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Star edge-coloring of square grids

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ABSTRACT

A star edge-coloring of a graph G is a proper edge-coloring without bichromatic paths or cycles of length four. The smallest integer k such that G admits a star edge-coloring with k colors is the *star chromatic index* of G. In the seminal paper on the topic, Dvořák, Mohar, and Šámal asked if the star chromatic index of complete graphs is linear in the number of vertices and gave an almost linear upper bound. Their question remains open, and consequently, to better understand the behavior of the star chromatic index, this parameter has been studied for a number of other classes of graphs. In this paper, we consider star edge-colorings of square grids; namely, the Cartesian products of paths and cycles and the Cartesian products of two cycles. We improve previously established bounds and, as a main contribution, we prove that the star chromatic index of graphs in both classes is either 6 or 7 except for prisms. Additionally, we give a number of exact values for many considered graphs.

1. Introduction

A proper edge-coloring of a graph G is called a *star edge-coloring* if there is neither bichromatic path nor bichromatic cycle of length four. The minimum number of colors for which G admits a star edge-coloring is called the *star chromatic index* and we denote it by $\chi'_{\mathrm{st}}(G)$.

The star edge-coloring was defined in 2008 by Liu and Deng [8], and was motivated by the vertex version introduced by Grünbaum [4]. Despite a number of papers have already been published about this coloring, we have a very limited knowledge about it. In particular, the exact value of the star chromatic index of complete graphs is still not known, although some relatively strong lower and upper bounds have been determined by Dvořák et al. in their seminal paper [3].

Theorem 1 (Dvořák, Mohar, Šámal, 2013)**.** *The star chromatic index of the complete graph K_n satisfies*

$$2n(1+o(1)) \le \chi'_{\mathrm{st}}(K_n) \le n\frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{1/4}}.$$

In particular, for every $\epsilon > 0$ there exists a constant C such that $\chi'_{\mathrm{st}}(K_n) \le C n^{1+\epsilon}$ for every $n \ge 1$.

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They proved the upper bound using a nontrivial result about sets without arithmetic progressions, and up till now, it is still the best known. For the lower bound, they used an elegant double counting approach. The authors of [1] observed a small improvement in their proof and obtained the bound $\chi'_{st}(K_n) \geq 3n(n-1)/(n+4)$ (see [10] for a proof), which gives the exact values for the chromatic index of K_n, for $n \in \{1, 2, 3, 4, 8\}$. However, despite all efforts, the asymptotic behavior of the star chromatic index of complete graphs is not known, and in [3] the following question has been asked.

Question 1 (Dvořák, Mohar, Šámal, 2013)**.** What is the true order of magnitude of $\chi'_{st}(K_n)$? Is $\chi'_{st}(K_n) = O(n)$?

Another class of graphs with highly regular structure are complete bipartite graphs. They are important for better understanding of the coloring already on their own, and also, as Dvořák et al. [3] observed, the bounds for their star chromatic index provide bounds for the index of complete graphs.

$$\chi'_{st}(K_{n,n}) - n \leq \chi'_{st}(K_n) \leq \sum_{i=1}^{\lceil \log_2 n \rceil} 2^{i-2}\, \chi'_{st}(K_{\lceil n/2^i \rceil, \lceil n/2^i \rceil}).$$

Recently, Casselgren et al. [2] considered complete bipartite graphs and proved the tight upper bound for $K_{3,r}$, $r \geq 5$, derived a lower and upper bound for $K_{4,s}$, $s \geq 4$, and, using computer, they also determined the star chromatic index for some complete bipartite graphs of small order.

Star edge-coloring has been studied also for other classes of graphs, e.g., graphs with maximum degree 3 [3,6,7] and 4 [14], subcubic Halin graphs [2], outerplanar graphs [1,13], and planar graphs with various constraints [13]. Moreover, the list version of the star edge-coloring has also been investigated (see, e.g., [5,9]). Finally, there is also a complexity result on the topic; namely, it is NP-complete to decide whether 3 colors suffice for a star edge-coloring of a subcubic multigraph [6].

Since most of the obtained upper bounds for the star chromatic index are not tight and many questions remain open, we focus our attention to graphs with a relatively simple structure, i.e. to the Cartesian products of graphs.

The star edge-coloring of the Cartesian products of graphs has already been considered by Omoomi and Dastjerdi [11]. They established an upper bound for the star chromatic index of the Cartesian product of two arbitrary graphs, proved its exact values for the Cartesian product of two paths (Theorem 4), and they started investigation on the Cartesian products of a path and a cycle, and the Cartesian product of two cycles (i.e., square grids). They further proved upper bounds for d-dimensional grids and d-dimensional hypercubes.

Motivated by the results presented in [11], in this paper, we consider star edge-coloring of square grids; in particular, the Cartesian products of two cycles and the Cartesian products of paths and cycles. Apart from the usual combinatorial methods, due to the complexity of the problems considered in this paper, we have used computer to obtain star edge-colorings of small graphs and to establish some of the lower bounds. Standard (formal) mathematical proofs would require enourmous amount of case analysis, while their contribution to the theory would be minimal. We establish exact bounds for the star chromatic index of many graphs from the two considered classes, and show that the upper bound for the chromatic index of both Cartesian products is 7.

The paper is structured as follows. We give our notation and prove some auxiliary results in Section 2. Section 3 contains the algorithm used in our computations and describes the preprocessing procedures used in them. In Section 4, we present the main results of this paper, and we list some open problems in Section 5.

2. Preliminaries

In this section, we present some additional terminology used in the paper and give auxiliary results. We abbreviate a 'star edge-coloring with k colors' to a 'star k-edge-coloring', and, if it is clear from the context, sometimes we just write 'coloring' instead of 'star edge-coloring'.

The *Cartesian product of graphs G and H*, denoted by $G \square H$, is the graph with the vertex set $V(G) \times V(H)$ and edges between the vertices (u, v) and (u', v') if:

- $uu' \in E(G)$ and $v = v'$ (a G-*edge*), or
- $u = u'$ and $vv' \in E(H)$ (an H-*edge*).

We call the graphs G and H the *factor graphs*. The G-*fiber* with respect to $v \in V(H)$, denoted by G_v, is the copy of G in $G \square H$ induced by the vertices having v as the second component. Analogously, the H-*fiber* with respect to $u \in V(G)$, denoted by H_u, is the copy of H in $G \square H$ induced by the vertices having u as the first component.

Since the Cartesian product of two paths is a subgraph of the Cartesian product of a path and a cycle, and the Cartesian product of a path and a cycle is a subgraph of the Cartesian products two cycles, we have the following sequence of inequalities.

Observation 1. For every pair of positive integers m and n, where $m \geq 3$ and $n \geq 3$, we have

$$\chi'_{st}(P_m \square P_n) \leq \chi'_{st}(C_m \square P_n) \leq \chi'_{st}(C_m \square C_n).$$

Having a star edge-coloring of the Cartesian product of an n-cycle and a graph H, we can extend it to a coloring of the Cartesian product of a cycle of length $k \cdot n$ and H.

Lemma 1. *For every integers k and m, where $k \geq 2$ and $m \geq 3$, and for every graph H, we have*

$$\chi'_{st}(C_{k \cdot m} \,\square\, H) \leq \chi'_{st}(C_m \,\square\, H).$$

Proof. Let $C_m = u_1 \dots u_m u_1$, $C_{k \cdot m} = v_1 \dots v_{k \cdot m} v_1$, and $V(H) = \{w_1, \dots, w_n\}$. Moreover, let $p : \{1, \dots, k \cdot m\} \to \{1, \dots, m\}$ be an assignment given by $p(t) = s$ if and only if $(t - s)$ is divisible by m.

Let σ be a star edge-coloring of $C_m \,\square\, H$. Consider an edge $e = (v_i, w_a)(v_j, w_b)$ of $C_{k \cdot m} \,\square\, H$. By the definition of the Cartesian product, we have $i = j$ or $a = b$. Note that if $a = b$, then $|i - j| = 1$, and thus we may assume $j = i + 1$. We define a proper edge-coloring τ of $C_{k \cdot m} \,\square\, H$ as follows. If $i = j$, then set $\tau((v_i, w_a)(v_i, w_b)) = \sigma((u_{p(i)}, w_a)(u_{p(i)}, w_b))$. In the case $a = b$, we set $\tau((v_i, w_a)(v_j, w_a)) = \sigma((u_{p(i)}, w_a)(u_{p(j)}, w_a))$.

Now we show that τ is also a star edge-coloring. For an integer s, where $1 \leq s \leq k \cdot m$, let G_s be the graph induced by the vertices $\{(v_\ell, w)\}$, where $\ell \in \{s + 1, \dots, s + m\}$ (the values $s + 1, \dots, s + m$ are taken modulo $k \cdot m$) and all $w \in V(H)$, i.e., G_s is the graph induced on m consecutive H-fibers. Observe that the coloring τ on G_s corresponds to a coloring σ of the subgraph of $C_m \square H$ without the edges $(u_{p(s+m)}, w)(u_{p(s+1)}, w)$, for all $w \in V(H)$. Therefore, every 4-path and every 4-cycle in $C_{k \cdot m} \,\square\, H$, contained in some G_s, is not bichromatic.

Finally, if a 4-path or a 4-cycle is not contained in any G_s, then it contains at least m edges of type $(v_i, w_a)(v_j, w_a)$ (i.e., only when $m \in \{3, 4\}$). However, in the case of $m = 3$, three consecutive edges on every C_m-fiber receive three distinct colors, and hence no 4-path with three consecutive edges on a C_m-fiber is bichromatic. If a 4-path has two consecutive edges on a C_m-fiber, an edge in an H-fiber, and the fourth edge in another C_m-fiber, then its coloring corresponds to a coloring of some 4-path in $C_m \,\square\, H$, which is not bichromatic. In the case of $m = 4$, we only have 4-cycles, whose colorings correspond to a coloring of a 4-cycle by σ in some C_m-fiber, and hence they are not bichromatic. □

We continue by showing how star edge-colorings of two Cartesian products, each having at least one cycle as a factor, can be combined. Let m and n be a pair of integers, where $3 \leq m < n$, and let $v_1, \dots, v_n$ be consecutive vertices of the cycle C_n. We say that a star edge-coloring σ of $C_n \,\square\, H$ *includes* a star edge-coloring of $C_m \,\square\, H$ if the coloring σ^* of the subgraph of $C_n \,\square\, H$ induced by the vertices of m consecutive H-fibers $H_{v_1}, \dots, H_{v_m}$, together with the additional edges $e_w = (v_1, w)(v_m, w)$, for all $w \in V(H)$, where we set $\sigma^*(e_w) = \sigma((v_1, w)(v_n, w))$, is a star edge-coloring.

Symmetrically, we can say that a star k-edge-coloring of $H \,\square\, C_n$ includes a star edge-coloring of $H \,\square\, C_m$. Note that the star edge-coloring of $C_{k \cdot m} \,\square\, H$, constructed in the proof of Lemma 1, includes a star edge-coloring of $C_m \,\square\, H$.

Lemma 2. *If for a pair of positive integers m and n, where $m < n$, a star k-edge-coloring of $C_n \,\square\, H$ includes a star edge-coloring of $C_m \,\square\, H$, then, for every pair of non-negative integers p and q, we have*

$$\chi'_{st}(C_{p \cdot m + q \cdot n} \,\square\, H) \leq k.$$

Proof. Let σ be a star k-edge-coloring of $C_n \,\square\, H$ which includes a star edge-coloring σ^* of $C_m \,\square\, H$. Let $C_n = v_1 \dots v_n v_1$, $C_m = v_1 \dots v_m v_1$, and $C_{pm+qn} = u_1 \dots u_{pm+qn} u_1$. Furthermore, we define an assignment $r : \{1, \dots, pm + qn\} \to \{1, \dots, n\}$ such that, if $t \leq pm$, then $r(t) \leq m$ and $t - r(t)$ is divisible by m, and, if $t > pm$, then $t - pm - r(t)$ is divisible by n.

Now, similarly as in the proof of Lemma 1, we define an edge-coloring τ of $C_{pm+qn} \,\square\, H$. We combine p copies of σ^* followed by q copies of σ. More precisely, for $w_a, w_b \in H$,

$$\tau((u_i, w_a)(u_j, w_b)) = \begin{cases} \sigma^*((v_{r(i)}, w_a)(v_{r(i)}, w_b)), & \text{for } i = j \leq pm; \\ \sigma((v_{r(i)}, w_a)(v_{r(i)}, w_b)), & \text{for } i = j > pm; \\ \sigma^*((v_{r(i)}, w_a)(v_{r(j)}, w_a)), & \text{for } i = j - 1 \leq pm; \\ \sigma((v_{r(i)}, w_a)(v_{r(j)}, w_a)), & \text{for } i = j - 1 > pm. \end{cases}$$

Note that, by the definition of σ^*, we have $\sigma^*((v_m, w)(v_1, w)) = \sigma((v_n, w)(v_1, w))$.

It remains to show that τ is a star edge-coloring. For an integer s, where $1 \leq s \leq pm + qn$, let G_s be the graph induced by the vertices of $m + 1$ consecutive H-fibers $H_{u_{s+1}}, \dots, H_{u_{s+m+1}}$ (the indices $s + i$ are taken modulo $pm + qn$). Since the coloring of each G_s is a part of two consecutive copies of σ^* or a part of two consecutive copies of σ, no 4-path and no 4-cycle in G_s is bichromatic (using Lemma 1 for $k = 2$).

Finally, if a 4-path is not contained in any G_s, then it contains 4 edges of type $(v_i, w_a)(v_j, w_a)$ and $m = 3$. Moreover, such a 4-path traverses the H_{v_i}-fibers, for $i \in \{pm, \dots, pm + 4\}$ or $i \in \{pm + qn, 1, \dots, 4\}$. In both cases, colors of three consecutive edges of the 4-path correspond to colors of a C_3-fiber of σ^* and therefore they are distinct. This completes the proof. □

We will use Lemma 2 to prove results for arbitrary lengths of cycles. To do that, we will use the following result on Frobenious numbers [12].

Theorem 2 (Sylvester, 1882). *Let positive integers n and m be relatively prime. Then for every integer $k \geq (n - 1)(m - 1)$ there exist non-negative integers α and β such that*

$$k = \alpha \cdot n + \beta \cdot m.$$

We also recall the result of Dvořák et al. [3] about star edge-coloring of subcubic graphs, which we will use when considering prisms.

Theorem 3 (Dvořák, Mohar, Šámal, 2013).

(a) *If G is a subcubic graph, then* $\chi'_{st}(G) \leq 7$.
(b) *If G is a simple cubic graph, then* $\chi'_{st}(G) \geq 4$, *and the equality holds if and only if G covers the graph of the 3-dimensional hypercube.*

Here, a graph G is said to *cover* a graph H if there is a graph homomorphism from G to H that is locally bijective. In other words, there is a mapping f: $V(G) \to V(H)$ such that whenever uv is an edge of G, the image $f(u)f(v)$ is an edge of H, and, for each vertex $v \in V(G)$, f is a bijection between the neighbors of v and the neighbors of $f(v)$.

At this point, we remark the following, somehow hidden, corollary of the above result. Hexagonal grids are subcubic graphs and they cover the graph of the 3-dimensional hypercube. Thus:

Corollary 1. *For an infinite hexagonal grid G, we have*

$$\chi'_{st}(G) = 4.$$

3. Computer computations and algorithm

For our computations, we used a simple backtracking algorithm (see Algorithm 1), which, together with some preprocessing, enabled us to compute exact lower bounds for some important cases on one hand, and on the other hand, provided star edge-colorings with required properties for some graphs.

Algorithm 1 Star edge-coloring algorithm.

```
procedure StarColor(G, k, P)                         ▷ Graph G, number of colors k, precolored edges P
    edgeOrder ← GetEdgeOrdering(G, P)
    edgeColors ← InitEdgeColors(P,edgeOrder)
    triedColors ← InitTriedColors(edgeOrder)           ▷ Dictionary of empty lists for all edges
    for i in 1.edgeOrder.Count do                       ▷ Try to color edges according to the ordering
        e ← edgeOrder[i]
        isColored ← false
        for color c in {1.k} \ triedColors[e] do
            edgeColors[e] ← c
            if Conflict(c, G, edgeColors) then          ▷ Check if a conflict occurs
                edgeColors[e] ← ∅
            else
                add c to triedColors[e]
                isColored ← true
                goto 18
            end if
        end for
        if !isColored and i > 1 then                    ▷ If no color is found, continue if not at first edge
            edgeColors[edgeOrder[i − 1]] ← ∅
            i = i − 2                                   ▷ Step up, −2 handles automatic loop increment
        else if !isColored and i == 1 then
            return "No coloring found"
        else if isColored and i == edgeOrder.Count then
            return edgeColors                           ▷ A star edge-coloring is found
        end if
    end for
end procedure
```

The main coloring algorithm takes three input parameters: the graph to be colored, the number of colors, and a possible precoloring of some edges, in order to avoid testing some isomorphic partial colorings; e.g., one may fix the colors on the edges incident to a vertex of maximum degree. Note also that before calling the function `StarColor`, we first verify that the precoloring of the edges is a star edge-coloring.

Another important part of our algorithm is determining the order of edges (the function `GetEdgeOrdering`), in which it tries to color them. We order the edges (ignoring the precolored edges) by the number of precolored neighbors (incident edges) and the number of neighbors appearing earlier in the ordering in a descending order.

The function `Conflict` checks if assigning a color to the current edge introduces a conflict, namely, it checks if two adjacent edges receive the same color, and if a bichromatic 4-path or 4-cycle appears. In some cases, we manually controlled the different cases of precolored edges. If we established some additional property of a required coloring, e.g., that no 4-path can be colored with just three colors, we included that in the procedure.

Table 1
The star chromatic index of the Cartesian products of two paths $\chi'_{st}(P_m \,\square\, P_n)$.

$m\backslash n$	2	3	4	5^+
2	3	4	4	4
3	4	5	5	6
4	4	5	6	6
5^+	4	6	6	6

Finally, we also adopted the algorithm to output all possible colorings of a given graph, and in the case of symmetric graphs, e.g., cycles, we eliminated isomorphic colorings. The remaining colorings were used to test if they can be extended to graphs on more vertices.

4. Cartesian products of paths and cycles

4.1. Cartesian products of paths

In a recent paper, Omoomi and Dastjerdi [11] established tight bounds for the star chromatic index of two paths (see Table 1).

Theorem 4 (Omoomi and Dastjerdi, 2019)**.** *For the graph $P_m \,\square\, P_n$, where m and n are integers with $2 \le m \le n$, we have*

$$\chi'_{st}(P_m \,\square\, P_n) = \begin{cases} 3, & \text{if } m = n = 2; \\ 4, & \text{if } m = 2 \text{ and } n \ge 3; \\ 5, & \text{if } m = 3 \text{ and } 3 \le n \le 4; \\ 6, & \text{otherwise.} \end{cases}$$

As a corollary, we establish the lower bound of 6 colors for the Cartesian products, where one factor is a cycle and the other is a path of length at least 2.

Corollary 2. *For every pair of integers m and n, where $m \ge 3$ and $n \ge 3$, we have*

$$\chi'_{st}(C_m \,\square\, P_n) \ge 6.$$

Proof. We first note that the graph $C_3 \,\square\, P_2$ is one of the two known examples of simple bridgeless cubic graphs that have star chromatic index equal to 6 [9]. Then, for $m = 3$, by Observation 1, we have $\chi'_{st}(C_3 \,\square\, P_n) \ge \chi'_{st}(C_3 \,\square\, P_2) = 6$.

If $m = 4$, we proceed by a contradiction. Suppose that $\chi'_{st}(C_4 \,\square\, P_n) \le 5$. Then, by Lemma 1, $\chi'_{st}(C_{4\ell} \,\square\, P_n) \le 5$, for any integer ℓ, and hence also $\chi'_{st}(P_5 \,\square\, P_n) \le 5$, a contradiction. Finally, if $m \ge 5$, then we have $\chi'_{st}(C_m \,\square\, P_n) \ge \chi'_{st}(P_5 \,\square\, P_3) = 6$ by Theorem 4 and Observation 1. □

4.2. Cartesian products of cycles

Having the Cartesian products of paths resolved, the logical direction of research is consideration of cylinders and toroidal grids, i.e., the Cartesian products of cycles and paths, and the Cartesian products of two cycles. We begin by giving some results about the latter.

Corollary 2 implies that the Cartesian product of any two cycles will need at least 6 colors for a star edge-coloring. On the other hand, as we will show in this section, the star chromatic index of the Cartesian product of two cycles is at most 7. We first investigate the Cartesian products of C_3 with another cycle.

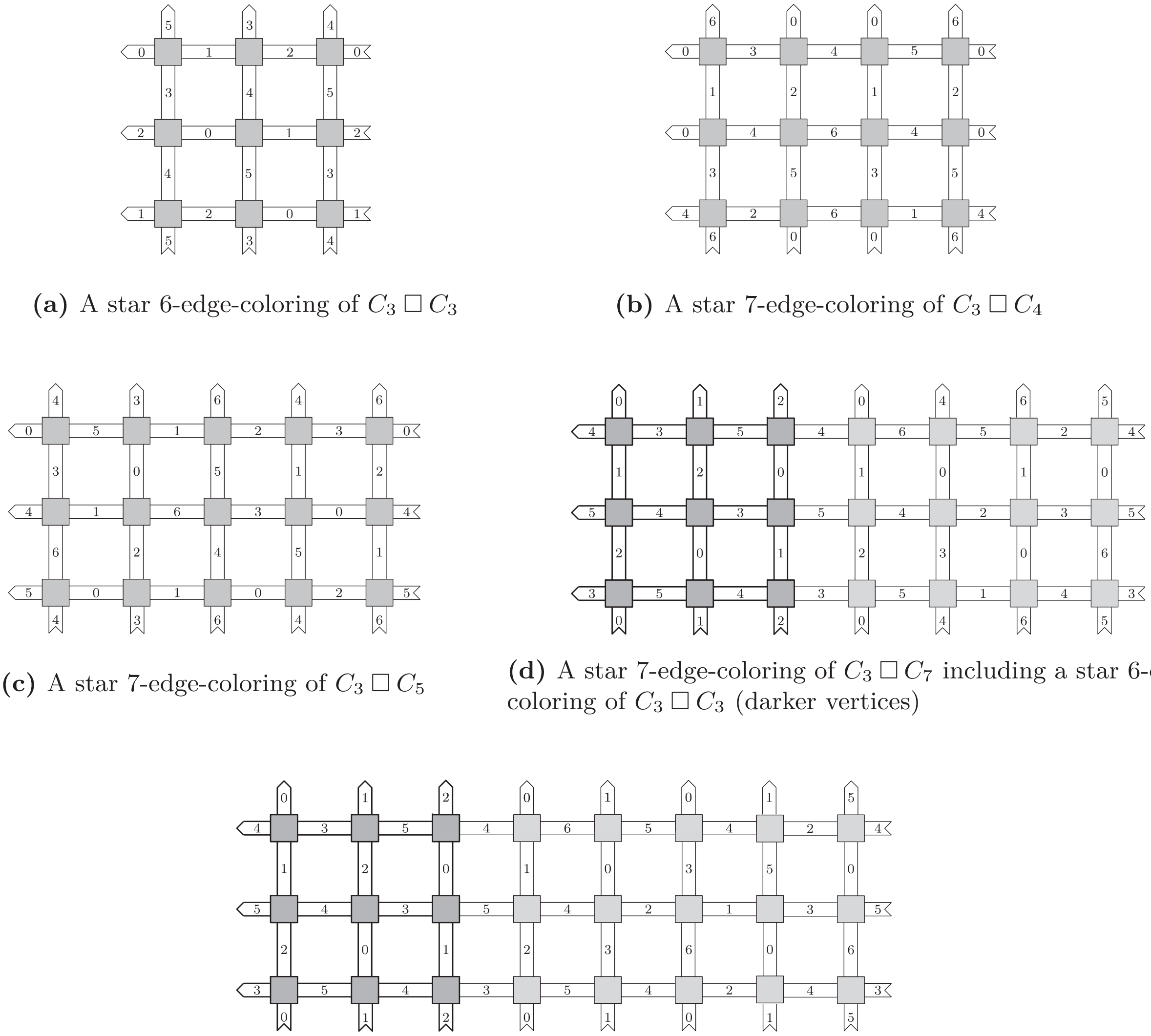
Theorem 5. *For every integer n, where $n \ge 3$, we have*

$$\chi'_{st}(C_3 \,\square\, C_n) = \begin{cases} 6, & \text{if } n = 3k, \\ 7, & \text{otherwise} \end{cases}$$

Proof. By Corollary 2, $\chi'_{st}(C_3 \,\square\, C_n) \ge 6$. Now suppose that $n = 3k$ for some integer $k \ge 1$. If $k = 1$, then there is a star 6-edge-coloring of $C_3 \,\square\, C_3$ (one is depicted in Fig. 1(a)). Next, by Lemma 1, we have $\chi'_{st}(C_3 \,\square\, C_n) = 6$.

Using Algorithm 1, we infer that the Cartesian product $C_3 \,\square\, P_3$ has only one star 6-edge-coloring up to a permutation of colors. Namely, three colors, say 0, 1, and 2, appear on C_3-fibers, and the colors 4, 5, and 6 on the P_3-fibers. Since $C_3 \,\square\, P_3$ is a subgraph of every graph $C_3 \,\square\, C_n$, it follows that such Cartesian products admit a star 6-edge-coloring only when n is divisible by 3.

Therefore, if $n \ne 3k$ for every integer k, then $\chi'_{st}(C_3 \,\square\, C_n) \ge 7$. In Fig. 1(d) and 1(e), a star 7-edge-coloring of $C_3 \,\square\, C_7$ and $C_3 \,\square\, C_8$, respectively, is depicted. Observe that, in both colorings, a star 6-edge-coloring of $C_3 \,\square\, C_3$ is included. Hence, by



(a) A star 6-edge-coloring of $C_3 \,\square\, C_3$

(b) A star 7-edge-coloring of $C_3 \,\square\, C_4$

(c) A star 7-edge-coloring of $C_3 \,\square\, C_5$

(d) A star 7-edge-coloring of $C_3 \,\square\, C_7$ including a star 6-edge-coloring of $C_3 \,\square\, C_3$ (darker vertices)

(e) A star 7-edge-coloring of $C_3 \,\square\, C_8$ including a star 6-edge-coloring of $C_3 \,\square\, C_3$ (darker vertices)

Fig. 1. Cartesian products of C_3 with cycles.

Lemma 2 and Theorem 2, we have $\chi'_{st}(C_3 \,\square\, C_n) = 7$ for every n, $n \geq 7$, not divisible by 3. The remaining two cases, namely $n = 4$ and $n = 5$, are depicted in Fig. 1(b) and 1(c), respectively. □

Similarly as in the proof of Theorem 5, we can use Lemma 1 (twice) to extend the star edge-coloring of $C_3 \,\square\, C_3$ to products of cycles of lengths divisible by 3.

Corollary 3. *For every pair of positive integers k and ℓ, we have*

$$\chi'_{st}(C_{3k} \,\square\, C_{3\ell}) = 6.$$

We proceed with a result about the Cartesian products of C_4 with another cycle.

Theorem 6. *For every pair of positive integers k and ℓ, where $k \geq 1$ and $\ell \geq 2$, we have*

$$\chi'_{st}(C_{4k} \,\square\, C_{2\ell}) = 6.$$

Proof. We use the star 6-edge-coloring σ_{10} of $C_4 \,\square\, C_{10}$ depicted in Fig. 3. Note that σ_{10} includes a star 6-edge-coloring $C_4 \,\square\, C_4$, and a star 6-edge-coloring $C_4 \,\square\, C_6$. Therefore, by Lemma 2 and Theorem 2, we have $\chi'_{st}(C_4 \,\square\, C_{2\ell}) = 6$ for every integer $\ell \geq 4$. Finally, we use Lemma 1 to infer $\chi'_{st}(C_{4k} \,\square\, C_{2\ell}) = 6$ for every integer k. □

Proposition 1. *For $n \in \{5, 7, 9, 11\}$, we have*

$$\chi'_{st}(C_4 \square C_n) = 7.$$

Proof. Using Algorithm 1, we established that $\chi'_{st}(C_4 \square C_n) > 6$ for every $n \in \{5, 7, 9\}$. The bounds $\chi'_{st}(C_4 \square C_n) = 7$ follow from the star 7-edge-colorings depicted in Fig. 2(a), 2(c), and 2(d).

In the case of $n = 11$, we split the computation in two steps. First, using Algorithm 1, we determined that if the edges of some C_{11}-fiber are colored in such a way that a same color appears twice on some 4-path, then the coloring cannot be extended to a star 6-edge-coloring of $C_4 \square C_{11}$. In the second step, the algorithm checked only the colorings in which every 4-path in each C_{11}-fiber had four colors on its edges. It turned out that such a coloring does not exist. Therefore, $\chi'_{st}(C_4 \square C_{11}) = 7$ by the star edge-coloring depicted in Fig. 2(c) and Lemma 1. □

Theorem 7. *For any odd integer n, where $n \geq 13$, we have*

$$\chi'_{st}(C_4 \square C_n) \leq 7.$$

Proof. In Fig. 2(c), we present a star 7-edge-coloring of $C_4 \square C_7$ with a star 7-edge-coloring of $C_4 \square C_4$ included. Thus, by Lemma 2 and Theorem 2, we infer that $\chi'_{st}(C_4 \square C_n) \leq 7$ for every odd n, where $n > 18$. Colorings for $n \in \{13, 15, 17\}$ can be obtained by using Lemma 2 and the colorings depicted in Fig. 2(a) (for $n = 15$) and 2(d) (for $n \in \{13, 17\}$). □

Theorem 8. *For every integer n, where $n \geq 3$, we have*

$$\chi'_{st}(C_5 \square C_n) = 7.$$

Proof. The lower bounds $\chi'_{st}(C_5 \square C_n) > 6$ for $3 \leq n \leq 6$, were established using Algorithm 1. For $n \geq 7$, using Algorithm 1, we infer that $\chi'_{st}(C_5 \square P_n) \geq 7$. Therefore, by Observation 1, we have $\chi'_{st}(C_5 \square C_n) \geq 7$.

Star 7-edge-colorings of $C_5 \square C_m$, for $m \in \{3, 4, 5, 7, 11\}$, are depicted in Figs. 1(c), 2(a), 4(a), 4(b), and 4(c), respectively. By Lemma 1, we also infer star 7-edge-colorings of $C_5 \square C_m$ for $m \in \{6, 8, 9, 10\}$. Finally, since in Fig. 4(b), a star 7-edge-coloring of $C_5 \square C_3$ is included, by Lemma 2 and Theorem 2, we obtain $\chi'_{st}(C_5 \square C_n) = 7$ for every integer $n \geq 12$. □

Theorem 9. *For every integer n, where $n \geq 3$, we have*

$$\chi'_{st}(C_6 \square C_n) = \begin{cases} 6, & \text{if } n \equiv 0 \bmod 3 \text{ or } n \equiv 0 \bmod 4, \\ 7, & \text{otherwise.} \end{cases}$$

Proof. By the star 6-edge-colorings depicted in Figs. 1(a) and 2(b), and by Lemma 1, we have $\chi'_{st}(C_6 \square C_n) = 6$, for every integer n divisible by 3 or 4.

Now we show that $\chi'_{st}(C_6 \square C_n) > 6$ if n is not divisible by 3 or 4. First, we consider the graph $C_6 \square P_{31}$, where $P_{31} = v_1 \dots v_{31}$. We start with a precolored C_6-fiber at the vertex v_{16} (i.e., the middle C_6-fiber) using each of the nine possible star 6-edge-colorings of C_6 (up to symmetries and permutations of colors). Using Algorithm 1, we tried to extend such a precoloring to the whole $C_6 \square P_{31}$. For five colorings of the C_6-fiber, namely for (0,1,0,2,0,3), (0,1,0,2,1,2), (0,1,0,2,1,3), (0,1,2,0,1,3), and (0,1,2,0,3,4), we obtain that such precolorings cannot be extended.

For the remaining four precolorings, namely (0,1,0,2,3,2), (0,1,0,2,3,4), (0,1,2,0,1,2), and (0,1,2,3,4,5), we obtain 27 078 colorings of $C_6 \square P_{31}$ in total. Some of them are either 4-, or 6-periodical, i.e., the initial coloring repeats on every 4-th or 6-th fiber, except at the final three fibers on both sides, where the coloring restrictions are relaxed.

The remaining 26 ,448 colorings correspond to the precoloring (0,1,2,3,4,5), and moreover, all C_6-fibers are colored by shifts of this precoloring, and every pair of adjacent C_6-fibers is either colored with the same sequence of colors, or the coloring of one is the coloring of the other shifted by 1. In a more detailed analysis of these colorings, we find that, if they are periodic, then the period must be a multiple of 6.

Thus, for the graphs $C_6 \square C_n$, it follows that they are star 6-edge-colorable if n is divisible by 3 or 4. Otherwise they are not star 6-edge-colorable. □

Theorem 10. *For every integer n, where $n \geq 3$, we have*

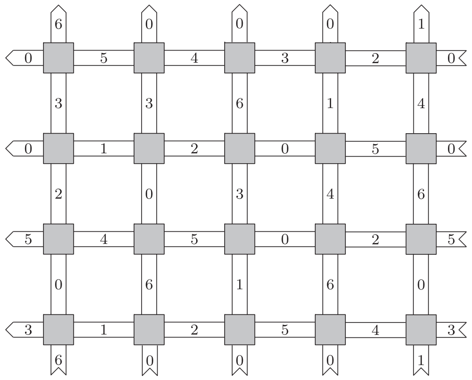
$$\chi'_{st}(C_7 \square C_n) = 7.$$

Proof. The lower bounds $\chi'_{st}(C_7 \square C_n) > 6$, for $n \in \{3, 4, 5, 6\}$, were established using Algorithm 1. For $n \geq 7$, using Algorithm 1, we infer that $\chi'_{st}(C_7 \square P_n) \geq 7$. Therefore, by Observation 1, we have $\chi'_{st}(C_7 \square C_n) \geq 7$.

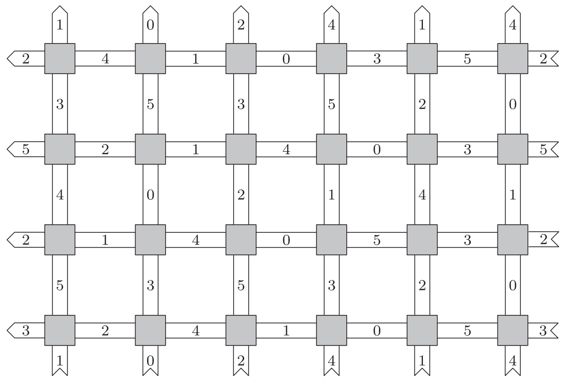
Star 7-edge-colorings of $C_7 \square C_n$, for $n \in \{3, 4, 5, 7\}$, are depicted in Figs. 1(d), 2(c), 4(b), and 5, respectively. By Lemma 1, from these colorings, we also infer star 7-edge-colorings of $C_7 \square C_n$ for $n \in \{6, 8, 9, 10\}$. Moreover, in the coloring depicted in Fig. 6, a star 7-edge-coloring of $C_7 \square C_{11}$ is included, and hence we also have $\chi'_{st}(C_7 \square C_{11}) = 7$. Finally, since in the coloring depicted in Fig. 5, a star 7-edge-coloring of $C_7 \square C_3$ is included, by Lemma 2 and Theorem 2, we have $\chi'_{st}(C_7 \square C_n) = 7$ for every $n \geq 12$. □

Proposition 2. *For $C_8 \square C_9$, we have*

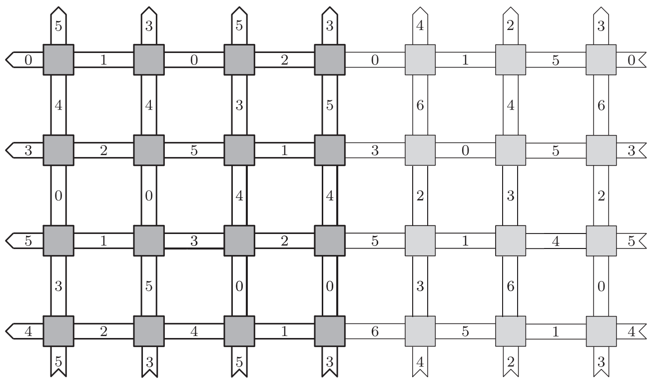
$$\chi'_{st}(C_8 \square C_9) = 7.$$



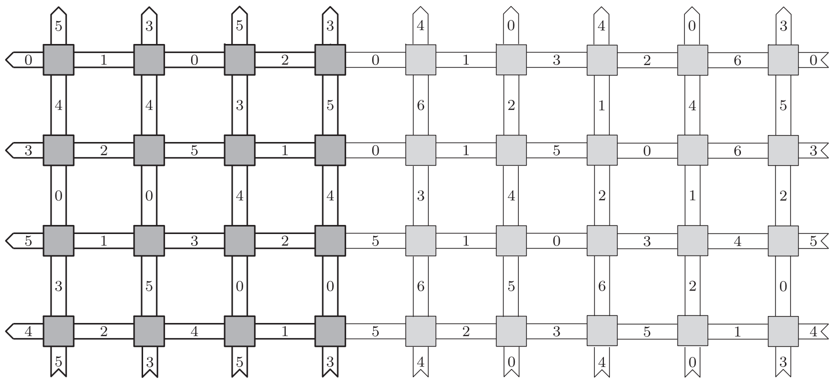
(a) A star 7-edge-coloring of $C_4 \,\square\, C_5$



(b) A star 6-edge-coloring of $C_4 \,\square\, C_6$



(c) A star 7-edge-coloring of $C_4 \,\square\, C_7$ including a star edge-coloring of $C_4 \,\square\, C_4$ (darker vertices)



(d) A star 7-edge-coloring of $C_4 \,\square\, C_9$ including a star edge-coloring of $C_4 \,\square\, C_4$ (darker vertices)

Fig. 2. Cartesian products of C_4 with cycles.

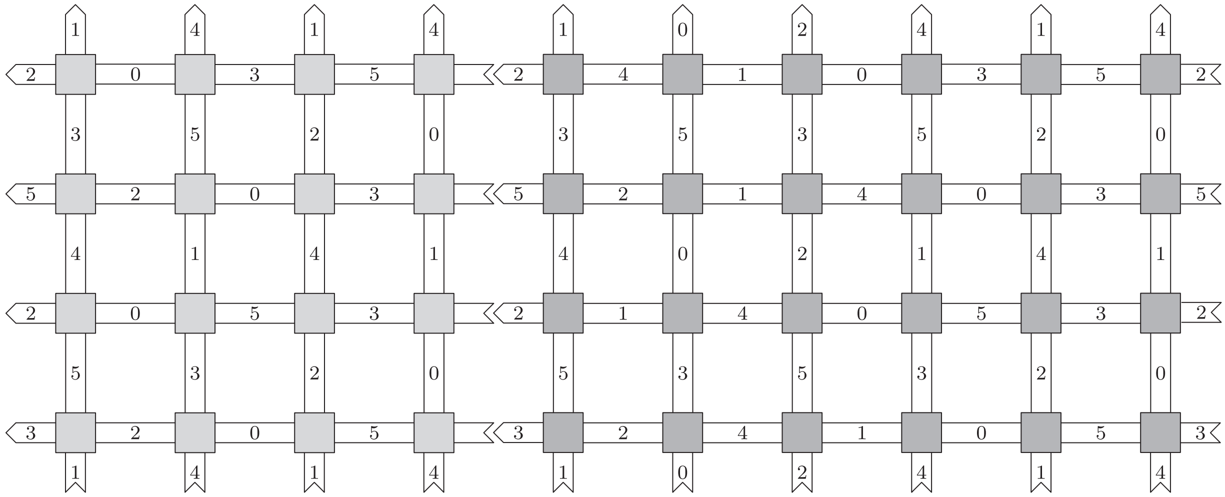
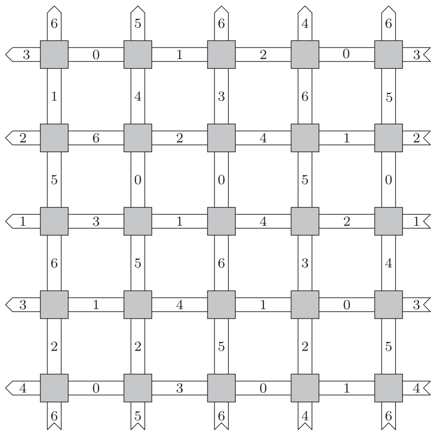
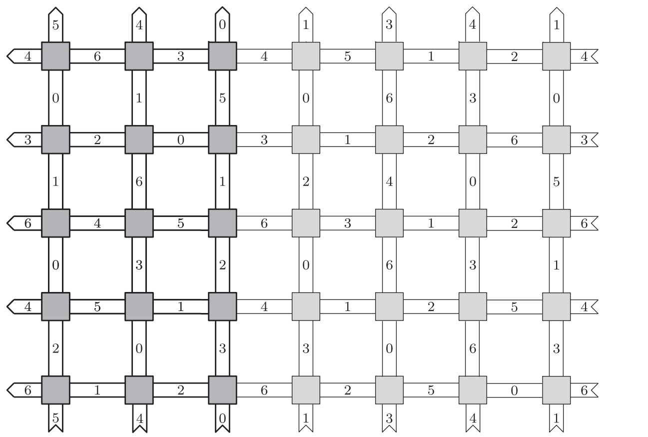


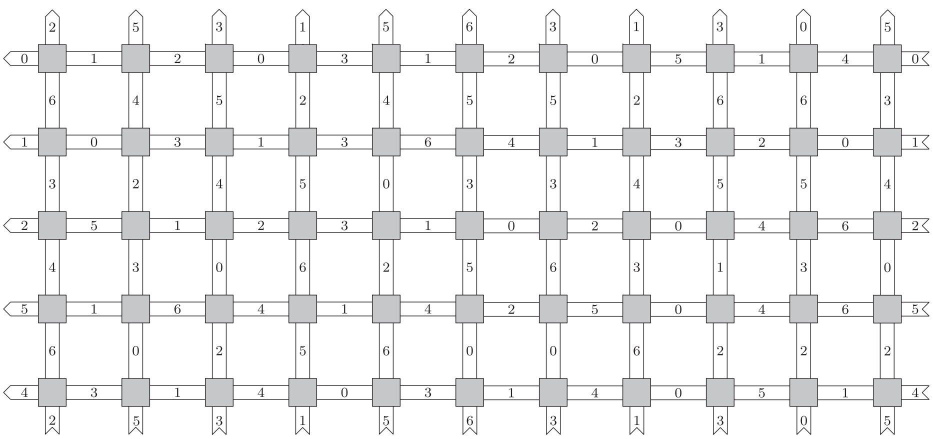
Fig. 3. A star 6-edge-coloring of $C_4 \,\square\, C_{10}$ combined of star edge-colorings of $C_4 \,\square\, C_4$ (lighter vertices) and $C_4 \,\square\, C_6$ (darker vertices).



(a) A star 7-edge-coloring of $C_5 \,\square\, C_5$



(b) A star 7-edge-coloring of $C_5 \,\square\, C_7$, including a star edge-coloring of $C_5 \,\square\, C_3$ (darker vertices)



(c) A star 7-edge-coloring of $C_5 \,\square\, C_{11}$

Fig. 4. Cartesian products of C_5 with cycles.

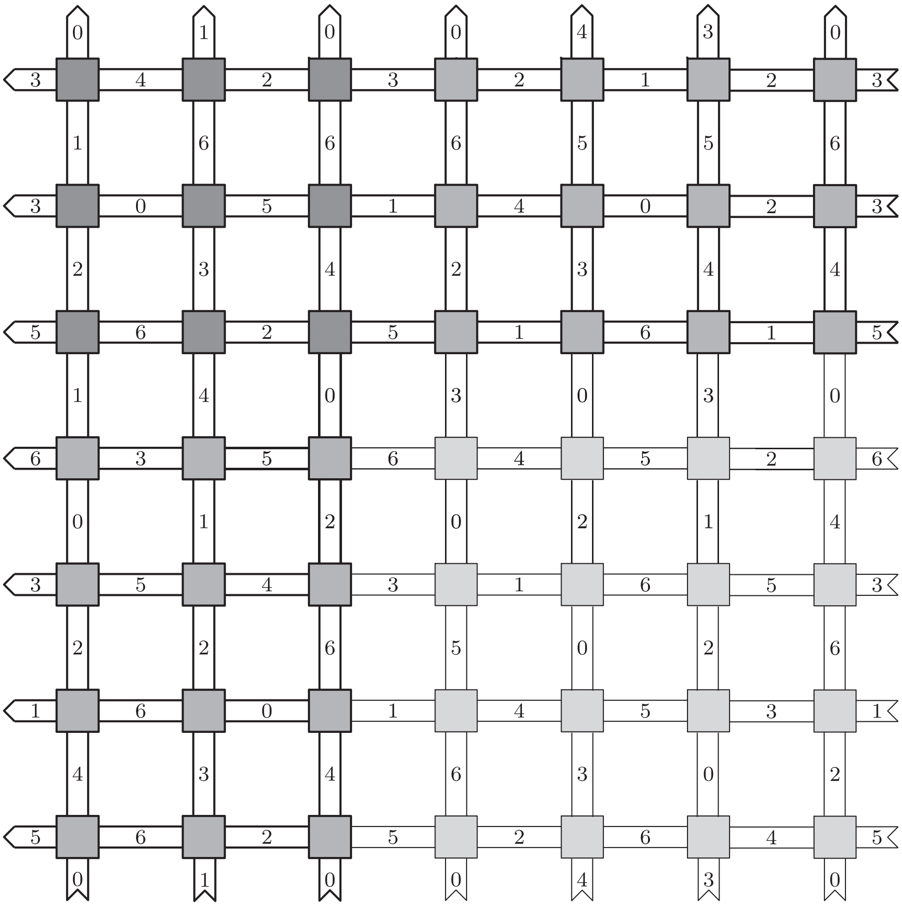


Fig. 5. A star 7-edge-coloring of $C_7 \square C_7$ including a star edge-coloring of $C_3 \square C_7$ (darker vertices in horizontal direction) and a star edge-coloring of $C_7 \square C_3$ (darker vertices in vertical direction).

Proof. We determined that $\chi'_{st}(C_8 \square C_9) \geq 7$ by exhaustive computer search. Namely, we generated all 147 non-isomorphic star 6-edge-colorings of C_9 and tried to extend each of them to the graph $C_8 \square C_9$. None of them could be extended, thus $\chi'_{st}(C_8 \square C_9) \geq 7$. The equality follows from Lemma 1 and the fact that $\chi'_{st}(C_4 \square C_9) = 7$. □

Finally, we give a general result, showing that 7 is the upper bound for the star chromatic index of the Cartesian products of any two cycles.

Theorem 11. *For every pair of positive integers m and n, where $3 \leq m \leq n$, we have*

$$\chi'_{st}(C_m \square C_n) \leq 7.$$

Proof. By Theorems 5–10, we have $\chi'_{st}(C_m \square C_n) \leq 7$ for $3 \leq m \leq 7$ and $n \geq 3$. Furthermore, by Lemma 1, we can use Theorem 7 to obtain a star 7-edge-coloring of $C_8 \square C_n$, Theorem 5 to obtain a star 7-edge-coloring of $C_9 \square C_n$, and Theorem 8 to obtain a star 7-edge-coloring of $C_{10} \square C_n$, for every $n \geq 3$. The star 7-edge-coloring of $C_{11} \square C_{11}$ is depicted in Fig. 6.

We complete the proof by showing that $\chi'_{st}(C_m \square C_n) \leq 7$ if $m, n \geq 12$. Note that the star 7-edge-coloring of $C_7 \square C_7$ depicted in Fig. 5, includes a 7-edge-coloring of $C_3 \square C_7$ and a 7-edge-coloring of $C_7 \square C_3$. Furthermore, the latter two colorings include a common star 7-edge-coloring of $C_3 \square C_3$. This fact enables us to use Lemma 2 and Theorem 2 to obtain $\chi'_{st}(C_m \square C_n) \leq 7$ for $m, n \geq 12$. □

The above results are summarized in Table 2.

4.3. Cartesian products of cycles and paths

In the last part of this section, we give results about the Cartesian products of paths and cycles. We begin with proving the cases for specific lengths of cycles.

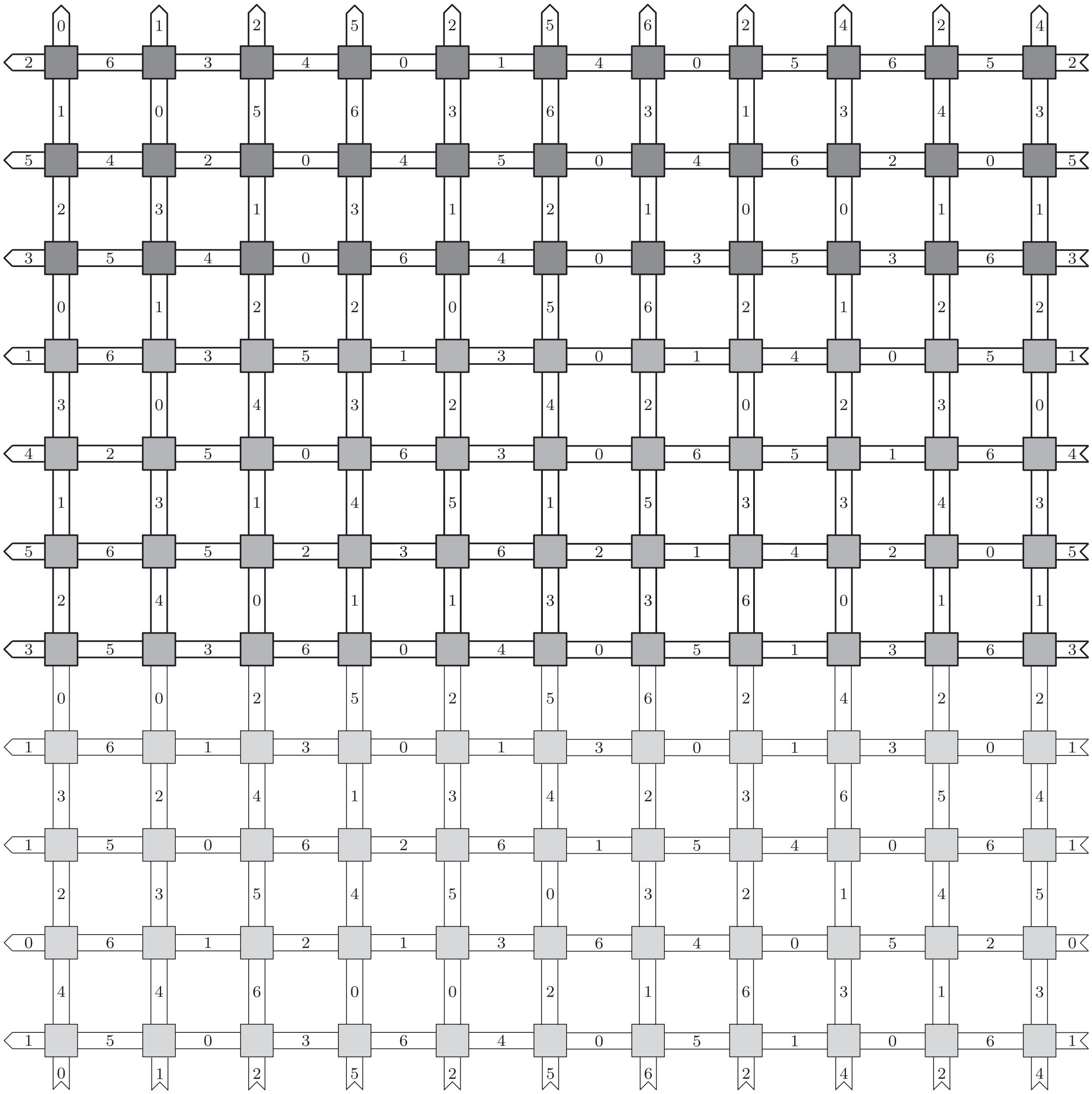


Fig. 6. A star 7-edge-coloring of $C_{11} \square C_{11}$ including a star edge-coloring of $C_7 \square C_{11}$ (all darker vertices), which furthermore includes a star edge-coloring of $C_3 \square C_{11}$ (the darkest vertices above).

Table 2
The star chromatic index of the Cartesian products of cycles $\chi'_{st}(C_m \square C_n)$. In red, we denote the cases, where the exact bounds are not established yet. The value 7^- means that the exact value of the star chromatic index is either 6 or 7.

$m \backslash n$	3	4	5	6	7	8	9	10	11	12
3	6	7	7	6	7	7	6	7	7	6
4	7	6	7	6	7	6	7	6	7	6
5	7	7	7	7	7	7	7	7	7	7
6	6	6	7	6	7	6	6	7	7	6
7	7	7	7	7	7	7	7	7	7	7
8	7	6	7	6	7	6	7	6	7^-	6
9	6	7	7	6	7	7	6	7^-	7^-	6
10	7	6	7	7	7	6	7^-	7^-	7^-	6
11	7	7	7	7	7	7^-	7^-	7^-	7^-	7^-
12	6	6	7	6	7	6	6	6	7^-	6

Theorem 12. *For every pair of integers k and n, where $k \geq 2$ and $n \geq 3$, we have*

$$\chi'_{st}(C_{2k} \,\square\, P_n) = 6.$$

Proof. By Corollary 2, we have $\chi'_{st}(C_{2k} \,\square\, P_n) \geq 6$. On the other hand, 6 colors also suffice by Theorem 6, since $C_{2k} \,\square\, P_n$ is a subgraph of $C_{2k} \,\square\, C_{4\ell}$ for every $\ell \geq n/4$. □

Theorem 13. *For every pair of integers k and n, where $n \geq 3$, we have*

$$\chi'_{st}(C_{3k} \,\square\, P_n) = 6.$$

Proof. By Corollary 2, we have $\chi'_{st}(C_{3k} \,\square\, P_n) \geq 6$. On the other hand, 6 colors also suffice by Corollary 3, since $C_{3k} \,\square\, P_n$ is a subgraph of $C_{3k} \,\square\, C_{3\ell}$ for every $\ell \geq n/3$. □

Theorem 14. *For every integer n, where $n \geq 3$, we have*

$$\chi'_{st}(C_5 \,\square\, P_n) = \begin{cases} 6, & \text{if } n \in \{3, 4, 5, 6\}, \\ 7, & \text{if } n \geq 7. \end{cases}$$

Proof. Let $G = C_5 \,\square\, P_n$ for some integer $n \geq 3$. Suppose first that $n \in \{3, 4, 5, 6\}$. By Theorem 4, we have $\chi'_{st}(P_3 \,\square\, P_5) = 6$, and thus, since $P_5 \,\square\, P_3$ is a subgraph of G, it follows that $\chi'_{st}(G) \geq 6$. On the other hand, in Fig. 7(a), we give a star 6-edge-coloring of $C_5 \,\square\, P_6$, hence establishing $\chi'_{st}(C_5 \,\square\, P_n) = 6$ for every $n \in \{3, 4, 5, 6\}$. Now, suppose that $n \geq 7$. Using Algorithm 1, we infer that $\chi'_{st}(C_5 \,\square\, P_n) \geq 7$. The upper bound $\chi'_{st}(C_5 \,\square\, P_n) \leq 7$ follows from the fact that $\chi'_{st}(C_5 \,\square\, C_{5k}) = 7$ for every positive integer k (see Theorem 8 and Fig. 4(a)). □

Theorem 15. *For every integer n, where $n \geq 3$, we have*

$$\chi'_{st}(C_7 \,\square\, P_n) = \begin{cases} 6, & \text{if } n \in \{3, 4, 5, 6\}, \\ 7, & \text{if } n \geq 7. \end{cases}$$

Proof. Let $G = C_7 \,\square\, P_n$ for some integer $n \geq 3$. Suppose first that $n \in \{3, 4, 5, 6\}$. Since $\chi'_{st}(P_3 \,\square\, P_5) = 6$, we again have $\chi'_{st}(G) \geq 6$. On the other hand, in Fig. 7(b), we give a star 6-edge-coloring of G, and hence $\chi'_{st}(G) = 6$ for every $n \in \{3, 4, 5, 6\}$. Suppose now that $n \geq 7$. Using Algorithm 1, we infer that $\chi'_{st}(C_7 \,\square\, P_7) = 7$, and hence $\chi'_{st}(G) \geq 7$. The equality is established by Fig. 1(d), where a pattern for a star 7-edge-coloring of $C_7 \,\square\, C_{3k}$ is presented. Since G is a subgraph of $C_7 \,\square\, C_{3k}$, for k large enough, the statement follows. □

We now turn our attention to the Cartesian products of cycles and paths P_m, for $m \in \{2, 3, 4\}$.

Theorem 16. *For every integer m, where $m \geq 3$, we have*

$$\chi'_{st}(C_m \,\square\, P_2) = \begin{cases} 6, & \text{if } m = 3, \\ 4, & \text{if } m \equiv 0 \bmod 4, \\ 5, & \text{otherwise.} \end{cases}$$

Proof. For $m = 3$, recall that $\chi'_{st}(C_3 \,\square\, P_2) = 6$. If $m \equiv 0 \bmod 4$, then the graph $C_m \,\square\, P_2$ covers the graph Q_3, and hence its star chromatic index equals 4 by Theorem 3. Finally, if $m \not\equiv 0 \bmod 4$, by Theorem 3, we have $\chi'_{st}(C_m \,\square\, P_2) \geq 5$. The equality follows from Theorem 7 in [11]. □

Theorem 17. *For every pair of integers m and n, where $m \geq 3$ and $n \in \{3, 4, 5, 6\}$, we have*

$$\chi'_{st}(C_m \,\square\, P_n) = 6.$$

Proof. First, recall that 6 colors are needed in all the cases by Corollary 2. Next, since $C_m \,\square\, P_6$ contains all the graphs $C_m \,\square\, P_n$, for $n \in \{3, 4, 5\}$, it suffices to show that there exists a star 6-edge-coloring of $C_m \,\square\, P_6$.

By Lemma 1, Theorems 5 and 6, we have $\chi'_{st}(C_{3k} \,\square\, P_6) = \chi'_{st}(C_{4k} \,\square\, P_6) = 6$, for any positive integer k. Similarly, the star 6-edge-colorings depicted in Fig. 7(a) and 7(b) together with Lemma 1 imply $\chi'_{st}(C_{5k} \,\square\, P_6) = \chi'_{st}(C_{7k} \,\square\, P_6) = 6$. Furthermore, using the star 6-edge-coloring of $C_{10} \,\square\, P_6$ depicted in Fig. 7(c), which includes a star 6-edge-coloring of $C_3 \,\square\, P_6$, we obtain $\chi'_{st}(C_m \,\square\, P_6) = 6$ for every $m \in \{10, 13, 16\}$. Moreover, together with Lemma 2 and Theorem 2, this coloring implies $\chi'_{st}(C_m \,\square\, P_6) = 6$ for every $m \geq 18$. For the remaining two cases, namely $m \in \{11, 17\}$, the corresponding colorings are depicted in Fig. 8(a) and 8(b) (note that, in fact, we have even more, namely, we give a star 6-edge-coloring of $C_{11} \,\square\, P_8$ and a star 6-edge-coloring of $C_{14} \,\square\, P_8$ which includes a star 6-edge-coloring of $C_3 \,\square\, P_8$). □

We collect known and our new results in Table 3.

5. Conclusion

In this paper, we have established tight upper bounds for the Cartesian product of cycles and paths, and the Cartesian products of two cycles. We proved that 7 colors always suffice for star edge-colorings of these graphs, which is, in a way, a surprising bound, especially because at least 6 colors are always needed as soon as one of the factors is not isomorphic to

Table 3
The star chromatic index of the Cartesian products of cycles and paths $\chi'_{st}(C_m \square P_n)$. In red, we denote the cases, where the exact bounds are not established yet. The value 7^- means that the exact value of the star chromatic index is either 6 or 7.

$m \backslash n$	2	3	4	5	6	7	8	9^+
3	6	6	6	6	6	6	6	6
4	4	6	6	6	6	6	6	6
5	5	6	6	6	6	7	7	7
6	5	6	6	6	6	6	6	6
7	5	6	6	6	6	7	7	7
8	4	6	6	6	6	6	6	6
9	5	6	6	6	6	6	6	6
10	5	6	6	6	6	6	6	6
11	5	6	6	6	6	6	6	7^-
12	4	6	6	6	6	6	6	6
13	5	6	6	6	6	7^-	7^-	7^-
14	5	6	6	6	6	6	6	6
15	5	6	6	6	6	6	6	6
16	4	6	6	6	6	6	6	6
17	5	6	6	6	6	6	6	7^-
18	5	6	6	6	6	6	6	6
19^+ and $m \equiv 0 \bmod 4$	4	6	6	6	6	6	6	6
19^+ and $m \equiv r \bmod 12$, $r \in \{2, 3, 6, 9, 10\}$	5	6	6	6	6	6	6	6
19^+ and $m \equiv r \bmod 12$, $r \in \{1, 5, 7, 11\}$	5	6	6	6	6	7^-	7^-	7^-

P_2. Although we improved existing bounds and proved a number of exact values, there are still some open questions. We are very confident that the following conjecture is true.

Conjecture 1. *There exist constants K_1 and K_2 such that for every pair of integers m and n, where $m \geq K_1$ and $n \geq K_2$, we have*

$$\chi'_{st}(C_m \square P_n) = 6.$$

Note that Conjecture 1 is equivalent to the following.

Conjecture 2. *There exists a constant K such that for every integer m, where $m \geq K$, there exists an integer n such that*

$$\chi'_{st}(C_m \square C_n) = 6.$$

In fact, we believe (with a bit lower confidence) that the following stronger version of Conjecture 1 can also be confirmed.

Conjecture 3. *There exists a constant L such that for every pair of integers m and n, where $m, n \geq L$, we have*

$$\chi'_{st}(C_m \square C_n) = 6.$$

The above conjectures seem to be challenging, since we were not able to observe any straightforward pattern in colorings of different Cartesian products, despite the help of computer in our constructions.

There are also some (maybe) less complicated open questions regarding the Cartesian product of the cycles C_m, $m \in \{11, 13, 17\}$, with paths of arbitrary lengths.

Question 2. What is the star chromatic index of $C_m \square P_n$ for $m \in \{11, 13, 17\}$ and $n \geq 7$?

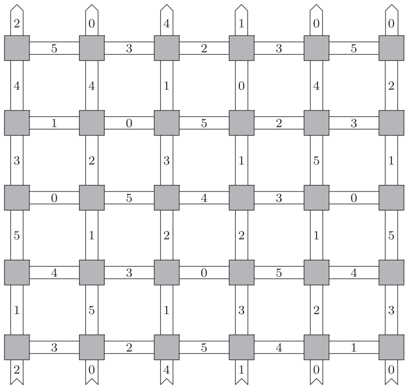
As we observed in Theorems 14 and 15, the star chromatic index increases for the Cartesian products of the cycles C_5 and C_7 with paths on at least 7 vertices. We expect this phenomenon will repeat also for some cycle C_m, where $m \in \{11, 13, 17\}$, although, in the case of C_{11} and C_{17}, we found star 6-edge-colorings of $C_{11} \square P_8$ and $C_{17} \square P_8$, which could indicate that 6 colors are sufficient in these cases.

To conclude, as in the case of complete graphs, also for the Cartesian products of paths and cycles (and two cycles), it is hard to determine their chromatic index with our current methods. However, it seems that the latter will be easier to resolve as the former.

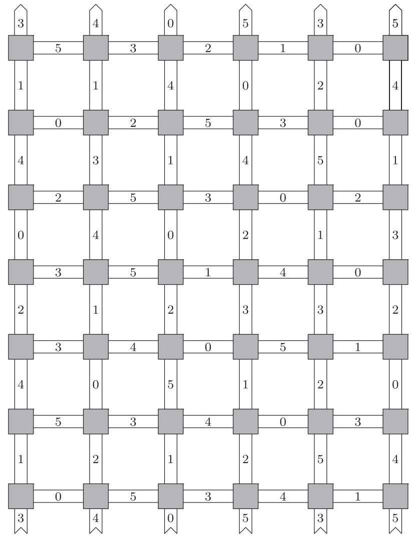
Acknowledgment

PH and MM acknowledge the financial support from the project GA20–09525S of the Czech Science Foundation. BL was partly supported by the Slovenian Research Agency Program P1–0383 and the project J1–1692. EM and RS were supported by the Slovak Research and Development Agency under the Contract No. APVV–15–0116 and by the Slovak VEGA Grant 1/0368/16.

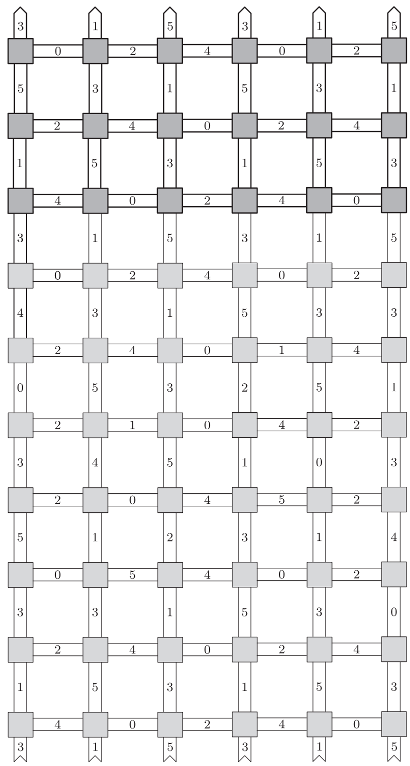
Appendix A. Cartesian products of two cycles



(a) A star 6-edge-coloring of $C_5 \,\square\, P_6$



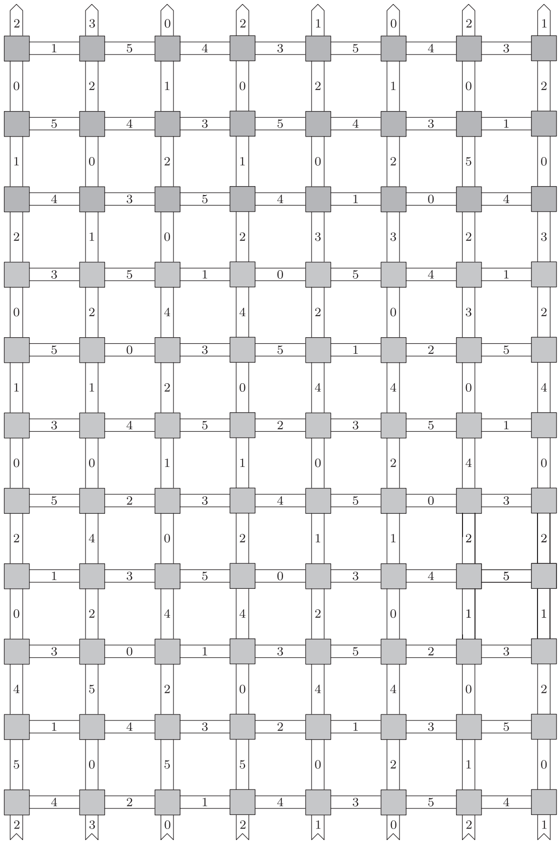
(b) A star 6-edge-coloring of $C_7 \,\square\, P_6$



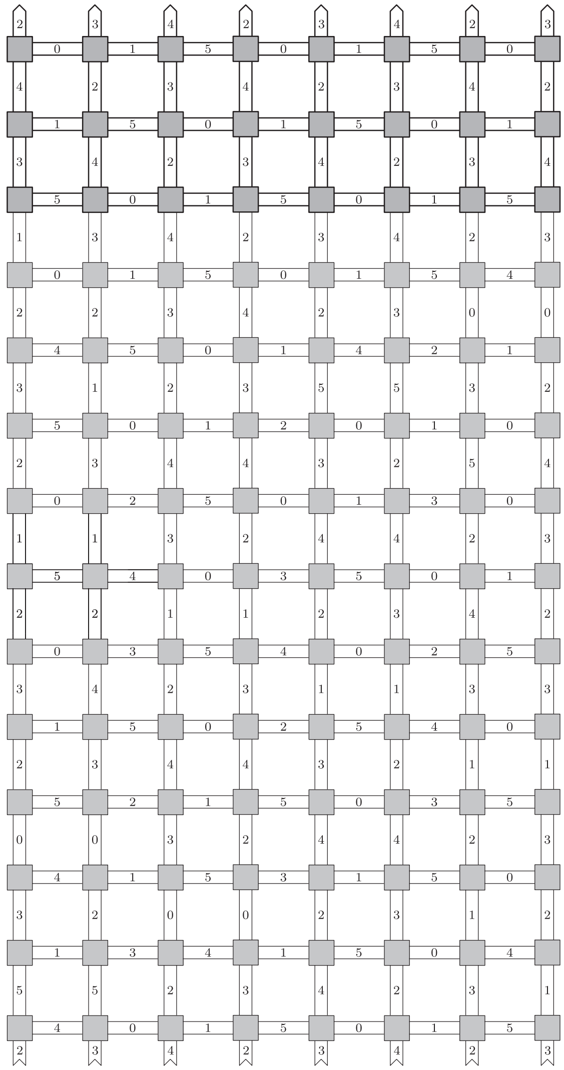
(c) A star 6-edge-coloring of $C_{10} \,\square\, P_6$ including a star edge-coloring of $C_3 \,\square\, P_6$ (darker vertices)

Fig. 7. Cartesian products of cycles and P_6.

Appendix B. Cartesian products of paths and cycles



(a) A star 6-edge-coloring of $C_{11} \,\square\, P_8$



(b) A star 6-edge-coloring of $C_{14} \,\square\, P_8$ including a star edge-coloring of $C_3 \,\square\, P_8$ (darker vertices)

Fig. 8. Cartesian products of cycles and P_8.

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