# Model Predictive Control with Linear Models

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*This article discusses the existing linear model predictive control concepts in a unified theoretical framework based on a stabilizing, infinite horizon, linear quadratic regulator. In order to represent unstable as well as stable multivariable systems, the standard state-space formulation is used for the plant model. The incorporation of a nominally stabilizing constrained regulator eliminates the current requirement of tuning for nominal stability. Output feedback is addressed in the well-established framework of the linear quadratic state-estimation problem. This framework allows the flexibility to handle nonsquare systems, noisy inputs and outputs, and nonzero input, output, and state disturbances. This formulation subsumes the integral control schemes designed to remove steady-state offset currently in industrial use. The on-line implementation of the controller requires the solution of a standard quadratic program that is no more computationally intensive than existing algorithms.*

## Introduction

Linear model predictive control refers to a class of control algorithms that compute a manipulated variable profile by utilizing a *linear* process model to optimize a linear or quadratic open-loop performance objective subject to linear constraints over a future time horizon. The first move of this open loop optimal manipulated variable profile is then implemented. This procedure is repeated at each control interval with the process measurements used to update the optimization problem.

This class of control algorithms, which is also referred to as receding horizon control or moving horizon control, has several advantages for application in chemical process control. The controller uses a linear transfer function, state space, or convolution plant model. These models can be obtained from process tests using time series analysis techniques that do not require a significant fundamental modeling effort. Multivariable processes can easily be handled by superposition of the linear models. Optimization of the open-loop performance objective is performed by either linear or quadratic programming algorithms. These algorithms are efficient and robust, which is essential for on-line applications. Constraints on the manipulated and controlled variables are incorporated into the performance objective optimization. This allows operation close to process constraints, which is necessary for economically optimal control of chemical processes.

A number of implementations of linear model predictive control have been developed by industry to address constrained, multivariable processes. The emphasis in the development of these controllers was a robust algorithm with acceptable performance that could be implemented on-line. Therefore, several aspects of these controllers were designed based on a heuristic approach with little theoretical justification. This produced controllers that performed very well for a specific class of plants, but were unable to adequately address others. These remarks are not intended to minimize the significance of the contribution made by industry. Were it not for the willingness of the process industries to develop and implement these approaches, there would be little need for a sound theoretical framework to further their development.

The industrial implementations began with model algorithmic control (MAC) developed by Richalet et al. (1978) and dynamic matrix control (DMC) developed by Cutler and Ramaker (1980). The implementation by Richalet et al. is also referred to as IDCOM. Linear dynamic matrix control (LDMC), which uses a linear objective function and incorporates constraints explicitly, is outlined by Morshedi et al. (1985). García and Morshedi (1986) discuss quadratic dynamic matrix control (QDMC), which is an extension of DMC incorporating a quadratic performance function and explicit incorporation of constraints. Grosdidier et al. (1988) present IDCOM-M, which is an extension of IDCOM using a quadratic programming algorithm to replace the iterative solution technique of the original implementation. Marquis and Broustail (1988) discuss Shell multivariable optimizing control (SMOC), which is a state-space implementation.

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In addition to the industrially developed controllers, there have been other implementations of linear model predictive control commonly cited in the literature. These include a constrained, multivariable algorithm similar to quadratic dynamic matrix control discussed by Ricker (1985) and receding horizon tracking control (RHTC) presented by Kwon and Byun (1989). Implementations developed for use in adaptive control include extended horizon adaptive control (EHAC) presented by Ydstie (1984), extended prediction self-adaptive control (EPSAC) presented by De Keyser and Van Cauwenberge (1985), and Generalized Predictive Control (GPC) presented by Clarke et al. (1987a).

A more complete discussion of model predictive control implementations is contained in the review articles by De Keyser et al. (1988), Byun and Kwon (1988), and García et al. (1989). These articles present comparisons of several of the implementations listed previously. The differences between these implementations are in the form of the linear model and performance objective, the choice of horizon, and the tuning parameters. However, these controllers all share the same general structure with many of the features that originated with the industrial controllers.

Impulse or step response models are used in several of these implementations including model algorithmic control and dynamic matrix control. The advantage of these convolution models is the ability to represent any stable dynamic response. One of the disadvantages is that unstable plants cannot be represented. Morari and Lee (1991) and Eaton and Rawlings (1992) present a finite step response model that can represent an integrating process. Integrating processes can also be represented by using an impulse or step response model for the derivative of the process dynamics. In order to use these implementations on an unstable plant, the plant must be modeled as an integrator with one of the approaches above. This imposes a limitation on the performance that can be achieved by the controller due to the structural error in the model.

A serious limitation to the model predictive controllers outlined above is that they must be tuned for *nominal* stability. The stability results available for these controllers require restrictions on either the tuning parameters or the plant models that can be considered. The following results are also limited to the unconstrained controller. Rouhani and Mehra (1982) discuss stability of model algorithmic control for stable systems. García and Morari (1982) discuss stability of dynamic matrix control in the framework of internal model control for stable systems. For the finite receding horizon linear quadratic regulator, Kwon and Byun (1989) and Bitmead et al. (1990) discuss sufficient conditions on the horizon length and terminal penalty weights to ensure stability. Clarke et al. (1987b), Clarke and Mohtadi (1989), and Clarke (1991) discuss stability of the Generalized Predictive Control algorithm by the choice of both tuning parameters and horizon length. Scattolini and Bittanti (1990) discuss stability of both generalized predictive control and extended horizon adaptive control by the choice of horizon length for stable systems. Byun and Kwon (1988) discuss sufficient conditions for stability of generalized predictive control and extended horizon adaptive control based on tuning parameters. Maurath et al. (1988) present a necessary condition for the stability of a SISO model predictive controller for stable systems.

For the constrained controller, there are fewer stability results. Gutman and Hagander (1985) present a stabilizing saturated linear state feedback controller. Zafiriou (1990) and Zafiriou and Marchal (1991) discuss the contraction properties of quadratic dynamic matrix control subject to output constraints. Sznaier and Damborg (1990) present a modified receding horizon formulation that is stable for certain classes of constraints. Rawlings and Muske (1992) present a constrained receding horizon regulator that is stabilizing for both stable and unstable plants for all choices of tuning parameters.

This article presents a model predictive controller formulation that addresses the stability and plant modeling issues discussed above. In order to represent unstable as well as stable plants, the state-space formulation is used as the plant model. The incorporation of the stabilizing constrained regulator design of Rawlings and Muske eliminates the requirement to tune for nominal stability. Output feedback is performed with the use of linear quadratic filtering theory. This allows flexibility in the design of the noise model for the system within a well-established framework that extends to the output feedback schemes of the industrial implementations. Target tracking and integral action in the controller are obtained by using results from standard linear quadratic regulatory theory.

## Receding Horizon Regulator Formulation

The discrete dynamical system model used by the controller is the state-space formulation shown below in which $y$ is the vector of outputs, $u$ is the vector of inputs, and $x$ is the vector of states.

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \quad k=0,\ 1,\ 2,\ \ldots \\ y_k &= Cx_k \end{aligned} \tag{1}$$

Discrete transfer function and convolution models are easily transformed into an equivalent discrete state-space model as discussed by Prett and García (1988). Li et al. (1989) discuss a reduced order state-space form of the step response convolution model. Dead time can be added to a state-space model with state augmentation as shown by Franklin and Powell (1980).

The receding horizon regulator is based on the minimization of the following infinite horizon open-loop quadratic objective function at time $k$.

$$\min_{u^N} \sum_{j=0}^{\infty} (y_{k+j}^T Q y_{k+j} + u_{k+j}^T R u_{k+j} + \Delta u_{k+j}^T S \Delta u_{k+j}) \tag{2}$$

$Q$ is a symmetric positive semidefinite penalty matrix on the outputs with $y_{k+j}$ computed from Eq.1. $R$ is a symmetric positive definite penalty matrix on the inputs in which $u_{k+j}$ is the input vector at time $j$ in the open-loop objective function. $S$ is a symmetric positive semidefinite penalty matrix on the rate of change of the inputs in which $\Delta u_{k+j} = u_{k+j} - u_{k+j-1}$ is the change in the input vector at time $j$. The vector $u^N$ contains the $N$ future open-loop control moves as shown below.

$$u^N = \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+N-1} \end{bmatrix} \tag{3}$$

At time $k+N$, the input vector $u_{k+j}$ is set to zero and kept at this value for all $j \geq N$ in the open-loop objective function value calculation.

The receding horizon regulator computes the vector $u^N$ that optimizes the open-loop objective function in Eq. 2. The first input value in $u^N$, $u_k$, is then injected into the plant. This procedure is repeated at each successive control interval with feedback incorporated by using the plant measurements to update the state vector at time $k$.

The infinite horizon open-loop objective function in Eq. 2 can be expressed as the finite horizon open-loop objective shown below.

$$\min_{u^N} \Phi_k = x_{k+N}^T \overline{Q} x_{k+N} + \Delta u_{k+N}^T S \Delta u_{k+N}$$

$$+ \sum_{j=0}^{N-1} (x_{k+j}^T C^T Q C x_{k+j} + u_{k+j}^T R u_{k+j} + \Delta u_{k+j}^T S \Delta u_{k+j}) \tag{4}$$

The output penalty term in Eq. 2 has been replaced with the corresponding state penalty term in Eq. 4. Determination of the terminal state penalty matrix, $\overline{Q}$, depends on the stability of the plant model.

### *Stable systems*

For stable systems, $\overline{Q}$ in Eq. 4 is defined as the infinite sum in Eq. 5.

$$\overline{Q} = \sum_{i=0}^{\infty} \mathrm{A}^{T^i} \mathrm{C}^T \mathrm{QC}\, \mathrm{A}^i \tag{5}$$

This infinite sum can be determined from the solution of the following discrete Lyapunov equation.

$$\overline{Q} = C^T Q C + A^T \overline{Q} A \tag{6}$$

There are standard methods available for the solution of this equation.

Straightforward algebraic manipulation of the quadratic objective presented in Eq. 4 results in the following quadratic program for $u^N$.

$$\min_{u^N} \Phi_k = (u^N)^T H u^N + 2(u^N)^T (G x_k - F u_{k-1}) \tag{7}$$

The matrices $H$, $G$, and $F$ are computed as shown below with $\overline{Q}$ determined from Eq. 6.

$$H = \begin{bmatrix} B^T\overline{Q}B + R + 2S & B^T A^T \overline{Q} B - S & \cdots & B^T A^{T^{N-1}} \overline{Q} B \\ B^T \overline{Q} A B - S & B^T \overline{Q} B + R + 2S & \cdots & B^T A^{T^{N-2}} \overline{Q} B \\ \vdots & \vdots & \ddots & \vdots \\ B^T \overline{Q} A^{N-1} B & B^T \overline{Q} A^{N-2} B & \cdots & B^T \overline{Q} B + R + 2S \end{bmatrix}$$

$$G = \begin{bmatrix} B^T\overline{Q}A \\ B^T\overline{Q}A^2 \\ \vdots \\ B^T\overline{Q}A^N \end{bmatrix}, \quad F = \begin{bmatrix} S \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

### *Unstable systems*

The discussion of unstable systems begins with partitioning the Jordan form of the $A$ matrix into stable and unstable parts in which the unstable eigenvalues of $A$ are contained in $J_u$.

$$A = VJV^{-1} = [V_u \; V_s] \begin{bmatrix} J_u & 0 \\ 0 & J_s \end{bmatrix} \begin{bmatrix} \tilde{V}_u \\ \tilde{V}_s \end{bmatrix} \tag{8}$$

The stable and unstable modes, $z^s$ and $z^u$ respectively, then satisfy the following relationships.

$$\begin{bmatrix} z^u \\ z^s \end{bmatrix} = \begin{bmatrix} \tilde{V}_u \\ \tilde{V}_s \end{bmatrix} x \tag{9}$$

$$\begin{bmatrix} z_{k+1}^u \\ z_{k+1}^s \end{bmatrix} = \begin{bmatrix} J_u & 0 \\ 0 & J_s \end{bmatrix} \begin{bmatrix} z_k^u \\ z_k^s \end{bmatrix} + \begin{bmatrix} \tilde{V}_u \\ \tilde{V}_s \end{bmatrix} B u_k \tag{10}$$

For unstable plants, the finite horizon open-loop objective function in Eq. 4 is subject to the following equality constraint on the unstable modes at time $k+N$.

$$z_{k+N}^u = \tilde{V}_u x_{k+N} = 0 \tag{11}$$

This equality constraint is required if the unstable modes are not brought to zero at time $k+N$, they evolve uncontrolled after this time and do not converge to zero. Therefore, the optimal solution to Eq. 4 must be a vector $u^N$ that zeroes the unstable modes at time $k+N$.

With the equality constraint ensuring that only the stable modes contribute to the value of $\Phi_k$ after time $k+N-1$, $\overline{Q}$ for unstable systems can be computed from the stable modes in a manner similar to Eq. 5.

$$\overline{Q} = \tilde{V}_s^T \Sigma \tilde{V}_s \tag{12}$$

$$\Sigma = \sum_{i=0}^{\infty} J_s^{T^i} V_s^T C^T Q C V_s J_s^i \tag{13}$$

The infinite sum in Eq. 13 can be obtained from the solution of the following discrete Lyapunov equation.

$$\Sigma = V_s^T C^T Q C V_s + J_s^T \Sigma J_s \tag{14}$$

For unstable plants, $u^N$ is then determined as the solution to the quadratic program in Eq. 7 subject to the equality constraint in Eq. 11. This equality constraint can be represented as the following matrix equation in $u^N$.

$$\tilde{V}_u [A^{N-1}B, A^{N-2}B, \ldots, B] u^N = -\tilde{V}_u A^N x_k \tag{15}$$

The matrices $H$ and $G$ in Eq. 7 for unstable systems consist of the sum of the contribution from the finite horizon terms in Eq. 4 and the contribution from the terminal state penalty on the stable modes. The contribution from the finite horizon terms, $H_1$ and $G_1$, is computed as shown below.

$$H_1=\begin{bmatrix} B^TK_{N-1}B+R+2S & B^TA^TK_{N-2}B-S & \cdots & B^TA^{T^{N-1}}K_0B \\ B^TK_{N-2}AB-S & B^TK_{N-2}B+R+2S & \cdots & B^TA^{T^{N-2}}K_0B \\ \vdots & \vdots & \ddots & \vdots \\ B^TK_0A^{N-1}B & B^TK_0A^{N-2}B & \cdots & B^TK_0B+R+2S \end{bmatrix}$$

$$G_1=\begin{bmatrix} B^TK_{N-1}A \\ \vdots \\ B^TK_0A^N \end{bmatrix}, \quad K_N=\sum_{i=0}^{N} A^{T^i}QA^i$$

Computation of the contribution from the terminal state penalty, $H_2$ and $G_2$, is shown below with $\overline{Q}$ determined from Eq. 12.

$$H_2=\begin{bmatrix} B^TL_{N-1}B & B^TA^TL_{N-2}B & \cdots & B^TA^{T^{N-1}}L_0B \\ B^TL_{N-2}AB & B^TL_{N-2}B & \cdots & B^TA^{T^{N-2}}L_0B \\ \vdots & \vdots & \ddots & \vdots \\ B^TL_0A^{N-1}B & B^TL_0A^{N-2}B & \cdots & B^TL_0B \end{bmatrix}$$

$$G_2=\begin{bmatrix} B^TL_{N-1}A \\ \vdots \\ B^TL_0A^N \end{bmatrix}, \quad L_N=A^{T^N}\overline{Q}A^N$$

The matrix $F$ in Eq. 7 for unstable systems is the same as that presented for stable systems.

Implementation of the receding horizon regulator based on the quadratic program in Eqs. 7 and 15 requires feasibility of the equality constraint for an optimal solution to exist. Therefore, the regulator must be restricted to stabilizable systems with $N \geq r$, in which $r$ is the number of unstable modes in the system. This ensures that the equality constraint is feasible for every $x_k$.

If the system is not stabilizable, then there exist uncontrollable unstable modes that cannot be brought to zero. If the number of control moves is less than the number of unstable modes, then the unstable modes cannot all be brought to zero from an arbitrary initial condition. Both of these cases will result in infeasibility of the equality constraint in Eq. 15, which allows the regulator to detect that the system cannot be stabilized.

### *Nominal stability of the infinite horizon regulator*

Muske and Rawlings (1992) show that this regulator formulation guarantees nominal stability for all choices of tuning parameters satisfying the conditions outlined in the previous sections. Nominal stability comes from the evaluation of the state penalty on an infinite horizon even though there are a finite number of decision variables. Previous model predictive controller formulations are finite horizon. The absence of nominal stability in these implementations is a direct consequence of the finite horizon formulation of the control algorithm. Bitmead et al. (1990) demonstrate that nominal stability cannot be guaranteed for a finite receding horizon regulator.

Kwon and Pearson (1978) propose a nominally stabilizing receding horizon regulator based on a finite horizon objective subject to a terminal state constraint. The terminal constraint forces all of the modes of the system to be zero at the end of the horizon instead of only the unstable modes. This constraint leads to aggressive control action with small values of $N$ for both stable and unstable systems since the regulator approaches a deadbeat controller. Feasibility of this terminal constraint also requires that the system be completely controllable. The example below demonstrates the limitations imposed by this stronger controllability condition.

*Example 1.* Consider the isothermal CSTR example presented by Ray (1981) with the following irreversible first-order reactions.

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

It is required to control the concentration of $B$ in the reactor, $C^B$, by adjusting the inlet concentration of $B$, $C^{B_f}$. The discrete time modeling equations are presented below in which $\theta$ is the residence time of the reactor and $\Delta$ is the sample time. It is assumed that $k_1 \neq k_2$.

$$\begin{bmatrix} C^A_{k+1} \\ C^B_{k+1} \end{bmatrix} = \begin{bmatrix} e^{-\Delta(\theta^{-1}+k_1)} & 0 \\ \dfrac{k_1}{k_2-k_1}e^{\Delta(k_2-k_1)} & e^{-\Delta(\theta^{-1}+k_2)} \end{bmatrix} \begin{bmatrix} C^A_k \\ C^B_k \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{\theta^{-1}}{\theta^{-1}+k_2}(1-e^{-\Delta(\theta^{-1}+k_2)}) \end{bmatrix} C^{B_f}$$

The controllability matrix of this equation is given below.

$$\frac{\theta^{-1}}{\theta^{-1}+k_2}\begin{bmatrix} 0 & 0 \\ (1-e^{-\Delta(\theta^{-1}+k_2)}) & (1-e^{-\Delta(\theta^{-1}+k_2)})e^{-\Delta(\theta^{-1}+k_2)} \end{bmatrix}$$

Since the controllability matrix is singular, the system is not completely controllable and the approach of Kwon and Pearson cannot be used. However, the uncontrollable mode is stable. The system is therefore stabilizable and the regulator presented in this article can be implemented on this example.

## Constraints

Input and output constraints of the following form are considered.

$$u_{\min} \leq u_{k+j} \leq u_{\max}, \quad j=0,\ 1,\ \ldots,\ N-1 \tag{16}$$

$$y_{\min} \leq y_{k+j} \leq y_{\max}, \quad j=j_1,\ j_1+1,\ \ldots,\ j_2 \tag{17}$$

$$\Delta u_{\min} \leq \Delta u_{k+j} \leq \Delta u_{\max}, \quad j=0,\ 1,\ \ldots,\ N \tag{18}$$

The output constraints are applied from time $k+j_1$, $j_1 \geq 1$, through time $k+j_2$, $j_2 \geq j_1$. The value of $j_2$ is chosen such that feasibility of the output constraints up to time $k+j_2$ implies feasibility of these constraints on the infinite horizon. The value of $j_1$ is chosen such that the output constraints are feasible at time $k$. The constrained regulator will remove the output constraints at the beginning of the horizon up to time $k+j_1$ in order to obtain feasible constraints and a solution to the quadratic program. Rawlings and Muske (1992) show the existence of finite values for both $j_1$ and $j_2$.

Equations 16, 17, and 18 can be expressed as the following constraint on $u^N$.

$$\begin{bmatrix} I \\ -I \\ D \\ -D \\ W \\ -W \end{bmatrix} u^N \leq \begin{bmatrix} i_1 \\ i_2 \\ d_1 \\ d_2 \\ w_1 \\ w_2 \end{bmatrix} \tag{19}$$

The matrices $D$ and $W$ are computed as shown below with $A^{j-i}$ defined to be 0 for all $j<i$.

$$D=\begin{bmatrix} A^{j_1-1}B & \cdots & A^{j_1-N}B \\ \vdots & & \vdots \\ A^{j_2-1}B & \cdots & A^{j_2-N}B \end{bmatrix}, \quad W=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 1 \\ 0 & \cdots & 0 & -1 \end{bmatrix}$$

The values of the right-hand side vectors in Eq. 19 are the following.

$$i_1=\begin{bmatrix} u_{\max} \\ \vdots \\ u_{\max} \end{bmatrix}, \quad i_2=\begin{bmatrix} -u_{\min} \\ \vdots \\ -u_{\min} \end{bmatrix},$$

$$d_1=\begin{bmatrix} y_{\max}-CA^{j_1}x_k \\ \vdots \\ y_{\max}-CA^{j_2}x_k \end{bmatrix}, \quad d_2=\begin{bmatrix} -y_{\min}+CA^{j_1}x_k \\ \vdots \\ -y_{\min}+CA^{j_2}x_k \end{bmatrix},$$

$$w_1=\begin{bmatrix} \Delta u_{\max}+u_{k-1} \\ \Delta u_{\max} \\ \vdots \\ \Delta u_{\max} \end{bmatrix}, \quad w_2=\begin{bmatrix} \Delta u_{\min}-u_{k-1} \\ \Delta u_{\min} \\ \vdots \\ \Delta u_{\min} \end{bmatrix}$$

In order to ensure that a consistent constraint set is specified, the following restrictions are imposed on the constraints. These restrictions guarantee feasibility of the origin.

$$\begin{bmatrix} u_{\max} \\ -u_{\min} \\ y_{\max} \\ -y_{\min} \\ \Delta u_{\max} \\ \Delta u_{\min} \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

## *Nominal stability of the constrained regulator*

Muske and Rawlings (1992) prove that the feasibility of the quadratic program in Eqs. 7 and 19 for stable systems or Eqs. 7, 15 and 19 for unstable systems guarantees nominal stability of the constrained receding horizon regulator. For stable systems, the input constraints are feasible independent of $x_k$ and the output constraints can be made feasible by the choice of $j_1$. Since feasibility implies stability of the regulator, this formulation relaxes the output constraints at the beginning of the horizon to retain feasibility and, therefore, stability of the constrained regulator. The importance of specifying the output constraints on the infinite horizon is demonstrated in the following example.

*Example 2.* Consider the SISO plant with the following discrete transfer function that has an unstable zero at $z=3/2$.

$$G(z)=\frac{-2z+3}{3z^2-4z+2}$$

A minimal state-space realization of the discrete transfer function is shown below.

$$A=\begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix}, \quad B=\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C=[-2/3 \quad 1]$$

In this example, the input is unconstrained with the following regulator tuning parameters.

$$Q=1, \quad R=1, \quad S=0, \quad N=5$$

The output target is zero with a maximum output constraint of 0.5. At time $k=0$, a state disturbance of $[3, 3]^T$ enters the system. This results in a disturbance of unity magnitude in the output. The figures below demonstrate the unconstrained and constrained responses for both the finite horizon and infinite horizon regulators.

With the finite horizon regulator, the output constraint is enforced $N$ sample periods into the future at each execution. Forcing the output to meet this constraint causes the controller to invert the unstable zero of the plant. The input then increases without bound as the output remains at the maximum constraint value as shown in Figures 1 and 2. There are no choices of the regulator tuning parameters, $N$, $Q$, $R$, and $S$, that can eliminate the instability in this example. Note that the unconstrained regulator is stable. Further examples of instability due to output constraints with the QDMC algorithm are presented by Zafiriou (1990) and Zafiriou and Marchal (1991).

Enforcing the constraint on the infinite horizon results in the stable response shown in Figures 3 and 4. The constraint is infeasible at time $k=0$ for $j_1=1$ due to the limitation on the speed of response imposed by the nonminimum phase plant. To achieve feasibility at time $k=0$, $j_1$ must be increased to 2. The constraint is then violated at time $k=1$. After this time, the constraint is feasible for $j_1=1$ and it is enforced for all $k\geq 2$.

As shown in Figure 3, the magnitude of the constraint violation at time $k=1$ is greater for the constrained regulator than for the unconstrained regulator. Although the constraint is violated for only one sample period, the magnitude of the violation may not be acceptable. A method for influencing the magnitude of the constraint violation is to minimize a weighted norm of the violation as discussed by Ricker et al. (1988). However, this procedure is not stabilizing. In this formulation, the magnitude of the constraint violations are influenced by the value of $j_1$. As shown in Figure 5, the magnitude of the

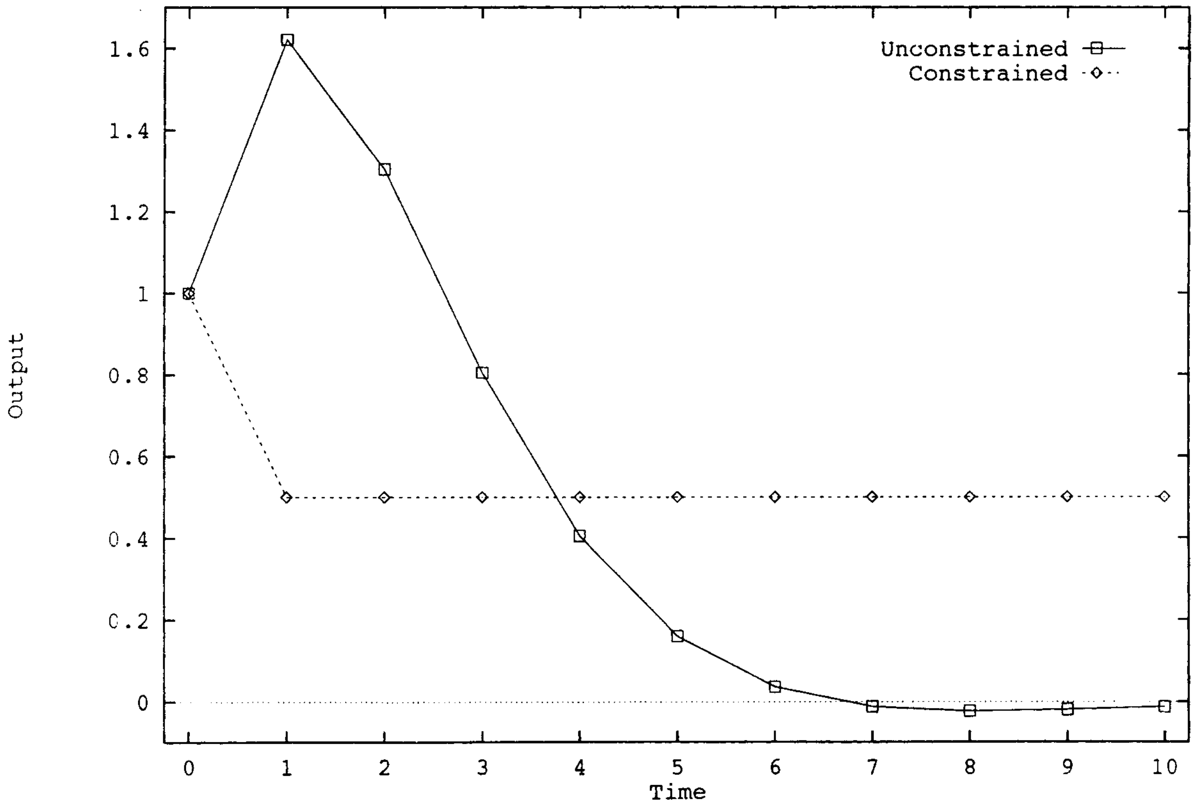


**Figure 1. Output response for the finite horizon controller.**

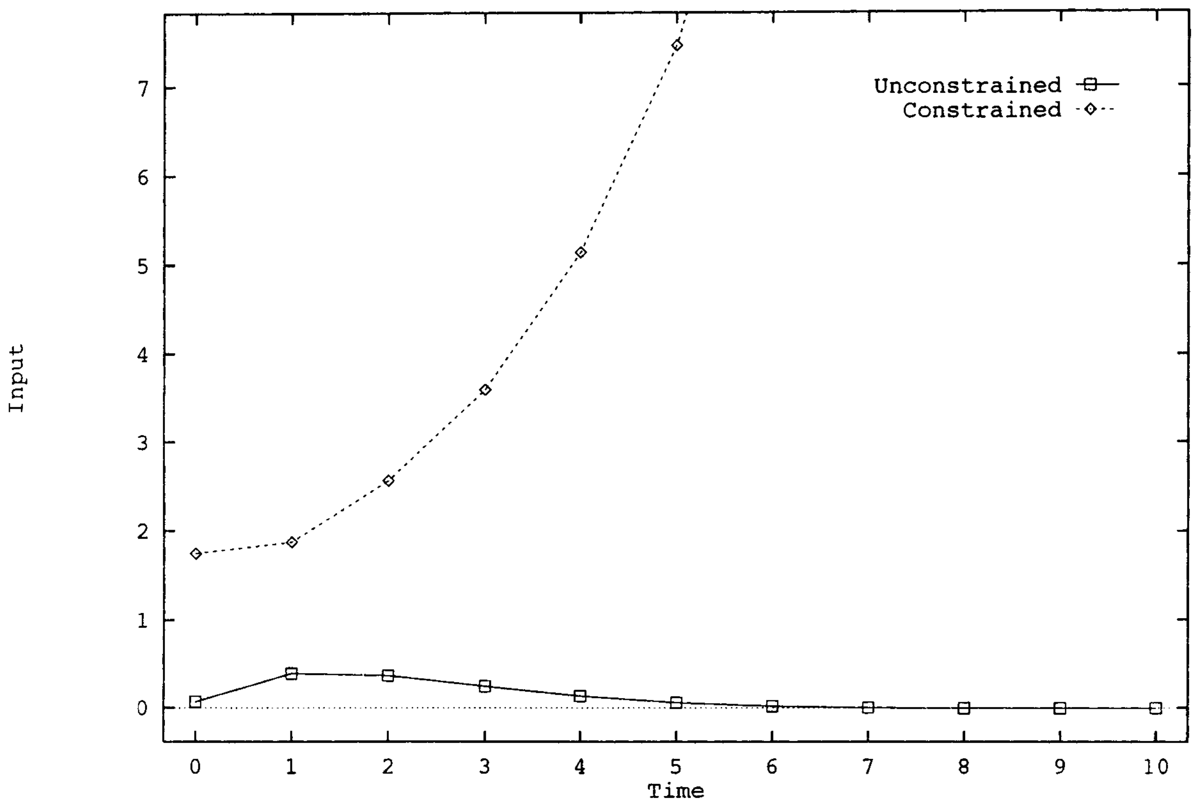


**Figure 2. Input response for the finite horizon controller.**

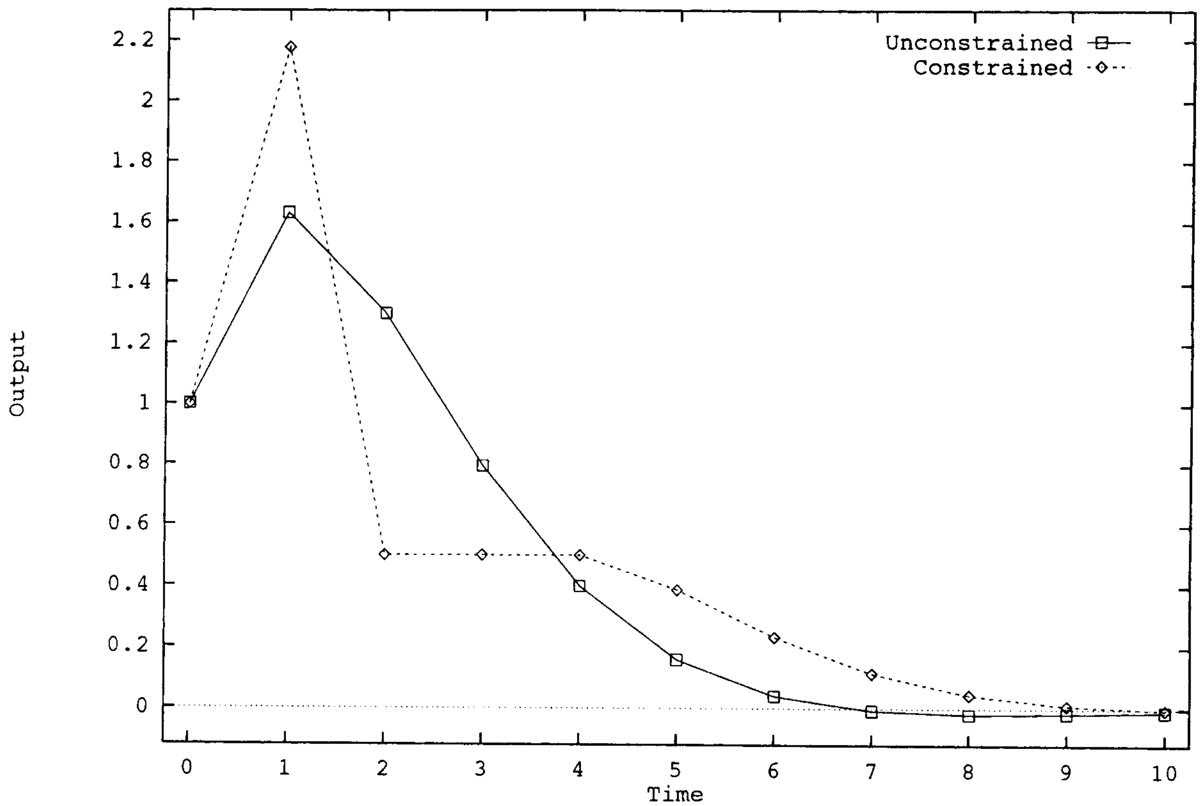


**Figure 3. Output response for the infinite horizon controller.**

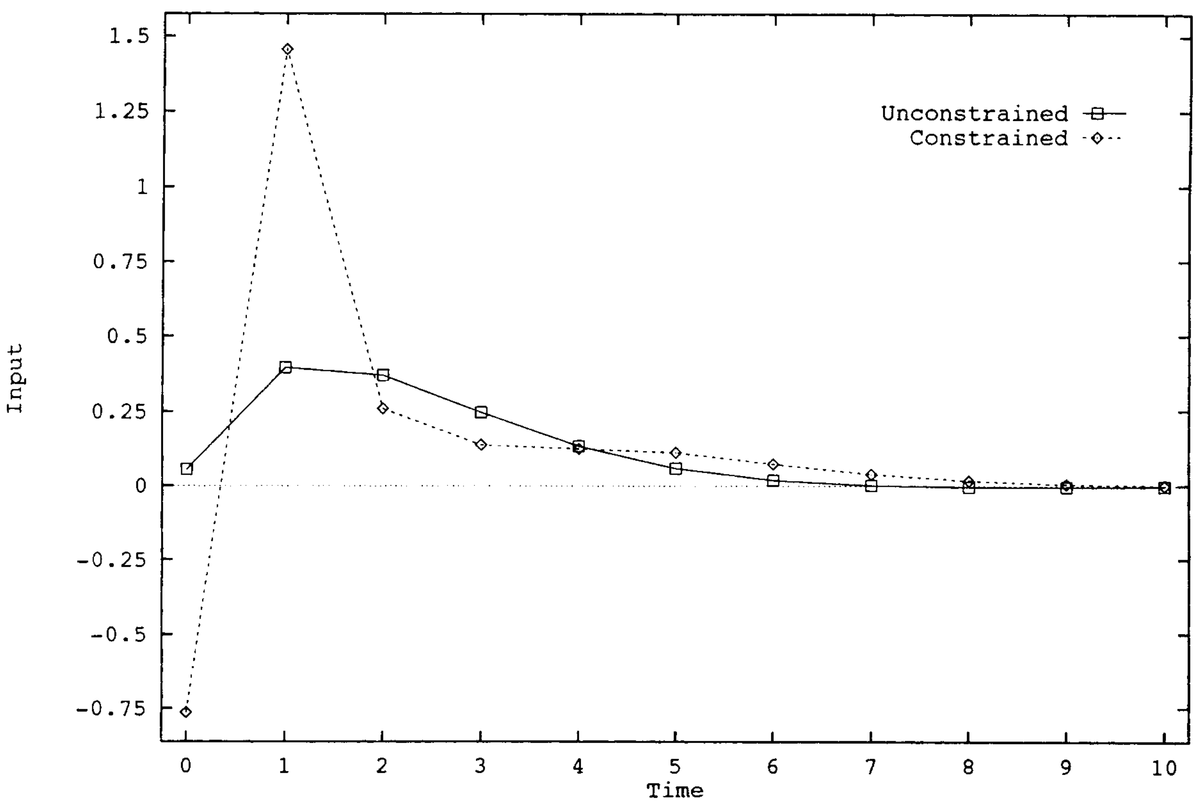


**Figure 4. Input response for the infinite horizon controller.**

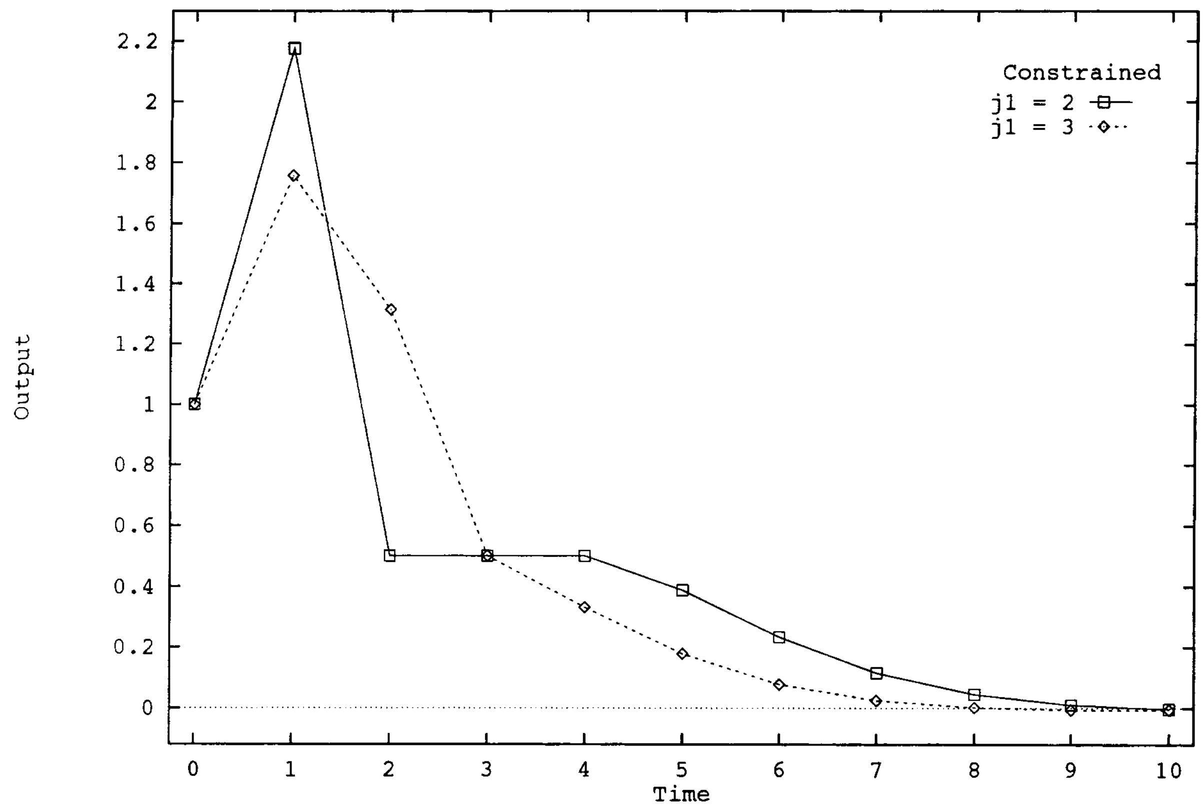


**Figure 5. Output response for the infinite horizon regulator varying $j_1$.**

constraint violation can be decreased by increasing the value of $j_1$ from 2 to 3 at time $k=0$. This results in the constraint being violated for two sample periods instead of only a single sample period, but the magnitude of the violations are reduced. This procedure guarantees stability of the constrained regulator for all choices of $j_1$ that result in a feasible quadratic program.

### *Constrained stabilizability*

For unstable systems, the constraints in Eqs. 15 and 19 may not be feasible. If the input constraints in Eqs. 16 and 18 are too restrictive for a given initial condition and value of $N$, it will not be possible to zero the unstable modes of a stabilizable system at time $k+N$. Since the input constraints represent physical limits on the plant and cannot be changed arbitrarily, feasibility can only be achieved by increasing $N$. However, a bounded value of $N$ that makes the constraints feasible does not always exist. If the unstable modes grow faster than the constrained input can reduce them at each time $k$, then there is no bounded value of $N$ that can stabilize the system. In this case, the system is not constrained stabilizable.

A system is constrained stabilizable if the unstable modes can be brought asymptotically to the origin by an admissible input sequence. When a stabilizable system is not constrained stabilizable, there are unstable modes that cannot be controlled by any regulator. This has the same implications as an unstabilizable system. Infeasibility of the equality constraint in Eq. 15 allows the constrained regulator presented in this article to detect that a system is not constrained stabilizable.

Constrained stabilizability is a function of the plant, input constraints, and initial state. Since it depends only on these factors, there are options available to stabilize the system. These include increasing the manipulated variable action, decreasing the operating range, and decreasing the magnitude of disturbances entering the system. If none of these are possible, the plant must be redesigned to be stabilizable.

*Example 3.* Consider the nonisothermal, nonadiabatic CSTR example with an irreversible first-order reaction presented by Uppal et al. (1974). The dimensionless modeling differential equations are shown below where $\chi_1$ is the conversion, $\chi_2$ is the reactor temperature, $\beta$ is the heat-transfer coefficient, $\chi_{2c}$ is the heat-transfer medium temperature, $B$ is the heat of reaction, and $Da$ is the Damköhler number.

$$\frac{d\chi_1}{dt} = -\chi_1 + Da(1-\chi_1)e^{\chi_2}$$

$$\frac{d\chi_2}{dt} = -\chi_2 + Da(1-\chi_1)Be^{\chi_2} - \beta(\chi_2 - \chi_{2c})$$

The parameter values used in this example are taken from Patwardhan et al. (1990) and result in an open-loop unstable steady state. A SISO discrete linear system is obtained from the CSTR model above by linearization of the modeling equations about the unstable steady state with 0.1 as the sampling interval, $\beta$ as the manipulated variable, and $\chi_2$ as the controlled variable. The linearized model is used as the plant in the following discussion. This plant has an unstable pole at $z=1.166$. A minimal state-space realization is shown below.

$$A=\begin{bmatrix} 1.0759 & 0.1382 \\ 0.1036 & 1.0068 \end{bmatrix}, \quad B=\begin{bmatrix} 0.1036 \\ 0.0051 \end{bmatrix}, \quad C=[-3 \quad -6]$$

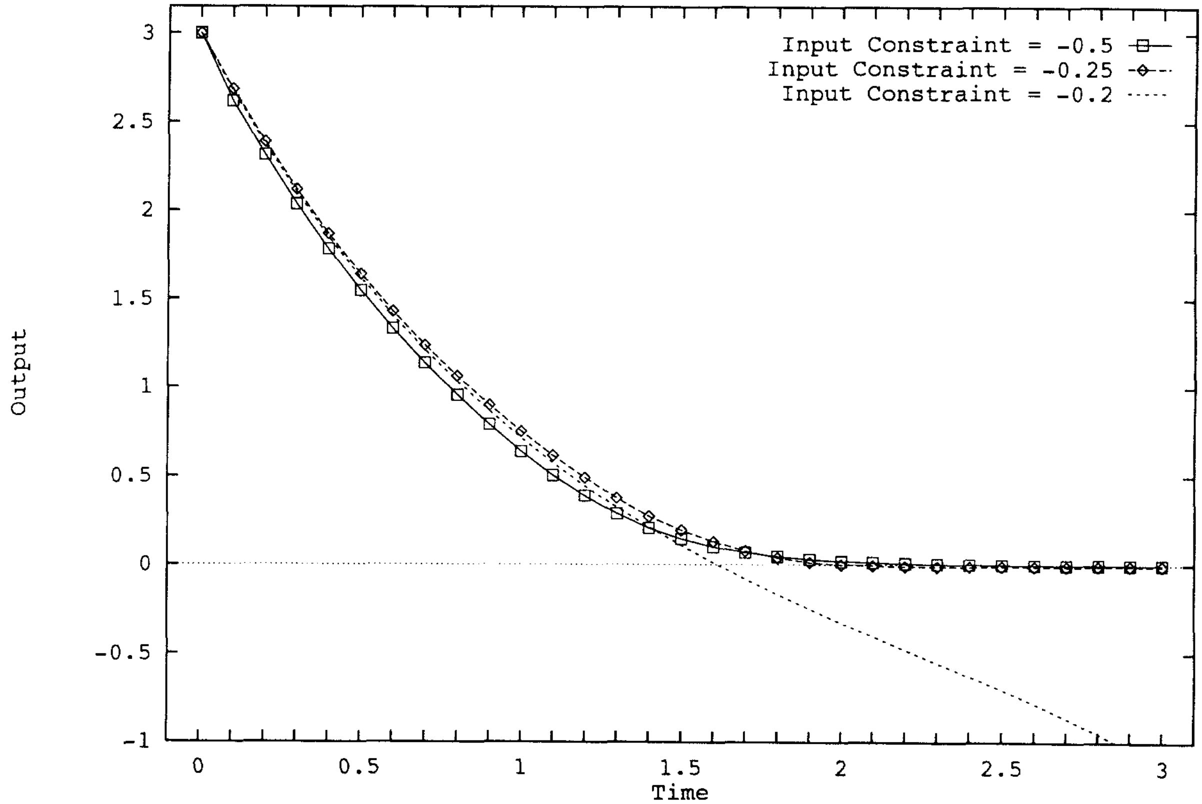


**Figure 6. Temperature responses for the unstable CSTR.**

At time $k=0$, a state disturbance of $[1, -1]^T$ enters the plant. The regulator is to maintain the output target at zero with no output constraints, a minimum input constraint, and the following regulator tuning parameters.

$$Q=1, \quad R=1, \quad S=0, \quad N=5$$

Figures 6 and 7 show the closed-loop response of the system for three values of the minimum input constraint. When this constraint is $-0.5$, the regulator is able to reject the disturbance with the tuning shown above. In this case, the input does not reach the input constraint since it is never the first value calculated in $u^N$. This illustrates the sometimes nonintuitive behavior of model predictive control caused by the moving horizon. When the minimum input constraint is increased to $-0.25$, the constraints in Eqs. 15 and 16 are infeasible for $N=5$. Increasing $N$ from 5 to 11 makes these constraints feasible and the regulator is able to reject the disturbance. When the minimum input constraint is further increased to $-0.2$, the system is no longer constrained stabilizable. In this case, the input constraint is too restrictive to control the unstable mode of the system excited by the state disturbance. The dashed line in Figure 6 shows the unstable response of the system from attempting to control the unstable mode by saturating the input at the minimum constraint value.

## Target Tracking

The presentation of the regulator in the previous sections was for a zero target. If the controller is to track step changes in a nonzero output target vector, $y_t$, then state and input vectors, $x_s$ and $u_s$, are required which bring the system to $y_t$ at steady state. These vectors can be determined from the output target vector by the following quadratic program.

$$\min_{[x_s,\ u_s]^T} \Psi = (u_s - \bar{u})^T R_s (u_s - \bar{u}) \tag{20}$$

subject to:

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_t \end{bmatrix} \tag{21}$$

$$u_{\min} \leq u_s \leq u_{\max}$$

In this quadratic program, $\bar{u}$ is the desired value of the input vector at steady state and $R_s$ is a positive definite weighting matrix for the deviation of the input vector from $\bar{u}$. The equality constraints in Eq. 21 guarantee a steady-state solution and offset free tracking of the target vector.

If there are not enough degrees of freedom to track the output target vector without offset, then the quadratic program in Eqs. 20 and 21 will be infeasible. In this case, $x_s$ and $u_s$ can be determined from the quadratic program below in which $Q_s$ is a positive definite weighting matrix for the output tracking error.

$$\min_{[x_s,\ u_s]^T} \Psi = (y_t - Cx_s)^T Q_s (y_t - Cx_s) \tag{22}$$

subject to:

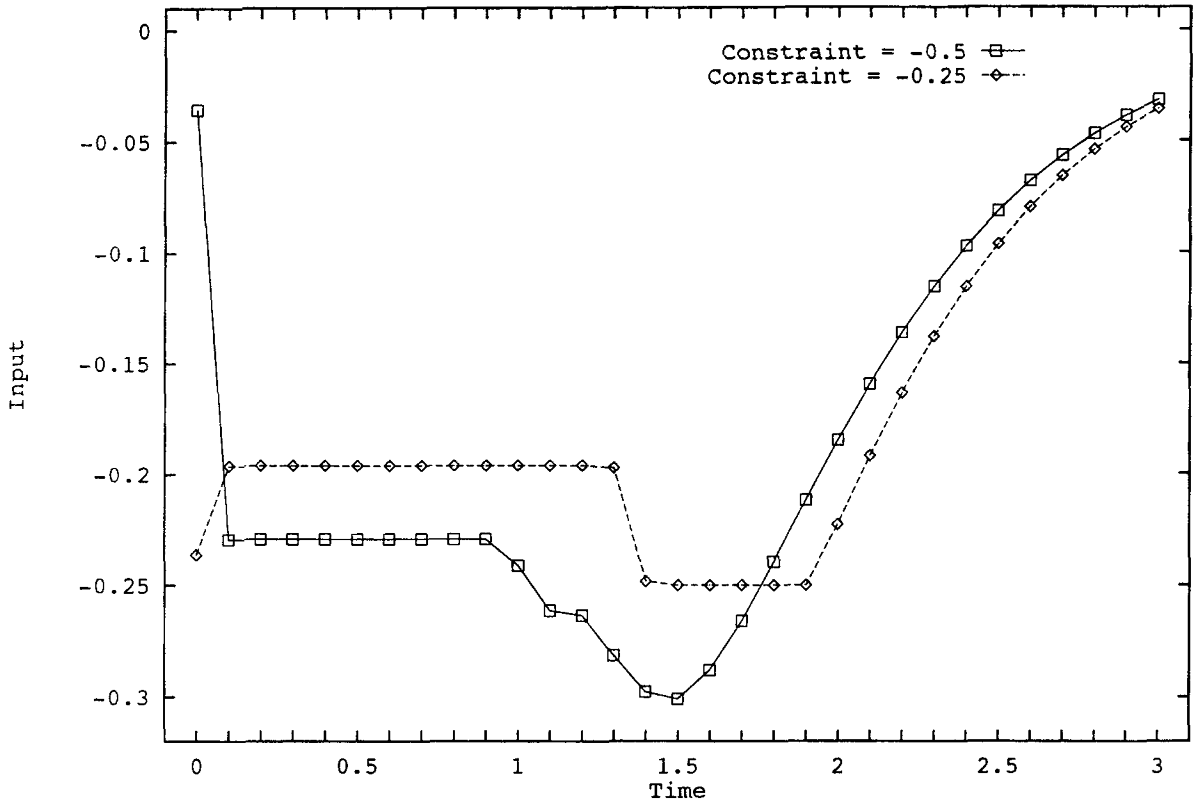


**Figure 7. Input responses for the unstable CSTR.**

$$[I-A \quad -B]\begin{bmatrix} x_s \\ u_s \end{bmatrix} = 0 \tag{23}$$

$$u_{\min} \le u_s \le u_{\max}$$

This quadratic program will track the output target in a least-squares sense. A steady-state solution is guaranteed by the equality constraints in Eq. 23. The actual value of the output that will be achieved at steady state is $y_s = Cx_s$.

The approach outlined above yields similar results to the procedure used in the IDCOM-M controller as described by Grossdidier et al. (1988). Both implementations perform least-squares control for nonsquare or constrained systems. In this implementation, $\bar{u}$ is equivalent to the IDCOM-M ideal resting value for inputs. It is used to move the input toward a desired steady-state value when there are degrees of freedom present in the system.

*Example 4.* Consider the discrete transfer function matrix given below.

$$G(z) = \begin{bmatrix} \dfrac{z}{2z-1} & \dfrac{z}{2.5z-1.5} \\ \dfrac{0.5z}{2z-1} & \dfrac{1.5z}{2.5z-1.5} \end{bmatrix}$$

A minimal state-space realization of this transfer function matrix is the following.

$$A = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \\ 0.25 & 0 \\ 0 & 0.6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The output target is $y_t = [1, -1]^T$. When both $u_1$ and $u_2$ are available, the quadratic program in Eqs. 20 and 21 is feasible. However, there are no degrees of freedom and the unique values of $x_s$ and $u_s$ are shown below.

$$x_s = \begin{bmatrix} 2.5 \\ -1.5 \\ 1.25 \\ -2.25 \end{bmatrix}, \quad u_s = \begin{bmatrix} 2.5 \\ -1.5 \end{bmatrix}, \quad y_s = \begin{bmatrix} 1.0 \\ -1.0 \end{bmatrix}$$

When only the first input, $u_1$, is available, the quadratic program in Eqs. 20 and 21 is infeasible. The target tracking error is then minimized using the quadratic program in Eqs. 22 and 23. With $Q_s = I$, the following values of $x_s$, $u_s$, and $y_s$ are obtained.

$$x_s = \begin{bmatrix} 0.4 \\ 0 \\ 0.2 \\ 0 \end{bmatrix}, \quad u_s = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}, \quad y_s = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}$$

When both inputs are available, but only the first output is to be controlled to $y_t^1 = 1$, the quadratic program in Eqs. 20 and

21 results in the following values of $x_s$ and $u_s$ for $R_s=I$ and $\bar{u}=0$.

$$x_s=\begin{bmatrix}0.5\\0.5\\0.25\\0.75\end{bmatrix},\quad u_s=\begin{bmatrix}0.5\\0.5\end{bmatrix},\quad y_s=\begin{bmatrix}1.0\\1.0\end{bmatrix}$$

### *Target tracking regulator objective function*

When tracking a nonzero target vector, the following quadratic objective function is used for the regulator.

$$\min_{u^N}\Phi_k=(x_{k+N}-x_s)^T\overline{Q}(x_{k+N}-x_s)+\Delta u_{k+N}^T S\Delta u_{k+N}$$
$$+\sum_{j=0}^{N-1}[(x_{k+j}-x_s)^T C^T QC(x_{k+j}-x_s)$$
$$+(u_{k+j}-u_s)^T R(u_{k+j}-u_s)+\Delta u_{k+j}^T S\Delta u_{k+j}] \quad (24)$$

The terminal state penalty matrix, $\overline{Q}$, is determined from Eq. 6 for stable systems or Eq. 12 for unstable systems. The steady-state vectors $x_s$ and $u_s$ are computed from the quadratic program in Eq. 20 or Eq. 22. This results in the discrete nonzero setpoint optimal regulator discussed by Kwakernaak and Sivan (1972).

The input and state penalties in this objective function penalize deviations from the steady-state target values. Therefore, the $u-u_s$ input penalty term is required along with the $x-x_s$ state penalty term to prevent offset in the regulator. This is equivalent to shifting the origin of the system to the steady state described by $x_s$ and $u_s$ and using the regulator presented earlier. The input vector $u_{k+j}$ is then set to $u_s$ for all $j \geq N$ in the open-loop objective function calculation. Stability of the target tracking regulator then follows from the same arguments as the zero target regulator.

Tuning of the target tracking regulator represents a tradeoff between deviation of the state and of the input from their steady-state values. In the limit as $Q\rightarrow 0$ and $S\rightarrow 0$, the regulator approaches a steady-state controller since only the input deviation is penalized. In the limit as $R\rightarrow 0$ and $S\rightarrow 0$, the regulator approaches a deadbeat controller since only the state deviation is penalized. The $\Delta u$ penalty matrix, $S$, is used to penalize rapid movement of the input. This prevents the regulator from taking overly aggressive control action whenever the output target is changed as demonstrated in the following example.

*Example 5.* Consider the system from Example 4 with the following output target vector and tuning parameters.

$$y_t=\begin{bmatrix}1\\-1\end{bmatrix},\quad Q=I,\quad R=I,\quad N=2$$

Figures 8 and 9 show the closed-loop response of the outputs and inputs for an output target change from zero to $y_t$. When $S=0$ the input moves to nearly the steady-state value at time zero. If this input action is unacceptably fast, the input velocity penalty can be increased to slow down the input response. This is shown in Figures 8 and 9 for $S=5I$.

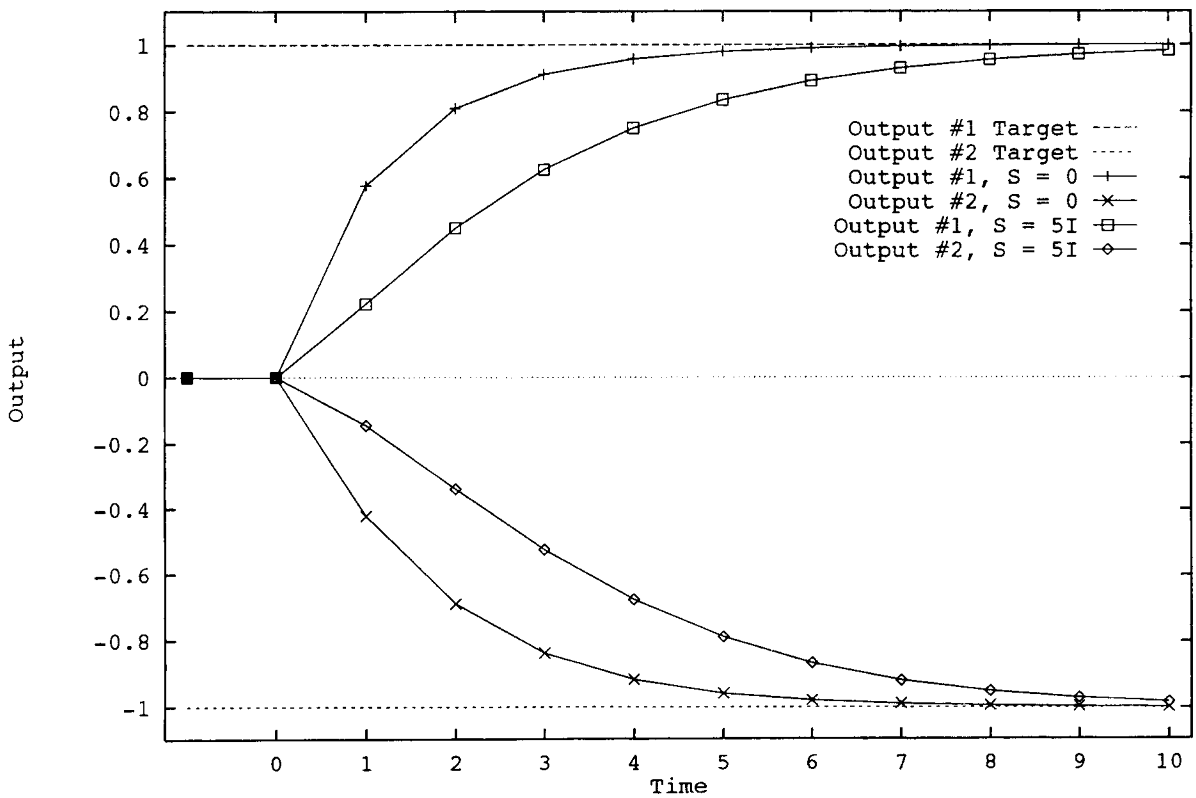


**Figure 8. Output response for target change.**

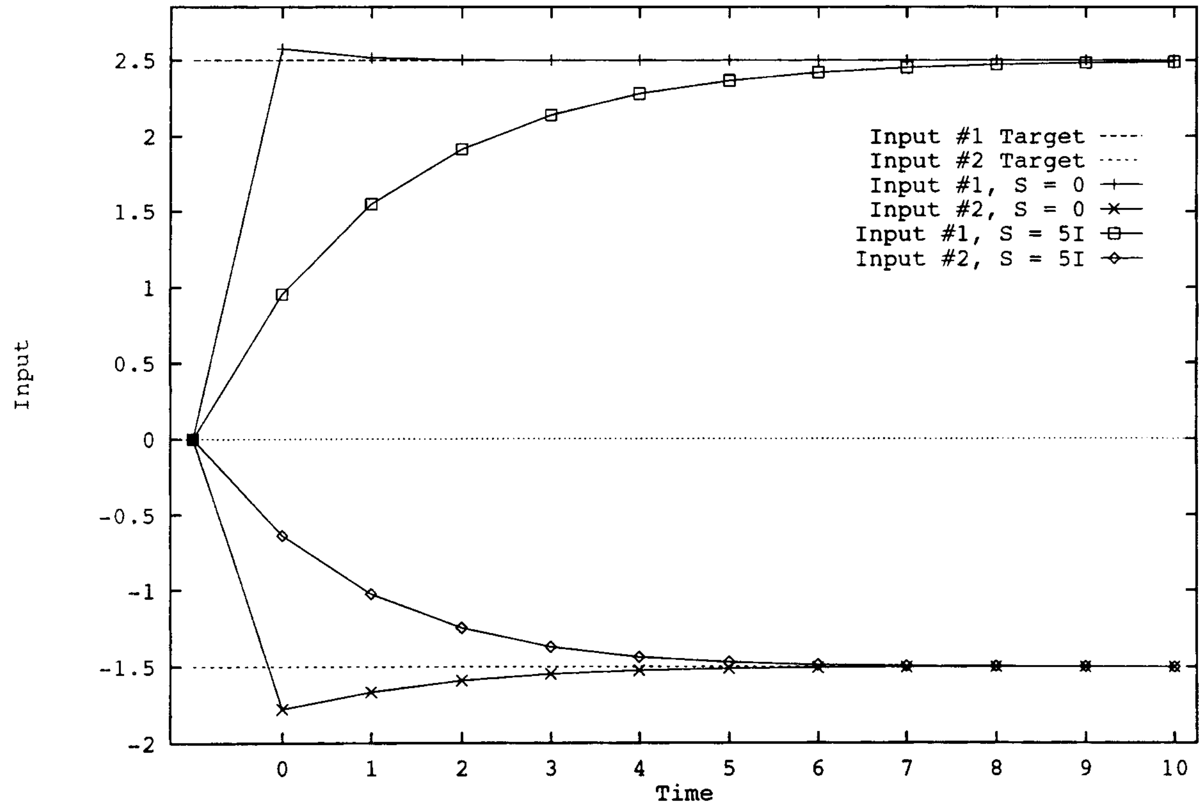


**Figure 9. Input response for target change.**

### *Reference trajectory tracking regulator*

The target tracking regulator presented in the previous section is designed to track step changes in the output target vector. In order to track a reference trajectory from the current state to the output target vector, the following quadratic objective function is used for the regulator.

$$\min_{u^N} \Phi_k = \sum_{j=0}^{\infty} [(Cx_{k+j} - C^r x^r_{k+j})^T Q (Cx_{k+j} - C^r x^r_{k+j}) + (u_{k+j} - u_s)^T R (u_{k+j} - u_s) + \Delta u^T_{k+j} S \Delta u_{k+j}] \quad (25)$$

In this objective function, the state penalty penalizes deviations from the specified reference trajectory over the infinite horizon. The reference states, $x^r_{k+j}$, are computed from the dynamic system below in which $A^r$ and $C^r$ describe the desired trajectory of the output from the initial state at time $k$ to the origin. As in the previous section, the origin of the system is shifted to the steady state described by $x_s$ and $u_s$.

$$\begin{aligned} x^r_{k+1} &= A^r x^r_k \\ y^r_k &= C^r x^r_k \end{aligned} \quad (26)$$

As discussed by Kwakernaak and Sivan (1972) and Bitmead et al. (1990), the reference trajectory dynamics can be combined with the plant dynamics to form an augmented system model. The reference trajectory tracking regulator can then be implemented as the zero target regulator presented previously for the following augmented system.

$$\overline{A} = \begin{bmatrix} A & 0 \\ 0 & A^r \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \overline{C} = [C \quad C^r]$$

The initial condition for the augmented state vector is shown below in which the matrix $J^r$ is specified when the order of the reference trajectory model differs from the plant model.

$$\overline{x}_k = \begin{bmatrix} x_k - x_s \\ J^r (x_s - x_k) \end{bmatrix}$$

The target tracking regulator can be recovered from this formulation by setting both $A^r$ and $C^r$ to zero.

Stabilizability of the augmented system requires that $A^r$ be a stable matrix. This restriction prevents the regulator from tracking reference trajectories, such as ramps, that are not norm bounded. However, an unbounded reference trajectory specified on an infinite horizon cannot actually be implemented. A ramp is used in a finite horizon regulator to move the process from one operating point to another. Once the process has reached the new operating point, the ramp is replaced with some bounded reference. In this implementation, the reference trajectory that the process is to follow to the new operating point is specified on the infinite horizon by $A^r$ and $C^r$.

*Example 6.* In this example, the system from Example 4 is reconsidered with the following output target vector and tuning parameters.

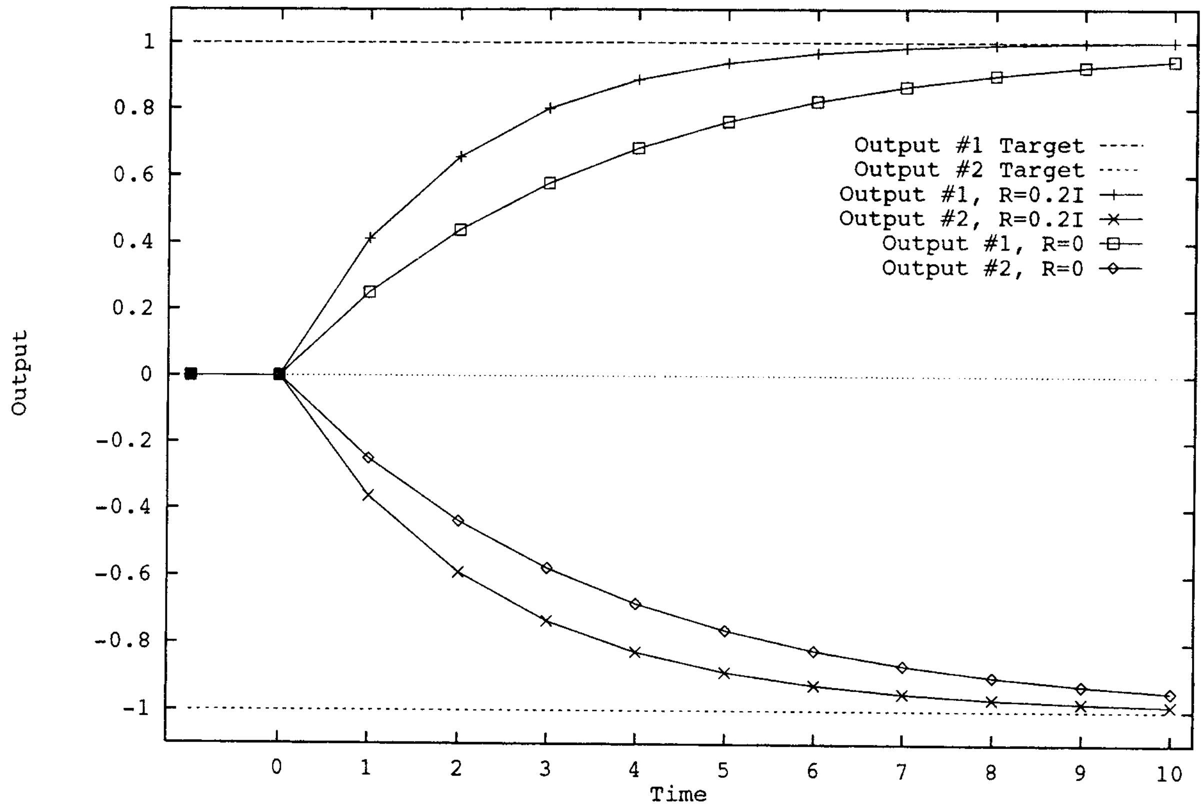


**Figure 10. Output response for the trajectory tracking regulator.**

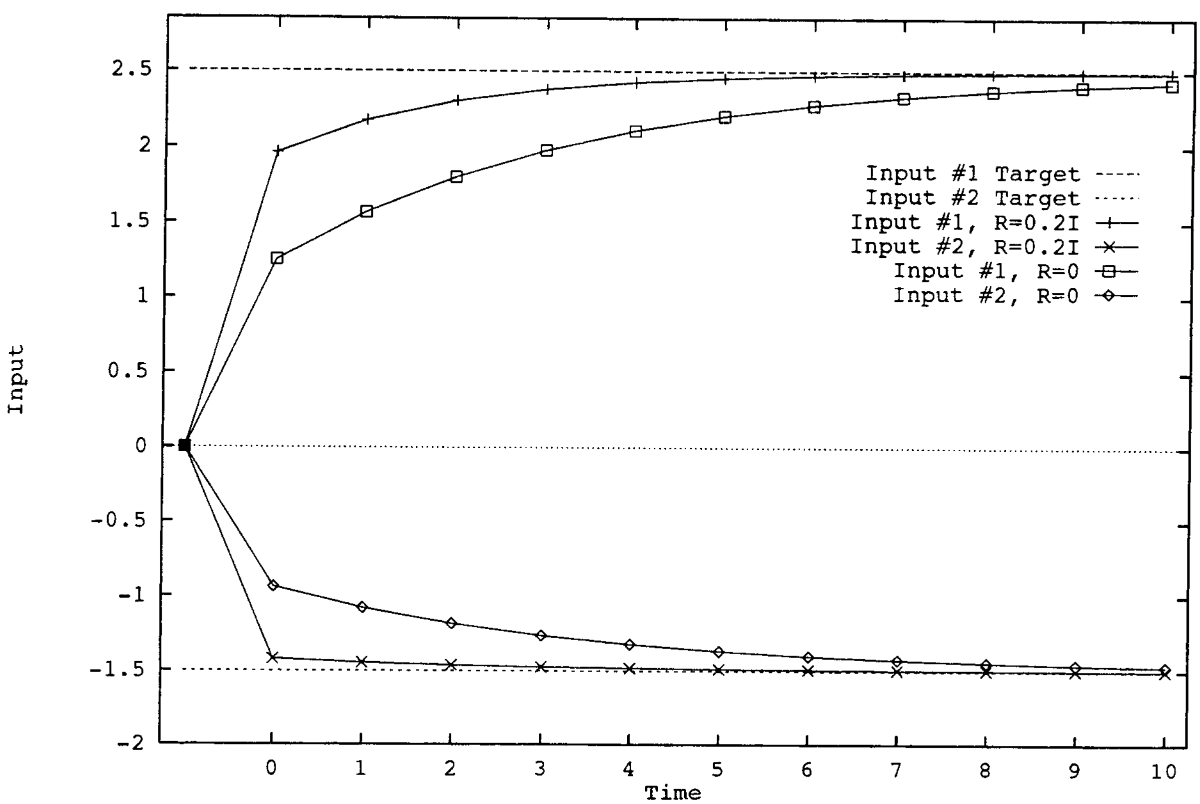


**Figure 11. Input response for the trajectory tracking regulator.**

$$y_t = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad Q = I, \quad S = 0, \quad N = 2$$

Both outputs are to track the first-order trajectory described by the following system.

$$x^r_{k+1} = \begin{bmatrix} .75 & 0 \\ 0 & .75 \end{bmatrix} x^r_k$$

$$y^r_k = [1 \quad 1] x^r_k$$

The reference trajectory part of the augmented trajectory tracking dynamic system model is shown below.

$$A^r = \begin{bmatrix} .75 & 0 \\ 0 & .75 \end{bmatrix}, \quad C^r = I, \quad J^r = C$$

Figures 10 and 11 show the closed-loop response of the outputs and inputs for an output target change from zero to $y_t$ with $R = 0$ and $R = 0.2I$. When $R = 0$, both outputs follow the reference trajectory exactly due to the deadbeat regulatory tuning. Since the dynamics of the trajectory are slower than the plant, the input response is less aggressive. When $R = 0.2I$, the input response is more aggressive because of the input penalty. Consequentially, the output response is faster than the reference trajectory.

## Output Feedback

The previous discussion assumed that the states are measured at each time $k$. In most applications, however, the states are not directly measured. If the state-space model came from a discrete transfer function, then the states will usually have no physical meaning and not be measurable. Even if the states are physically meaningful, sensors may not be available to measure each state. In these cases, output feedback must be performed using an observer that reconstructs the states from the output measurements.

The industrial implementations of model predictive control outlined in the first section are all based on one simple output feedback method. In these controllers, the difference between the model prediction and the measured output at the current time is assumed to be caused by a step output disturbance that remains constant in the future. This disturbance model has the advantage of being very easy to implement with convolution models and also yields integral action in the controller. The disadvantage is that it is unrealistic for most processes and, therefore, cannot adequately address many practical applications without external signal processing or controller detuning. Li et al. (1989), Ricker (1990), and Morari and Lee (1991) discuss more general disturbance modeling while retaining the convolution model for these implementations.

Since the controller presented in this article is in the state-space linear quadratic framework, it can take direct advantage of the results from linear quadratic filtering theory. This allows for the specification of a number of output feedback schemes for a given process. Included in these schemes is the output feedback method of the industrial controllers, which retains its simplicity and ease of implementation.

### *Optimal linear observer*

The standard linear observer is constructed for the system below in which $w_k$ and $v_k$ are zero-mean, uncorrelated, normally distributed, stochastic variables appended to the plant model in Eq. 1.

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + G_w w_k, \quad k = 0, 1, 2, \ldots \\ y_k &= Cx_k + v_k \end{aligned} \tag{27}$$

The optimal linear observer for this system in which $\hat{x}_{k+1|k}$ is the estimate of the state vector at time $k+1$ given output measurements up to time $k$ is (Åström, 1970):

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + Bu_k + L(y_k - C\hat{x}_{k|k-1}) \tag{28}$$

The discrete Kalman filter gain, $L$, minimizes the mean-square error of the state estimate $\hat{x}_{k+1|k}$. It is computed from the solution of the following discrete filtering steady-state Riccati equation with $Q_w$ and $R_v$ the covariance matrices of $w_k$ and $v_k$ respectively.

$$P = A[P - PC^T(CPC^T + R_v)^{-1}CP]A^T + G_w Q_w G_w^T \tag{29}$$

$$L = APC^T(CPC^T + R_v)^{-1} \tag{30}$$

This observer optimally reconstructs the states from the output measurements given the noise assumptions above. The steady-state Riccati formulation guarantees nominal stability of the filter in Eq. 28 provided $R_v > 0$, $[C, A]$ detectable, and $[A, G_w Q_w^{1/2}]$ stabilizable.

Since the states are stochastic variables that are not directly measured, the objective function for the regulator becomes the minimization of the expected value of $\Phi_k$. The expected value can be computed by replacing $x_{k+j}$ by $\hat{x}_{k+j|k}$ in the $\Phi_k$ calculation in which $\hat{x}_{k+j|k}$ is the expected value of $x_{k+j}$ given output measurements up to time $k$. It is computed by the following recursion starting with Eq. 28.

$$\hat{x}_{k+j+1|k} = A\hat{x}_{k+j|k} + Bu_{k+j}, \quad j \geq 1 \tag{31}$$

### *Step disturbance observer*

If a step disturbance enters the system, the combined observer/regulator discussed in the previous section will exhibit offset from the output target. In order to eliminate this offset, the observer must be redesigned to incorporate the step disturbance. Even when step disturbances are not expected to be present in the process, this modification can be made to obtain integral action in the controller. The presence of integral action leads to zero steady-state tracking error which can compensate for mismatch between the plant and the model.

*Output Step Disturbance.* The most common method to obtain integral action in the controller is to include a step disturbance in the output as discussed previously. This approach can be represented within the state-space framework by augmenting the system with a vector of additional states, $p$, that represent the output step disturbance.

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, 2, \ldots$$
$$p_{k+1} = p_k \tag{32}$$
$$y_k = Cx_k + p_k$$

The state estimates are then updated as shown below in which $\hat{p}$ is the estimate of the output step disturbance.

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + Bu_k \tag{33}$$

$$\hat{p}_{k+1|k} = y_k - C\hat{x}_{k|k-1} \tag{34}$$

The Kalman filter gain for the augmented system in Eq. 32 using this approach is $L = [0, I]^T$. This output step disturbance filter results in a deadbeat observer for the output disturbance vector, $\hat{p}$, and an open-loop observer for the model states. This formulation is equivalent to the observer in SMOC presented by Marquis and Broustail (1988).

Although this is the standard method for feedback in model predictive control formulations, it is optimal only for output step disturbances. Example 8 demonstrates the performance of this method in the presence of measurement noise. This filter cannot be used with unstable plants since the observer poles contain the plant poles.

*Input Step Disturbance.* Another method to obtain integral action is to include a step input disturbance. In this implementation, it is assumed that the difference between the predicted output and the measurement is caused by an input step disturbance. This is analogous to the assumption made above for the step disturbance in the output. The input step disturbance can be viewed as a particular step disturbance to the states. The corresponding filter can be used for unstable plants.

This approach can be represented by augmenting the system with a vector of additional states, $z$, to represent the input step disturbance.

$$x_{k+1} = Ax_k + B(u_k + z_k), \quad k = 0, 1, 2, \ldots$$
$$z_{k+1} = z_k \tag{35}$$
$$y_k = Cx_k$$

The state estimates are then updated as shown below in which $\hat{z}$ is the estimate of the input step disturbance and $[L_x, L_z]^T$ is the partitioned Kalman filter gain. Equation 30 is used to calculate this filter gain with the augmented system in Eq. 35 assuming a nonzero covariance matrix for the $z$ states only.

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + B(\hat{z}_{k|k-1} + u_k) + L_x(y_k - C\hat{x}_{k|k-1}) \tag{36}$$

$$\hat{z}_{k+1|k} = \hat{z}_{k|k-1} + L_z(y_k - C\hat{x}_{k|k-1}) \tag{37}$$

The input step disturbance filter results in a deadbeat observer for both the input disturbance vector, $\hat{z}$, and the model state vector. This is a special case of the observer presented by Kwakernaak and Sivan (1972) for the zero steady-state error discrete linear quadratic regulator.

### *Step disturbance regulator*

Augmenting the system with a step disturbance vector as discussed in the previous section includes states that are not asymptotically stable. Since these additional states are also uncontrollable, the augmented system is not stabilizable. Therefore, the regulator presented in this article cannot be implemented on the augmented system. However, these states are observable and the corresponding observer can be made stabilizing. The estimate of these states can be used to remove the disturbance from the nominal system with the constant disturbance regulator formulation discussed by Kwakernaak and Sivan (1972). The resulting control law is identical to that obtained with the nonzero target tracking regulator presented earlier.

The input and state target vectors, $u_s$ and $x_s$, that remove the step disturbance at steady state can be determined from the quadratic program below in which $y_t$ is the output target, $\hat{p}$ is the estimate of the output step disturbance, and $\hat{z}$ is the estimate of the input step disturbance.

$$\min_{[x_s, u_s]^T} \Psi = (u_s - \bar{u})^T R_s (u_s - \bar{u}) \tag{38}$$

subject to:

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} B\hat{z} \\ y_t - \hat{p} \end{bmatrix} \tag{39}$$
$$u_{\min} \le u_s \le u_{\max}$$

Depending on which step disturbance model is chosen, one of the step disturbance vectors will not be present in the system and is set to zero. If both step disturbance vectors are set to zero, then this quadratic program reduces to the quadratic program of Eqs. 20 and 21.

If the quadratic program in Eqs. 38 and 39 is infeasible, then the output target vector cannot be tracked without offset. The output tracking error is then minimized with the following quadratic program.

$$\min_{[x_s, u_s]^T} \Psi = (y_t - Cx_s - \hat{p})^T Q_s (y_t - Cx_s - \hat{p}) \tag{40}$$

subject to:

$$[I-A \quad -B] \begin{bmatrix} x_s \\ u_s \end{bmatrix} = B\hat{z} \tag{41}$$
$$u_{\min} \le u_s \le u_{\max}$$

This quadratic program is analogous to Eqs. 22 and 23.

*Example 7.* Consider two plants with the discrete transfer function and state-space models shown below.

**Table 1. Observer Filter Gain Matrices**

| Observer | Plant A | Plant B |
|---|---|---|
| Zero-Mean State Noise | $L = [0.5]$ | $L = [0.9]$ |
| Output Step Disturbance | $L = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ | $L = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ |
| Input Step Disturbance | $L = \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$ | $L = \begin{bmatrix} 1.9 \\ 10 \end{bmatrix}$ |

Plant A: $G(z)=\frac{z}{2z-1}$; $A=0.5$, $B=0.5$, $C=1$

Plant B: $G(z)=\frac{z}{10z-9}$; $A=0.9$, $B=0.1$, $C=1$

Both plants are unity gain and first-order with the dynamic response of Plant A faster than Plant B. The following regulator tuning, which results in deadbeat control action, is used for each plant.

$$Q=1,\quad R=0,\quad S=0,\quad N=2$$

A zero-mean state noise, output step disturbance, and input step disturbance filter is designed for each plant. The zero-mean state noise filter is computed from Eq. 30 assuming no measurement noise. The output step disturbance filter is presented in Eqs. 33 and 34. The input step disturbance filter is presented in Eqs. 36 and 37. The filter gain matrices for each augmented plant are shown in Table 1.

A unity magnitude state disturbance, output step disturbance, and input step disturbance enters each plant at time $k=0$. The output response from each of these feedback schemes is shown in Figures 12 through 17. As expected, each observer/regulator perfectly rejects the disturbance that the observer was designed for. The performance of each observer/regulator for the other two disturbances is discussed below.

The state noise observer/regulator results in offset for both the output and input step disturbances. As shown in Figures 14 through 17, the magnitude of this offset is a function of the plant dynamics. The larger offset is associated with the faster plant since the observer design assumes a state disturbance that will decay with the plant dynamics.

The output step observer/regulator has the slowest disturbance rejection response for the disturbances that it was not designed for as shown in Figures 12, 13, 16, and 17. This is due to the open-loop observer design for the model states. Since the disturbance passes through the plant dynamics, the response is worse for the slower plant although the magnitude of the disturbance is less.

The input step observer/regulator is able to reject the other disturbances in two time periods since both the observer and the regulator are deadbeat and there are two states in the augmented system. However, it has the largest target deviations of the three observer/regulator pairs. This is expected due to the input step disturbance observer being the most aggressive. The deviations are a function of the plant dynamics only for the output step as shown in Figures 14 and 15.

### *Measurement noise*

The observers in Example 7 were designed assuming no measurement noise. This assumption, common to most implementations of model predictive control, is unrealistic for practical applications. The framework presented in this article allows for the explicit design of noise in the system by appending the step disturbance states discussed earlier to the stochastic plant model presented in Eq. 27.

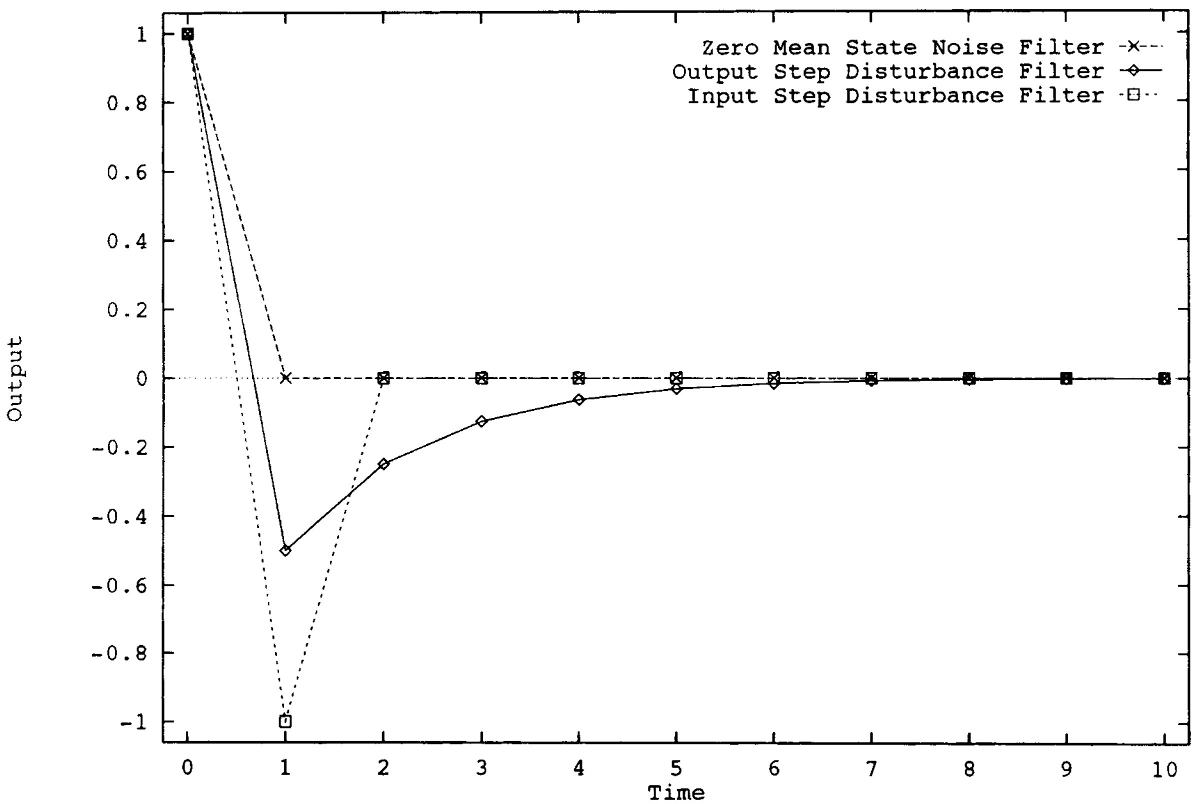


**Figure 12. Output response for a state disturbance in Plant A.**

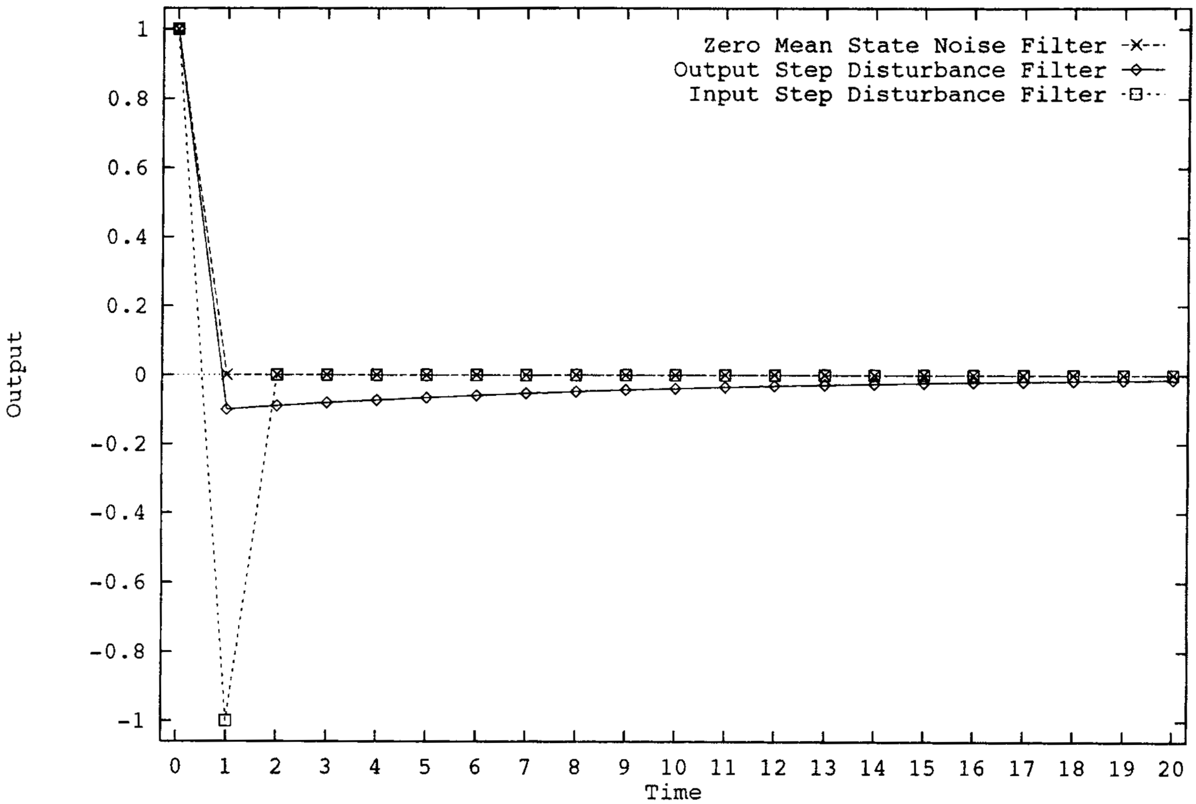


**Figure 13. Output response for a state disturbance in Plant B.**

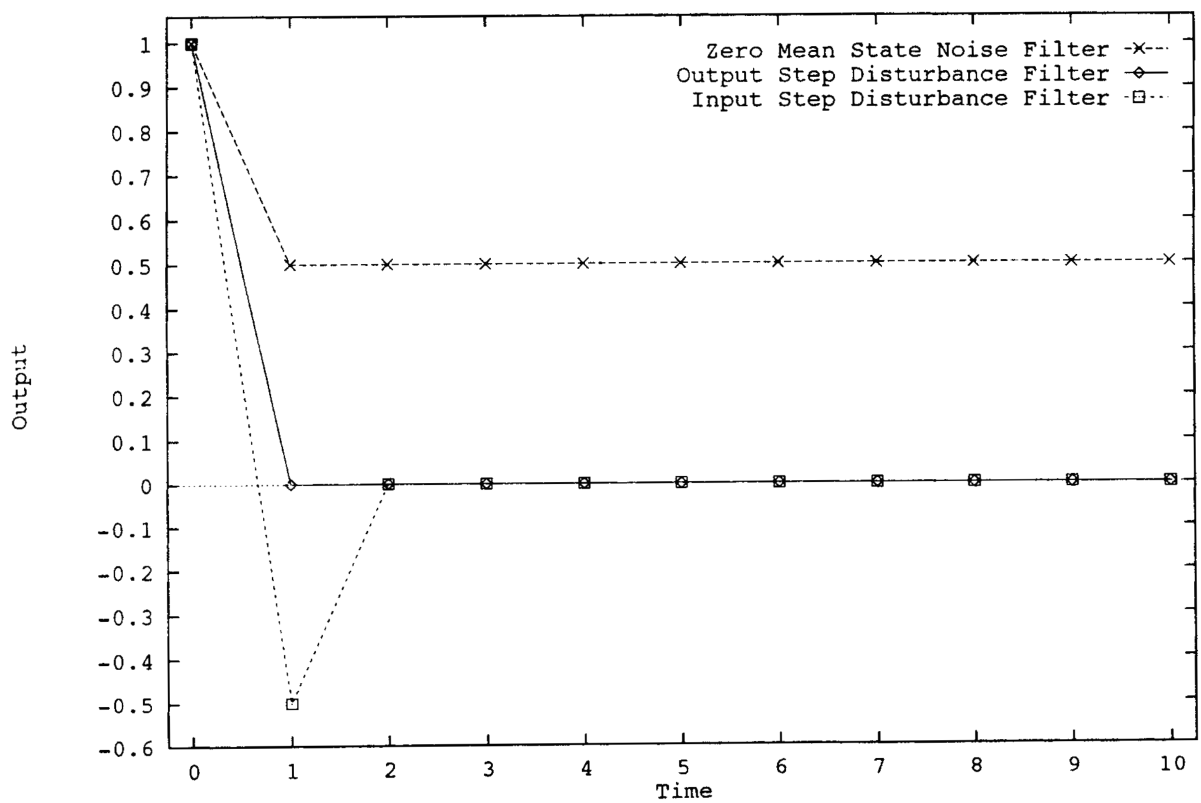


**Figure 14. Output response for an output step disturbance in Plant A.**

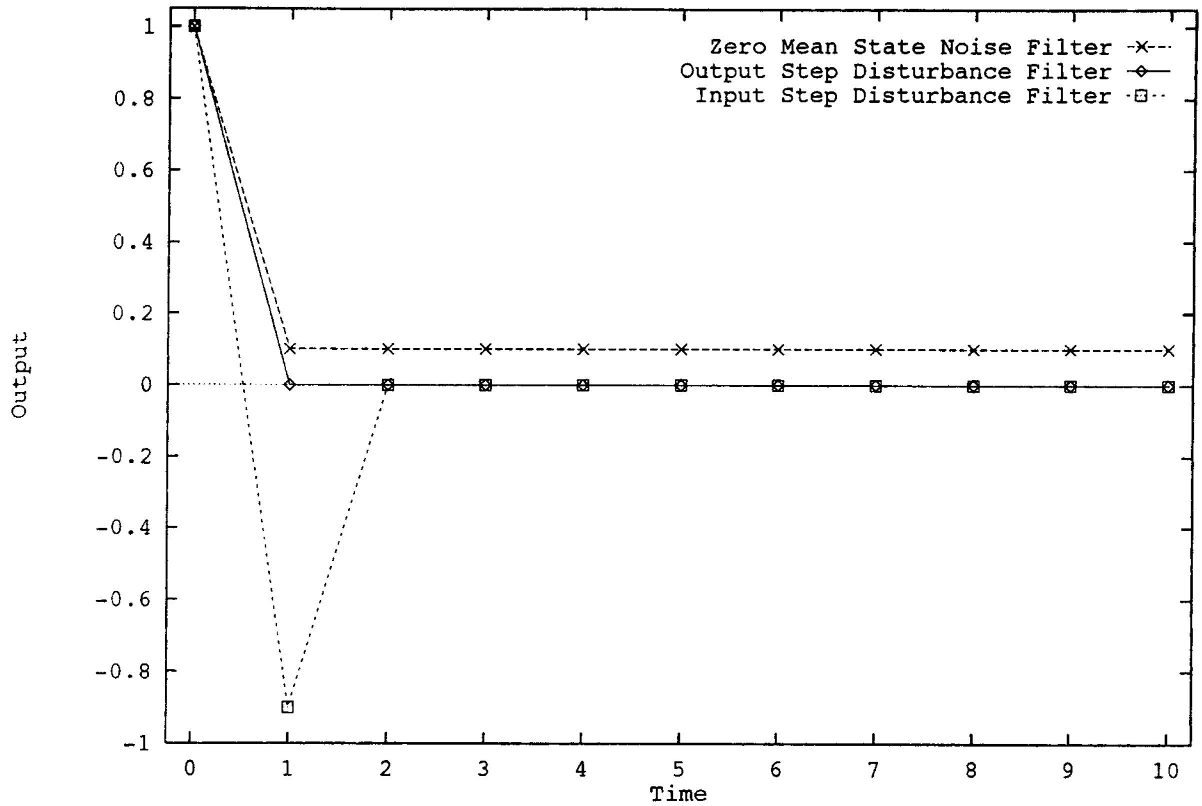


**Figure 15. Output response for an output step disturbance in Plant B.**

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + G_z z_k + G_w w_k, \quad k=0, 1, 2, \ldots \\
z_{k+1} &= z_k + \omega_k \\
p_{k+1} &= p_k + \nu_k \\
y_k &= Cx_k + G_p p_k + v_k
\end{aligned} \tag{42}
$$

In this representation, $w_k$, $\omega_k$, $\nu_k$, and $v_k$ are zero-mean, uncorrelated, normally distributed, stochastic variables with covariance matrices $Q_w$, $Q_\omega$, $Q_\nu$, and $R_v$ respectively. The dynamics of the output step disturbance vector, $p_k$, are contained in $G_p$. When $G_p = I$, the disturbance becomes the output step disturbance presented earlier. The dynamics of the state step disturbance vector, $z_k$, are contained in $G_z$. When $G_z = B$, the disturbance becomes the input step disturbance presented earlier.

The system in Eq. 42 can be represented by the following augmented state-space matrices.

$$
\tilde{A} = \begin{bmatrix} A & 0 & G_z \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix},
$$

$$
\tilde{C} = [C \quad G_p \quad 0], \quad \tilde{G}_w = \begin{bmatrix} G_w & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}
$$

The observer gain is computed using these augmented matrices in Eq. 30 along with the following augmented covariance matrices.

$$
\tilde{Q}_w = \begin{bmatrix} Q_w & 0 & 0 \\ 0 & Q_\nu & 0 \\ 0 & 0 & Q_\omega \end{bmatrix}, \quad \tilde{R}_v = R_v
$$

The state estimates are then computed as shown below in which the filter gain is partitioned into a state filter, $L_x$, an output step disturbance filter, $L_p$, and a state step disturbance filter, $L_z$.

$$
\begin{bmatrix} \hat{x}_{k+1|k} \\ \hat{p}_{k+1|k} \\ \hat{z}_{k+1|k} \end{bmatrix} = \tilde{A} \begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{p}_{k|k-1} \\ \hat{z}_{k|k-1} \end{bmatrix} + \tilde{B}u_k + \begin{bmatrix} L_x \\ L_p \\ L_z \end{bmatrix} \left( y_k - \tilde{C} \begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{p}_{k|k-1} \\ \hat{z}_{k|k-1} \end{bmatrix} \right) \tag{43}
$$

The step disturbance regulator for this system is constructed as shown previously with the following modifications to the quadratic programs. In Eqs. 39 and 41, $B\hat{z}$ is replaced with $G_z\hat{z}$. In Eqs. 39 and 40, $\hat{p}$ is replaced with $G_p\hat{p}$.

Ranging from the simple output step disturbance model of the industrial model predictive controllers to the combined step disturbance and noise model presented in this section, this framework allows for a great deal of flexibility in the design of the disturbance model for the process. However, an important limitation in this design is the detectability restriction on $[\tilde{C}, \tilde{A}]$. This restriction requires that both the $p$ and $z$ vectors

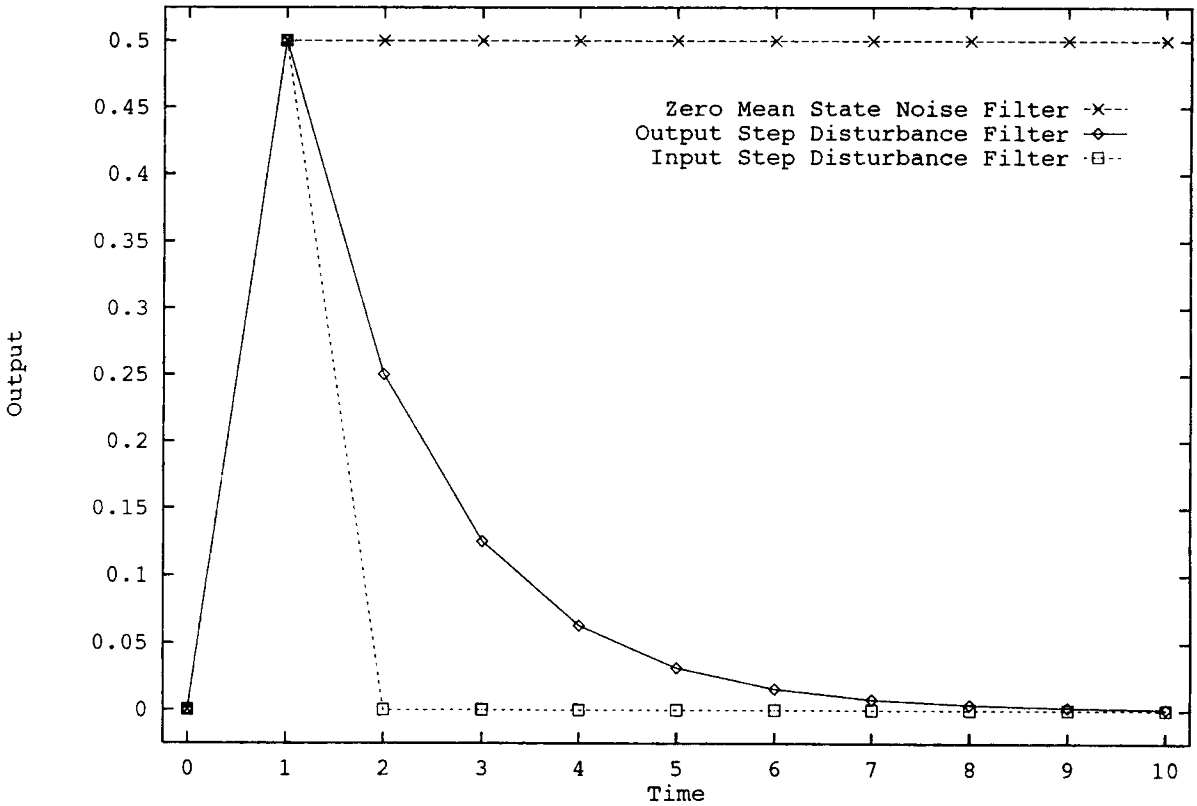


**Figure 16. Output response for an input step disturbance in Plant A.**

be completely observable in the augmented system. This imposes a limitation on the dimension of these vectors. In general, the combined number of states in the $p$ and $z$ vectors cannot be greater than the number of outputs.

*Example 8.* Consider the SISO plant with the following discrete transfer function.

$$G(z) = \frac{4z}{9z^2 - 4z - 1}$$

A minimal state-space realization of this discrete transfer function is shown below.

$$A = \begin{bmatrix} 4/9 & 1/9 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [4/9 \quad 0]$$

In this example, there are no constraints with the following regulator tuning parameters.

$$Q = 1, \quad R = 0, \quad S = 0, \quad N = 2$$

Measurement noise with a variance of 0.1 and output step disturbances with a variance of 0.001 enter this process. At time $k = 0$, the output target is changed from zero to one. The output is to track the following first-order reference trajectory dynamical system to the new target.

$$A^r = 0.7, \quad C^r = 1, \quad J^r = C$$

The performance of the standard output feedback filter discussed earlier is compared to the optimal filter computed using Eq. 43. In this example, the optimal filter for the noise entering the process is $L = [0,\ 0,\ 0.095]^T$. Figures 18 and 19 show the response of the standard output feedback method and the optimal output feedback. As shown in Figure 19, there is much more manipulated variable action using the standard output feedback. This is due to the standard method's assumption that all of the prediction error is attributable to output step disturbances that must be removed by control action. The optimal output feedback design incorporates measurement noise in the observer. As shown in Figure 18, this allows the optimal method to track the output reference trajectory without excessive manipulated variable action. The manipulated variable action could be reduced with the standard output feedback method by increasing the input penalty $S$, however, this will decrease the performance of the regulator.

### *Constrained optimal observer*

The observers outlined in the previous sections are optimal for zero-mean, uncorrelated, normally distributed noise. Although process noise rarely follows these assumptions, they are used since the actual stochastic process is generally unknown. However, spurious measurements can significantly effect the estimate when using these observers. This section presents a constrained optimal observer to prevent physically unrealistic estimates. The observer is a finite quadratic program formulation for the steady-state Kalman filter when there are no constraints.

Consider the minimization of the following quadratic objective function at time $k$ in which $Q_w$ is the covariance of the

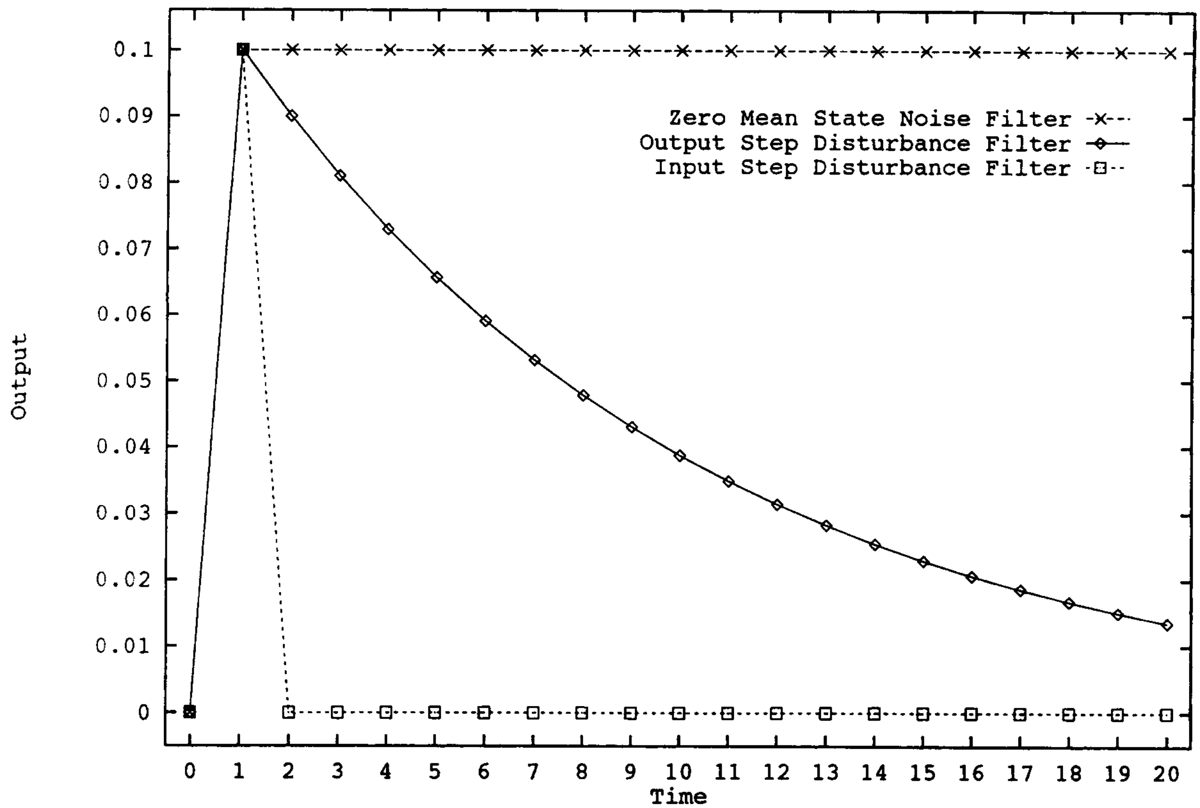


**Figure 17. Output response for an input step disturbance in Plant B.**

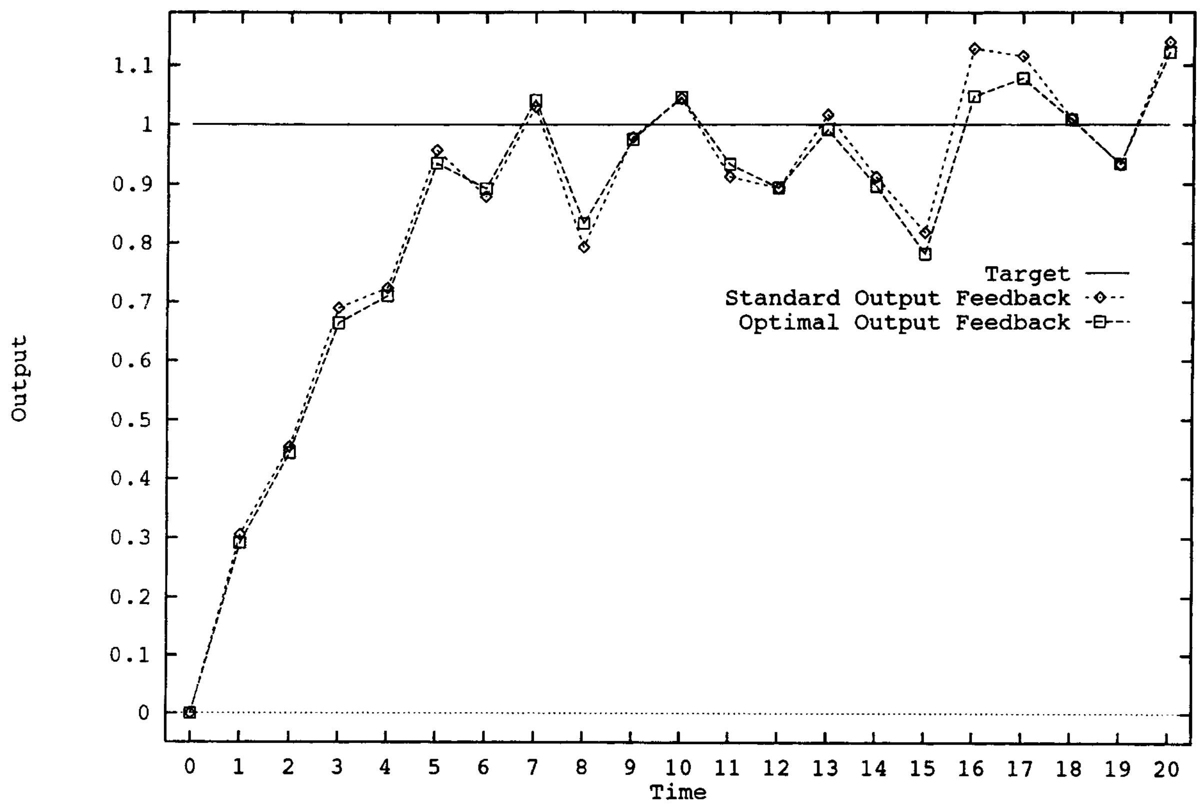


**Figure 18. Output response for a set point change.**

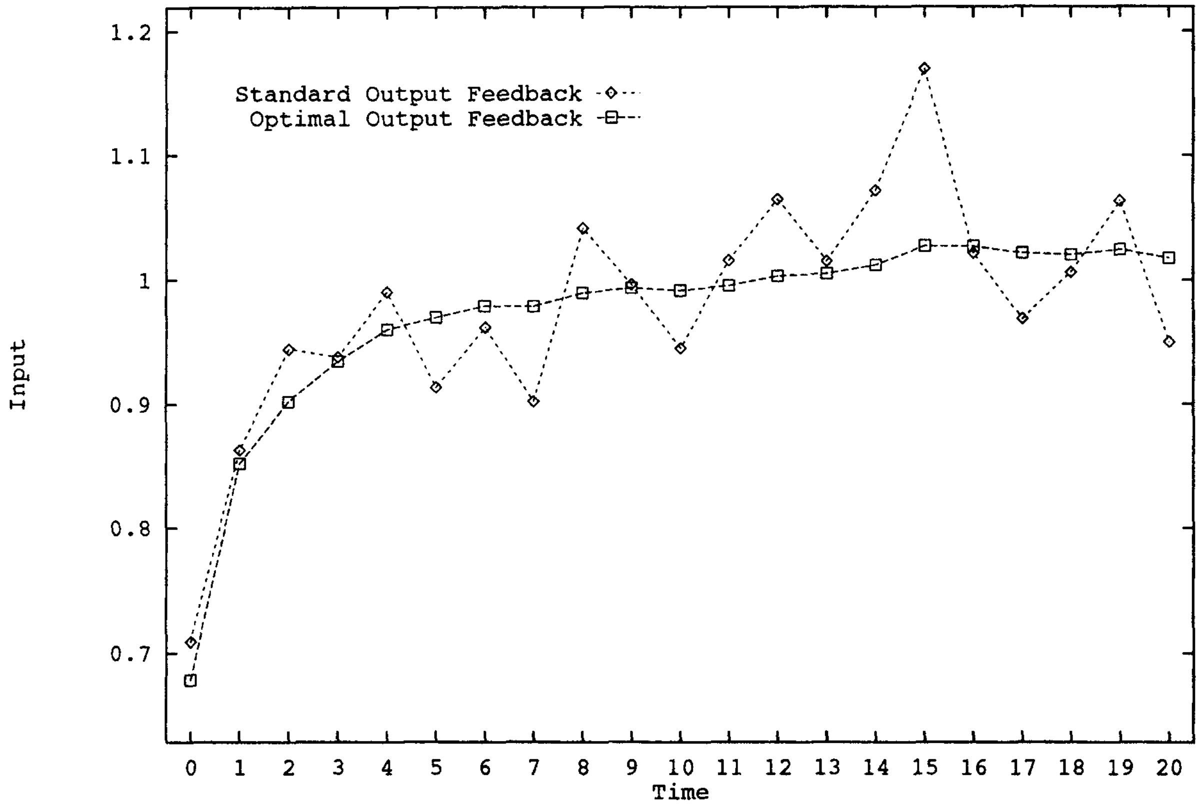


**Figure 19. Input response for a set point change.**

state noise vector, $R_v$ is the covariance of the measurement noise vector, and $P$ is the steady-state discrete filtering Riccati matrix determined from Eq. 29. Note that both $Q_w$ and $R_v$ must be positive definite for the inverses to exist.

$$\min_{\hat{w}^N} \Theta_k = \hat{w}_{k-N-1}^T P^{-1} \hat{w}_{k-N-1} + \sum_{j=k-N}^{k-1} (\hat{w}_j^T Q_w^{-1} \hat{w}_j) + \sum_{j=k-N}^{k} (\hat{v}_j^T R_v^{-1} \hat{v}_j) \quad (44)$$

Subject to:

$$\hat{x}_{j+1|k} = A\hat{x}_{j|k} + Bu_j + \hat{w}_j$$
$$y_j = C\hat{x}_{j|k} + \hat{v}_j$$

The initial condition for the state estimate is the following.

$$\hat{x}_{k-N|k} = \hat{x}_{k-N|k-N-1} + \hat{w}_{k-N-1} \quad (45)$$

The solution to this quadratic program is a series of estimated state noise vectors, $\hat{w}_j$, that are used to compute the smoothed state estimates at time $k$, $\hat{x}_{j|k}$. The estimated state noises are contained in the vector $\hat{w}^N$ as shown below.

$$\hat{w}^N = \begin{bmatrix} \hat{w}_{k-N-1} \\ \hat{w}_{k-N} \\ \vdots \\ \hat{w}_{k-1} \end{bmatrix} \quad (46)$$

Estimated state noise vector and smoothed state constraints of the following form are considered.

$$\hat{w}_{\min} \le \hat{w}_j \le \hat{w}_{\max}, \quad j = k-N-1,\ k-N,\ \ldots,\ k-1 \quad (47)$$

$$\hat{x}_{\min} \le \hat{x}_{j|k} \le \hat{x}_{\max}, \quad j = k-N,\ k-N+1,\ \ldots,\ k+1 \quad (48)$$

The constrained observer is implemented as the quadratic program in Eqs. 44, 47, and 48. Construction of the quadratic program matrices is analogous to those presented for the constrained regulator.

## Measured Disturbances

The discussion of output feedback in the preceding section was concerned with the construction of a noise model and the estimation of unmeasured disturbances. This section addresses feedforward control of measured disturbance. A linear system is first presented to represent the measured disturbance. This allows for the optimal estimation of the disturbance in the presence of measurement noise. The feedforward linear system model that describes the effect of the measured disturbance on the output of the plant is then presented. Feedforward control of the measured disturbance is discussed for stable disturbances.

### *Measured disturbance models*

In this formulation, measured disturbances are described by the following linear system in which $d_k$ is the measured disturbance vector, $x_k^d$ is the measured disturbance model state vector, and $w_k^d$, $v_k^d$ are zero-mean, uncorrelated, normally distributed, stochastic vectors.

$$\begin{aligned} x_{k+1}^d &= A^d x_k^d + G_w^d w_k^d \\ d_k &= C^d x_k^d + v_k^d \end{aligned} \tag{49}$$

An optimal estimate of the measured disturbance model state vector given measurements up to time $k$ can be determined as shown below in which $L^d$ is the discrete Kalman filter gain for the linear system in Eq. 49.

$$\hat{x}_{k+1|k}^d = A^d \hat{x}_{k|k-1}^d + L^d (d_k - C^d \hat{x}_{k|k-1}^d) \tag{50}$$

The filter gain in Eq. 50 can be determined using Eq. 30 with the system in Eq. 49 and the covariance matrices of $v^d$ and $w^d$.

The expected future values of the measured disturbance vector given measurements up to time $k$ can be computed by the following recursion starting with Eq. 50.

$$\begin{aligned} \hat{x}_{k+j+1|k}^d &= A^d \hat{x}_{k+j|k}^d \\ \hat{d}_{k+j|k} &= C^d \hat{x}_{k+j|k}^d \end{aligned} \tag{51}$$

The standard assumption that the measured disturbance remains constant in the future can be represented in this framework by setting $A^d = I$ and $C^d = I$.

The effect of the measured disturbance on the output of the plant is described by the following dynamic system.

$$\begin{aligned} x_{k+1}^m &= A^m x_k^m + B^m \hat{d}_k \\ y_k &= C^m x_k^m \end{aligned} \tag{52}$$

The input to this feedforward model is the measured disturbance computed from Eq. 51.

### *Measured disturbance regulator*

Combining the feedforward model in Eq. 52 with the plant model and the measured disturbance dynamics results in the following augmented system that describes the output response of both the input and measured disturbance.

$$\acute{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & A^m & B^m C^d \\ 0 & 0 & A^d \end{bmatrix}, \quad \acute{B} = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}, \quad \acute{C} = [C \quad C^m \quad 0]$$

The initial condition of $x^m$ and $x^d$ in the augmented state vector is computed using Eqs. 50 and 52 respectively. Feedforward control of the measured disturbance and feedback control are implemented as the standard controller formulation on this augmented system. This is analogous to the trajectory tracking controller presented earlier.

If it is assumed that the measured disturbance remains constant in the future, then $A^d = I$ and the augmented system is not stabilizable. In this case, the estimate of the measured disturbance can be used to remove its effect on the output in a manner entirely analogous to the step disturbance regulator presented earlier. The input and state target vectors, $u_s$ and $x_s$, that remove the step disturbance at steady state can be determined from the following quadratic programs in which $y_t$ is the output target, $\hat{p}$ is the estimate of the unmeasured output step disturbance, $\hat{z}$ is the estimate of the unmeasured state step disturbance, and $x_s^m$ is the steady-state value of the feedforward model state vector.

$$\min_{[x_s,\ u_s]^T} \Psi = (u_s - \bar{u})^T R_s (u_s - \bar{u}) \tag{53}$$

subject to:

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} G_z \hat{z} \\ y_t - G_p \hat{p} - C^m x_s^m \end{bmatrix} \tag{54}$$
$$u_{\min} \le u_s \le u_{\max}$$

If this quadratic program is infeasible, then the output target vector cannot be tracked without offset. The output tracking error is then minimized with the following quadratic program.

$$\min_{[x_s,\ u_s]^T} \Psi = (y_t - Cx_s - G_p \hat{p} - C^m x_s^m)^T Q_s \times (y_t - Cx_s - G_p \hat{p} - C^m x_s^m) \tag{55}$$

subject to:

$$[I-A \quad -B] \begin{bmatrix} x_s \\ u_s \end{bmatrix} = G_z \hat{z} \tag{56}$$
$$u_{\min} \le u_s \le u_{\max}$$

The steady-state value of the feedforward model state vector is computed as shown below.

$$x_s^m = (I - A^m)^{-1} B^m \hat{d} \tag{57}$$

*Example 9.* Consider the SISO plant and reference trajectory presented in Example 8. A measured disturbance that evolves with $A^d = 0.5$ and $C^d = 1$ enters this plant. The effect of the measured disturbance on the output of the plant is described by the nonminimum phase system presented in Example 3. The following augmented system represents the plant model, feedforward model, measured disturbance dynamics, and reference trajectory dynamics.

$$\acute{A} = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & A^m & B^m C^d & 0 \\ 0 & 0 & A^d & 0 \\ 0 & 0 & 0 & A^r \end{bmatrix}, \quad \acute{B} = \begin{bmatrix} B \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \acute{C} = [C \quad C^m \quad 0 \quad C^r]$$

Each of the individual state-space models are shown below.

$$A = \begin{bmatrix} 4/9 & 1/9 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [4/9 \quad 0]$$

$$A^m = \begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix}, \quad B^m = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C^m = [-2/3 \quad 1]$$

$$A^r = 0.7, \quad C^r = 1, \quad J^r = [4/9 \quad 0]$$

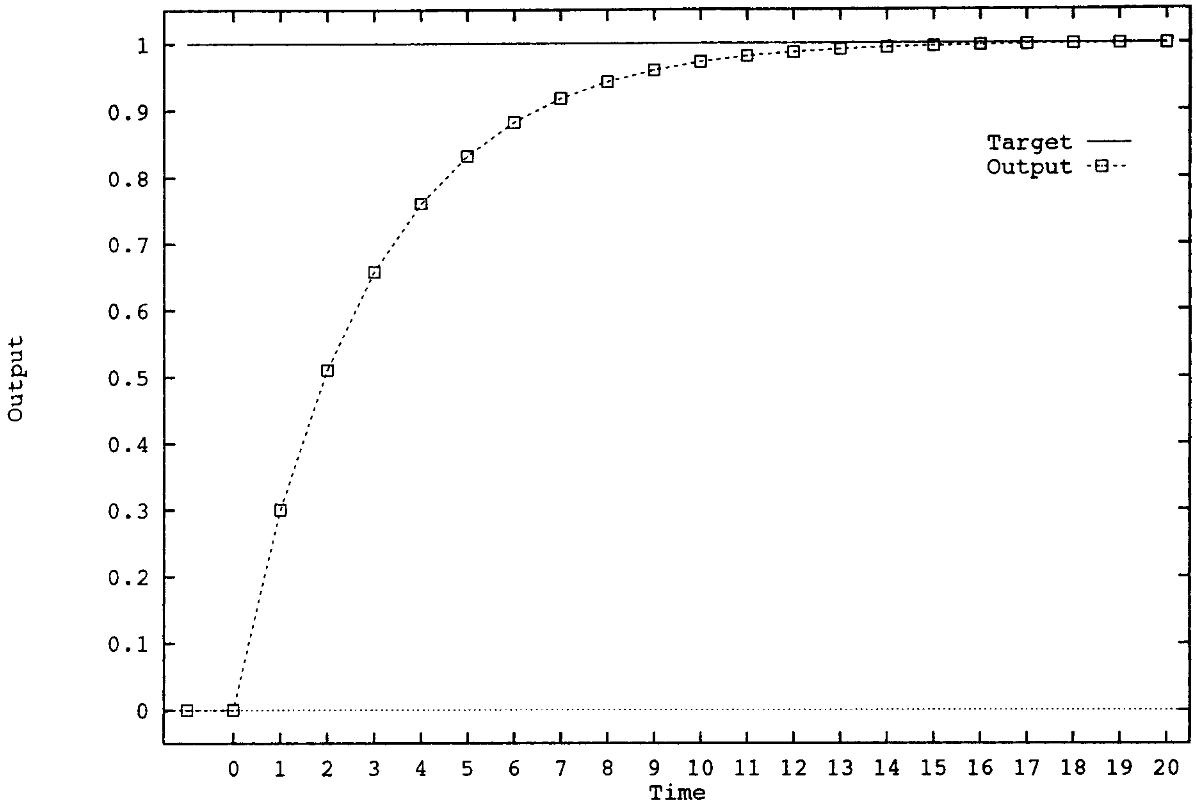


**Figure 20. Output response for feedforward control of a measured disturbance.**

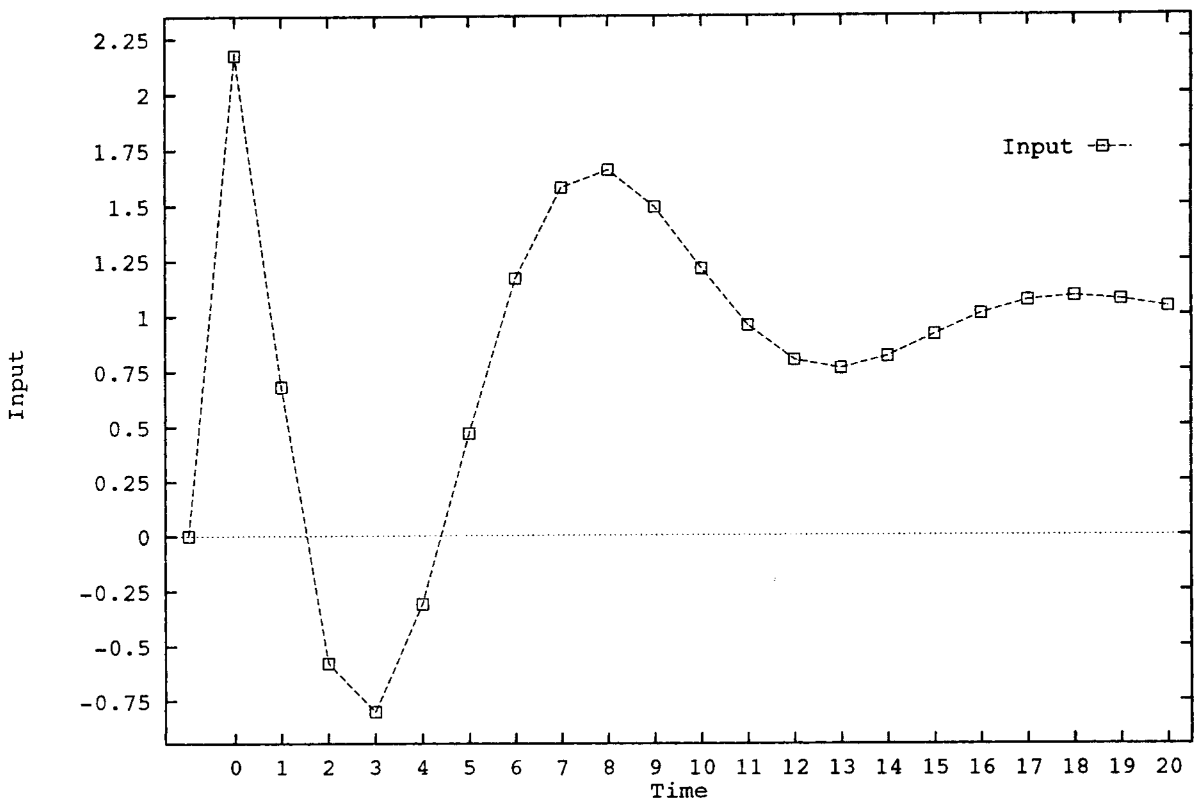


**Figure 21. Input response for feedforward control of a measured disturbance.**

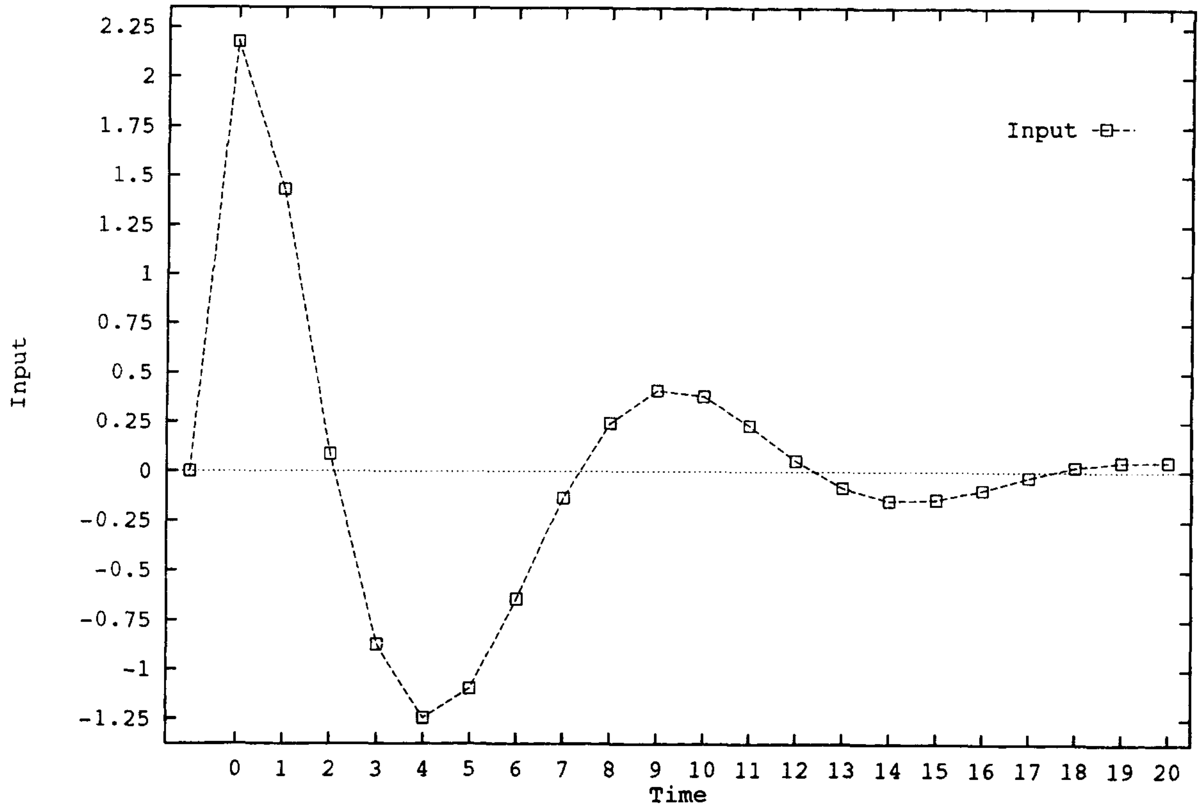


**Figure 22. Input response for feedforward control of a constant measured disturbance.**

In this example, there are no constraints and no measurement noise with the following regulator tuning parameters.

$$Q=1,\quad R=0,\quad S=0, N=5$$

At time $k=0$, the output target is changed from zero to one. Also at time $k=0$, a measured disturbance of unity magnitude enters the plant. Figures 20 and 21 show the response of the combined feedforward/feedback controller. Since the regulator is tuned as a deadbeat controller and there is no noise, the controller is able to exactly reject the measured disturbance and track the reference trajectory perfectly.

If the measured disturbance remains constant in the future, then the measured disturbance model dynamics are removed from the augmented system and the modified target tracking regulator formulation presented in this section is used. Figure 22 shows the input response of the controller in this case, which eventually settles to zero since the measured disturbance to the plant has unity gain. The output still tracks the reference trajectory perfectly as shown in Figure 20.

## Conclusions

Model predictive control has become one of the dominant methods of process control in terms of successful industrial applications and as a focus of academic research. The control papers presented at recent AIChE meetings, American Control Conferences, and the Chemical Process Control Conferences (CPC III & IV) clearly illustrate this fact. This article describes the implementation of a new model predictive control theory. By building the control algorithm around an infinite horizon, nominally stabilizing constrained regulator, nominal stability has been guaranteed independent of the controller tuning. The features of model predictive control necessary to treat difficult applications have been retained or in several instances expanded. Some of the features of many implementations that have unnecessarily restricted the approach have been removed. By exploiting the strong connections of linear model predictive control to the extensive linear quadratic regulator/estimator theory, many of the extensions are straightforward consequences of known results in that immense body of literature.

After cursorily scanning the plethora of articles that have been written on model predictive control, the reader naturally may wonder about the need for, and significance of, yet another approach. We would only comment that one of the primary functions of academic research in this area is to ensure that the implementations available for industrial use are founded on a rigorous and flexible theory. The technical machinery and mathematical detail required for that purpose may at times obscure the rather intuitive, simple, and appealing ideas that lie at the core of model predictive control. If a useful theory can be developed, however, it more than compensates with a simpler, more intuitive controller that requires fewer case-by-case exceptions and fixes. Our hope is that these results are a step in the right direction and will allow model predictive control to be further developed so that it can address several remaining challenges. These include off-line selection of tuning parameters ($N$, $Q$, $R$, $S$) to make the controller insensitive to plant modeling errors, and the control of plants described by nonlinear models.

## Notation

$A, B, C$ = state-space model matrices
$A^d, C^d$ = measured disturbance state-space model matrices
$A^m, B^m, C^m$ = feedforward state-space model matrices
$A^r, C^r$ = reference trajectory state-space model matrices
$\acute{A}, \acute{B}, \acute{C}$ = measured disturbance augmented state-space model matrices
$\bar{A}, \bar{B}, \bar{C}$ = trajectory tracking augmented state-space model matrices
$\tilde{A}, \tilde{B}, \tilde{C}$ = output feedback augmented state-space model matrices
$D, W$ = quadratic program constraint matrices
$\hat{d}$ = estimated measured disturbance vector
$F, G, H$ = quadratic program matrices
$G_1$ = finite horizon contribution to matrix $G$
$G_2$ = terminal state penalty contribution to matrix $G$
$G_w$ = state noise dynamics matrix
$G_w^d$ = disturbance model state noise dynamics matrix
$\tilde{G}_w$ = output feedback augmented state noise dynamics matrix
$H_1$ = finite horizon contribution to matrix $H$
$H_2$ = terminal state penalty contribution to matrix $H$
$J$ = eigenvalue matrix of $A$
$J^r$ = trajectory tracking initial condition matrix
$J_s$ = stable eigenvalue matrix of $A$
$J_u$ = unstable eigenvalue matrix of $A$
$L$ = filter gain matrix
$L_d$ = disturbance model filter gain matrix
$L_p$ = output step disturbance filter gain matrix
$L_x$ = state filter gain matrix
$L_z$ = state step disturbance filter gain matrix
$N$ = number of future input moves to compute
$P$ = steady-state discrete filtering Riccati matrix
$\hat{p}$ = estimated output step disturbance vector
$Q$ = output penalty matrix
$Q_\nu$ = covariance matrix of $\nu$
$Q_\omega$ = covariance matrix of $\omega$
$Q_s$ = target tracking output error penalty matrix
$Q_w$ = covariance matrix of $w$
$\bar{Q}$ = terminal state penalty matrix
$\tilde{Q}_w$ = output feedback augmented state noise covariance matrix
$R$ = input penalty matrix
$R_s$ = target tracking input penalty matrix
$R_v$ = covariance matrix of $v$
$\tilde{R}_v$ = output feedback augmented output noise covariance matrix
$S$ = input rate of change penalty matrix
$u$ = input vector
$u^N$ = vector of $N$ future input vectors
$u_{\max}$ = maximum input constraint vector
$u_{\min}$ = minimum input constraint vector
$\bar{u}$ = desired input vector at steady state
$u_s$ = steady-state input vector
$V$ = eigenvector matrix of $A$
$V_s$ = stable eigenvector matrix of $A$
$V_u$ = unstable eigenvector matrix of $A$
$v$ = zero-mean, normal output noise vector
$\hat{v}$ = estimated output noise vector
$w$ = zero-mean, normal state noise vector
$\hat{w}^N$ = vector of $N+1$ estimated state noise vectors
$\hat{w}_{\max}$ = maximum state noise constraint vector
$\hat{w}_{\min}$ = minimum state noise constraint vector
$\hat{w}$ = estimated state noise vector
$x$ = model state vector
$x^d$ = disturbance model state vector
$x^m$ = feedforward model state vector
$x^r$ = reference trajectory state vector
$x_s$ = steady-state state vector
$x_s^m$ = steady-state feedforward model state vector
$\bar{x}$ = trajectory tracking augmented state vector
$\hat{x}$ = estimated state vector
$\hat{x}^d$ = estimated disturbance model state vector
$\hat{x}_{\max}$ = maximum estimated state constraint vector
$\hat{x}_{\min}$ = minimum estimated state constraint vector
$y$ = output vector
$y_{\max}$ = maximum output constraint vector
$y_{\min}$ = minimum output constraint vector
$y^r$ = reference trajectory output vector
$y_t$ = output target vector
$z^s$ = stable modes of $A$
$z^u$ = unstable modes of $A$
$\hat{z}$ = estimated state step disturbance vector

### Greek letters

$\Delta u$ = change in the input vector
$\Delta u_{\max}$ = maximum input rate of change constraint vector
$\Delta u_{\min}$ = minimum input rate of change constraint vector
$\Theta_k$ = constrained observer objective function at time $k$
$\Phi_k$ = regulator objective function value at time $k$
$\Psi$ = target tracking objective function value
$\nu$ = zero-mean, normal output step disturbance noise vector
$\omega$ = zero-mean, normal state step disturbance noise vector

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## Literature Cited

Åström, K. J., *Introduction to Stochastic Control Theory*, Academic Press, San Diego, CA (1970).

Bitmead, R. R., M. Gevers, and V. Wertz, *Adaptive Optimal Control, The Thinking Man's GPC*, Prentice-Hall, Englewood Cliffs, NJ (1990).

Byun, D. G., and W. H. Kwon, "Predictive Control: A Review and Some New Stability Results," T. J. McAvoy, Y. Arkun, and E. Zafiriou, eds., *Proc. IFAC Workshop on Model Based Process Control*, p. 81, Pergamon Press, Oxford (1988).

Clarke, D. W., "Adaptive Generalized Predictive Control," *Proc. Int. Conf. Chemical Process Control*, p. 395 (1991).

Clarke, D. W., and C. Mohtadi, "Properties of Generalized Predictive Control," *Automat.*, **25**(6), 859 (1989).

Clarke, D. W., C. Mohtadi, and P. S. Tuffs, "Generalized Predictive Control: I. The Basic Algorithm," *Automatica*, **23**(2), 137 (1987a).

Clarke, D. W., C. Mohtadi, and P. S. Tuffs, "Generalized Predictive Control: II. Extensions and Interpretations," *Automatica*, **23**(2), 149 (1987b).

Cutler, C. R., and B. L. Ramaker, "Dynamic Matrix Control—A Computer Control Algorithm," *Proc. Joint Automatic Control Conf.* (1980).

De Keyser, R. M. C., and A. R. Van Cauwenberghe, "Extended Prediction Self-Adaptive Control," H. A. Barker and P. C. Young, eds., *Proc. IFAC Symp. on Identification and System Parameter Estimation*, p. 1255, Pergamon Press, Oxford (1985).

De Keyser, R. M. C., G. A. Van de Velde, and F. A. G. Dumortier, "A Comparative Study of Self-Adaptive Long-Range Predictive Control Methods," *Automatica*, **24**(2), 149 (1988).

Eaton, J. W., and J. B. Rawlings, "Model-Predictive Control of Chemical Processes," *Chem. Eng. Sci.*, **47**(4), 705 (1992).

Franklin, G. F., and J. D. Powell, *Digital Control of Dynamic Systems*, Addison-Wesley, Reading, MA (1980).

García, C. E., and M. Morari, "Internal Model Control: 1. A Unifying Review and Some New Results," *Ind. and Eng. Chem. Proc. Design and Develop.*, **21**, 308 (1982).

García, C. E., and A. M. Morshedi, "Quadratic Programming Solution of Dynamic Matrix Control (QDMC)," *Chem. Eng. Commun.*, **46**, 73 (1986).

García, C. E., D. M. Prett, and M. Morari, "Model Predictive Control: Theory and Practice—A Survey," *Automat.*, **25**(3), 335 (1989).

Grosdidier, P., B. Froisy, and M. Hammann, "The IDCOM—M Controller," T. J. McAvoy, Y. Arkun, and E. Zafiriou, eds., *Proc. IFAC Workshop on Model Based Process Control*, pp. 31-36, Pergamon Press, Oxford (1988).

Gutman, P., and P. Hagander, "A New Design of Constrained Controllers for Linear Systems," *IEEE Transactions on Automatic Control*, **30**(1), 22 (1985).

Kwakernaak, H., and R. Sivan, *Linear Optimal Control Systems*, Wiley, New York (1972).

Kwon, W. H., and D. G. Byun, "Receding Horizon Tracking Control as a Predictive Control and its Stability Properties," *Int. J. of Control*, **50**(5), 1807 (1989).

Kwon, W. H., and A. E. Pearson, "On Feedback Stabilization of Time-Varying Discrete Linear Systems," *IEEE Trans. on Automatic Control*, **23**(3), 479 (1978).

Li, S., K. Y. Lim, and D. G. Fisher, "A State Space Formulation of Model Predictive Control," *AIChE J.*, **35**(2), 241 (1989).

Marquis, P., and J. P. Broustail, "SMOC, A Bridge Between State Space and Model Predictive Controllers: Application to the Automation of a Hydrotreating Unit," T. J. McAvoy, Y. Arkun, and E. Zafiriou, eds., *Proc. IFAC Workshop on Model Based Process Control*, p. 37, Pergamon Press, Oxford (1988).

Maurath, P. R., D. A. Mellichamp, and D. E. Seborg, "Predictive Controller Design for Single-Input/Single-Output (SISO) Systems," *Ind. and Eng. Chem. Res.*, **27**, 956 (1988).

Morari, M., and J. H. Lee, "Model Predictive Control: The Good, the Bad, and the Ugly," Y. Arkun and W. H. Ray, eds., *Chemical Process Control—CPC IV*, p. 419, *Int. Conf. on Chemical Process Control*, Elsevier, Amsterdam (1991).

Morshedi, A. M., C. R. Cutler, and T. A. Skrovanek, "Optimal Solution of Dynamic Matrix Control with Linear Programming Techniques (LDMC)," *Proc. Amer. Control Conference*, p. 199 (1985).

Muske, K. R., and J. B. Rawlings, "Linear Model Predictive Control of Unstable Processes," *J. Process Control*, submitted (1992).

Patwardhan, A. A., J. B. Rawlings, and T. F. Edgar, "Nonlinear Model Predictive Control," *Chem. Eng. Comm.*, **87**, 123 (1990).

Prett, D. M., and C. E. Garcia, *Fundamental Process Control*, Butterworths, Boston (1988).

Rawlings, J. B., and K. R. Muske, "The Stability of Constrained Receding Horizon Control," *IEEE Trans. on Automatic Control*, accepted (1992).

Ray, W. H., *Advanced Process Control*, McGraw-Hill, New York (1981).

Richalet, J., A. Rault, J. L. Testud, and J. Papon, "Model Predictive Heuristic Control: Applications to Industrial Processes," *Automat.*, **14**, 413 (1978).

Ricker, N. L., "Use of Quadratic Programming for Constrained Internal Model Control," *Ind. and Eng. Chem. Process Des. and Dev.*, **24**, 925 (1985).

Ricker, N. L., "Model Predictive Control with State Estimation," *Ind. and Eng. Chem. Res.*, **29**, 374 (1990).

Ricker, N. L, T. Subrahmanian, and T. Sim, "Case Studies of Model-Predictive Control in Pulp and Paper Production," T. J. McAvoy, Y. Arkun, and E. Zafiriou, eds., *Proc. IFAC Workshop on Model Based Process Control*, p. 13, Pergamon Press, Oxford (1988).

Rouhani, R, and R. K. Mehra, "Model Algorithmic Control (MAC): Basic Theoretical Properties," *Automat.*, **18**(4), 401 (1982).

Scattolini, R., and S. Bittanti, "On the Choice of the Horizon in Long-Range Predictive Control—Some Simple Criteria," *Automat.*, **26**(5), 915 (1990).

Sznaier, M., and M. J. Damborg, "Heuristically Enhanced Feedback Control of Constrained Discrete-time Linear Systems," *Automat.*, **26**(3), 521 (1990).

Uppal, A., W. H. Ray, and A. B. Poore, "On the Dynamic Behavior of Continuous Stirred Tank Reactors," *Chem. Eng. Sci.*, **29**, 967 (1974).

Ydstie, B. E., "Extended Horizon Adaptive Control," J. Gertler and L. Keviczky, eds., *Proc. IFAC World Cong.*, p. 911, Pergamon Press, Oxford (1984).

Zafiriou, E., "Robust Model Predictive Control of Processes with Hard Constraints," *Comp. & Chem. Eng.*, **14**(4/5), 359 (1990).

Zafiriou, E., and A. L. Marchal, "Stability of SISO Quadratic Dynamic Matrix Control with Hard Output Constraints," *AIChE J.*, **37**(10), 1550 (1991).


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