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Analysis of van der Pol equation on slow time scale for combined random and harmonic excitation

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1. Introduction

Vortex shedding represents one of the most important processes that constantly attract the attention of experimental and theoretical research. A number of non-linear effects arise from the fluid-structure interaction. The non-stationary response in the vicinity of the lock-in region has a quasi-periodic character, beating frequency of which varies considerably with the distance from the lock-in frequency. This property is significantly affected by the assumption of combined random and harmonic excitation. This paper describes several details that contribute to the probabilistic characteristics of the system on a time-slow scale using partial response amplitudes.

2. Mathematical model

The problem is defined by a strongly nonlinear SDOF oscillator with additive excitation combining deterministic and random components, see Fig. 1.

The nonlinear response properties can be captured by means of the van der Pol equation. The assumed configuration makes the trivial solution unstable and the limit cycle stable. Thus it can represent the beating effects and stabilization due to a stable limit cycle. The corresponding Stochastic Differential Equation (SDE) can be written in the normal form

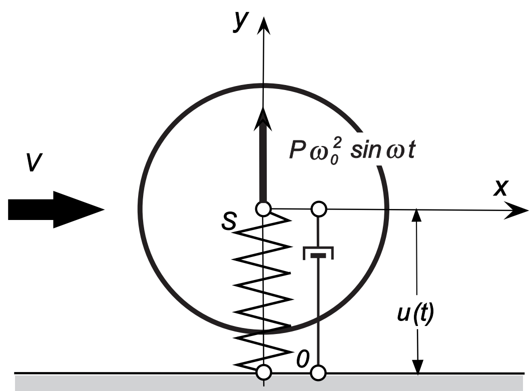

Fig. 1. SDOF system outline

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= (\eta - \nu u^2)v - \omega_0^2 u + P\omega^2 \cos\omega t + h\xi(t), \end{aligned} \tag{1}$$

where

$u, v -$ the displacement, [m], and velocity, [m s^{-1}];

$\eta, \nu -$ parameters of the damping, [s^{-1}], [s^{-1}m^{-2}];

$\omega_0, \omega -$ the eigen-frequency of the linear SDOF system and frequency of the vortex shedding, [s^{-1}];

$P\omega^2, \xi(t) -$ the amplitude of the harmonic excitation force, [ms^{-2}], and the broadband Gaussian random process, [1];

$h -$ the multiplicative constant, [m s^{-2}].

3. Slow-time system

In order to apply the stochastic averaging method [1, 3], displacement and velocity $u(t), v(t)$ can be written in the form of the first harmonic approximate

$$u(t) = a_c \cos\omega t + a_s \sin\omega t\,, \qquad v(t) = -a_c\omega\sin\omega t + a_s\omega\cos\omega t \tag{2}$$

together with an auxiliary condition, which reflects the fact that an additional variable was introduced

$$\dot{a}_c \cos\omega t + \dot{a}_s \sin\omega t = 0\,. \tag{3}$$

Such an approximation leads to an SDE system for amplitudes $a_c(\tau), a_s(\tau)$ slowly variable in time

$$\begin{aligned}\dot{a}_c = &\frac{\omega_0^2 - \omega^2}{\omega}\ \sin\omega t(a_c\cos\omega t + a_s\sin\omega t) - P\omega\sin\omega t\cos\omega t - \frac{h}{\omega}\sin\omega t\cdot\xi(t)\\ &- \sin\omega t[\eta - \nu(a_c\cos\omega t + a_s\sin\omega t)^2](-a_c\sin\omega t + a_s\cos\omega t),\end{aligned} \tag{4a}$$

$$\begin{aligned}\dot{a}_s = &-\frac{\omega_0^2 - \omega^2}{\omega}\ \cos\omega t(a_c\cos\omega t + a_s\sin\omega t) + P\omega\cos^2\omega t + \frac{h}{\omega}\cos\omega t\cdot\xi(t)\\ &+ \cos\omega t[\eta - \nu(a_c\cos\omega t + a_s\sin\omega t)^2](-a_c\sin\omega t + a_s\cos\omega t)\,,\end{aligned} \tag{4b}$$

which gives rise to the Itô averaged system

$$\mathrm{d}a_c = \frac{\pi}{\omega}\left[\ \eta a_c + 2\Delta a_s - \frac{1}{4}\nu\cdot a_c(a_c^2 + a_s^2)\right]\,\mathrm{d}t + \left(\frac{\pi}{\omega^2}\Phi_{\xi\xi}(\omega)\right)^{1/2}\mathrm{d}B_c(t), \tag{5a}$$

$$\mathrm{d}a_s = \frac{\pi}{\omega}\left[-2\Delta a_c + \eta a_s - \frac{1}{4}\nu\cdot a_s(a_c^2 + a_s^2)\right]\,\mathrm{d}t + \frac{\pi}{\omega}P\omega\,\mathrm{d}t + \left(\frac{\pi}{\omega^2}\Phi_{\xi\xi}(\omega)\right)^{1/2}\mathrm{d}B_c(t), \tag{5b}$$

where $B_c(t)$ is the Wiener process related with input excitation $\xi(t)$.

The closed form solution to Eq. (5) is available for vanishing detuning, $\Delta = 0$, see [2].

4. Fokker-Planck equation

The reduced FPE for the stationary cross PDF $p(a_c, a_s)$ (left side of the FPE is put to zero) can be written in the form

$$\begin{aligned}&\frac{\partial}{\partial a_c}\left(\left[\quad \eta a_c + \Delta a_s - \frac{1}{4}\nu\cdot a_c(a_c^2 + a_s^2)\right]p\right) \qquad -\frac{1}{2\omega^2}\Phi_{\xi\xi}(\omega)\frac{\partial^2 p}{\partial a_c^2}\\ +&\frac{\partial}{\partial a_s}\left(\left[-\Delta a_c + \eta a_s - \frac{1}{4}\nu\cdot a_s(a_c^2 + a_s^2) + P\omega\right]p\right) -\frac{1}{2\omega^2}\Phi_{\xi\xi}(\omega)\frac{\partial^2 p}{\partial a_s^2} = 0\end{aligned} \tag{6}$$

with zero boundary conditions at the infinity. The unknown PDF is assumed to have the form

$$p(a_c, a_s) = p_0(a_c, a_s)\sum_{k,l=0}^{M,k} q_{kl}\cdot a_c^{k-l}\cdot a_s^l\,. \tag{7}$$

In this expression, $p_0(a_c, a_s)$ represents the weight function and is selected in the form of the FPE solution for zero detuning, see [2],

$$p_0(a_c, a_s) \;=\; C\cdot\exp\left(\frac{\eta}{2S}\left[\left(a_s + \frac{P\omega}{\eta}\right)^2 + a_c^2 - \frac{\nu}{8\eta}\left(a_c^2 + a_s^2\right)^2\right]\right)\,. \tag{8}$$

The normalizing factor C is to be determined numerically for a particular setting of parameters, it can be considered $C = 1$. The powers of a_c, a_s are assembled to form stochastic moments of k-th order sequentially up to the M-th level; they function as correction terms.

In order to deretmine coefficients $q_{k,l}$ using the Galerkin-Petrov orthogonalization, the approximate solution Eq. (7) is introduced into the FPE, Eq. (6), multiplied by the factor $\varphi_{rs} = a_c^{r-s} \cdot a_s^s$ and integrated in the whole plane $\mathbb{R}$

$$\begin{aligned}
&\iint_{-\infty}^{\infty} a_c^{r-s} a_s^s \frac{\partial}{\partial a_c}\left(\left(\eta a_c + \Delta a_s - \frac{1}{4}\nu a_c(a_c^2 + a_s^2)\right) p_0(a_c, a_s) \sum_{k,l=0}^{M,k} q_{kl} a_c^{k-l} a_s^l\right) \mathrm{d}a_c \mathrm{d}a_s \\
+ &\iint_{-\infty}^{\infty} a_c^{r-s} a_s^s \frac{\partial}{\partial a_s}\left(\left(-\Delta a_c + \eta a_s - \frac{1}{4}\nu a_s(a_c^2 + a_s^2) + P\omega\right) p_0(a_c, a_s) \sum_{k,l=0}^{M,k} q_{kl} a_c^{k-l} a_s^l\right) \mathrm{d}a_c \mathrm{d}a_s \quad (9) \\
- &\iint_{-\infty}^{\infty} a_c^{r-s} a_s^s S\left[\frac{\partial^2}{\partial a_c^2} + \frac{\partial^2}{\partial a_s^2}\right]\left(p_0(a_c, a_s) \sum_{k,l=0}^{M,k} q_{kl} a_c^{k-l} a_s^l\right) \mathrm{d}a_c \mathrm{d}a_s = 0, \quad S = \frac{1}{2\omega^2}\Phi_{\xi\xi}(\omega)\,.
\end{aligned}$$

Here, M is the upper limit of stochastic moments we want to include into the analysis.

Several steps of the per-partes procedure and usage of homogeneous boundary conditions and particular forms of the $p_o(a_c, a_s)$ partial derivatives lead to a formula, which is applicable for the combined analytical-numerical integration

$$\begin{aligned}
0 = &\iint_{-\infty}^{\infty} \Bigg\{ \left[a_c^{\varrho-2} a_s^{s-2} \left(\varrho(\varrho-1) a_s^2 - s(s-1) a_c^2\right) S + \Delta a_c a_s \left(\varrho a_s^2 - s a_c^2\right)\right] \sum_{k,l=0}^{M,k} q_{kl} a_c^{k-l} a_s^l \\
&- S\left[s \frac{\mathrm{d}}{\mathrm{d}a_s}\left(a_c^{\varrho} a_s^{s-1} \sum_{k,l=0}^{M,k} q_{kl} a_c^{k-l} a_s^l\right) - \varrho \frac{\mathrm{d}}{\mathrm{d}a_c}\left(a_c^{\varrho-1} a_s^{s} \sum_{k,l=0}^{M,k} q_{kl} a_c^{k-l} a_s^l\right)\right]\Bigg\} p_0(a_c, a_s) \mathrm{d}a_c \mathrm{d}a_s\,, \quad (10) \\
&s = 0, .., r\,, \quad r = 0, .., M\,, \quad \varrho = (r - s)\,.
\end{aligned}$$

Further simplification of the expression (10) follows from the symmetry of the problem, so that the terms involving even powers of a_c vanish during integration. Eq. (10) represents a linear homogeneous algebraic system for $1/2 \cdot (M + 1)(M + 2)$ unknown coefficients $q_{kl}, k, l = 0, .., M; k + l \leq M$.

Eq. (10) degenerates for $s = 0, r = 0$. This missing condition can be replaced by Eq. (7), where setting $M = 0$ implies $q_{00} = 1$. This is equivalent to the condition of normalization of the resulting PDF.

Performance of the proposed procedure is shown in Fig. 2. Both partial amplitudes a_c, a_s are shown for six values $M = 0, \ldots, 5$. In each plot, the estimated PDF $p(a_c, a_s)$ is shown on the left and the value of the correction Galerkin term on the right. It can be seen that the value of the correction term within the selected domain of (a_c, a_s) increases for increasing M from approx. $\pm 5\%$ to approx. $\pm 10\%$ for $M = 5$.

5. Conclusions

The proposed procedure for estimation of PDF based on the partial amplitudes was shown on the example of the van der Pol equation, which was used for description of the vibrational effects based on the flow-structure interaction and vortex shedding. Sinmilar procedure is applicable to a variety of similar problems, namely those connected to traffic induced almost-resonant vibrations, identification problems and other system which work in a regime close to resonance.

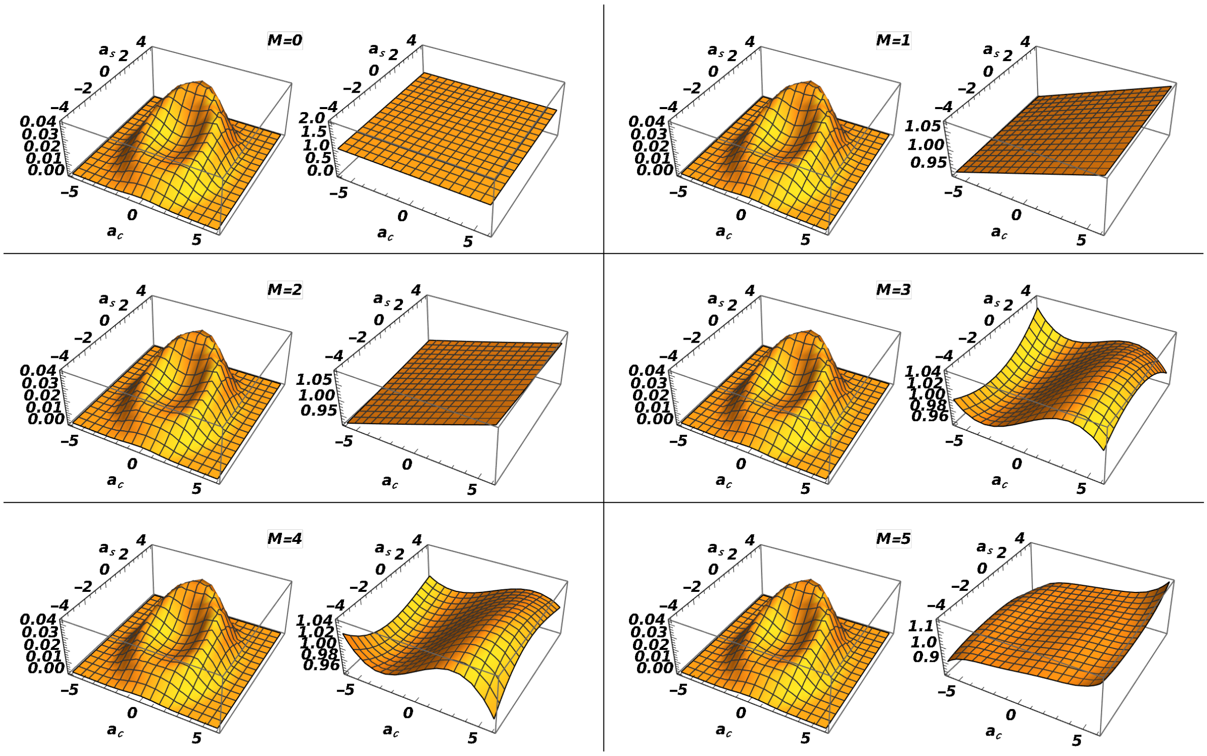

Fig. 2. Values of probability density function Eq. (7) for increasing value of M. Each pair of plots shows the Galerkin solution $p(a_c, a_s)$ on the left and the correction term value $\sum_{k,l=0}^{M,k} q_{kl} a_c^{k-l} a_s^l$ on the right. Values used: $\Delta = 0.05, \eta = 1.6, \nu = 1, P = 1, S = 4$

The approach based on the partial amplitudes, however, is based on a knowledge of the stationary solution of the corresponding FPE. For a really general approach, the dependence on the original time coordinate must be respected. For this purpose, the correction terms used for the Galerkin approximation has to encompass the time dependency. This topic is going to be further elaborated in the future.

Acknowledgements

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