



A best possible result for the square of a 2-block to be hamiltonian

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ABSTRACT

It is shown that for any choice of four different vertices x_1, \dots, x_4 in a 2-block G of order $p > 3$, there is a hamiltonian cycle in G^2 containing four different edges $x_i y_i$ of $E(G)$ for certain vertices y_i , $i = 1, 2, 3, 4$. This result is best possible.

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1. Introduction

As for standard terminology, we refer to the book by Bondy and Murty, [2], and to the papers quoted in the references.

The *square* of a graph G , denoted G^2 , is the graph obtained from G by joining any two nonadjacent vertices which have a common neighbor, by an edge. Fairly recent development in hamiltonian graph theory has shown a resurgence of interest in hamiltonian cycles and paths in the square of 2-connected graphs (which we call 2-blocks for short). In particular, short proofs have been found for two results of the second author of the present paper, [10,11]. And more recently, in [1] the authors develop algorithms which are linear in $|E(G)|$ and produce a hamiltonian cycle, a hamiltonian path joining arbitrary vertices u and v respectively, in G^2 . Moreover, they develop an algorithm running in $O(|V(G)|^2)$ time and producing cycles of arbitrary length from 3 to $|V(G)|$.

Also very recently it was shown in [3] and [8] that a 2-block has the \mathcal{F}_4 property; that is, given vertices x_1, x_2, x_3, x_4 in the 2-block G , there is a hamiltonian path in G^2 joining x_1 and x_2 and traversing distinct edges $x_3 y_3$ and $x_4 y_4$ of G (see Theorem 7). The proof of this result is very long and is based on techniques developed by Fleischner in [5–7] and by Fleischner and Hobbs in [9]. It remains to be shown whether one can find a much shorter proof of this result. However, this result will be of importance in the proof of the main result of the current paper.

We start with a definition.

Definition 1. A graph G is said to have the \mathcal{H}_k property if for any given vertices x_1, \dots, x_k there is a hamiltonian cycle in G^2 containing distinct edges $x_1 y_1, \dots, x_k y_k$ of G .

We note in passing that G having the \mathcal{F}_4 property implies that G has the \mathcal{H}_3 property; clearly, choose x_1, x_2, x_3 arbitrarily and a different x_4 adjacent to some x_1 for $i \in \{1, 2, 3\}$ in G , say $i = 1$. A hamiltonian path in G^2 joining x_1 and x_4 and containing edges $x_2 y_2$ and $x_3 y_3$ of G yields a hamiltonian cycle containing these two edges of G and $x_1 x_4$ which lies also in G .

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The main result of this paper is the following.

Theorem 2. *Given a 2-block G on at least four vertices, then G has the \mathcal{H}_4 property, and there are 2-blocks of arbitrary order greater than 4 without the \mathcal{H}_5 property.*

This theorem and the \mathcal{F}_4 property of 2-blocks are key to describe the most general block-cut vertex structure a graph G may have in order to guarantee that G^2 is hamiltonian, hamiltonian connected, respectively. This will be done in follow-up papers.

Moreover [Theorem 2](#) gives the positive answer to Conjecture 5.4 stated in [\[4\]](#) as an immediate corollary.

Corollary 3. *Let G be a connected graph such that its block-cutvertex graph $bc(G)$ is homeomorphic to a star in which the center c corresponds to a block B_c of G . If B_c contains at most 4 cutvertices, then G^2 is hamiltonian.*

2. Preliminaries

However, before proving [Theorem 2](#) we mention several concepts and results which we need to make use of, and we prove a lemma.

A graph G is an edge-critical block, if $\kappa(G) = 2$ and $\kappa(G - e) = 1$ for any edge e of G . Let $D(G)$ be the set of edges uv where both $d_G(u) \geq 3$ and $d_G(v) \geq 3$. If $D(G) = \emptyset$, then every edge of G is incident to a vertex of degree 2; we call such a graph a *DT-graph*.

Theorem 4 ([\[6\]](#)). *Let G be an edge-critical block. Then exactly one of the following two statements is true:*

- (1) G is a DT-block.
- (2) There is an edge f in $D(G)$ such that at least one of the endblocks of $G - f$ is a DT-block.

The basic result about hamiltonicity of the square of a 2-block is given by the following theorem.

Theorem 5 ([\[7\]](#)). *Suppose v and w are two arbitrarily chosen vertices of a 2-block G . Then G^2 contains a hamiltonian cycle C such that the edges of C incident to v are in G and at least one of the edges of C incident to w is in G . Furthermore, if v and w are adjacent in G , then these are three different edges.*

Let $bc(G)$ denote the block-cutvertex graph of G . Blocks corresponding to leaves of $bc(G)$ are called *endblocks*. Note that a block in a graph G is either a 2-block or a bridge of G . The graph G is called *blockchain* if $bc(G)$ is a path. Let G be a blockchain. We denote its blocks B_1, B_2, \dots, B_k and cutvertices c_1, c_2, \dots, c_{k-1} such that $c_i \in V(B_i) \cap V(B_{i+1})$, for $i = 1, 2, \dots, k - 1$. A blockchain G is called *trivial*, if $E(bc(G)) = \emptyset$, otherwise it is called *non-trivial*. Note that only B_1 and B_k are endblocks of a non-trivial blockchain G . An *inner block* is a block of G containing exactly 2 cutvertices. An *inner vertex* is a vertex in G which is not a cutvertex of G .

The first author proved in [\[4\]](#) the following theorem dealing hamiltonicity of the square of a blockchain graph.

Theorem 6 ([\[4\]](#)). *Let G be a blockchain and let u_1, u_2 be arbitrary inner vertices which are contained in different endblocks of G .*

Then G^2 contains a hamiltonian cycle C such that, for $i = 1, 2$,

- if u_i is contained in a 2-block, then both edges of C incident with u_i are in G , and
- if u_i is not contained in a 2-block, then exactly one edge of C incident with u_i is in G .

Let G be a connected graph. By a *uv-path* we mean a path from u to v in G . If a *uv-path* is hamiltonian, we call it a *uv-hamiltonian path*. Let $A = \{x_1, x_2, \dots, x_k\}$ be a set of k (≥ 3) distinct vertices in G . An x_1x_2 -hamiltonian path in G^2 which contains $k - 2$ distinct edges $x_iy_i \in E(G)$, $i = 3, \dots, k$, is said to be \mathcal{F}_k . A graph G is said to have the \mathcal{F}_k property if, for any set $A = \{x_1, x_2, \dots, x_k\} \subseteq V(G)$, there is an \mathcal{F}_k x_1x_2 -hamiltonian path in G^2 .

Theorem 7 ([\[8\]](#)). *Let G be a 2-block. Then G has the \mathcal{F}_4 property.*

A graph G is said to have the strong \mathcal{F}_3 property if, for any set of 3 vertices $\{x_1, x_2, x_3\}$ in G , there is an x_1x_2 -hamiltonian path in G^2 containing distinct edges $x_3z_3, x_iz_i \in E(G)$ for a given $i \in \{1, 2\}$. Such an x_1x_2 -hamiltonian path in G^2 is called a strong \mathcal{F}_3 x_1x_2 -hamiltonian path.

Theorem 8 ([\[8\]](#)). *Every 2-block has the strong \mathcal{F}_3 property.*

The following lemma is frequently used in the proofs below.

Lemma 9. *Let G be a non-trivial blockchain. We choose*

- $c_0 \in V(B_1)$, $c_k \in V(B_k)$ which are not cutvertices;
- $u_i \in V(B_i)$ (if any) which is not a cutvertex and $v_i \in V(B_i)$ such that $u_i \neq v_i$, $u_1 \neq c_0$ and $u_k \neq c_k$, for $i = 1, 2, \dots, k$.

Then G^2 contains a c_0c_k -hamiltonian path P such that there exist distinct edges $u_iu'_i, v_iv'_i \in E(B_i) \cap E(P)$ (if u_i exists), $i = 1, 2, \dots, k$.

Proof. If B_i is 2-connected, then let P_i be an \mathcal{F}_4 $c_{i-1}c_i$ -hamiltonian path in B_i^2 containing 2 distinct edges $u_iu'_i, v_iv'_i \in E(B_i)$ for $v_i \notin \{c_{i-1}, c_i\}$ by [Theorem 7](#); and let P_i be a strong \mathcal{F}_3 $c_{i-1}c_i$ -hamiltonian path in B_i^2 containing 2 distinct edges $u_iu'_i, v_iv'_i \in E(B_i)$ for $v_i \in \{c_{i-1}, c_i\}$ by [Theorem 8](#), respectively.

If $B_i = c_{i-1}c_i$, then we set $P_i = B_i$. Note that in this case u_i does not exist and $v_i \in \{c_{i-1}, c_i\}$.

Then $P = \cup_{i=1}^k P_i$ is a c_0c_k -hamiltonian path in G^2 as required. \square

The concept of EPS-graphs plays a central role in proofs of hamiltonicity in the square of a DT-graph (see [5]). We use this concept also in one part of the proof of [Theorem 2](#). Let G be a graph. An EPS-graph is a spanning connected subgraph S of G which is the edge-disjoint union of an Eulerian graph E (which may be disconnected) and a linear forest P . For $S = E \cup P$, let $d_E(v), d_P(v)$ denote the degree of v in E, P , respectively.

Fleischner and Hobbs introduced in [9] the concept of W -soundness of a cycle. Let W be a set of vertices of G . A cycle K is called W -maximal if $|V(K') \cap W| \leq |V(K) \cap W|$ for any cycle K' of G . Let K be a cycle of G and let W be a set of vertices of G . A blockchain P of $G - K$ is a W -separated K -to- K blockchain based on vertex x if a vertex of W is a cut vertex of P , both endblocks B and B' of P include vertices of K , $V(B) \cap V(K) = \{x\}$, no vertex of K is a cutvertex of P , and $(V(P) \cap V(K)) - \{x\} \subseteq V(B')$. For a given path $p = v_1, v_2, \dots, v_{n-1}, v_n$ we let $F(p) = v_1, L(p) = v_n$.

Definition 10. A cycle K in G is W -sound if it is W -maximal, $|W| = 5$ and the following hold:

- (1) $|V(K) \cap W| \geq 4$; or
- (2) $|V(K) \cap W| = 3$ and the following situation does not prevail; there are two W -separated K -to- K blockchains P and Q of $G - K$ based on a vertex w of W such that $V(P) \cap V(Q) = \{w\}$ and if p is a shortest path in P from w to a vertex of K different from w and q is the same for Q , then there is a subsequence $w, w', L(p), L(q), w'', w$ of K where w' and w'' are in $W - \{w\}$; or
- (3) $|V(K) \cap W| = 2$ and the following situation does not prevail; there are three W -separated K -to- K blockchains P_1, P_2 and P_3 of $G - K$ based on a single vertex a of $V(K) - W$, such that $V(P_i) \cap V(P_j) = \{a\}$ whenever i and j are distinct elements of $\{1, 2, 3\}$, and if p_i is a shortest path in P_i from a to a vertex of K different from a for each $i \in \{1, 2, 3\}$, then there is a subsequence $a, w', L(p_1), L(p_2), L(p_3), w'', a$ of K where $\{w', w''\} = V(K) \cap W$.

We observe that [Definition 10](#) is basically the content of Lemma 1 in [9]. That is, said lemma guarantees that for every choice $W \subseteq V(G)$ with $|W| = 5$ in a 2-block G of order at least 5, there is a W -sound cycle in G .

Theorem 11 ([9]). Let G be a 2-block and W a set of five distinct vertices in G , and let K be a W -sound cycle in G . Then there is an EPS-graph $S = E \cup P$ of G such that $K \subseteq E$ and $d_P(w) \leq 1$ for every $w \in W$.

3. Proof of [Theorem 2](#)

Proof. First we prove that G has the \mathcal{H}_4 property. We proceed by contradiction supposing that $|V(G)| + |E(G)|$ is minimal. It follows that G is an edge-critical block and in particular $|V(G)| \geq 5$. We distinguish cases by the number of edges in $D(G)$. The reader is advised to draw figures where he/she deems it necessary to follow our case distinctions.

Case 1. $|D(G)| > 0$. By [Theorem 4](#), let $f = x'x \in D(G)$ be an edge where both $d_G(x') \geq 3$ and $d_G(x) \geq 3$. Then $G - f$ is a blockchain and both endblocks B', B of $G - f$ are 2-blocks. Set $X = \{x_1, x_2, x_3, x_4\}$. Without loss of generality assume that $|X \cap (V(B) - y)| \leq 2$ (otherwise we consider B' instead of B); i.e., at most $x_1, x_2 \in V(B) - y$, say, where $x, y \in V(B)$ and y is a cutvertex of $G - f$. We distinguish the following 3 subcases.

Subcase 1.1: $|X \cap (V(B) - y)| = 2$; i.e., $x_1, x_2 \in V(B) - y$.

Then B^2 has an xy -hamiltonian path P_1 containing different edges x_1y_1, x_2y_2 of $E(G)$ for certain y_1, y_2 by [Theorem 7](#) or by [Theorem 8](#) if $x_1 = x$ or $x_2 = x$; and $(G - B)^2$ has an xy -hamiltonian path P_2 containing different edges x_3y_3, x_4y_4 of $E(G)$ for certain y_3, y_4 by [Lemma 9](#). Now $P_1 \cup P_2$ is a required hamiltonian cycle in G^2 , a contradiction. Note that $x_3, x_4 \in V(B') - y'$ where $y' \in V(B')$ is a cutvertex of $G - f$, otherwise we can use B' instead of B and x_3 or x_4 instead of x_1 or x_2 (see *Subcase 1.2* or *Subcase 1.3* below).

Subcase 1.2: $|X \cap (V(B) - y)| = 1$; i.e., $x_1 \in V(B) - y$ and $x_2 \notin V(B) - y$.

(1.2.1) Assume that x_2, x_3, x_4 are not inner vertices of G in the same block of $G - B$. We proceed very similar as in *Subcase 1.1*; we use only the strong \mathcal{F}_3 property in B , and $G - B$ is a non-trivial blockchain. Hence we can apply [Lemma 9](#) except if $x = x_1$, some $x_i = y$ for $i \in \{2, 3, 4\}$, say $i = 2$, and x_3, x_4 are inner vertices in the same endblock of $G - B$ which also contains x_2 .

If $x = x_1, x_2 = y$, and x_3, x_4 are inner vertices in the same endblock of $G - B$ which also contains x_2 , then B^2 has an x_2x_1 -hamiltonian path P_1 containing different edges x_2y_2, uv of $E(G)$ for certain y_2, u, v by [Theorem 8](#), and $(G - B)^2$ has an x_2x_1 -hamiltonian path P_2 containing different edges x_1x', x_3y_3, x_4y_4 of $E(G)$ for certain y_3, y_4 by [Lemma 9](#). Again, $P_1 \cup P_2$ is a required hamiltonian cycle in G^2 , a contradiction.

(1.2.2) Assume that x_2, x_3, x_4 are inner vertices of G in the same block B^* of $G - B$.

Clearly, B^2 contains a hamiltonian cycle H_B containing 3 different edges $y'y, x'_1x_1, x''x$ of $E(B)$ for certain vertices y', x'_1, x'' by [Theorem 7](#) (starting with a corresponding \mathcal{F}_4 $x''x$ -hamiltonian path in B^2) if $x \neq x_1$, and $y'y, x'_1x, x''x$ of $E(B)$ for certain vertices y', x'_1, x'' by [Theorem 5](#) if $x = x_1$.

Let G_1 be the component of $G - B^* - xx'$ containing B and $y^* = V(B^*) \cap V(G_1)$. Note that G_1 is a trivial or non-trivial blockchain.

(a) If $y^* = y$, then $G_1 = B$ and we set $H_{G_1} = H_B$ (see above).

(b) If $y^* \neq y$, then either $G_1 - B = y^*y$ or $(G_1 - B)^2$ contains a hamiltonian cycle C containing edges $y_1^*y^*, y''y$ of $E(G_1 - B)$ for certain y_1^*, y'' by applying [Theorem 5](#) or [Theorem 6](#).

Now we set

$$H_{G_1} = (H_B - y'y) \cup y'y^*$$

and $y_1^* = y$ if $G_1 - B = y^*y$; and

$$H_{G_1} = (H_B \cup C - \{y'y, y''y\}) \cup y'y''$$

if $G_1 - B \neq y^*y$.

Note that the edge $y_1^*y^* \in E(G_1)$ is contained in H_{G_1} in both cases.

Clearly, $|V(B^*)| + |E(B^*)| < |V(G)| + |E(G)|$. Hence $(B^*)^2$ contains a hamiltonian cycle H_{B^*} containing four different edges $y_2^*y^*, x_2x'_2, x_3x'_3, x_4x'_4$ of $E(B^*)$ for certain vertices $y_2^*, x'_i, i = 2, 3, 4$.

Let $z \in V(B^*)$ be the cutvertex of $G - x'x$ different from y^* .

(A) $x' = z$. Then

$$(H_{G_1} \cup H_{B^*} - \{y_2^*y^*, y_1^*y^*\}) \cup \{y_1^*y_2^*\}$$

is a required hamiltonian cycle in G^2 containing four different edges $x_ix'_i$, of $E(G)$, $i = 1, 2, 3, 4$, a contradiction.

(B) $x' \neq z$

If $d_{G-B^*}(z) = 1$, then we set $G_2 = G - G_1 - B^* - z_1z$ where z_1 is the unique neighbor of z in $G - B^*$; otherwise we set $G_2 = G - G_1 - B^*$. Note that G_2 is a trivial or non-trivial blockchain and $G_2 = x'x$ is not possible because of $d_G(x') > 2$.

We apply [Theorem 6](#) such that either $(G_2)^2$ contains a hamiltonian cycle H_{G_2} with $x'x \in E(H_{G_2})$ if $z \notin V(G_2)$, or $(G_2)^2$ contains a hamiltonian cycle H containing the edge $x'x$ and different edges z_1z, z_2z of G_1 for certain z_1, z_2 if $z \in V(G_2)$. In the latter case we set $H_{G_2} = (H - \{z_1z, z_2z\}) \cup z_1z_2$. Then

$$(H_{G_1} \cup H_{G_2} \cup H_{B^*} - \{y_2^*y^*, y_1^*y^*, x'x, x''x\}) \cup \{y_1^*y_2^*, x''x'\}$$

is again a hamiltonian cycle in G^2 containing four different edges $x_ix'_i$ of $E(G)$, $i = 1, 2, 3, 4$, a contradiction.

Subcase 1.3: $|X \cap (V(B) - y)| = 0$; i.e., $x_1, x_2 \notin V(B) - y$.

Let G_1 be a graph which arises from G by replacing B with a path p of length 3, say $p = x, a, b, y$. Then $|V(G_1)| + |E(G_1)| < |V(G)| + |E(G)|$ since B is not a triangle because G is edge-critical. Hence $(G_1)^2$ contains a hamiltonian cycle H_1 containing four different edges x_iy_i of $E(G_1)$ for certain vertices $y_i, i = 1, 2, 3, 4$, and as many edges as possible of G_1 .

In the following we shall proceed in a manner very similar to the proof in [6] that the square of a 2-block is hamiltonian. However, in order to avoid total dependence of the reader on the knowledge or study of [6], we shall describe and partially repeat the procedure employed in that paper. In particular, we shall quote the cases with the numbering of [6].

This yields the consideration of 13 cases on how the hamiltonian cycle H_1 traverses vertices of the path p . As in [6], Cases 3, Case 4, Case 12, and Case 13 are contradictory to the maximality of the number of edges of G_1 belonging to H_1 ; and Case 6 can be reduced to Case 10, Case 8 to Case 7, Case 10 to Case 9 and Case 11 to Case 5. Note that by the reductions we preserve the existence of the edges x_iy_i even if $x_i \in \{x', y\}$ for $i \in \{1, 2, 3, 4\}$.

The remaining 5 cases are (using the labeling of vertices x', x, a, b, y instead of x, w, a, b, v in [6]):

Case 1. $H_1 = \dots, x, a, b, y, \dots$

Case 2. $H_1 = \dots, x, a, b, y', \dots$

Case 5. $H_1 = \dots, x', a, b, x, \dots$

Case 7. $H_1 = \dots, x', a, y, \dots, y', b, x$

Case 9. $H_1 = \dots, x', a, y, b, x, \dots$;

and $y'y$ is an edge of G .

In order to extend H_1 to H in G^2 in these five cases with H having the required property, one can proceed in the same way as it has been done in [6]. However, we deem it necessary to show explicitly that no problems arise under the stronger condition of this theorem (similarly as in [7]).

Case 1. By [Theorem 8](#), B^2 has an xy -hamiltonian path P starting with an edge yy^* of $E(B)$ and containing an edge uv of B for certain vertices u, v . Replace in H_1 the path p with a hamiltonian path P and we get a hamiltonian cycle H as required.

Case 2. Take P as in Case 1 and replace in H_1 the path x, a, b, y' with $(P - yy^*) \cup y'y^*$ and again we get a hamiltonian cycle H as required. Note that H contains all edges of G belonging to H_1 .

Case 5. By Theorem 5, B^2 contains a hamiltonian cycle H_B such that both edges of H_B incident to y (say yy^* , yy^{**}) are in B and at least one of the edges of H_B incident to x (say xx^*) is in B . We set

$$H^* = (H_B - \{yy^*, yy^{**}\}) \cup y^*y^{**}$$

which does not contain y , and replace in H_1 the path x', a, b, x with $(H^* - xx^*) \cup x'x^*$, thus obtaining a hamiltonian cycle H in G^2 which has the same behavior in all vertices of $G_1 - \{a, b\} \subset G$ as H_1 .

Case 7. Take H_B as in Case 5 and replace in H_1 the path x', a, y with the path $P_1 \cup x^*x'$ where $P_1 \subset H_B$ is the path from y to x^* and does not contain x ; and replace in H_1 the path y', b, x with the path $P_2 \cup y't$ where $t \in \{y^*, y^{**}\}$ and $P_2 \subset H_B$ is the path from x to t and does not contain any of y, x^* . Again we get a hamiltonian cycle H as required.

Case 9. Take H_B as in Case 5 and replace in H_1 the path x', a, y, b, x with $(H_B - xx^*) \cup x'x^*$, thus obtaining a hamiltonian cycle H in G^2 which has the same behavior in all vertices of $G_1 - \{a, b, y\} \subset G$ as H_1 and both edges of H incident to y are in G .

In all cases we obtained a hamiltonian cycle H in G^2 containing four different edges $x_i x'_i$, of $E(G)$ (in most cases we have $x'_i = y_i$; see the first paragraph of this subcase 1.3), $i = 1, 2, 3, 4$, a contradiction.

Case 2. $|D(G)| = 0$. That is, G is a DT -graph.

(a) Suppose $N(x_i) \subseteq V_2(G)$ for every $i = 1, 2, 3, 4$.

Set $W' = \{x_1, x_2, x_3, x_4\}$ and let K be a W' -maximal cycle in G . Observe that $|V(K)| \geq 4$ since an edge-critical block on at least 4 vertices cannot contain a triangle.

If $|W' \cap V(K)| = 4$, then we choose x_5 arbitrary in $V(G) - W'$. If $|W' \cap V(K)| = 3$, then we choose x_5 arbitrary in $V(K) - W'$. If $|W' \cap V(K)| = 2$, then we choose an arbitrary 2-valent vertex x_5 in $V(K) - W'$ which exists because all neighbors of x_i are 2-valent.

We set $W = W' \cup \{x_5\}$. Then K is W -sound in G unless $|W \cap V(K)| = 3$ and forbidden situation (2) in Definition 10 arises. That is, without loss of generality $x_1, x_2 \in V(K)$ and there exist W -separated K -to- K blockchains P, Q based on $x_i, i \in \{1, 2\}$, $P \cap Q = x_i$, and paths p, q in P, Q , respectively, such that there is a subsequence $x_i, w', L(p), L(q), w'', x_i$, where $\{w', w''\} = \{x_{3-i}, x_5\}$ and $x_3, x_4 \in V(p) \cup V(q)$. Then there is a cycle K' containing x_i, x_3, x_4 , a contradiction to the W' -maximality of K .

By Theorem 11, G contains an EPS-graph $S = E \cup P$ such that $K \subseteq E$ and $d_p(w) \leq 1$ for every $w \in W$. If there is no adjacent pair x_i, x_j for $i, j \in \{1, 2, 3, 4\}$, we use S and an algorithm in [5] to obtain a hamiltonian cycle in G^2 with the required properties, a contradiction. However, if there is an adjacent pair, say x_1, x_2 , then $d_G(x_1) = d_G(x_2) = 2$ and $d_p(x_1) = d_p(x_2) = 0$ and we can proceed with the cycle K containing x_1, x_2, x_3 to obtain a required hamiltonian cycle in G^2 as before, a contradiction.

(b) Without loss of generality suppose that $N(x_4) \not\subseteq V_2(G)$.

Hence $\deg_G(x_4) = 2$. Let $P_4 = y_4 x_4 z_1 \dots z_k$ be a unique path in G such that $d_G(y_4) > 2, d_G(z_k) > 2$ and $d_G(z_i) = 2$, for $i = 1, 2, \dots, k - 1$. We set $G^- = G - \{x_4, z_1, \dots, z_{k-1}\}$, where $z_0 = x_4$ if $k = 1$.

(b1) Assume that G^- is 2-connected.

If $x_i \in V(G^-) - \{y_4, z_k\}$ for $i = 1, 2, 3$, then $|V(G)| + |E(G)| > |V(G^-)| + |E(G^-)|$ and hence $(G^-)^2$ has a hamiltonian cycle H^- containing different edges $x_i y_i, z_k w_4 \in E(G), i = 1, 2, 3$. It is easy to see that we can extend H^- to a hamiltonian cycle H in G^2 such that H contains edges $x_i y_i, x_4 z_1$, for $i = 1, 2, 3$, a contradiction.

Suppose $x_3 \notin V(G^-) - \{y_4, z_k\}$. If $\{x_1, x_2, x_3\} \cap \{y_4, z_k\} \neq \emptyset$, then without loss of generality $x_3 \in \{y_4, z_k\}$. By Theorem 7 or Theorem 8, $(G^-)^2$ contains a $y_4 z_k$ -hamiltonian path P^- and P^- contains distinct edges $x_i y_i$ of G if $x_i \in V(G^-)$ for $i = 1, 2$. Then $P^- \cup P_4$ is a hamiltonian cycle in G^2 with the required properties, a contradiction.

(b2) Assume that G^- is not 2-connected.

Then G^- is a non-trivial blockchain with y_4, z_k in distinct endblocks and y_4, z_k are not cutvertices.

Assume not all x_1, x_2, x_3 are inner vertices in the same block. Then we apply Lemma 9 to get a $y_4 z_k$ -hamiltonian path P^- in $(G^-)^2$ with distinct edges $x_i y_i \in E(G^-), i = 1, 2, 3$. Note that x_i could be y_4 or z_k . Then again $P^- \cup P_4$ is a hamiltonian cycle in G^2 with the required properties, a contradiction.

Now assume that x_1, x_2, x_3 are inner vertices in the same block B . Then there exists an end block B^* of G^- such that $x_i \notin V(B^*), i = 1, 2, 3$. A graph G' arises from G by the replacement of B^* by a path p of length 3. Hence $|V(G)| + |E(G)| > |V(G')| + |E(G')|$ and we denote by H' a hamiltonian cycle in $(G')^2$ containing edges $x_i w_i, i = 1, 2, 3, 4$, and as many edges of G' as possible.

We proceed in the same manner as in Subcase 1.3 (note that in this case none of $x_i, i = 1, 2, 3, 4$, is on p) to get a hamiltonian cycle in G^2 with required properties, a contradiction.

Finally we want to show that Theorem 2 is best possible, i.e., we construct an infinite family of graphs which do not satisfy the \mathcal{H}_5 property. For this purpose start with an arbitrary 2-block G and fix different vertices $x_1, x_2 \in V(G)$.

Define

$$H = G \cup \{y_1, y_2, \dots, y_t; t \geq 3\} \cup \{x_i y_j : 1 \leq i \leq 2, 1 \leq j \leq t\},$$

where $\{y_1, \dots, y_t\} \cap V(G) = \emptyset$. Then H is a 2-block. However, H does not have the \mathcal{H}_5 property: indeed, there is no hamiltonian cycle C in H^2 containing edges of H incident to x_1, x_2, y_1, y_2, y_3 because of the neighbors of y_1, y_2, y_3 in H which are x_1 and x_2 only; that is x_1 or x_2 would be incident to three edges of $C \cap H$, which is impossible. \square

4. Conclusion

We introduced the concept of the \mathcal{H}_k property and proved that every 2-block has the \mathcal{H}_4 property but not the \mathcal{H}_5 property in general. Similarly in [8] it is proved that every 2-block has the \mathcal{F}_4 property but not the \mathcal{F}_5 property in general. Moreover, a 2-block G having the \mathcal{F}_k property implies that G has the \mathcal{H}_{k-1} property for $k = 3, 4, \dots$. Hence we conclude that [Theorems 2](#) and [7](#) are best possible with respect to hamiltonicity and hamiltonian connectedness in the square of a 2-block.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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