue measure of the set K. This term measures the average of ψ over the set K and if this average drops bellow the threshold (again in this case the zero), then the source is switched on and is acting again on the set K.

We will work with more general terms e.g., several sources (and sinks) and we will correctly describe properties of K, Ω etc. later. Our inspiration are partially papers about unilateral terms mentioned above and also [19], in which integral terms are used in some sense in boundary conditions.

We consider the reaction-diffusion system

$$\begin{aligned}
&\frac{\partial u}{\partial t} = d_1\Delta u + f(u,v) + \sum_{i=1}^{n}\chi^{M_i^-}(x)f_-^i\left(\left(\int_{K_i^-}\frac{u-\overline{u}}{|K_i^-|}\,dK_i^-\right)^-\right) - \sum_{j=1}^{m}\chi^{M_j^+}(x)f_+^j\left(\left(\int_{K_j^+}\frac{u-\overline{u}}{|K_j^+|}\,dK_j^+\right)^+\right),\\
&\frac{\partial v}{\partial t} = d_2\Delta v + g(u,v) \qquad \text{in } \Omega\times[0,+\infty)
\end{aligned} \tag{2}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary, d_1 and d_2 are positive diffusion parameters, $f,g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are real differentiable functions and there exist constants $\overline{u},\overline{v}>0$ such that

$$f(\overline{u},\overline{v}) = g(\overline{u},\overline{v}) = 0, \tag{3}$$

i.e., $[\overline{u},\overline{v}]$ is a spatially homogeneous steady state. Furthermore we assume that $f_-^i, f_+^j:\mathbb{R}\to\mathbb{R}$ are real functions such that

$$f_-^i(0) = f_+^j(0) = 0 \quad \text{for every } i=1,\dots n, j=1,\dots,m, \tag{4}$$

and there exist

$$\tau_-^i := \frac{\partial f_-^i}{\partial\xi}(\xi)|_{\xi=0}\in\mathbb{R}_0^+, \quad \tau_+^j := \frac{\partial f_+^j}{\partial\xi}(\xi)|_{\xi=0}\in\mathbb{R}_0^+ \qquad \text{for all } i=1,\dots,n, j=1,\dots,m. \tag{5}$$

We suppose that $K_i^-, K_j^+\subseteq\Omega$ and functions $\chi^{K_i^-}(x)$ and $\chi^{K_j^+}(x)$ are characteristic functions of sets K_i^- and K_j^+, respectively. We will assume that sets K_i^- are connected and disjoint. The same is assumed for sets K_j^+.

The system (2) will be completed by boundary conditions

$$\begin{aligned}
u = \overline{u}, v = \overline{v} \quad &\text{on } \Gamma_D,\\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad &\text{on } \Gamma_N,
\end{aligned} \tag{6}$$

where n is a unit out-ward pointing normal vector of the boundary $\partial\Omega$ and Γ_D, Γ_N are open disjoint subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma_D}\cup\overline{\Gamma_N}$. We will distinguish two cases $\Gamma_D\neq\emptyset$, i.e., mixed boundary conditions, and $\Gamma_D=\emptyset$, i.e., pure Neumann boundary conditions. The case of pure Dirichlet boundary conditions is included in the case $\Gamma_D\neq\emptyset$ and we do not treat it separately, because the analysis and the results are the same. We will also discuss problems with unilateral integral terms in boundary conditions as well as the case of periodic boundary conditions

$$\begin{aligned}
u(\mathbf{x}) &= u(\mathbf{x}_P),\\
-\frac{\partial u}{\partial n}(\mathbf{x}) &= \frac{\partial u}{\partial n}(\mathbf{x}_P),
\end{aligned} \tag{7}$$

on special domains. We will describe this case in more detail in the next section.

Our goal is to show that we can get similar results concerning bifurcations for reaction-diffusion systems with unilateral integral terms as we did for unilateral terms without integral in the activator equation in [15]. Hence, the set of couples of diffusion parameters, for which the bifurcation occurs and a bifurcating pattern exists, is smaller than in the classical case. In the second part of the paper we use numerical experiments to investigate the influence of unilateral terms from [15] and unilateral integral terms presented in this paper on the pattern formation. We study the case of Neumann and periodic boundary conditions on the square domain Ω and we use well-known Schnakenberg kinetics as functions f and g. The goal is to study the shape of patterns, to compare Neumann and periodic boundary conditions and to show that with increasing strength of unilateral (integral) terms the set of $[d_1, d_2]$ such that patterns are produced and the spatially homogeneous steady state seems to be unstable is getting smaller.

Main ideas are in some sense similar to those in [15] and [14]. We will take the stationary problem of the reaction-diffusion system (2), (6) and rewrite it into the weak formulation in Sobolev space and further to the system of operator equations. Then we will extract the variable v from the second equation and substitute it to the first one. This way the system of operator equation will be reduced to a single operator equation. In the paper [15] we used a variational characterization of the maximal eigenvalue. However, in this case it is not possible. Therefore, we will use different, maybe more simple, approach to get our results, similar to the one in [14]. The numerical experiments are inspired by analytical results from [15] and this paper in some sense follows the paper [17].

The text is divided in the following manner. Section 2 consists of essential definitions, general assumptions and the summary of important known facts. We formulate our main results in Section 3. In Section 4 we define operators and prove their properties and in Section 5 we reduce our system to a single operator equation and present proofs of theorems from Section 3. Section 6 concerns numerical experiments.

2. Turing's instability, basic assumptions and periodic boundary conditions

It is possible without the loss of generality to shift the reference spatially homogeneous steady state $[\overline{u}, \overline{v}]$ to the origin for the sake of simpler analysis. Therefore we will assume from the start that $[\overline{u}, \overline{v}] = [0, 0]$. In such case, functions u and v do not describe concentrations of the chemicals, but rather the difference of these concentrations from the original positive reference steady state $[\overline{u}, \overline{v}]$. Hence, our boundary conditions (6) are transformed to

$$\begin{aligned} u = v = 0 \quad & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad & \text{on } \Gamma_N. \end{aligned} \tag{8}$$

Remark 2.1. *By solutions we will mean weak solutions usually in the space*

$$H^1_D(\Omega) := \{\phi \in W^{1,2}(\Omega) : \phi = 0 \text{ on } \Gamma_D \text{ in the sense of traces}\}. \tag{9}$$

If $\Gamma_D = \emptyset$, then the space H^1_D is actually the whole Sobolev space $W^{1,2}$ equipped with the standard inner product

$$(u, \varphi)_{H^1_D} = (u, \varphi)_{W^{1,2}} = \int_\Omega (\nabla u \nabla \varphi + u\varphi) \ d\Omega \tag{10}$$

and the Sobolev norm $\|u\|_{W^{1,2}} = \left(\int_\Omega (\nabla u)^2 + u^2 \ d\Omega\right)^{\frac{1}{2}}$. If $\Gamma_D \neq \emptyset$, then we will use the inner product

$$(u, \varphi)_{H^1_D} = \int_\Omega \nabla u \nabla \varphi \ d\Omega \tag{11}$$

and the norm $\|u\|_{H^1_D} = \left(\int_\Omega (\nabla u)^2 \ d\Omega\right)^{\frac{1}{2}}$ equivalent to the classical Sobolev norm.

We will now define the functional $T_X^{\mp} : H_D^1 \to \mathbb{R}_0^+$ by

$$T_X^{\mp}(\psi) = \left(\int_X \frac{\psi}{|X|}\, dX \right)^{\mp}, \tag{12}$$

where X is some subset of Ω. It will allow us to write many upcoming formulas in more compact way.

We can write stationary system corresponding to the system (2) and to the classical system without unilateral integral terms (1) as

$$\begin{aligned} 0 &= d_1\Delta u + b_{1,1}u + b_{1,2}v + n_1(u,v) + \sum_{i=1}^{n} \chi^{K_i^-}(x) f_-^i \left(T^-_{K_i^-}(u) \right) - \sum_{j=1}^{m} \chi^{K_j^+}(x) f_+^j \left(T^+_{K_j^+}(u) \right), \\ 0 &= d_2\Delta v + b_{2,1}u + b_{2,2}v + n_2(u,v), \end{aligned} \tag{13}$$

and

$$\begin{aligned} 0 &= d_1\Delta u + b_{1,1}u + b_{1,2}v + n_1(u,v), \\ 0 &= d_2\Delta v + b_{2,1}u + b_{2,2}v + n_2(u,v), \end{aligned} \tag{14}$$

respectively, where $b_{i,j}, i,j = 1,2$ are elements of Jacobi matrix of mappings f, g at $[0,0]$ and n_1, n_2 are higher order terms, i.e.,

$$n_{1,2}(u,v) = o(|u| + |v|) \text{ as } |u| + |v| \to 0. \tag{15}$$

We will always assume that the following set of inequalities is satisfied:

$$b_{1,1} + b_{2,2} < 0,\ b_{1,1}b_{2,2} - b_{1,2}b_{2,1} > 0,\ b_{1,1} > 0,\ b_{2,2} < 0,\ b_{1,2}b_{2,1} < 0. \tag{16}$$

The first two conditions of (16) guarantee that the reference steady state is stable as a solution of the classical system in the absence of diffusion. The third and fourth condition correspond to the fact that we expect that the first equation of the system (2) is the activator equation while the second one is the inhibitor equation. The last condition decides whether the system is in activator-inhibitor form (for $b_{1,2} < 0$, $b_{2,1} > 0$) or substrate-depletion form (for $b_{1,2} > 0$, $b_{2,1} < 0$) as was specified in [2].

Further we will suppose that there exists $c \in \mathbb{R}$ such that

$$|n_j(\chi,\xi)| \le c(1 + |\chi|^{q-1} + |\xi|^{q-1}) \quad \text{for all } \chi,\xi \in \mathbb{R},\ j = 1,2, \tag{17}$$

with some $q > 2$ if $N = 2$ or $2 < q < \frac{2N}{N-2}$ if $N > 2$. In the dimension $N = 1$ no growth assumptions are necessary.

We can homogenize the system (13) and linearize the classical system (14) to get

$$\begin{aligned} 0 &= d_1\Delta u + b_{1,1}u + b_{1,2}v + \sum_{i=1}^{n} \chi^{K_i^-}(x) \tau_-^i T^-_{K_i^-}(u) - \sum_{j=1}^{m} \chi^{K_j^+}(x) \tau_+^j T^+_{K_j^+}(u), \\ 0 &= d_2\Delta v + b_{2,1}u + b_{2,2}v, \end{aligned} \tag{18}$$

and

$$\begin{aligned} 0 &= d_1\Delta u + b_{1,1}u + b_{1,2}v, \\ 0 &= d_2\Delta v + b_{2,1}u + b_{2,2}v. \end{aligned} \tag{19}$$

Definition 2.1 (Critical point)**.**
A parameter $d = [d_1, d_2] \in \mathbb{R}_+^2$ will be called a critical point of (19), (8) or (18), (8) if there exists a non-trivial (weak) solution of (19), (8) or (18), (8), respectively.

Definition 2.2 (Bifurcation point)**.**
A parameter $d^0 = [d_1^0, d_2^0] \in \mathbb{R}_+^2$ will be called a bifurcation point of (14), (8) or (13), (8) if in any neighbourhood of $[d^0, 0, 0] \in \mathbb{R}_+^2 \times H_D^1 \times H_D^1$ there exists $[d, W] = [d, u, v]$, $\|W\| \neq 0$ satisfying (14), (8) or (13), (8), respectively.

Remark 2.2. *Let us consider the problem*

$$\begin{aligned} -\Delta u &= \kappa u, \\ u &= 0 \text{ on } \Gamma_D, \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_N. \end{aligned} \tag{20}$$

The eigenvalues of (20) *form a non-negative non-decreasing sequence* κ_k *with* $k = 1, 2, \ldots$ *(for* $\Gamma_D \neq \emptyset$*) or* $k = 0, 1, 2, \ldots$ *(for* $\Gamma_D = \emptyset$*). The first eigenvalue is always simple. In the case* $\Gamma_D \neq \emptyset$*, the eigenfunction* e_1 *corresponding to the first eigenvalue* κ_1 *does not change the sign on the domain* Ω*. In the case* $\Gamma_D = \emptyset$*, the eigenfunction* e_0 *corresponding to the first eigenvalue* $\kappa_0 = 0$ *is constant. Other eigenfunctions change the sign in both cases. We can choose an orthonormal basis* e_k *in* H^1_D*,* $k = 1, 2, \ldots$ *(for* $\Gamma_D \neq \emptyset$*) or* $k = 0, 1, 2, \ldots$ *(for* $\Gamma_D = \emptyset$*) composed of the eigenfunctions of* (20).

The eigenvalue problem

$$\begin{aligned} \lambda u &= d_1 \Delta u + b_{1,1} u + b_{1,2} v, \\ \lambda v &= d_2 \Delta v + b_{2,1} u + b_{2,2} v. \end{aligned} \tag{21}$$

with boundary conditions (8) determines the stability of the reference stationary solution $[0, 0]$ of the evolutionary problem corresponding to (14), (8). From the dispersion relation

$$b_{1,1} b_{2,2} - b_{1,2} b_{2,1} + \kappa^2 d_1 d_2 - \kappa (b_{1,1} d_2 + b_{2,2} d_1) = 0 \tag{22}$$

we can derive sets of points $[d_1, d_2]$ for which λ from the system (21) is zero. These sets can be written with respect to d_1 as

$$C_k := \left\{ [d_1, d_2] \in \mathbb{R}^2_+ : \; d_1 = \frac{1}{\kappa_k} \left(\frac{b_{1,2} b_{2,1}}{d_2 \kappa_k - b_{2,2}} + b_{1,1} \right) \right\}, \; k = 1, 2, \ldots \tag{23}$$

where κ_k are the eigenvalues of the problem (20). These sets are actually hyperbolas, but since we consider only positive d_1, d_2, we have only parts of these hyperbolas (see Figure 1). There is no hyperbola for the zero eigenvalue κ_0 (see (22) and (16)). Let us define the envelope

$$C_E := \left\{ d = [d_1, d_2] \in \mathbb{R}^2_+ : \; d_1 = \max_{\tilde{d}_1 \in \mathbb{R}_+} \left\{ \tilde{d}_1 : [\tilde{d}_1, d_2] \in \bigcup_{k=1}^{\infty} C_k \right\} \right\}, \tag{24}$$

which divides the positive quadrant $\mathbb{R}^2_+$ on two sets D_U and D_S (see Figure 1). Moreover, we define two sets, which will be important in the formulation of our main results. Let $r, R, \varepsilon \in \mathbb{R}_+$ and $r < R$. We define

$$C_r^R := \{ d = [d_1, d_2] \in C_E : d_2 \in [r, R] \}, \tag{25}$$

$$C_r^R(\varepsilon) := \{ d = [d_1, d_2] \in C_E \cup D_U : d_2 \in [r, R] \wedge \operatorname{dist}(d, C_E) < \varepsilon \}. \tag{26}$$

Remark 2.3. *If all eigenvalues of* (20) *are simple, i.e.,* $\kappa_k < \kappa_{k+1}$ *for all* $k \in \mathbb{N}$*, then* $C_k \neq C_{k+1}$ *for all* $k \in \mathbb{N}$*. If an eigenvalue* κ_k *has a multiplicity* l*, then* $\kappa_{k-1} < \kappa_k = \ldots = \kappa_{k+l-1} < \kappa_{k+l}$ *and* $C_{k-1} \neq C_k = \ldots = C_{k+l-1} \neq C_{k+l}$*. The sets*

$$\begin{aligned} D_U &:= \{ d = [d_1, d_2] \in \mathbb{R}^2_+ : d \text{ is on the left of } C_E \}, \\ D_S &:= \{ d = [d_1, d_2] \in \mathbb{R}^2_+ : d \text{ is on the right of } C_E \} \end{aligned}$$

are called the domain of instability and the domain of stability. It is known that if $[d_1, d_2] \in D_S$*, then all eigenvalues* λ *of the problem* (21), (8) *have negative real parts and if* $[d_1, d_2] \in D_U$*, then there is a positive eigenvalue* λ *(for the particular case see [20],[21] and for the general case [10]). In particular, the trivial solution of* (14), (8) *is linearly stable for* $[d_1, d_2] \in D_S$ *and unstable for* $[d_1, d_2] \in D_U$ *(see e.g., Chapter 11 in [22]).*

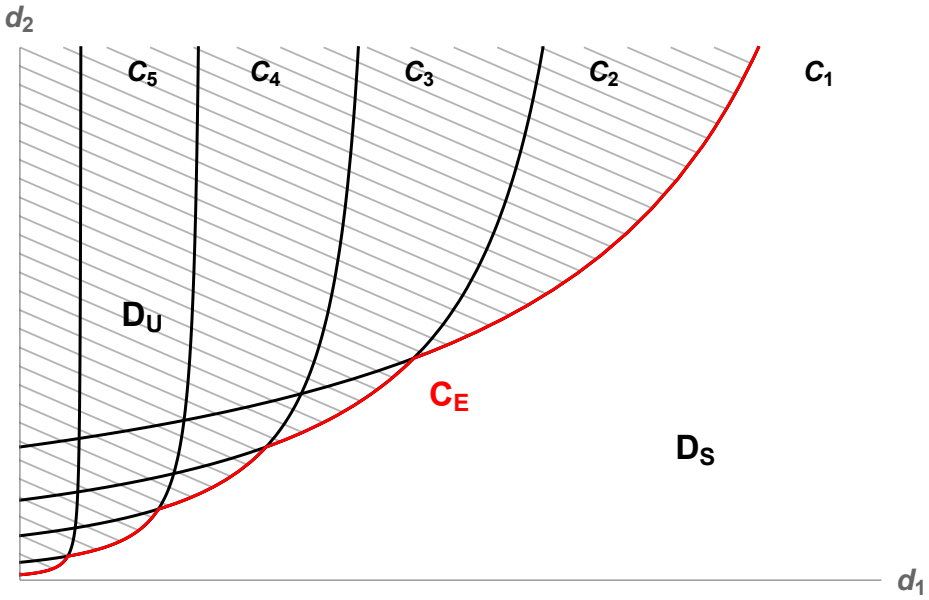


Figure 1: The illustration of hyperbolas C_k and the envelope C_E. The case when all eigenvalues κ_k are simple. The hatched region is the domain of instability D_U.

Remark 2.4. *The following properties of the curves C_k are known, see e.g., [21],[20] for the particular case, or [10] for the general case.*

- *A point $d = [d_1, d_2]$ is a critical point of (19), (8) if and only if there exists k such that $d \in C_k$. In particular, the domain of stability D_S does not contain any critical point of (19), (8) or bifurcation point of (14), (8). Under some additional assumptions, e.g., if the eigenvalue κ_k is simple or of odd multiplicity, the points on C_k are simultaneously bifurcation points (see e.g., [21]).*
- *If $d \in C_n$ for $n = k, \dots, k+l-1$ (either l is the multiplicity of the eigenvalue κ_k or d is in the intersection of two hyperbolas C_k,C_m and l is the sum of multiplicities of κ_k, κ_m, see Remark 2.3), then span $\left(\left[\frac{d_2\kappa_k - b_{2,2}}{b_{2,1}} e_k, e_k \right]_{n=k}^{k+l-1} \right)$ is the set of the solutions of (19), (8).*

Now let us assume that $K_i^-, K_j^+ \subseteq \Gamma_N$. We will also consider a problem with unilateral integral terms in boundary conditions, i.e., systems (14) and (19) with boundary conditions

$$\begin{aligned} u = v = 0 \quad &\text{on} \quad \Gamma_D, \\ \frac{\partial u}{\partial n} = \sum_{i=1}^{n} \chi^{K_i^-}(x) \tau_-^i T_{K_i^-}^-(u) - \sum_{j=1}^{m} \chi^{K_j^+}(x) \tau_+^j T_{K_j^+}^+(u) \quad &\text{on} \quad \Gamma_N, \\ \frac{\partial v}{\partial n} = 0 \quad &\text{on} \quad \Gamma_N. \end{aligned} \tag{27}$$

2.1. Periodic boundary conditions

Probably the most common boundary conditions considered in the theory of Turing's DDI and pattern formation are homogeneous Neumann boundary conditions. While they are more natural for these kind of problems than Dirichlet boundary conditions, they are not always the best option. Let us for example study the pattern formation on the skin of some animal (e.g., cheetah). We would like to assume that our domain Ω is actually a cut out part of its fur and we could construct his fur by repeating this domain Ω. For such case periodic boundary conditions are better option than Neumann.

The basic idea of periodic boundary conditions is that in paired points the solution has the same value and the same derivative in the corresponding direction (see bellow). The definition is simple in dimension $N = 1$, let us assume i.e., $\Omega = (a, b)$ with $a < b$. In this case periodic boundary conditions are

$$\begin{aligned} u(a) &= u(b), \\ -\frac{\partial u}{\partial n}(a) &= \frac{\partial u}{\partial n}(b), \end{aligned} \tag{28}$$

and the same for v.

The setting of periodic boundary conditions in the higher dimension is not so easy. We will suppose here that the domain Ω satisfies the following properties:

$$\begin{aligned} &\textit{for } N = 2 : \Omega \textit{ is convex and its boundary is composed of } n \textit{ pairs of edges } \Gamma^i \textit{ and } \Gamma^i_P \\ &\qquad \textit{with } i = 1, \dots n, \textit{ which are parallel and of the same length.} \\ &\textit{for } N \neq 2 : \Omega \textit{ is a hypercube with } N \textit{ pairs of parallel facets } \Gamma^i \textit{ and } \Gamma^i_P \textit{ with } i = 1, \dots N. \end{aligned} \tag{29}$$

The dimension $N = 2$ is the most interesting for us, because patterns are often studied by numerical experiments in this dimension and it is probably the most important one for applications. That is why we assume more general Ω in (29) despite the fact that we could just settle here with a hypercube (square) too. We mention that periodic boundary conditions could be defined on even more general domains.

Now in general, we denote $\overrightarrow{p_i}$ the vector of the line connecting the center of Γ^i and the center of Γ^i_P. For every point $\mathbf{x} \in \Gamma^i$ there exists a point $\mathbf{x}_P \in \Gamma^i_P$ such that $\mathbf{x}_P$ lies in the intersection of Γ^i_P and the line given by $\overrightarrow{p_i}$ and $\mathbf{x}$. By periodic boundary conditions we will mean boundary conditions of the type

$$u(\mathbf{x}) = u(\mathbf{x}_P), \tag{30a}$$

$$-\frac{\partial u}{\partial n}(\mathbf{x}) = \frac{\partial u}{\partial n}(\mathbf{x}_P), \tag{30b}$$

for every pair of $\mathbf{x} \in \Gamma^i, \mathbf{x}_P \in \Gamma^i_P$ (and the same for v). The illustration of periodic boundary conditions on hexagon and rhomboid are on Figure 2.

In the following remark, we discuss a special type of the shape of Ω in the dimension $N = 2$ suitable for construction of the animal skin.

Remark 2.5. *As we mentioned above, the motivation for periodic boundary conditions is to simulate pattern formation on Ω and we aim to construct something larger by repeatedly folding Ω ("as a puzzle"). This leads to the tesselation (or tilling) of the plane, in particular the tesselation by a single geometrical object, i.e., we cannot combine e.g., squares and octagons. The plane can be tesselated by equilateral triangles, squares and hexagons (and their distorted variants). Since triangles do not satisfy the condition (29), we exclude them. Hence, we can use only convex parallelogons (e.g., square, rhomboid, hexagon etc.) for these purposes.*

Remark 2.6. *When we consider reaction-diffusion system with periodic boundary conditions described in Remark 2.5, by solution we will always mean weak solution in the space of periodic functions*

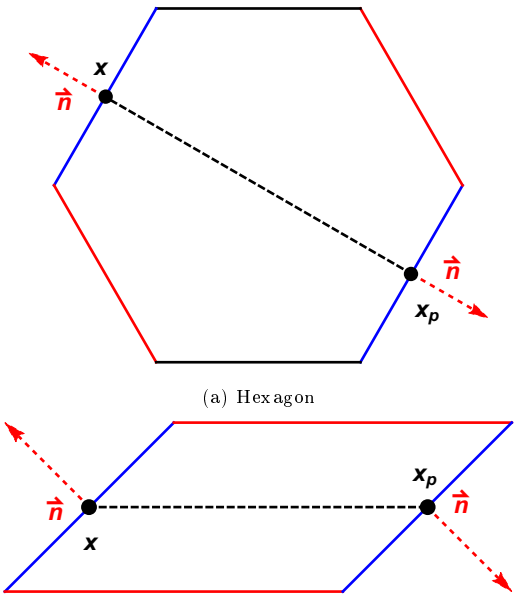
$$H^1_{per}(\Omega) := \left\{ \varphi \in W^{1,2}(\Omega) : \varphi \textit{ satisfies } (30a) \right\}. \tag{31}$$

The space is equipped with the same norm and inner product as $W^{1,2}$ (see Remark 2.1).

The Laplace eigenvalue problem

$$-\Delta u = \kappa u \tag{32}$$

with periodic boundary conditions has similar structure of eigenvalues and eigenfunctions as if pure Neumann boundary conditions were considered. Hence, there is the eigenvalue $\kappa_0 = 0$ and the corresponding eigenfunction e_0 is constant. Other eigenvalues are positive. Let us note that hyperbolas and their properties are the same.



(a) Hexagon

(b) Rhomboid

Figure 2: Illustration of periodic boundary conditions. Every two edges, where we define periodic boundary conditions, have the same color.

3. Main results

We will use notation from previous sections and we assume that conditions (4), (5), (15), (16), (17) are satisfied. Proofs of results presented here are postponed to the end of Section 5.

Theorem 3.1.

(i) The domain of stability D_S contains neither critical points of (18), (8) nor bifurcation points of (13), (8).

(ii) Let $0 < r < R$. Let $C_k, \dots, C_{k+l-1}$ be all hyperbolas which have a non-empty intersection with C_r^R. Let any linear combination e of the eigenfunctions of (20) corresponding to $\kappa_k, \dots, \kappa_{k+l-1}$ satisfy

$$\sum_{i=1}^{n} \chi^{K_i^-}(x)\tau_-^i T_{K_i^-}^-(e) - \sum_{j=1}^{m} \chi^{K_j^+}(x)\tau_+^j T_{K_j^+}^+(e) \not\equiv 0. \tag{33}$$

Then there exists $\varepsilon > 0$ such that there are neither critical points of (18), (8) nor bifurcation points of (13), (8) in $C_r^R(\varepsilon)$.

Remark 3.1. *If the condition (33) is not satisfied, then (18) becomes (19) and every point $[d_1, d_2] \in C_r^R$ is a critical point of (18), (8) due to Remark 2.4. Let e.g., C_r^R has non-empty intersection with exactly two non-coinciding hyperbolas C_1 and C_2. Now it is possible that both e_1 and e_2 satisfy the condition (33), hence there are no critical points on C_1 and C_2 excluding their intersection. In the same time this intersection can be a critical point due to the fact that a linear combination of e_1 and e_2 does not have to satisfy this condition. However, the opposite case that there would be critical points on C_1 and C_2, but not in their intersection, is not possible. In the scenario in which $C_1 = C_2$ all linear combinations of e_1 and e_2 must satisfy (33) so that there would not be any critical points on C_1 or C_2.*

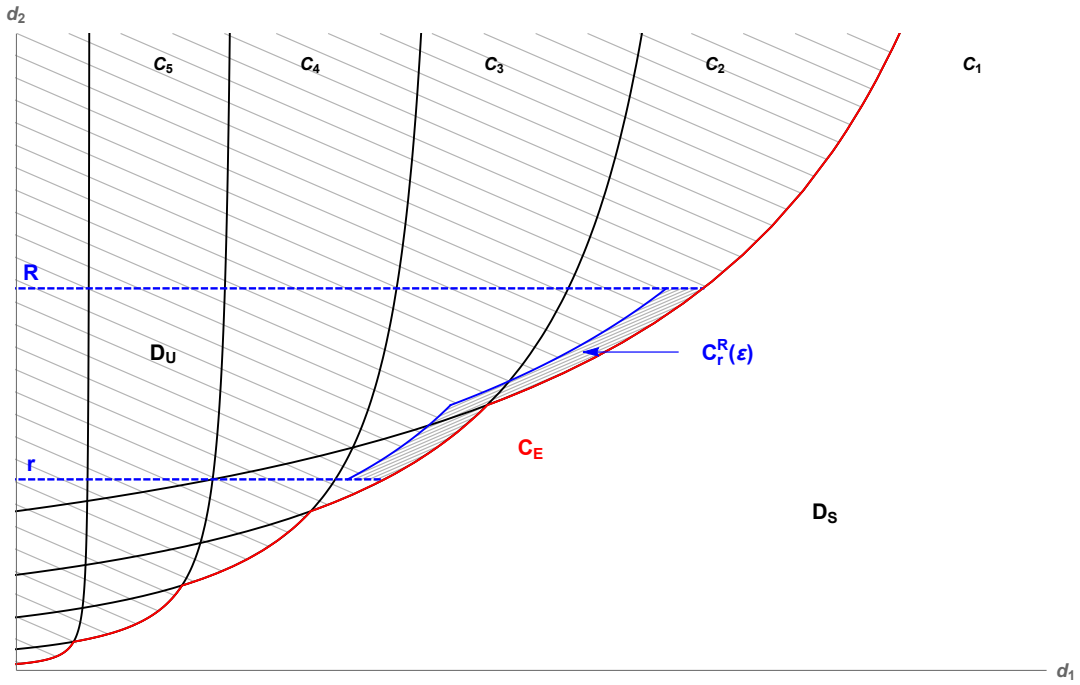


Figure 3: The illustration of the main result and $C_r^R(\varepsilon)$ neighbourhood. The case when all eigenvalues κ_k are simple.

For the case of unilateral terms of the type u^-, u^+, if only sources or sinks were present in the system, it was possible to show the existence of the ε-neighbourhood along the whole envelope C_E (excluding C_1 in the mixed boundary conditions case) free of critical points of (18), (8) without satisfying the condition similar to (33) (see [15, Theorems 3.2, 3.3]). It was based on the fact that the eigenfunctions e_k of (20) for $k > 1$ change the sign. This is not enough in the case of unilateral integral terms and it is necessary to satisfy condition (33).

Theorem 3.2. *Let $\Gamma_D \neq \emptyset$. Let either $\tau_-^i = 0$ and $\tau_+^j > 0$ or $\tau_-^i > 0$ and $\tau_+^j = 0$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$ (that means we have either sources or sinks in the system). Let d_2^I be the second coordinate of the intersection point of C_1 and C_2. Any $d \in C_1$, in particular any $d \in C_r^R$ with $d_2^I \leq r < R$, is a critical point of (18),(8).*

Theorem 3.3. *Let $\Gamma_D = \emptyset$ and $K_i^- = K_j^+ = \Omega$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$. The condition (33) from Theorem 3.1 can never be satisfied and any point $[d_1, d_2] \in C_E$ is a critical point of (18), (8).*

Remark 3.2. *The same formulation of Theorem 3.3 holds in the case that we consider Ω with property (29) and systems with periodic boundary conditions instead of Neumann.*

Theorem 3.4. *Let $K_i^-, K_j^+ \subseteq \Gamma_N$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$.*

(i) The domain of stability D_S contains neither critical points of (19), (27) nor bifurcation points of (14), (27).

(ii) Let $0 < r < R$. Let $C_k, \dots, C_{k+l-1}$ be all hyperbolas which have a non-empty intersection with C_r^R. Let any linear combination e of the eigenfunctions of (20) corresponding to $\kappa_k, \dots, \kappa_{k+l-1}$ satisfy

$$\sum_{i=1}^{n} \chi^{K_i^-}(x) \tau_-^i T_{K_i^-}^-(e) - \sum_{j=1}^{m} \chi^{K_j^+}(x) \tau_+^j T_{K_j^+}^+(e) \not\equiv 0 \quad \text{on } \Gamma_N. \tag{34}$$

Then there exists $\varepsilon > 0$ such that there are neither critical points of (19), (27) nor bifurcation points of (14), (27) in $C_r^R(\varepsilon)$.

Remark 3.3. *The fact that there are no bifurcation points in D_S or even in $C_r^R(\varepsilon) \cup D_S$ implies that for any compact part M of D_S or $C_r^R(\varepsilon) \cup D_S$, respectively, there exists $\delta > 0$ such that for any $[d_1, d_2] \in M$ there are no non-trivial solution of (18), (8) (or (19), (27)) with $0 < \|u\|_{H_D^1} + \|v\|_{H_D^1} < \delta$.*

Theorem 3.5. *Let us suppose that the domain Ω satisfies (29).*

(i) The domain of stability D_S contains neither critical points of (18), (30) nor bifurcation points of (13), (30).

(ii) Let $0 < r < R$. Let $C_k, \dots, C_{k+l-1}$ be all hyperbolas which have a non-empty intersection with C_r^R. Let any linear combination e of the eigenfunctions of (32), (30) corresponding to $\kappa_k, \dots, \kappa_{k+l-1}$ satisfy (33). Then there exists $\varepsilon > 0$ such that there are neither critical points of (18), (30) nor bifurcation points of (13), (30) in $C_r^R(\varepsilon)$.

Remark 3.4. *The same assertion considering periodic boundary conditions as in Theorem 3.5 could be stated for the case of unilateral sources and sinks studied in [15]. Also one could assume a combination of periodic and Neumann boundary conditions, e.g., periodic boundary conditions on two parallel edges of the square and Neumann boundary conditions on the rest of the boundary.*

4. Abstract setting

We define the operator $A : H_D^1 \to H_D^1$ as

$$(A\psi, \varphi) = \int_\Omega \psi\varphi \, d\Omega \quad \text{for all } \psi, \varphi \in H_D^1(\Omega). \tag{35}$$

Remark 4.1. *The operator A defined by (35) is linear, bounded, symmetric and compact due to compact embedding $W^{1,2} \hookrightarrow\hookrightarrow L^2$. Simple calculation gives that the eigenvalues of the operator A are $\mu_k = \frac{1}{\kappa_k}, k = 1, 2, \dots$ for $\Gamma_D \neq \emptyset$ and $\mu_k = \frac{1}{\kappa_k + 1}, k = 0, 1, 2, \dots$ for $\Gamma_D = \emptyset$, and the corresponding eigenvectors of A coincide with the eigenfunctions of (20). Also in the case $\Gamma_D = \emptyset$ we have $((I - A)u, u) \geq 0$ for all u and the equality holds only for $u \in span\{e_0\}$, i.e., u constant.*

We define operators $\beta^-, \beta^+ : H_D^1 \to H_D^1$ by

$$\begin{aligned} (\beta^-(\psi), \varphi) &= -\int_\Omega \left(\sum_{i=1}^n \chi^{K_i^-}(x) \tau_-^i T^-_{K_i^-}(\psi) \right) \varphi \, d\Omega \quad \text{for all } \psi,\ \varphi \in H_D^1, \\ (\beta^+(\psi), \varphi) &= \int_\Omega \left(\sum_{j=1}^m \chi^{K_j^+}(x) \tau_+^j T^+_{K_j^+}(\psi) \right) \varphi \, d\Omega \quad \text{for all } \psi,\ \varphi \in H_D^1 \end{aligned} \tag{36}$$

and the operator $\beta : H_D^1 \to H_D^1$ as

$$\beta = \beta^- + \beta^+. \tag{37}$$

We also define operators $F^-, F^+ : H_D^1 \to H_D^1$ by

$$\begin{aligned} (F^-(\psi), \varphi) &= -\int_\Omega \left(\sum_{i=1}^n \chi^{K_i^-}(x) f_-^i \left(T^-_{K_i^-}(\psi) \right) \right) \varphi \, d\Omega \quad \text{for all } \psi, \varphi \in H_D^1, \\ (F^+(\psi), \varphi) &= \int_\Omega \left(\sum_{j=1}^m \chi^{K_j^+}(x) f_+^j \left(T^+_{K_j^+}(\psi) \right) \right) \varphi \, d\Omega \quad \text{for all } \psi, \varphi \in H_D^1 \end{aligned} \tag{38}$$

and $F : H_D^1 \to H_D^1$ as

$$F := F^- + F^+. \tag{39}$$

Lemma 4.1. *Functionals $T_X^{\mp}$ and operators β, F have the following properties:*

$$\textit{(i)} \quad \beta(t\psi) = t\beta(\psi) \text{ for all } t > 0,\ \psi \in H_D^1, \tag{40}$$

$$\textit{(ii)} \quad (\beta(\psi), \psi) \geq 0, \tag{41}$$

$$\textit{(iii)} \quad \psi_k \rightharpoonup \psi \implies T_X^{\mp}(\psi_k) \to T_X^{\mp}(\psi), \tag{42}$$

$$\textit{(iv)} \quad \psi_k \rightharpoonup \psi \implies \beta(\psi_k) \to \beta(\psi), \tag{43}$$

$$\textit{(v)} \quad \psi_k \to 0, \frac{\psi_k}{\|\psi_k\|_{H_D^1}} \rightharpoonup w \implies \frac{F(\psi_k)}{\|\psi_k\|_{H_D^1}} \to \beta(w). \tag{44}$$

Proof.

(i) The property described in *(i)* is positive homogeneity and it is apparent.

(ii) For any $\psi \in H_D^1$ we have

$$\begin{aligned}(\beta(\psi), \psi) &= -\int_\Omega \left(\sum_{i=1}^n \chi^{K_-^i}(x) \tau_-^i T^-_{K_-^i}(\psi) \right) \psi \, d\Omega + \int_\Omega \left(\sum_{j=1}^m \chi^{K_j^+}(x) \tau_+^j T^+_{K_j^+}(\psi) \right) \psi \, d\Omega = \\ &= -\sum_{i=1}^n \tau_-^i T^-_{K_i^-}(\psi) \int_{K_i^-} \psi \, dK_i^- + \sum_{j=1}^m \tau_+^j T^+_{K_j^+}(\psi) \int_{K_j^+} \psi \, dK_j^+ = \\ &= \sum_{i=1}^n \tau_-^i \left(T^-_{K_i^-}(\psi) \right)^2 |K_i^-| + \sum_{j=1}^m \tau_+^j \left(T^+_{K_j^+}(\psi) \right)^2 |K_j^+| \geq 0.\end{aligned}$$

(iii) We will prove the assertion for T_X^-, the proof for T_X^+ is the same. Let us have a sequence $(\psi_k) \subset H_D^1$ such that $\psi_k \rightharpoonup \psi \in H_D^1$. Then by the compact embedding $W^{1,2} \hookrightarrow\hookrightarrow L^2$, we get $\psi_k \to \psi$ in L^2. One can see that

$$\begin{aligned}\left| T_X^-(\psi_k) - T_X^-(\psi) \right| &= \left| \left(\int_X \frac{\psi_k}{|X|} \, dX \right)^- - \left(\int_X \frac{\psi}{|X|} \, dX \right)^- \right| \leq \frac{1}{|X|} \int_X |\psi_k - \psi| \, dX \leq \\ &\leq \frac{1}{|X|} \int_\Omega |\psi_k - \psi| \, d\Omega \leq \frac{c}{|X|} \|\psi_k - \psi\|_{L^2} \to 0.\end{aligned}$$

(iv) We will show this property for β^-. Let us have a sequence $(\psi_k) \subset H_D^1$ such that $\psi_k \rightharpoonup \psi \in H_D^1$. Then by the compact embedding $W^{1,2} \hookrightarrow\hookrightarrow L^2$, we get $\psi_k \to \psi$ in L^2. We use the property (ii) of this lemma, the continuous embedding $W^{1,2} \hookrightarrow L^2$ and Hölder's inequality to get

$$\begin{aligned}\|\beta^-(\psi_k) - \beta^-(\psi)\|_{H_D^1} &= \sup_{\|\varphi\|_{H_D^1} \leq 1} |(\beta^-(\psi_k) - \beta^-(\psi), \varphi)| \leq \\ &\leq \sup_{\|\varphi\|_{H_D^1} \leq 1} \int_\Omega |\varphi| \cdot \sum_{i=1}^n \chi^{K_i^-}(x) \tau_i^- \left| T^-_{K_i^-}(\psi_k) - T^-_{K_i^-}(\psi) \right| \, d\Omega \leq \\ &\leq C \|\psi_k - \psi\|_{L^2} \to 0.\end{aligned}$$

The same can be shown for β^+ and the assertion follows.

(v) We will prove this property for F^- and β^-. We again use the continuous embedding $W^{1,2} \hookrightarrow$

L^2 with some embedding constant c_{emb} and Hölder's inequality to get

$$\left\| \frac{F^-(\psi_k)}{\|\psi_k\|_{H^1_D}} - \beta^-(w) \right\|_{H^1_D} \leq \sup_{\|\varphi\|_{H^1_D}\leq 1} \sum_{i=1}^{n} \left| \frac{f^i_-\left(T^-_{K_i^-}(\psi_k)\right)}{\|\psi_k\|_{H^1_D}} - \tau^i_- T^-_{K_i^-}(w) \right| \cdot \int_\Omega \left| \chi^{K_i^-}(x) \cdot \varphi \right| \, d\Omega \leq$$

$$\leq c_{emb} \sum_{i=1}^{n} \left| \frac{f^i_-\left(T^-_{K_i^-}(\psi_k)\right)}{T^-_{K_i^-}(\psi_k)} \cdot \frac{T^-_{K_i^-}(\psi_k)}{\|\psi_k\|_{H^1_D}} - \tau^i_- T^-_{K_i^-}(w) \right| \cdot \left\| \chi^{K_i^-}(x) \right\|_{L^2}$$

Since $\frac{\psi_k}{\|\psi_k\|_{H^1_D}} \rightharpoonup w$, we get $\frac{T^-_{K_i^-}(\psi_k)}{\|\psi_k\|_{H^1_D}} \to T^-_{K_i^-}(w)$ by properties (i) and (iii). From the assumption (5) we have

$$\lim_{\xi \to 0} \frac{f^i_-(\xi)}{\xi} = \tau^i_-, \tag{45}$$

and due to the fact that $\psi_k \to 0 \implies T^-_{K_i^-}(\psi_k) \to 0$ we have

$$\frac{f^i_-\left(T^-_{K_i^-}(\psi_k)\right)}{T^-_{K_i^-}(\psi_k)} \to \tau^i_- \quad \text{for every } i = 1, \ldots, n. \tag{46}$$

All this together yields

$$\left| \frac{f^i_-\left(T^-_{K_i^-}(\psi_k)\right)}{T^-_{K_i^-}(\psi_k)} \cdot \frac{T^-_{K_i^-}(\psi_k)}{\|\psi_k\|_{H^1_D}} - \tau^i_- T^-_{K_i^-}(w) \right| \to 0, \quad \text{for every } i = 1, \ldots, n, \tag{47}$$

which means $\frac{F^-(\psi_k)}{\|\psi_k\|_{H^1_D}} \to \beta^-(w)$. The proof for F^+ and β^+ is analogous and the assertion is proved.

□

Remark 4.2.
One could consider more general type of unilateral integral terms, e.g., $\chi^M(\mathbf{x}) \left(\int_K \frac{u(\mathbf{x})}{|K|} \, dK \right)^-$ *with* $M \neq K$. *While these terms could be interesting for applications, the crucial non-negativity condition* (41) *is not satisfied in such cases and therefore we exclude them.*

5. Reduction to a single operator equation and proofs of main results

Let us remind that we have two different inner products in two different cases $\Gamma_D = \emptyset$ and $\Gamma_D \neq \emptyset$. The system of operator equations is slightly different in these two cases, but the approach is the same. We will focus here on technically more difficult case $\Gamma_D = \emptyset$, i.e., pure zero Neumann boundary conditions. Hence, the space H^1_D is the identical with the space $W^{1,2}$ and we use the inner product $(u, \varphi) = \int_\Omega (\nabla u \nabla \varphi + u\varphi) \, d\Omega$. Now, a weak solution of the problem (18), (8) or (19), (8) is a pair of functions $u, v \in H^1_D$ satisfying

$$\begin{aligned} d_1(I - A)u - b_{1,1}Au - b_{1,2}Av + \beta(u) = 0, \\ d_2(I - A)v - b_{2,1}Au - b_{2,2}Av = 0 \end{aligned} \tag{48}$$

or

$$\begin{aligned} d_1(I - A)u - b_{1,1}Au - b_{1,2}Av = 0, \\ d_2(I - A)v - b_{2,1}Au - b_{2,2}Av = 0, \end{aligned} \tag{49}$$

respectively.

Let us suppose $d_2 > 0$ fixed. Now we can reduce each of these systems of operator equations

to one operator equation by expressing the variable v from the second equation and substituting it to the first one. This way we can transform systems (48) and (49) to

$$d_1(I-A)u - S_{d_2}u + \beta(u) = 0, \tag{50a}$$
$$v = (d_2 I - d_2 A - b_{2,2}A)^{-1} b_{2,1} A u \tag{50b}$$

and

$$d_1(I-A)u - S_{d_2}u = 0, \tag{51a}$$
$$v = (d_2 I - d_2 A - b_{2,2}A)^{-1} b_{2,1} A u \tag{51b}$$

with the new operator

$$S_{d_2} := b_{1,1}A + b_{1,2}A(d_2 I - d_2 A - b_{2,2}A)^{-1} b_{2,1} A. \tag{52}$$

The inverse in equations above always exists and the reduction is very similar in the case $\Gamma_D \neq \emptyset$ (see Section 5.1 and 5.2 in [15]). This idea of reduction can be found e.g., in [15] or [14]. In the next remark we will summarize known properties of S_{d_2}, present the form of eigenvalues d_1 of the problem (51a) and the connection of hyperbolas, critical points of (19), (8) and bifurcation points of (14), (8).

Remark 5.1. *The operator S_{d_2} defined by (52) is linear, continuous, symmetric and compact. The eigenvalues of the operator S_{d_2} are*

$$\lambda^k = \frac{1}{\kappa_k + 1}\left(\frac{b_{1,2}b_{2,1}}{d_2\kappa_k - b_{2,2}} + b_{1,1}\right), \ k = 0, 1, 2, \dots \tag{53}$$

and the eigenvectors of S_{d_2} corresponding to λ^k coincide with those of A corresponding to μ_k, i.e., with the eigenfunctions of (20) corresponding to κ_k. Also for u_0 constant we have

$$(S_{d_2}u_0, u_0) = (\lambda^0 u_0, u_0) = \left(b_{1,1} + \frac{b_{1,2}b_{2,1}}{-b_{2,2}}\right)\|u_0\|^2_{H^1_D} = \frac{-det(\mathbf{B})}{-b_{2,2}}\|u_0\|^2_{H^1_D} < 0. \tag{54}$$

The eigenvalues d_1^k of the problem (51a) are given by

$$d_1^k = \frac{1}{\kappa_k}\left(\frac{b_{1,2}b_{2,1}}{d_2\kappa_k - b_{2,2}} + b_{1,1}\right), \ k = 1, 2, \dots \tag{55}$$

There is no eigenvalue with $k = 0$ (c.f. (22)).

We will denote by d_1^{MAX} the maximal eigenvalue of the problem (51a). It is well know that we can characterize this maximal eigenvalue of (51a) as

$$d_1^{MAX} = \max_{\substack{u \notin Ker(I-A) \\ u \in H^1_D}} \frac{(S_{d_2}u, u)}{((I-A)u, u)}. \tag{56}$$

We can see from the form of eigenvalues d_1^k that $[d_1^k, d_2] \in C_k$, $[d_1^{MAX}, d_2] \in C_E$ and there is infinitely many positive eigenvalues d_1^k and only finite number of negative ones.

It can be shown that d_1 is an eigenvalue of (51a) or (50a) if and only if $[d_1, d_2]$ is a critical point of (19), (8) or (18), (8), respectively (see Remark 2.4 for the classical case or Lemma 5.1 of [15]). Also every bifurcation point $[d_1, d_2]$ of (14), (8) or (13), (8) is a critical point of (19), (8) or (18), (8), respectively. To prove this implication, one needs properties of β and F we proved in Lemma 4.1 and definition of nonlinear operators corresponding to higher order terms n_1, n_2 for which the growth conditions (17) are necessary. The proof is the same as the proof in the case of problems with unilateral terms and it can be found in appendix of [15] (see Lemma A.2).

In our previous paper [15], we used the variational characterization of the maximal eigenvalue d_1 of problems (50a) (with different β) and (51a) and compared them. The eigenvalue problem (50a) is non-linear, therefore we used [15, Theorem 5.1] (originally from [16]) to get the existence of the maximal eigenvalue and its variational characterization. In the case of unilateral integral terms this is not possible, because we are not able to verify the crucial potentiality condition (51) of [15, Theorem 5.1]. Hence, we use a different approach that does not guarantee the existence of critical points of (18), (8), but considering the fact that we aim to prove the non-existence of critical points on C_E, this is not an issue.

Theorem 5.1. *Let $d_2 > 0$ be arbitrary fixed. If $[d_1, d_2]$ is a critical point of (18), (8), then we always have $d_1 \leq d_1^{MAX}$. If $[d_1^{MAX}, d_2] \in C_n$ exactly for $n = k, \ldots, k+l-1$, all linear combinations e of $e_k, \ldots, e_{k+l-1}$ satisfy (33) and $[d_1, d_2]$ is a critical point of (18), (8), then $d_1 < d_1^{MAX}$.*

Proof.
Let $[d_1, d_2]$ be a critical point of (18), (8), i.e., d_1 is an eigenvalue of the problem (50a) with some eigenfunction u_0. The eigenfunction u_0 cannot be constant, i.e., it cannot be $u_0 \in Ker(I - A)$. If it were, we would have $-(S_{d_2}u_0, u_0) + (\beta(u_0), u_0) > 0$ due to (41) and (54). However, this is not possible, because an eigenfunction u_0 satisfies

$$((I - A)u_0, u_0) - (S_{d_2}u_0, u_0) + (\beta(u_0), u_0) = 0.$$

For any eigenfunction $u_0 \notin Ker(I - A)$ corresponding to the eigenvalue d_1 we have

$$d_1 = \frac{(S_{d_2}u_0, u_0) - (\beta(u_0), u_0)}{((I - A)u_0, u_0)} \leq \frac{(S_{d_2}u_0, u_0)}{((I - A)u_0, u_0)} \leq \max_{\substack{u \notin Ker(I-A) \\ u \in H_D^1}} \frac{(S_{d_2}u, u)}{((I - A)u, u)} = d_1^{MAX}$$

due to non-negativity of $(\beta(u), u)$ (see Lemma 4.1). Hence, the first assertion is proved.

Let e be as in the assumptions of the theorem and let us suppose that $[d_1^{MAX}, d_2] \in C_E$ is a critical point of (18), (8), i.e., d_1^{MAX} is an eigenvalue of (50a) with corresponding eigenfunction u_0. It means that

$$d_1^{MAX} = \frac{(S_{d_2}u_0, u_0) - (\beta(u_0), u_0)}{((I - A)u_0, u_0)} \leq \frac{(S_{d_2}u_0, u_0)}{((I - A)u_0, u_0)} \leq d_1^{MAX}$$

due to (41) and the fact that d_1^{MAX} is the maximal eigenvalue of (51a). Hence, the inequality is not possible, but in the same moment e is an eigenfunction corresponding to the maximal eigenvalue d_1^{MAX} too and $(\beta(e), e) > 0$ because of (33), which leads to the contradiction. Hence, $[d_1^{MAX}, d_2] \in C_E$ cannot be a critical point of (18), (8) and together with already proven first assertion of the theorem we get that every critical point $[d_1, d_2]$ of (18), (8) satisfies $d_1 < d_1^{MAX}$. □

Proof of Theorem 3.1

(i) By Theorem 5.1, any critical point $[d_1, d_2]$ of (18), (8) satisfies $d_1 \leq d_1^{MAX}$ for a fixed $d_2 > 0$. By Remark 5.1 there is $[d_1^{MAX}, d_2] \in C_E$. Hence, there are no critical points in D_S (see Figure 1 and Remark 2.4). Since any bifurcation point of (13), (8) is also a critical point of (18), (8) (see Remark 5.1), there are no bifurcation points of (13), (8) in D_S.

(ii) The idea of the following proof is the same as in our paper [15]. We will make the proof for the case $\Gamma_D = \emptyset$. The case $\Gamma_D \neq \emptyset$ is analogous.
Let us suppose the opposite, i.e., the assumptions of the second part of Theorem 3.1 are satisfied and there are critical points of (13), (8) in $C_r^R(\varepsilon)$ for every $\varepsilon > 0$. We can choose a sequence $d^n = [d_1^n, d_2^n] \in D_U$ and $W_n = [u_n, v_n]$ such that $d^n \to d^0 \in C_r^R$, $\|W_n\| = \|u\|_{H_D^1} + \|v\|_{H_D^1} \neq 0$ and d^n, W_n satisfy (48). We can assume that $\frac{W_n}{\|W_n\|} \rightharpoonup W = [w, z]$. Let us divide (48) by $\|W_n\|$ to get

$$\begin{aligned} d_1^n(I - A)\frac{u_n}{\|W_n\|} - b_{1,1}A\frac{u_n}{\|W_n\|} - b_{1,2}A\frac{v_n}{\|W_n\|} + \beta\left(\frac{u_n}{\|W_n\|}\right) &= 0, \\ d_2^n(I - A)\frac{v_n}{\|W_n\|} - b_{2,1}A\frac{u_n}{\|W_n\|} - b_{2,2}A\frac{v_n}{\|W_n\|} &= 0. \end{aligned} \tag{57}$$

By the compactness of A and (43), we get $A\frac{u_n}{\|W_n\|} \to Aw$ and $\beta\left(\frac{u_n}{\|W_n\|}\right) \to \beta(w)$, analogously for v_n and z. Hence, it follows easily from (57) that $\frac{u_n}{\|W_n\|} \to w$, $\frac{v_n}{\|W_n\|} \to z$ and

$$\begin{aligned} d_1^0(I-A)w - b_{1,1}Aw - b_{1,2}Az - \beta(w) &= 0, \\ d_2^0(I-A)z - b_{2,1}Aw - b_{2,2}Az &= 0. \end{aligned}$$

Therefore the point $d^0 = [d_1^0, d_2^0] \in C_r^R$ is a critical point of the problem (18), (8), which contradicts Theorem 5.1 for $d_2 = d_2^0$. Hence, there exists $\varepsilon > 0$ such that there are no critical points of (18), (8) and consequently no bifurcation points of (13), (8) in $C_r^R(\varepsilon)$, because every bifurcation point is also a critical point (see Remark 5.1).

□

Proof of Theorem 3.2

Since we assume $\Gamma_D \neq \emptyset$, the first eigenfunction e_1 of (20) does not change the sign by Remark 2.2. Hence, for either $e = e_1$ or $e = -e_1$ we have $\sum_{i=1}^n \chi^{K_i^-}(x)\tau_-^i T_{K_i^-}^-(e) - \sum_{j=1}^m \chi^{K_j^+}(x)\tau_+^j T_{K_j^+}^+(e) = 0$. Since any point $[d_1, d_2] \in C_1$ is a critical point of (19), (8) with the solution $\left[\frac{d_2\kappa_1 - b_{2,2}}{b_{2,1}}e_1, e_1\right]$ (see Remark 2.4), it is also a critical point of (18), (8).

□

Proof of Theorem 3.3

In the case $\Gamma_D = \emptyset$ any linear combination e of eigenfunctions e_k for $k > 0$ satisfies $\int_\Omega e \, d\Omega = 0$. Indeed, using Green's formula and zero Neumann boundary conditions we have

$$\kappa_k \int_\Omega e_k \, d\Omega = \int_\Omega -\Delta e_k \, d\Omega = -\int_{\partial\Omega} \frac{\partial e_k}{\partial n} \, d\partial\Omega = 0 \quad \text{for every } k > 0.$$

Since all κ_k for $k > 0$ are positive, we immediately get $\int_\Omega e \, d\Omega = 0$. This means that $T_\Omega^\mp(e) = 0$ and by the similar argumentation as in the proof of Theorem 3.2 the assertion follows.

□

Proof of Theorem 3.4

First let us define operators $\beta_N^-, \beta_N^+ : H_D^1 \to H_D^1$ as

$$\begin{aligned} (\beta_N^-(\psi), \varphi) &= -\int_{\Gamma_N} \left(\sum_{i=1}^n \chi^{K_i^-}(x)\tau_-^i T_{K_i^-}^-(\psi) \right) \varphi \, d\Gamma_N \quad \text{for all } \psi,\ \varphi \in H_D^1, \\ (\beta_N^+(\psi), \varphi) &= \int_{\Gamma_N} \left(\sum_{j=1}^m \chi^{K_j^+}(x)\tau_+^j T_{K_j^+}^+(\psi) \right) \varphi \, d\Gamma_N \quad \text{for all } \psi,\ \varphi \in H_D^1 \end{aligned} \tag{58}$$

and then the operator $\beta_N : H_D^1 \to H_D^1$ as

$$\beta_N = \beta_N^+ + \beta_N^-. \tag{59}$$

The operator β_N has the same properties as β (see Lemma 4.1). Now we can rewrite problem (19), (27) as

$$\begin{aligned} d_1(I-A)u - b_{1,1}Au - b_{1,2}Av + \beta_N(u) &= 0, \\ d_2(I-A)v - b_{2,1}Au - b_{2,2}Av &= 0. \end{aligned} \tag{60}$$

Then it is necessary to repeat the process of the reduction to one operator equation from the beginning of this section. Theorem 5.1 applies to this problem as well and the rest of the proof of Theorem 3.4 is the same as the proof of Theorem 3.1 presented above.

□

Proof of Theorem 3.5

In the case of periodic boundary conditions, we use the function space H^1_{per} from Remark 2.6 and by solution we mean a weak solution from this space. We can define the same operators A, β etc. from Section 4 using the inner product of H^1_{per} (let us remind that it is the same inner product as the inner product of $W^{1,2}$). These operators have again the properties described in Lemma 4.1. Eventually, we get the same systems of operator equations (48), (49) as we had in the case of Neumann boundary conditions. Then we can apply again the reduction to one operator equation. Theorem 5.1 still applies in this case and the proof of Theorem 3.5 is then exactly the same as the proof of Theorem 3.1.

□

6. Numerical experiments

In this section we will present a collection of results of our numerical experiments with specific reaction kinetics. In some sense we take inspiration in the paper [17], where Vejchodský et al. investigated the influence of unilateral sources of the type τv^- (and its modifications) in the inhibitor equation of the reaction-diffusion system on pattern formation. Analytical results for systems with unilateral terms suggest that the domain of instability could be bigger than in the classical case. Vejchodský et al. were looking for a critical value of the portion of diffusion parameters $D = \frac{d_1}{d_2}$ where Turing's instability occurs. Of course, one cannot talk here about true stability (or instability), because everything is just based on numerical experiments and one can only observe whether the solution evolving from initial perturbations of the homogeneous steady state converges to this state or not (i.e., it is evolving into something significantly bigger). We are investigating here the dual problem, i.e., unilateral terms in the activator equation. We focus both on numerical experiments considering unilateral terms of the type u^-, u^+ (analytical results were presented in [15]) and also unilateral integral terms presented in the analytic part of this paper. One goal is to investigate the influence of these terms on the "change of stability" of the reference steady state in the similar way as Vejchodský et al. did. Also we are interested in the shape of resulting patterns in different scenarios. We use here two types of boundary conditions, i.e., pure Neumann boundary conditions and periodic boundary conditions. Since we most commonly used Neumann boundary conditions before and we deem periodic boundary conditions more realistic, we want to compare results and shapes of patterns for these two types of boundary conditions and conclude that periodic boundary conditions are the better option for future experiments.

We use the well-known Schnakenberg reaction kinetics (see [23])

$$\begin{aligned} f(u,v) &= a - u + u^2 v, \\ g(u,v) &= b - u^2 v, \end{aligned} \tag{61}$$

where a, b are positive coefficients. The reaction-diffusion system with this kinetics and either Neumann or periodic boundary conditions has only one constant steady state $[\overline{u}, \overline{v}] = \left[a + b, \frac{b}{(a+b)^2}\right]$. We will assume values of coefficients

$$a = 0.2, \quad b = 2. \tag{62}$$

One can easily verify in such case that conditions (16) are satisfied and the kinetics is of the substrate-depletion type. The spatial domain is always the square $\Omega = [-25, 25]^2$.

The overall approach of numerical solving of our reaction-diffusion system is made in the sense of the method of lines. We use finite difference method for the spatial approximation, in particular the five-point scheme to approximate the Laplace operator. The evolution in time is managed by *ode15s* solver in the software MATLAB. We should mention that we use the idea of the ghost-point and the central difference to deal with Neumann boundary condition and periodic boundary conditions (see e.g., [24]). We adopt the trapezoidal rule to compute integral terms in the activator equation.

We use the random noise around $[\overline{u}, \overline{v}]$ as an initial condition with range $[-10^{-2}, 10^{-2}]$. We label the solution stationary, when it does not change too much, i.e., the difference of solutions in the maximum norm in two consecutive times is smaller than 10^{-3}. We say that $[\overline{u}, \overline{v}]$ is unstable if the solution $u(\mathbf{x})$ that evolved from perturbations of the reference homogeneous steady state satisfy

$$\frac{\max_{\mathbf{x}\in\Omega} |u(\mathbf{x}) - \overline{u}|}{\overline{u}} > 0.1. \tag{63}$$

The value 0.1 is ten times bigger than the range of initial perturbation, hence we assume that the solution has evolved enough and it suggests instability of $[\overline{u}, \overline{v}]$. We use the relative difference from $\overline{u}$ here, because the constant $\overline{u}$ is quite bigger that $\overline{v}$ and the stationary solution u is in general much bigger than v. It does not seem to be the best idea to look for some exact value of d_1, where the stability changes. We will rather look for a critical interval $I_{crit} := (d_1^U, d_1^S)$ such that $|d_1^U - d_1^S| < 0.01$ and $[\overline{u}, \overline{v}]$ is unstable for $[d_1^U, d_2]$ and stable for $[d_1^S, d_2]$ (in the sense of (63)).

The typical patterns produced by reaction-diffusion system with Schnakenberg kinetics and either periodic of Neumann boundary conditions can be seen on Figure 4. The pattern with spots typically appears for some $[d_1, d_2]$ deep in the domain of instability, while stripe patterns are produced for $[d_1, d_2]$ close to the envelope C_E (the eigenfunctions corresponding to the smaller eigenvalues have probably bigger influence here). One can see that the shape of patterns in solution

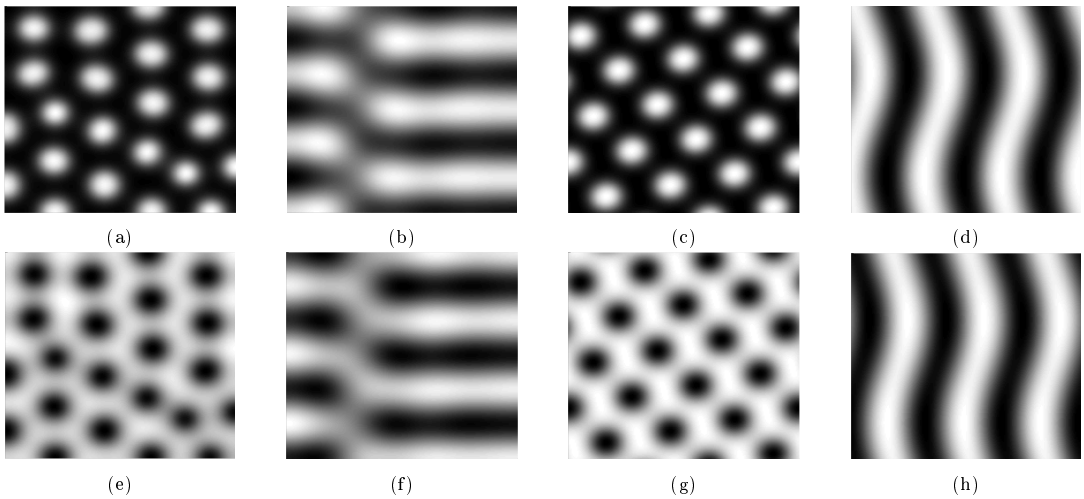


Figure 4: Examples of typical patterns in the classical case without unilateral terms. (a) sol. u, spots, Neumann b.c. (b) sol. u, stripes, Neumann b.c. (c) sol. u, spots, periodic b.c. (d) sol. u, stripes, periodic b.c. (e) sol. v, spots, Neumann b.c. (f) sol. v, stripes, Neumann b.c. (g) sol. v, spots, periodic b.c. (h) sol. v, stripes, periodic b.c.

u and v are the same, there is just inverted coloring (maximums and minimums). In [17], Vejchodský et al. always showed the solution u, but they also used inverted colormap in Matlab. Since we use standard colormap (grey), we will always show the solution u so that spots are always black.

6.1. Experiments with unilateral terms

We will consider the unilateral source

$$\tau(\mathbf{x})(u - \overline{u})^- \tag{64}$$

in the activator equation with $\tau(\mathbf{x}) \geq 0$ for every $\mathbf{x} \in \Omega$. We chose this simple unilateral source, because we can expect the existence of $C_r^R(\varepsilon)$ along the whole envelope C_E according to analytical results from [15] (for $\tau > 0$).

First, let us focus on the case of Neumann boundary conditions and τ constant on the whole

domain Ω. We experimentally found the critical interval I_{crit} for several values of d_2 and three values of τ. A sample of these experiments for $d_2 = 600$ can be found in Table 1, the full table for several values of d_2 is Table 7 in the appendix. One can see that as we increase τ, the critical interval shifts to zero. It would be interesting to find large enough value of τ, such that the critical interval reaches zero, but the numerical methods we used are not very reliable for d_1 close to zero. Hence, this remains open problem. The case $\tau = 0$ corresponds to the classical case without unilateral sources. Of course, we could just use d_1 computed analytically using the definition of C_E (in the second column). However, since everything is just approximate here, we should compare I_{crit} for positive τ with I_{crit} for $\tau = 0$. The patterns have usually the same shape when we use τu^- on the whole domain Ω as in the classical case (meaning in the problem without any unilateral terms).

d_2	d_1	I_{crit} for $\tau = 0$	I_{crit} for $\tau = 0.1$	I_{crit} for $\tau = 0.5$	I_{crit} for $\tau = 1$
600	14.73	(14.6149,14.6221)	(12.8456,12.8528)	(9.8967,9.9039)	(8.6452,8.6524)

Table 1: Critical intervals I_{crit} for different values of τ. The case of the unilateral source $\tau(u-\overline{u})^-$ and Neumann boundary conditions. The value of d_1 in the second column is analytically computed value of d_1 such that $[d_1, d_2] \in C_E$.

In the next series of experiments we investigate the influence of the unilateral source acting only on some subset of the domain Ω. We assume

$$\tau(\mathbf{x}) = \begin{cases} 0.5 & \dots \quad \mathbf{x} \in \Omega_S, \\ 0 & \dots \quad \text{otherwise}, \end{cases} \tag{65}$$

where Ω_S is a square with center in the center of the square Ω. The goal is once again to find the critical interval for different sizes of Ω_S. We can see in the sample in Table 2, that as we increase the size of Ω_S, the unilateral term has bigger influence and the critical interval is closer to zero (see full table in appendix). This is the expected result, but it is interesting that even for $|\Omega_S| = 30^2$, which is quite large square, the shift is very small. This suggests that using unilateral terms on small subset is not very effective in this sense.

d_2	I_{crit} for $\lvert\Omega_S\rvert = 50^2$	I_{crit} for $\lvert\Omega_S\rvert = 40^2$	I_{crit} for $\lvert\Omega_S\rvert = 30^2$	I_{crit} for $\lvert\Omega_S\rvert = 10^2$
600	(9.8967,9.9039)	(12.637,12.6442)	(14.1258,14.133)	(14.6005,14.6077)

Table 2: Critical intervals I_{crit} for $\tau(\mathbf{x}) = 0.5$ on square support Ω_S in the middle of Ω. The case of the unilateral source $\tau(\mathbf{x})(u-\overline{u})^-$ and Neumann boundary conditions.

The shape of patterns is more interesting in the case that the unilateral source is acting only on the part of the domain Ω. In Figure 5 we illustrate the dependence of the shape of patterns on the size of Ω_S and the value of d_1. One can see that for the higher value of d_1 or the larger Ω_S the pattern is not produced.

In the case of periodic boundary conditions, the situation considering the critical interval and the shape of patterns is very similar to the case of Neumann boundary conditions. The shift of the critical interval I_{crit} for $d_2 = 600$ and three values of τ is in Table 3, for the several values of d_2 see full Table 9. The patterns are more of less the same as typical patterns in Figure 4g, 4h.

d_2	I_{crit} for $\tau = 0$	I_{crit} for $\tau = 0.1$	I_{crit} for $\tau = 0.5$	I_{crit} for $\tau = 1$
600	(14.1258,14.133)	(12.6946,12.7017)	(9.7169,9.7241)	(8.6596,8.6668)

Table 3: Critical intervals I_{crit} for different values of τ. The case of the unilateral source $\tau(u-\overline{u})^-$ and periodic boundary conditions.

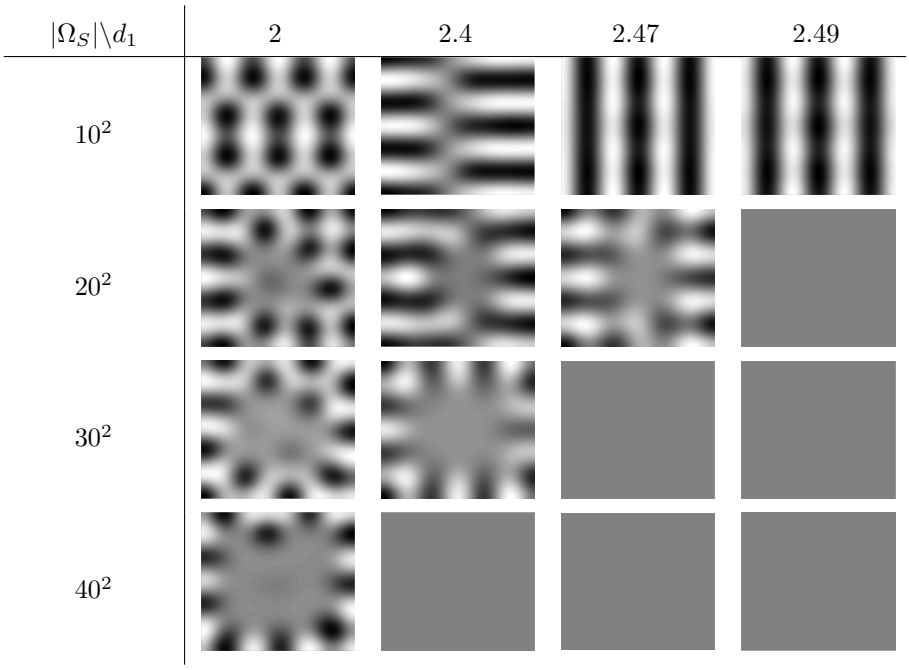


Figure 5: The dependence of the shape of patterns on the size of Ω_S and the diffusion parameter d_1- the case of Neumann boundary conditions and fixed $d_2 = 100$.

In the case that the unilateral source is active only on Ω_S the shift of the critical interval I_{crit} for $d_2 = 600$ and four sizes of Ω_S is in Table 4, for the several values of d_2 see full Table 10. The shape of patterns is again in this case more interesting than in the case that the unilateral source is active on the whole domain Ω (see Figure 6).

d_2	I_{crit} for $\|\Omega_S\| = 40^2$	I_{crit} for $\|\Omega_S\| = 30^2$	I_{crit} for $\|\Omega_S\| = 20^2$	I_{crit} for $\|\Omega_S\| = 10^2$
600	(12.1881,12.1934)	(14.0108,14.018)	(14.1412,14.148)	(14.1345,14.1412)

Table 4: Critical intervals I_{crit} for $\tau(\mathbf{x}) = 0.5$ on square support Ω_S in the middle of Ω. The case of the unilateral source $\tau(\mathbf{x})(u - \overline{u})^-$ and periodic boundary conditions.

The unilateral source $\tau(u - \overline{u})^-$ can be of course replaced with more complicated sources or we could use both the source and the sink. We repeated some of experiments for the unilateral source with saturation $\frac{\tau(u-\overline{u})^-}{1+(u-\overline{u})^-}$ and we got similar results. We should mention that this term is more natural due to the fact that it is bounded, while $\tau(u - \overline{u})^-$ is not.

6.2. Experiments with unilateral integral terms

We consider one unilateral integral source and one sink

$$\tau\chi_M(\mathbf{x})\left(\int_K \frac{u-\overline{u}}{|K|}\,dK\right)^- - \varepsilon\chi_M(\mathbf{x})\left(\int_K \frac{u-\overline{u}}{|K|}\,dK\right)^+ \tag{66}$$

with $\tau, \varepsilon > 0$. In the analytical part of the paper we always suppose $K = M$. Here, we will make some experiments even in the case that $K \neq M$. The convergence to the stationary solution takes much more time for unilateral integral terms. Hence, we look for I_{crit} for fewer values of d_2. Also we discovered that the shift of I_{crit} to the left is much smaller than in the case of unilateral terms, therefore we require that $|d_1^U - d_1^S| < 0.001$ in the definition of the critical interval I_{crit} instead of

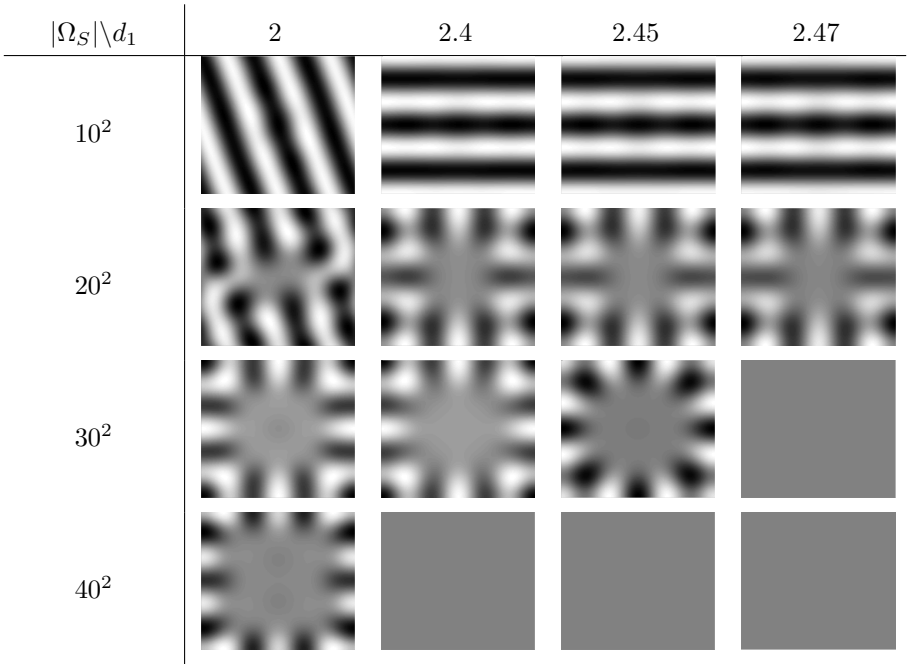


Figure 6: The dependence of the shape of patterns on the size of Ω_S and the diffusion parameter d_1- the case of periodic boundary conditions and fixed $d_2 = 100$.

the value 0.01 we used before.

In the case of Neumann boundary conditions, we tested several settings of parameters τ, ε and sets K, M. In Tables 5 and 6 we summarize computed critical intervals I_{crit} for two different values of d_2. One can see here that the shift of the critical interval is very small. We note here that columns (v) and (vi) are in Table 6. Looking at the case (ii) (only source) and (iii) (both source and sink), we can see that the shift is bigger, when we use both source and sink, not just the source. On the other hand taking small sets K, M does not necessarily result in smaller shift (compare (iii) and (iv)). This make sense, because the integral does not need to be larger if we take larger sets K, M. The same apparently is true, when we increase values of τ and ε (compare (iii) and (v)). The last case (vi) is the case, where the sets K and M are different, which is the case excluded in the analytic part this paper. We can observe here the shift to the right, which is something new.

The shape of patterns does not seem to be very influenced by unilateral integral terms. One

d_2	(i)	(ii)	(iii)	(iv)
600	(14.6218,14.6224)	(14.6166,14.6172)	(14.6218,14.6224)	(14.6189,14.6195)
500	(12.3099,12.3109)	(12.213,12.214)	(12.246,12.2469)	(11.9368,11.9377)

Table 5: Critical intervals I_{crit} in different cases for two values of d_2. (i)- the classical case (no unilateral integral terms), (ii)- the case $\tau = 0.8$, $\varepsilon = 0$, $K = M = [-20, 20]^2$, (iii)- the case $\tau = 0.8$, $\varepsilon = 0.7$, $K = M = [-20, 20]^2$, (iv)- the case $\tau = 0.8$, $\varepsilon = 0.7$, $K = M = [-10, 10]^2$.

could say that patterns are in some cases more blurry than in the classical case, but the difference is quite small. Also the case of periodic boundary conditions gives the same results as the case of Neumann boundary conditions.

d_2	(i)	(v)	(vi)
600	(14.6218,14.6224)	(14.6103,14.6109)	(15.5855,15.5862)
500	(12.3099,12.3109)	(12.0921,12.0931)	(13.6883,13.6891)

Table 6: Critical intervals I_{crit} in different cases for two values of d_2 (extension of Table 5). (i)- the classical case (no unilateral integral terms), (v)- the case $\tau = 1.5$, $\varepsilon = 1.2$, $K = M = [-20, 20]^2$ (vi)- the case $\tau = 0.8$, $\varepsilon = 0.7$, $K = [0, 20]^2$, $M = [-20, 0]^2$ (i.e., the case $K \neq M$).

6.3. Conclusion

The numerical investigation of unilateral and unilateral integral terms yielded several observations. Should we compare the influence of unilateral terms and unilateral integral terms, former ones have much bigger impact on the pattern formation. First we have seen that with the increasing strength of unilateral terms, the shift of the critical interval I_{crit} to the left is getting bigger and the same is true, when we increase the set, where these terms are active. We have observed that the pattern formation can be locally broken on the set, where unilateral terms are active and it leads to quite interesting patterns. This behaviour can be detected for the case of Neumann boundary conditions and also periodic boundary conditions. Hence, we conclude that we should focus more on periodic boundary conditions in the future, because they give qualitatively similar results as Neumann boundary conditions and we deem them more natural. On the other hand, in the case of unilateral integral terms, we observed that the shift of I_{crit} is small in several scenarios. The shift is especially small in comparison to the shift in the case of unilateral terms. Also the shape of patterns usually remains unchanged. The most interesting observation here is the fact that when $K \neq M$ (the set where we compute integral and the set where these terms are active is different), the critical interval can truly shift to the right. This is something new, which was not possible in the case of unilateral terms.

7. Acknowledgement

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A. Appendix

A.1. Tables

d_2	d_1	I_{crit} for $\tau = 0$	I_{crit} for $\tau = 0.1$	I_{crit} for $\tau = 0.5$	I_{crit} for $\tau = 1$
600	14.73	(14.6149,14.6221)	(12.8456,12.8528)	(9.8967,9.9039)	(8.6452,8.6524)
400	9.96	(9.9503,9.96)	(8.8706,8.8804)	(6.7211,6.7308)	(5.7776,5.7873)
300	7.47	(7.4262,7.4335)	(6.6821,6.6894)	(5.0919,5.0992)	(4.3332,4.3405)
200	5.01	(4.9904,5.0002)	(4.4718,4.4816)	(3.3661,3.3759)	(2.8866,2.8964)
150	3.75	(3.7207,3.728)	(3.3398,3.3472)	(2.5342,2.5415)	(2.168,2.1753)
100	2.51	(2.4806,2.4904)	(2.2257,2.2355)	(1.6962,1.706)	(1.4511,1.4609)
70	1.76	(1.7462,1.7531)	(1.5675,1.5744)	(1.1894,1.1963)	(1.0106,1.0175)
80	2.01	(1.9943,2.0021)	(1.798,1.8059)	(1.3583,1.3662)	(1.1542,1.162)
60	1.5	(1.4941,1.5)	(1.3359,1.3418)	(1.0195,1.0254)	(0.86719,0.87305)
50	1.25	(1.2402,1.25)	(1.1133,1.123)	(0.83984,0.84961)	(0.72266,0.73242)

Table 7: Critical intervals I_{crit} for several values of d_2. The case of the unilateral source $\tau(u-\overline{u})^-$ and Neumann boundary conditions.

d_2	I_{crit} for $\lvert\Omega_S\rvert = 50^2$	I_{crit} for $\lvert\Omega_S\rvert = 40^2$	I_{crit} for $\lvert\Omega_S\rvert = 30^2$	I_{crit} for $\lvert\Omega_S\rvert = 10^2$
600	(9.8967,9.9039)	(12.637,12.6442)	(14.1258,14.133)	(14.6005,14.6077)
400	(6.7211,6.7308)	(8.7053,8.715)	(9.7266,9.7363)	(9.9308,9.9405)
300	(5.0919,5.0992)	(6.6092,6.6165)	(7.2074,7.2147)	(7.3606,7.3679)
200	(3.3661,3.3759)	(4.6088,4.6186)	(4.9121,4.9219)	(4.9806,4.9904)
150	(2.5342,2.5415)	(3.457,3.4644)	(3.6401,3.6475)	(3.728,3.7354)
100	(1.6962,1.706)	(2.3727,2.3825)	(2.4512,2.461)	(2.4904,2.5002)
80	(1.3583,1.3662)	(1.9079,1.9158)	(1.9629,1.9707)	(1.9864,1.9943)
70	(1.1894,1.1963)	(1.7325,1.7394)	(1.7462,1.7531)	(1.6775,1.6844)
60	(1.0195,1.0254)	(1.4473,1.4531)	(1.4766,1.4824)	(1.4941,1.5)
50	(0.83984,0.84961)	(1.2109,1.2207)	(1.2305,1.2402)	(1.2402,1.25)

Table 8: Critical intervals I_{crit} for $\tau(\mathbf{x}) = 0.5$ on square support Ω_S in the middle of Ω. The case of the unilateral source $\tau(\mathbf{x})(u-\overline{u})^-$ and Neumann boundary conditions.

d_2	I_{crit} for $\tau = 0$	I_{crit} for $\tau = 0.1$	I_{crit} for $\tau = 0.5$	I_{crit} for $\tau = 1$
600	(14.1258,14.133)	(12.6946,12.7017)	(9.7169,9.7241)	(8.6596,8.6668)
400	(9.8919,9.9016)	(8.608,8.6177)	(6.6432,6.653)	(5.7484,5.7581)
300	(7.0615,7.0688)	(6.3466,6.3539)	(4.8949,4.9022)	(4.3259,4.3332)
200	(4.9415,4.9513)	(4.4718,4.4816)	(3.3661,3.3759)	(2.8866,2.8964)
150	(3.6548,3.6621)	(3.2153,3.2227)	(2.4829,2.4902)	(2.168,2.1753)
100	(2.4806,2.4904)	(2.2355,2.2453)	(1.6864,1.6962)	(1.4413,1.4511)
80	(1.9786,1.9864)	(1.7745,1.7823)	(1.3583,1.3662)	(1.1542,1.162)
70	(1.7394,1.7462)	(1.5675,1.5744)	(1.1825,1.1894)	(1.0106,1.0175)
60	(1.4824,1.4883)	(1.3184,1.3242)	(1.0195,1.0254)	(0.86719,0.87305)
50	(1.2402,1.25)	(1.1133,1.123)	(0.83984,0.84961)	(0.72266,0.73242)

Table 9: Critical intervals I_{crit} for several values of d_2. The case of the unilateral source $\tau(u - \overline{u})^-$ and periodic boundary conditions.

d_2	I_{crit} for $\lvert\Omega_S\rvert = 40^2$	I_{crit} for $\lvert\Omega_S\rvert = 30^2$	I_{crit} for $\lvert\Omega_S\rvert = 20^2$	I_{crit} for $\lvert\Omega_S\rvert = 10^2$
600	(12.1881,12.1934)	(14.0108,14.018)	(14.1412,14.148)	(14.1345,14.1412)
400	(8.2662,8.2733)	(9.7849,9.7946)	(9.8978,9.904)	(9.8915,9.8978)
300	(6.508,6.5157)	(6.9375,6.9448)	(6.9626,6.9716)	(7.0582,7.065)
200	(4.6114,4.6189)	(4.9062,4.9125)	(4.9286,4.9348)	(4.9411,4.9474)
150	(3.3953,3.401)	(3.5375,3.5438)	(3.6201,3.6293)	(3.6572,3.6665)
100	(2.3624,2.3716)	(2.4563,2.4625)	(2.4786,2.4849)	(2.4849,2.4912)
80	(1.8992,1.9066)	(1.9563,1.9625)	(1.9698,1.9748)	(1.9798,1.9849)
70	(1.6404,1.649)	(1.7062,1.7125)	(1.7248,1.7336)	(1.7424,1.7512)
60	(1.4394,1.4449)	(1.4688,1.475)	(1.4775,1.485)	(1.485,1.4925)
50	(1.2129,1.2191)	(1.2375,1.2437)	(1.2375,1.2437)	(1.2437,1.25)

Table 10: Critical intervals I_{crit} for $\tau(\mathbf{x}) = 0.5$ on square support Ω_S in the middle of Ω. The case of the unilateral source $\tau(\mathbf{x})(u - \overline{u})^-$ and periodic boundary conditions.

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