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Random response of a simple system with stochastic uncertainty and noise in parameters

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Abstract

The paper is concerned with the analysis of the simultaneous effect of a random perturbation and white noise in the coefficient of the system on its response. The excitation of the system of the 1st order is described by the sum of a deterministic signal and additive white noise, which is partly correlated with a parametric noise. The random perturbation in the parameter is considered statistics in a set of realizations. It reveals that the probability density of these perturbations must be limited in the phase space, otherwise the system would lose the stochastic stability in probability, either immediately or after a certain time. The width of the permissible zone depends on the intensity of the parametric noise, the extent of correlation with the additive excitation noise, and the type of probability density. The general explanation is demonstrated on cases of normal, uniform, and truncated normal probability densities.

Keywords: parametric imperfections, interaction of imperfections with input noises, stochastic stability

1. Introduction

The parameters of dynamic systems are usually burdened by random noises due to the imperfect function of the system's external factors, etc. These noises are random functions of time. The problem, however, may require the determination of the response statistics, if the parameters of the system have uncertain values thanks to the variance of production, ageing, wear or degradation of the system, etc. It is coming to light that the character of such formulated problem, i.e., when the statistical set consists of individual realizations, is completely different from the case of perturbations randomly variable in time. The problem concerns the investigation of response statistics of a typical system, and the probable limits within which the response will occur under these conditions.

In such a case, the coefficients have two sources of perturbations: a random noise, usually introduced as the time variable Gaussian white noise, and random imperfections, representing the statistics in the framework of realizations of systems of the same type, described by a certain density of probability. In this respect the fact of whether the non-zero density of probability is confined onto a limited area of the phase space is of fundamental significance. If this area is not limited (e.g., classic normal distribution), the stochastic stability of the system fails; either immediately or after a certain time. The width of the admissible zone of every phase variable is dependent on the parametric noise intensity, the extent of its correlation with the additive excitation noise, and the type of probability density of imperfections. It is obvious that the stochastic stability is a crucial issue, and the core of the investigation of the problem discussed.

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Stochastic stability is a widely elaborated area. Its essentials are discussed in many monographs, e.g., [2, 3, 7–9]. Furthermore, many papers can be cited investigating specific attributes of the stochastic stability phenomena related to the topic dealt with in this study, see [1, 11, 12] and many others. This paper extends the results cited in the references. It applies imperfections understood as a set of individual cases of the system characterized by a given probability density function, and interacting with multiplicative (internal), and additive (external) noises perturbing the signal evaluated. This problem definition leads to the emerging of a couple of new effects, which are worth examining, as for instance the existence and quantification of response asymmetry, necessary limitations of input imperfections PDF, possibilities in approximations, etc.

This paper is organized as follows: After this introduction, the paper describes the used mathematical model based on the Fokker-Planck equation. Three subsequent sections apply the model to individual cases of the probability distributions of imperfections, namely to the normal distribution, the uniform distribution, and the truncated normal distribution. The last section is the conclusion.

2. Mathematical model

Let us consider a simple system, the function of which can be described by a stochastic differential equation of the first order

$$\dot{u}(t) = -[C + p + w(t)]u(t) + f(t) + \varphi(t), \tag{1}$$

where
C – constant, nominal value of a system parameter,
p – deviation (imperfection) of the parameter from its nominal value; it represents a centered random variable with a prescribed probability density function $h(p)$ with respect to multiplicative and additive perturbations $w(t), \varphi(t)$,
$f(t)$ – useful signal (deterministic part of excitation),
$w(t), \varphi(t)$ – time variable parametric perturbation or additive perturbation of excitation; Gaussian white noises of constant intensities.

The initial condition of the response $u(0) = u_\circ$ is a random quantity of probability density $h_\circ(u_\circ)$. The imperfection p can be considered a constant. Within every individual realization of the system or time period of its service, the parameter is not subjected to differentiation or integration in time.

For every fixed value of p, we can consider in (1) the processes $w(t), \varphi(t), u(t)$ as Markov processes in time. With respect to (1), the Fokker-Planck equation (FPE) can be deduced. Using the Itô white noise definition, see [2, 3, 9, 13] and other monographs, the relevant equation can be written as

$$\begin{aligned} \frac{\partial h(u,t)}{\partial t} = & \frac{\partial}{\partial u}\left\{\left[\left(C + p - \frac{1}{2}s_w\right)u + \frac{1}{2}s_{w\varphi} - f(t)\right] h(u,t)\right\} + \\ & \frac{1}{2}\frac{\partial^2}{\partial u^2}\Big[\left(s_w u^2 - 2s_{w\varphi}u + s_\varphi\right) h(u,t)\Big], \end{aligned} \tag{2}$$

where
$h = h(u,t)$ – probability density function of the system response,
$s_w, s_{w\varphi}, s_\varphi$ – intensities or cross-intensity of parametric and additive noises $w(t), \varphi(t)$.

Using (2) for the construction of the equations for the first and second stochastic moments of the response, it can be obtained

$$\dot{u}_s(t) = -\Big(C + p^i - \frac{1}{2}s_w\Big)u_s(t) - \frac{1}{2}s_{w\varphi} + f(t), \qquad u_s^i(0) = u_{s\circ}^i, \tag{3}$$

$$\dot{D}_u(t) = -2(C + p^i - s_w)D_u(t) - 2s_{w\varphi}u_s(t) + s_\varphi, \qquad D_u^i(0) = D_{u\circ}^i, \tag{4}$$

where
$u_s(t)$ – expected value of the response for the i-th realization of the parameter imperfection,
$D_u(t)$ – variance of the response for the i-th realization of the parameter imperfection,
$u_{s\circ}^i, D_{u\circ}^i$ – random initial conditions (we will introduce the assumption of the statistical independence of initial conditions and parameter imperfections),
p^i – fixed value of the perturbation p.

Note that the expansion represented by (3) and (4) can be easily extended to higher moments. We would obtain a hierarchy, where each additional equation contains only one unknown higher moment, and all lower moments in linear expressions can be considered known from the previous analysis. For example, see (4) containing a linear expression with $u_s(t)$. The influence of higher moments, however, is related to multiplicative perturbation $w(t)$, and can be incorporated into results on a level of the two first moments approximately, provided the noise $w(t)$ remains within the scale of a small perturbation. For a detailed discussion, see [12].

The solution of (3) and (4) can be expressed by means of Green functions in the form of

$$u_s(t) = u_{sg}(t, 0, p^i) \cdot u_{s\circ} + \int_0^t u_{sg}(t, \tau, p^i)\Big(f(\tau) - \frac{1}{2}s_{w\varphi}\Big)\,\mathrm{d}\tau, \tag{5}$$

$$D_u(t) = D_{ug}(t, 0, p^i) \cdot D_{u\circ} + \int_0^t D_{ug}(t, \tau, p^i)(s_\varphi - 2s_{w\varphi}u_s(\tau))\,\mathrm{d}\tau, \tag{6}$$

where $u_{sg}(t, \tau, p^i)$, $D_{ug}(t, \tau, p^i)$ are the Green functions arising from (3) and (4) for annulled right-hand sides, and initial conditions of $u_{sg}(\tau, \tau, p^i) = 1$, $D_{ug}(\tau, \tau, p^i) = 0$ and $u_{sg}(\tau, \tau, p^i) = 0$, $D_{ug}(\tau, \tau, p^i) = 1$, respectively,

$$u_{sg}(t, \tau, p^i) = \mathrm{e}^{-\left(C+p^i-\frac{1}{2}s_w\right)(t-\tau)}, \tag{7}$$

$$D_{ug}(t, \tau, p^i) = \mathrm{e}^{-2\left(C+p^i-s_w\right)(t-\tau)}. \tag{8}$$

Let us recall that the FPE (2) and subsequent equations characterizing moments $u_s(t, \tau, p^i)$, $D_u(t, \tau, p^i)$ determine the system's behavior related to one element p^i of the set of perturbations p. To obtain the expected value $u_s(t)$ and variance of the response $D_u(t)$ on the set of realizations, we apply the expectation operator $\mathbf{E}\{\cdot\}$ to (5) and (6), using the facts that in the given case the expectation and integration operators are mutually commutable, and that the initial conditions are independent of imperfections. That means

$$u_s(t) = u_{sg}(t, 0) \cdot u_{s\circ} + \int_0^t u_{sg}(t, \tau)\Big(f(\tau) - \frac{1}{2}s_{w\varphi}\Big)\,\mathrm{d}\tau, \tag{9}$$

$$D_u(t) = D_{ug}(t, 0) \cdot D_{u\circ} + \int_0^t D_{ug}(t, \tau)(s_\varphi - 2s_{w\varphi}u_s(\tau))\,\mathrm{d}\tau, \tag{10}$$

$$u_{sg}(t, \tau) = \mathbf{E}\{u_{sg}(t, \tau, p^i)\}, \qquad D_{ug}(t, \tau) = \mathbf{E}\{D_{ug}(t, \tau, p^i)\}. \tag{11}$$

The kernels of integrals in (9) and (10) implicitly depend, in the meaning of (11), on the probability density of the imperfection p, while the influence of particularly the additive noise $\varphi(t)$ is expressed in (9) and (10) relatively distinctly.

Note that (10) implicitly includes an approximate relation

$$\mathbf{E}\{D_{ug}(t,\tau,p^i)u_s^i(\tau)\} \approx \mathbf{E}\{D_{ug}(t,\tau,p^i)\} \cdot u_s(\tau),$$

which is often used when analyzing the hierarchies of (9) and (10) type, see, e.g., [9, 13]. The aim of this step is to eliminate the secondarily emerging nonlinear term. Indeed, both sets we are working with consist of a mean value and a certain ϵ-scaled perturbation. The product of both perturbations results in a value with ϵ^2-scale (i.e., small of the higher order), and is therefore negligible. In general, this assumption can also be considered as the independence of $D_{ug}(t,\tau,p^i)$ and $u_s(\tau)$. Another interpretation is that the hierarchy (9) and (10) is solved sequentially, when the averaged $u_s(t)$ resulting from (9) is substituted into (10). A noteworthy fact is that (9) remains unaffected by this approximate step. A certain error resulting from this approximation is further reduced by the fact that the cross-correlation given by the coefficient $s_{w\varphi}$ is usually small, which many authors neglect by stating $s_{w\varphi} = 0$. Analogous steps can be taken when a broader expansion, and a higher hierarchy than that in (9) and (10) is dissected.

3. Normal distribution of imperfections

The influence of a discrete set on the stochastic characteristics of the response can be evaluated examining (7) and (8) as a stochastic set for elements p^i limiting the number of realizations to infinity. This limitation is entitled if the response process for individual cases of p^i is stable (see [5] and later editions of this book) and the theorems on stochastic convergence are applicable.

Evaluating the stability of the system with respect to noises $w(t)$, $\varphi(t)$, and perturbations p, we manipulate the first two moments. In so doing, we will consider the stability in the meaning of stability of the expected value $u_s(t)$, and variance $D_u(t)$ investigating their behavior starting from the initial conditions at $t = 0$ and limiting $t \to \infty$.

The uncertainty of the quality of individual parts of the system is usually approximated by the normal distribution or its expected value (zero in this case), and by variance D_p, see [4],

$$h(p) = \frac{1}{\sqrt{2\pi D_p}} \cdot \mathrm{e}^{-\frac{p^2}{2D_p}}. \tag{12}$$

If we substitute (7) and (12) into (11), we obtain after limitation for the number of realizations growing beyond all limits

$$u_{sg}(t,\tau) = \frac{1}{\sqrt{2\pi D_p}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{p^2}{2D_p} - (C+p-\frac{1}{2}s_w)(t-\tau)}\, \mathrm{d}p,$$

which yields

$$u_{sg}(t,\tau) = \mathrm{e}^{-(C-\frac{1}{2}s_w)(t-\tau)+\frac{1}{2}D_p(t-\tau)^2}. \tag{13}$$

Using (8), (12) and (11), we will arrive at the expression for $D_{sg}(t,\tau)$

$$D_{sg}(t,\tau) = \mathrm{e}^{-2(C-s_w)(t-\tau)+2D_p(t-\tau)^2}. \tag{14}$$

From the viewpoint of the analysis of a linear stochastic system and its properties, the results do not differ qualitatively for various histories of the useful signal $f(t)$, if it is square integrable within a finite time interval, see [10]. Thus, the useful signal can be introduced as a constant $f(t) = f_\circ > 0$. Therefore, we will substitute (13) in (9):

$$u_s(t) = u_{s\circ} \cdot \mathrm{e}^{-(C-\frac{1}{2}s_w)t+\frac{1}{2}D_p t^2} + \left(f_\circ - \frac{1}{2}s_{w\varphi}\right) \int_0^t \mathrm{e}^{-(C-\frac{1}{2}s_w)(t-\tau)+\frac{1}{2}D_p(t-\tau)^2}\,\mathrm{d}\tau. \tag{15}$$

For $t = 0$, the mean value of the response (15) equals the mean value of the initial condition $u_s(0) = u_{s\circ}$. For increasing time t, the stochastic stability in probability of the system is preserved if the exponent of the first term is negative. Hence it follows that

$$\text{a)} \qquad C > \tfrac{1}{2}s_w \tag{16}$$

For $0 < t < t_m = 2(C - \frac{1}{2}s_w)/D_p$, i.e., within a finite time interval, the system is stable in probability. For $t > t_m$, the first term in (15) increases exponentially. The same also holds for the second term. As both terms increase without limitations, also the whole expression (15) grows without limitations regardless of the sign of the second term, which is determined by a mutual relation of $f_\circ$, and $\frac{1}{2}s_{w\varphi}$. Even in the special case when $u_{s\circ} = -(f_\circ - \frac{1}{2}s_{w\varphi})$ the right-hand side of (15) diverges for $t \to \infty$.

$$\text{b)} \qquad C \leq \tfrac{1}{2}s_w \tag{17}$$

The system is unstable from the very beginning, i.e., for all $t > 0$. These conclusions would not change even if $f(t) \neq \mathrm{const}$. The loss of stability in both cases results from the non-zero variance of imperfections D_p. In case of $D_p \to 0$, the system would become stable in probability for all $t > 0$ if $C > \frac{1}{2}s_w$. However, in the case when $t \to \infty$, the expected value of the response is non-zero even for a zero useful signal if the mutual correlation of both noises differs from zero. The systematic deviation is non-zero for $f_\circ > 0$ even for independent noises.

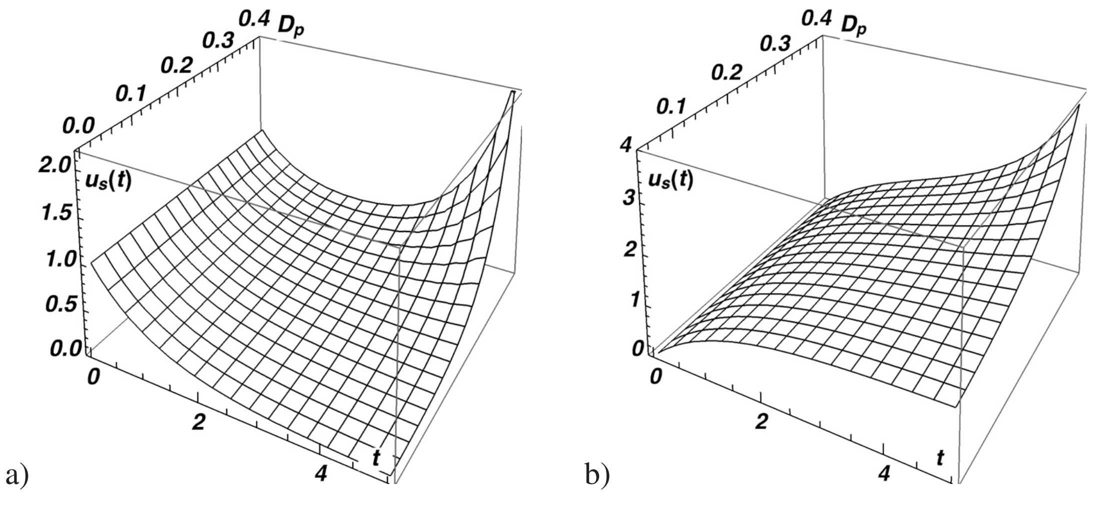

Fig. 1. Expectation of the response for the normal distribution of imperfections ($s_\varphi = 0$, $s_{w\varphi} = 0$, $C = 1$, $s_w = 0.3$): (a) $f_0 = 0.0$, $u_{s0} = 1.0$; (b) $f_0 = 1.0$, $u_{s0} = 0.0$

The character of the expected value $u_d(t)$ is demonstrated by the plots in Fig. 1. Either the temporary stability or no stability is obvious depending on D_p. In order to highlight the influence of imperfections, noises have been suppressed. The temporary stability is visible especially in Fig. 1a, when no useful signal is applied. However, even the solely useful signal is able to bring the system to an unstable state, despite the homogeneous initial condition.

4. Uniform distribution of imperfections

The conclusions of the preceding section show that the non-zero probability of imperfections p of unlimited magnitude yields unrealistic results. The possibility of a great deviation from the nominal value of the parameter C means that the system contains an element that practically eludes the basic quality requirements. For instance, a material would be so saturated with cracks that the system would collapse much earlier than expected. Although such a state, provided it occurs after a certain time, could be interpreted as a moment of the life time period exhaustion, although the present one-dimensional model is not probably suitable for this type of analysis, and does not form the object of investigation of this paper. What is substantial is that too large imperfections p must be eliminated in advance from our analysis. This can be attained by the limitation of the probability density of imperfections p on either side. Such limitation can be certainly assumed in practice for the most varied technological and operational reasons.

The simplest distribution of probability complying with these requirements is uniform distribution, see [3, 7] and Fig. 2,

$$h(p) = \begin{cases} \frac{1}{2\Delta} & \text{for } -\Delta < p < \Delta, \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

where Δ is the parameter determining the bounds of imperfections p.

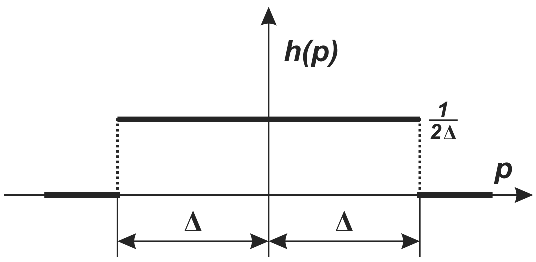

Fig. 2. Uniform distribution

For $u_{sg}(t,\tau)$, we can write according to (11), taking into account (7) and (18),

$$u_{sg}(t,\tau) = \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} \mathrm{e}^{-\left(C+p-\frac{1}{2}s_w\right)(t-\tau)}\,\mathrm{d}p, \tag{19}$$

from which we obtain by a simple integration

$$u_{sg}(t,\tau) = \mathrm{e}^{-(C-\frac{1}{2}s_w)(t-\tau)}\cdot\frac{\sinh\,(\Delta(t-\tau))}{\Delta(t-\tau)}, \quad 0 < \tau \le t. \tag{20}$$

For small values of $t-\tau$, we can use

$$u_{sg}(t,\tau) \approx \mathrm{e}^{-(C-\frac{1}{2}s_w)(t-\tau)}\cdot\left(1+\frac{1}{6}\Delta^2(t-\tau)^2\right). \tag{21}$$

Substituting (20) in (9), we come to the formula for the expectation of the response

$$u_s(t) = u_{s\circ}\,\mathrm{e}^{-(C-\frac{1}{2}s_w)t}\,\frac{\sinh\,\Delta t}{\Delta t} + \int_0^t \mathrm{e}^{-(C-\frac{1}{2}s_w)(t-\tau)}\,\frac{\sinh\,(\Delta(t-\tau))}{\Delta(t-\tau)}\left(f(\tau)-\frac{1}{2}s_{w\varphi}\right)\mathrm{d}\tau. \tag{22}$$

We will introduce the constant useful signal $f(t) = f_\circ$. At this state, the integral in (22) will turn into the integral-exponential. If the system complies with the condition (16) of the stochastic stability in probability, the first part of the expression diverges for $t \to \infty$ provided $0 < \Delta < C - \frac{1}{2}s_w$. This condition constitutes a necessary requirement on probability distribution $h(p)$, as it was mentioned in the first paragraph of this section. In the opposite case, i.e., when $\Delta > C - \frac{1}{2}s_w$, the first term vanishes and only the integral term remains. The stationary value can be obtained when limiting $t \to \infty$

$$\lim_{t\to\infty} u_s(t) = u_{sn} = \frac{f_\circ - \frac{1}{2}s_{w\varphi}}{2\Delta} \cdot \lg\left(\frac{C - \frac{1}{2}s_w + \Delta}{C - \frac{1}{2}s_w - \Delta}\right). \tag{23}$$

Let us consider some special cases. If $s_w = 0$, $s_{w\varphi} = 0$, i.e., there is no parametric noise in the system, the expected value of the response in the stationary state is as follows

$$u_{sn} = \frac{f_\circ}{2\Delta} \cdot \lg\left(\frac{C + \Delta}{C - \Delta}\right). \tag{24}$$

If equation (16) holds true, then the argument of the logarithm in (23) will be positive only if

$$C > \frac{1}{2}s_w + \Delta, \tag{25}$$

which can be considered as the condition of stochastic stability in probability for the uniform distribution of the imperfections, and, consequently, as a certain generalization of the condition (16). In other words, the random deviations p of the parameter C must remain within the interval $\pm(C - \frac{1}{2}s_w)$. If the parametric noise $w(t)$ is very powerful, it is possible to permit a smaller zone of permissible imperfections expressed by the width 2Δ of the uniform distribution, see equation (18).

If the perturbations p of parameter C disappear, that means $\Delta = 0$, the limitation of (23) for $\Delta \to 0$ yields a result corresponding to the parametric and additive noises only

$$u_{sn} = \frac{f_\circ - \frac{1}{2}s_{w\varphi}}{C - \frac{1}{2}s_w} \tag{26}$$

and the expectation of the response deviation from the response equals

$$u^d_{sn} = \frac{f_\circ - \frac{1}{2}s_{w\varphi}}{C - \frac{1}{2}s_w} - \frac{f_\circ}{C}. \tag{27}$$

If there is no parametric noise in the system, according to (27), it holds $u^d_{sn} = 0$. The character of the response expectation depending on the parameter Δ is obvious from Fig. 3. The thick curve separates the stable and unstable domains for $\Delta = C - \frac{1}{2}s_w = 0.85$. The individual plots show cases for excitation induced solely by the initial condition (Fig. 3a), and that with the homogeneous initial condition, and a constant useful signal (Fig. 3b). Both pictures demonstrate that the system for super-critical Δ is stable only temporarily. However, for sub-critical Δ it remains stable permanently.

Let us determine the variance of the system response. We will substitute in (10) according to (22), and will obtain the following expression

$$D_u(t) = D_{u\circ}\mathrm{e}^{-2t(C-s_w)}\,\frac{\sinh\,2\Delta t}{2\Delta t} + \\ \frac{1}{2\Delta}\int_0^t \mathrm{e}^{-2(C-s_w)(t-\tau)}\,\frac{\sinh\,(2\Delta(t-\tau))}{t-\tau}\,(s_\varphi - 2s_{w\varphi}u_s(\tau))\,\mathrm{d}\tau. \tag{28}$$

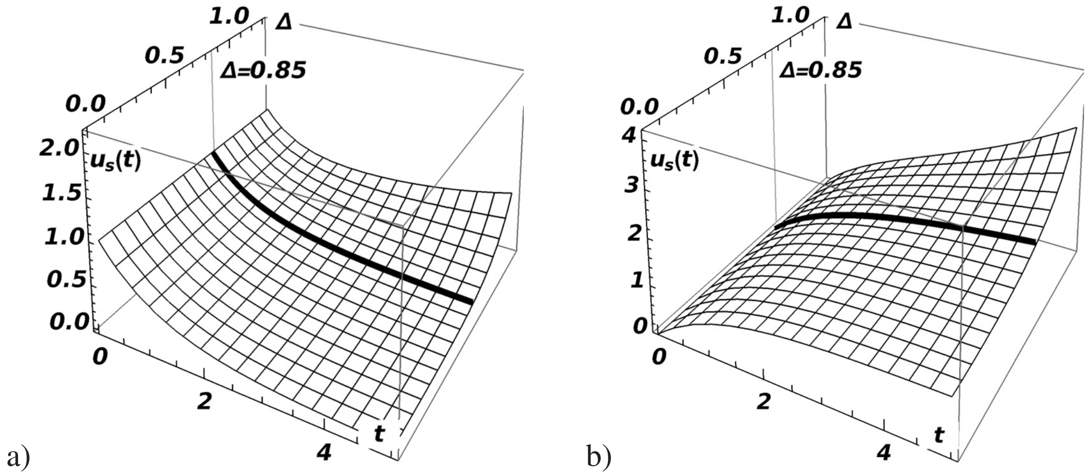

Fig. 3. Expectation of the response for the uniform distribution of imperfections; the parametric noise is not acting; $C = 1, s_w = 0.3$; initial conditions: (a) $f_0 = 0, u_{s0} = 1$; (b) $f_0 = 1, u_{s0} = 0$

The integral in (28) exists, if the condition

$$C > (s_w + \Delta) \tag{29}$$

is complied with, which case corresponds to the condition of stability in the 2nd stochastic moment, see [2] or [7], and if $u_s(t)$ is square integrable.

For a constant useful signal $f(\tau) = f_\circ$, the stationary expected value of the response is given by (23). Provided the condition (29) is satisfied, the stationary value of system variance may be expressed as

$$D_{un} = \frac{s_\varphi - 2s_{w\varphi}u_{sn}}{2\Delta} \lim_{t\to\infty} \int_0^t \mathrm{e}^{-2(C-s_w)(t-\tau)} \, \frac{\sinh\,(2\Delta(t-\tau))}{t-\tau} \, \mathrm{d}\tau, \tag{30}$$

which can be evaluated in a form of

$$D_{un} = \frac{s_\varphi - 2s_{w\varphi}u_{sn}}{4\Delta} \cdot \lg\left(\frac{C - s_w + \Delta}{C - s_w - \Delta}\right). \tag{31}$$

If the parameter C is not burdened with the perturbations of p, but merely with the noise $w(t)$, for $\Delta \to 0$ the last equation can be simplified to

$$D_{un} = \frac{\frac{1}{2}s_\varphi - s_{w\varphi}u_{sn}}{C - s_w}.$$

If, in addition, the parametric and additive noises are mutually independent, $s_{w\varphi} = 0$, equation (31) simplifies as follows:

$$D_{un} = \frac{s_\varphi}{2(C - s_w)}, \tag{32}$$

which corresponds to the solution of an analogous problem with parametric noises, see [10]. The variance of the response increases, provided the condition (29) has been complied with, faster with the increasing width 2Δ of the zone of perturbations, and slightly more slowly with the increasing intensity of the noise $w(t)$.

Fig. 4 illustrates the variance of the response corresponding the zero and constant useful signal $f(t) = 1$ in the left and right plots, respectively. The stability boundary, which is denoted by the thick curve, is shifted to the left to $\Delta = C - s_w = 0.7$. Comparing Figs. 3 and 4, it can be seen that the variance for unstable values of Δ diverges significantly faster than the expected value.

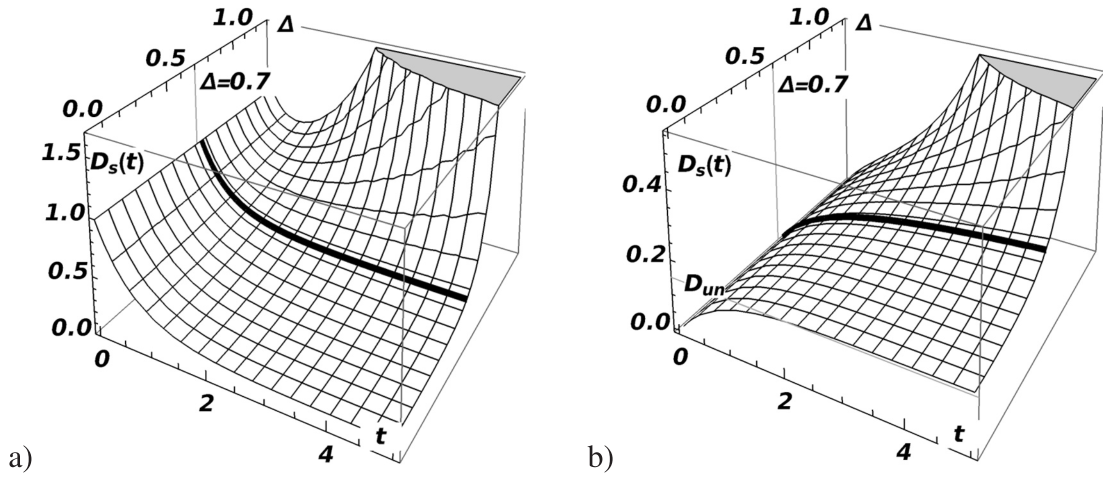

Fig. 4. Variance of the response for the uniform distribution of imperfections; the noises are assumed independent; $C = 1$, $s_w = 0.3$, $s_{w\varphi} = 0.2$; initial conditions: (a) $f_0 = 0$, $u_{s0} = 1.0$, $D_{u0} = 1$; (b) $f_0 = 1$, $u_{s0} = 0$, $D_{u0} = 0$

5. Truncated normal distribution of imperfections

Despite the analysis in Section 4, it is generally known that parameter imperfections are rather characterized by the normal distribution. We try to avoid the paradox, which emerged in Section 3, by means of the limitation of imperfection values. Theoretically considered, we gave up the Gaussian distribution again, but this approximation is probably the nearest to reality, when working with a selected set of realizations. Hence, let us assume that the probability density of imperfections p is described by the formulas, see, e.g., [4] and Fig. 5,

$$h(p) = \begin{cases} \frac{\mu}{\sqrt{2\pi D_p}} \cdot \mathrm{e}^{-\frac{p^2}{2D_p}} & \text{for } -\Delta < p < \Delta, \\ 0 & \text{otherwise,} \end{cases} \tag{33}$$

$$\mu^{-1} = 2\Phi\Big(\frac{\Delta}{\sqrt{2D_p}}\Big) = \frac{2}{\sqrt{2\pi}} \int_0^{\frac{\Delta}{\sqrt{D_p}}} \mathrm{e}^{-\frac{\xi^2}{2}}\, \mathrm{d}\xi. \tag{34}$$

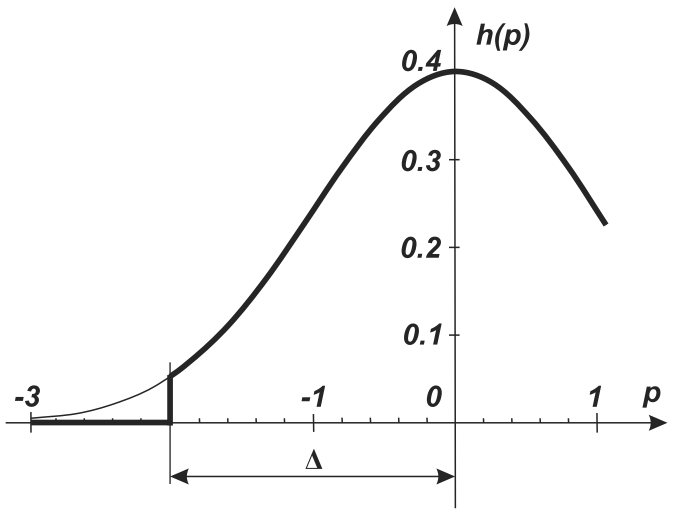

Fig. 5. Truncated normal distribution

In equations (33) and (34), D_p means the variance of the initial Gauss distribution. The variance of the distribution according to (33) has the value of

$$D_{pt} = D_p \left(1 - \frac{2\mu\Delta}{\sqrt{2\pi D_p}} \mathrm{e}^{-\frac{\Delta^2}{2D_p}}\right), \tag{35}$$

which for $\Delta \to \infty$ approaches the initial D_p.

As in the two preceding cases, we will substitute in (10) according to (7) and (33)

$$u_{sg}(t,\tau) = \frac{\mu}{\sqrt{2\pi D_p}} \int_{-\Delta}^{\Delta} \mathrm{e}^{-\frac{p^2}{2D_p} - (C+p-\frac{1}{2}s_w)(t-\tau)}\, \mathrm{d}p, \tag{36}$$

the integration of which yields

$$u_{sg}(t,\tau) = \mu \left[\Phi\left(\frac{\Delta + D_p(t-\tau)}{\sqrt{2D_p}}\right) + \Phi\left(\frac{\Delta - D_p(t-\tau)}{\sqrt{2D_p}}\right)\right] \mathrm{e}^{-(C-\frac{1}{2}s_w)(t-\tau)+\frac{1}{2}D_p(t-\tau)^2}. \tag{37}$$

Equation (37) is an expression similar to (13). For $(t-\tau) \to \infty$, however, it does not diverge as it can be proved easily. For large magnitudes of $t-\tau$, the asymptotic series applies, see [6],

$$2\,\Phi(z) \approx \begin{cases} 1 - \psi(z), & z \gg 0, \\ -1 - \psi(z), & z \ll 0, \end{cases} \qquad \psi(z) = \frac{\mathrm{e}^{-z^2}}{\sqrt{\pi}\cdot z}(1 - \frac{1}{2z^2} + \ldots). \tag{38}$$

After the substitution of (35) and (38) into (37) and the subsequent modification, we obtain

$$u_{sg}(t,\tau) \approx \frac{\sqrt{D_p}\,\mathrm{e}^{-\frac{\Delta^2}{2D_p}}}{2\sqrt{2\pi}\Phi\big(\frac{\Delta}{\sqrt{2D_p}}\big)\big(D_p^2(t-\tau)^2 - \Delta^2\big)} \cdot$$
$$\left[(\Delta - D_p(t-\tau))\,\mathrm{e}^{(-\Delta-C+\frac{1}{2}s_w)(t-\tau)} + (\Delta + D_p(t-\tau))\,\mathrm{e}^{(\Delta-C+\frac{1}{2}s_w)(t-\tau)}\right]. \tag{39}$$

For increasing $t-\tau$, the value of (39) approaches zero, if the condition (25) has been complied with, and if Δ is finite. For $\Delta \to \infty$ (the normal distribution without limitation), the fraction preceding the square bracket diverges, as the denominator decreases faster than the numerator, for further details, see [10].

The expression (37), consequently, is meaningful. We will substitute it in (9), and will ascertain immediately that under the conditions that $\Delta < \infty$, and (25) holds, the influence of initial conditions successively disappears with increasing t. If we use in the integral term, once again, the constant useful signal, we obtain for the stationary state

$$u_s(t)\Big|_{t\to\infty} = \frac{f_\circ - \frac{1}{2}s_{w\varphi}}{2\Phi\big(\frac{\Delta}{\sqrt{2D_p}}\big)} \int\limits_0^\infty \left[\Phi\left(\frac{\Delta + D_p\xi}{\sqrt{2D_p}}\right) + \Phi\left(\frac{\Delta - D_p\xi}{\sqrt{2D_p}}\right)\right] \cdot \mathrm{e}^{-(C-\frac{1}{2}s_w)\xi+\frac{1}{2}D_p\xi^2}\,\mathrm{d}\xi, \tag{40}$$

which is the final non-zero value influenced by the mutual correlation of both input noises, which increases without limitations for $\Delta \to \infty$. When $\Delta \to 0$ and $D_p \to 0$, the perturbations disappear and (40) changes continuously into (26).

If we compare (40) and (23) (numerically), we will ascertain that for identical Δ and excitation parameters, the expectation of the response is higher for the uniform distribution of the perturbations of parameter C.

The variance of the response may be ascertained by a similar method, using equations (8), (10), (11) and (33) with the assistance of (40) or a more general formula for $u_s(t)$. The variance first rises from zero, attains maximum, and then drops, and approaches asymptotically the constant value characterizing the stationary state. Other qualitative characteristics of the variance are similar to those of the expected value.

6. Conclusion

The stochastic perturbations of a system parameter, understood in the meaning of a set of realizations, result in a loss of the stochastic stability of the system, if their probability density differs from zero in the whole definition interval. For example, if the perturbations follow the normal probability distribution, the system becomes unstable in probability after a certain time, and its response in the meaning of the expected value rises exponentially. The physical reason for this result is the non-zero probability of such a large parameter deviation that brings the system into a permanently unstable state. This situation may occur either in connection with the additive noise or even without it, even in the case of a single realization.

The system's stability is preserved if we abandon the assumption of the "unlimited" Gaussian distribution of parameter imperfections, and if the standard conditions of the stochastic stability in probability or in the second stochastic moment are complied with. The width of the permissible zone of imperfections is determined by the white noise intensity in the coefficient; it decreases with increasing noise intensity, and vice versa. The width of the permissible zone is larger for the truncated normal distribution than for the uniform distribution.

The effect of the right-hand side with an additive noise for truncated distribution densities of parameter imperfections results in a constant stationary expectation of the response, the constant value of the variance, and, therefore, the stationary correlation function. The same applies to higher statistical moments, and the density of probability of the response.

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