FAKULTA APLIKOVANÝCH VĚD
ZÁPADOČESKÉ UNIVERZITY
V PLZNI

STUDENTSKÁ VĚDECKÁ KONFERENCE 2022

Reaction-diffusion equations in discrete space

Jakub Hesoun[1]

1 Introduction

A lattice Nagumo equation

$$\frac{dx_i}{dt} = D\left(x_{i-1} - 2x_i + x_{i+1}\right) + \lambda x_i\left(1 - x_i\right)\left(x_i - a\right), \quad i \in \mathbb{Z}, \tag{1}$$

same as its continuous version

$$u_t = Du_{xx} + \lambda u(1-u)(u-a), \quad x \in \mathbb{R},\, t > 0, \tag{2}$$

are subject of many studies for their richness of behaviour. While for sufficiently strong diffusion rate $D > 0$ they behave similarly and the existence of a traveling wave can be proven. For a small diffusion rate it has been shown by Keener (1987) that in the semi-discrete system (1) there exists a large number of stationary solutions which prevent propagation. This is not the case for continuous Nagumo equation (1), where a traveling wave exists for any $D > 0$.

Instead of the lattice, which can be interpreted as a special type of a graph - the infinite path, we can assume a finite graphs as in Stehlík (2017). The first part of our work is dedicated to complete bipartite graphs. In particular, main results here are obtained for a two patch Nagumo equation

$$\begin{cases} \dfrac{dx_1}{dt} &= D\left(x_2 - x_1\right) + \lambda_1 x_1\left(1 - \frac{x_1}{k_1}\right)\left(\frac{x_1}{k_1} - a\right), \\ \dfrac{dx_2}{dt} &= D\left(x_1 - x_2\right) + \lambda_2 x_2\left(1 - \frac{x_2}{k_2}\right)\left(\frac{x_2}{k_2} - a\right), \end{cases} \tag{3}$$

i.e., the complete bipartite graph with only two vertices.

Results obtained for the system (3) are further used for a two periodic semi-discrete Nagumo equation with different capacities on lattice

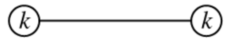
$$\begin{cases} \dfrac{dx_i}{dt} &= D\left(x_{i-1} - 2x_i + x_{i+1}\right) + \lambda_1 x_i\left(1 - \frac{x_i}{k_1}\right)\left(\frac{x_i}{k_1} - a\right), \quad i = 2k,\, k \in \mathbb{Z}, \\ \dfrac{dx_i}{dt} &= D\left(x_{i-1} - 2x_i + x_{i+1}\right) + \lambda_2 x_i\left(1 - \frac{x_i}{k_2}\right)\left(\frac{x_i}{k_2} - a\right), \quad i = 2k-1,\, k \in \mathbb{Z}. \end{cases} \tag{4}$$

In the both systems (3) and (4), $\lambda_1,\, \lambda_2 > 0$ stand for strength of reactions, $D > 0$ for the diffusion rate and $k_1,\, k_2 > 0$ for capacities of territories.

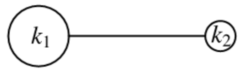
2 Results - finite graphs

While in the lattice Nagumo equation (1) and a homogeneous version of semi-discrete Nagumo equation on graph or two patches, i.e., the system (3) with $k_1 = k_2 = 1$, there always

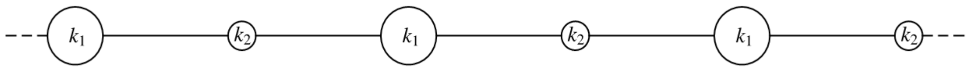
[1] student of the master's degree program Mathematics and its Application, Department of Mathematics, University of West Bohemia, e-mail: hesounj@students.zcu.cz



(a) Homogeneous K_2.



(b) Heterogeneous K_2.



(c) Heterogeneous 2-periodic lattice.

Figure 1: Examples of structures for the semi-discrete versions of Nagumo equation (2).

exist at least three stationary solution, for the heterogeneous systems with different capacities we derive different results.

For a sufficiently large difference of the capacities, the existence of a unique stationary solution of the system (3) is proven in three steps. First we derive two regions where a stationary solutions of this system may be located. Further, we eliminate a possibility that a stationary solution in each of these region does exist by applying additional assumptions on reaction terms.

Results obtained for the system (3) can be simply generalized for a complete bipartite graphs. This generalization is mainly used as intermediate step between the system (3) and the system (4), i.e., intermediate step between two dimensional case and infinite dimensional one.

3 Results - heterogeneous lattice

We show that almost identical assumptions as for two patches Nagumo equation (3) guarantee the existence of a unique non-negative bounded stationary solution of (4). This directly implies that in such a case, there is no traveling wave connecting two bounded stationary solution. One of the assumptions is a sufficiently strong diffusion rate, i.e., opposite boundary for diffusion as for homogeneous lattice system (1).

Last, we analyze unbounded stationary solutions (4). We show, that not only there are countable many non-negative unbounded stationary solutions based on a translation of a profile of such solution (the system (4) stays the same if we shift indexes i by even number) but in fact there are uncountable many of these solutions. Moreover, we prove that there exist two types of non-negative unbounded stationary solutions - both sided unbounded and one sided unbounded stationary solution, where the other side converges to zero.

Acknowledgement

I would like to thank my supervisor doc. RNDr. Petr Stehlík, Ph.D. for his support and kind guidance throughout the course of thesis preparation.

References

Keener J.P. (1987) Propagation and Its Failure in Coupled Systems of Discrete Excitable Cells. *SIAM Journal on Applied Mathematics*, Volume 47(3), pp. 556–572.

Stehlík, P. (2017) Exponential number of stationary solutions for Nagumo equations on graphs. *Journal of Mathematical Analysis and Applications*, Volume 455(2), pp. 1749–1764.