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# Analytical and numerical methods for modelling of acoustic streaming in homogenized rigid porous structures

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## 1. Introduction

In the paper, we recall the classical perturbation approach which enables to linearize the Navier-Stokes (N-S) equations governing the barotropic viscous fluid dynamics in pores in a rigid periodic structure. The obtained first and second order sub-problems are treated by the asymptotic homogenization to derive the macroscopic model of the porous medium describing the acoustic streaming (AS) phenomenon.

## 2. Successive approximations of the Navier-Stokes equations

The Acoustic Streaming (AS) appears due to inhomogeneities in viscous flow due to non-zero divergence of the Reynolds stress (due to the kinetic energy of the velocity fluctuations), or due to vibrating fluid-solid interface. It is observed at fluid boundary layers as the Rayleigh streaming due thermal and viscous phenomena, or in the bulk fluid as the high-frequency Eckart streaming.

To distinguish the phenomenon of the AS, pursuing the standard approach of the perturbation analysis, see [4], *cf.* [3], we consider the following approximation of the flow field expressed in terms of different order with respect to the small parameter $\alpha \approx v_0/c_0$, where $c_0$ is the reference sound speed and $v_0$ is a characteristic flow velocity, $v_0 \ll c_0$. The fluid velocity, pressure, and density denoted by $\boldsymbol{v}^f, p^f$ and $\rho^f$ are represented by expansions

$$\begin{aligned} \boldsymbol{v}^f &= \alpha \boldsymbol{v}_1 + \alpha^2 \boldsymbol{v}_2 + \dots \;, \\ p^f &= p_0 + \alpha p_1 + \alpha^2 p_2 + \dots \;, \\ \rho^f &= \rho_0 + \alpha \rho_1 + \alpha^2 \rho_2 + \dots \;, \end{aligned} \tag{1}$$

where $p_0, \rho_0$ are positive constants and $a_k$ denotes $k$-th order in $\alpha$ approximation of the quantity $a$. Moreover, we assume that $a_k$ quantity is $T$-periodic in time, such that the time average of the time derivative vanishes, $\overline{\partial_t a} = 0$. Using (1) substituted in the N-S equations, the 1st and 2nd order problems with respect to $\alpha$ can be distinguished. At the first order, $o(\alpha^1)$,

$$\begin{aligned} \frac{\partial}{\partial t}\rho_1 + \rho_0 \nabla \cdot \boldsymbol{v}_1 &= 0 \;, \\ \rho_0 \frac{\partial}{\partial t}\boldsymbol{v}_1 + \nabla p_1 &= \mu \nabla^2 \boldsymbol{v}_1 + (\mu/3 + \eta)\nabla(\nabla \cdot \boldsymbol{v}_1) \;, \\ p_1 &= c_0^2 \rho_1 \;, \end{aligned} \tag{2}$$

Using the time average over period $T$ of the second order terms, $o(\alpha^2)$, we get

$$\begin{aligned} \frac{\partial}{\partial t}\bar{\rho}_2 + \rho_0 \nabla \cdot \bar{\boldsymbol{v}}_2 &= -\nabla \cdot \overline{(\rho_1 \boldsymbol{v}_1)} \;, \\ \rho_0 \frac{\partial}{\partial t}\bar{\boldsymbol{v}}_2 + \nabla \bar{p}_2 - \mu \nabla^2 \bar{\boldsymbol{v}}_2 + (\mu/3 + \eta)\nabla(\nabla \cdot \bar{\boldsymbol{v}}_2) &= -\rho_0 \left( \overline{(\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{v}_1} + \overline{\boldsymbol{v}_1 (\nabla \cdot \boldsymbol{v}_1)} \right) \;, \\ \bar{p}_2 &= c_0^2 \bar{\rho}_2 + c_0 c_0' \overline{(\rho_1)^2} \;. \end{aligned} \tag{3}$$

The right hand side terms in (3) defined by the time average of the acoustic $T$-periodic fluctuations $\boldsymbol{v}_1$, *i.e.* the divergence of the Reynolds stress, present the driving force for the AS phenomenon described by $\bar{\boldsymbol{v}}_2$, the time average of the 2nd order velocity field. Different treatment allowing for the acoustic modulation due to multiple time scales, thus, respecting the fast and slow dynamics, was considered in [1].

## 3. AS in the homogenized medium

We consider the system (2) governing the acoustic waves in the fluid saturating periodic scaffolds represented by the fluid part $Y_f$ of the periodic unit cell, $Y_f \subset ]0,1[^2$. The asymptotic homogenization yields the macroscopic model which presents the Darcy flow. In the frequency domain, amplitudes of the acoustic pressure waves $\underset{\sim}{p}_1^0$ satisfying

$$\mathrm{i}\omega \frac{\phi_f}{c_0^2} \underset{\sim}{p}_1^0 - \nabla_x \cdot (\underset{\sim}{\mathcal{K}} \nabla_x \underset{\sim}{p}_1^0) = 0 \;, \tag{4}$$

where $\underset{\sim}{\mathcal{K}}(\mathrm{i}\omega)$ is the dynamic permeability and $\phi_f$ is the porosity. Since the associated velocity $\boldsymbol{v}_1(x,y)$ appears to be incompressible at the microscale, *i.e.* $\nabla_y \cdot \boldsymbol{v}_1 = 0$, the acoustic streaming force involved in the 2nd order system (3) is given by $\overline{\boldsymbol{v}_1(\nabla \cdot \boldsymbol{v}_1)}$. Homogenization of (3) leads to a flow model describing the acoustic streaming phenomenon. The macroscopic flow is described by solutions of

$$-\overline{\mathcal{K}} : \nabla_x \otimes \nabla_x p_2^0 = \nabla_x \cdot \underline{S}(p_1^0, \omega) \;, \quad \mathbf{W}_2 = -\overline{\mathcal{K}} \nabla_x p_2^0 - \underline{S}(p_1^0, \omega) \;, \tag{5}$$

where the "steady" permeability $\overline{\mathcal{K}} = \underset{\sim}{\mathcal{K}}(\mathrm{i}\omega = 0)$ is given by $\underset{\sim}{\mathcal{K}}$ obtained in the 1st order system homogenization, and $\underline{S}(p_1^0(x), \omega)$ depends on the streaming force $\overline{\boldsymbol{v}_1(\nabla \cdot \boldsymbol{v}_1)}$ expressed using $p_1^0$.

## 4. Example

To illustrate the acoustic streaming effect, we consider harmonic pressure waves in a 1D macroscopic domain $\Omega = ]0,1[$. The microstructure is generated as a periodic lattice by representative cell $Y = Y_f \cup \overline{Y}_s$ whereby the solid obstacle is non-symmetric, see Fig. 3For boundary conditions $\underset{\sim}{p}_1^0(x=0) = 0$ and $(\underset{\sim}{p}_1^0)'(x=1) = 0$, in Fig. 1 we display the analytic solution of (4) which is expressed in terms of Fourier series, such that

$$\underset{\sim}{p}_1^0 = \bar{p}\left(1 + \sum_k A_k \sin\left(\frac{(2k+1)\pi}{2}\frac{x}{L}\right)\right) \;, p_1^0(x,t) = \Re\{\underset{\sim}{p}_1^0(x)\exp(\mathrm{i}\omega t)\} \;, \tag{6}$$

where $A_k$ are the Fourier coefficients. The real response $p_1^0$ is needed to define the acoustic streaming vector $\underline{S}$ involved in (5). The second order macroscopic pressure distribution $\bar{p}_2^0$ is shown in Fig.2. Since no outflow condition at $x = 1$ is considered, the macro-streaming vanishes $\mathbf{W}_2 = 0$ in the whole of $\Omega$, see (5). Nevertheless, the micro-streaming $\mathbf{v}_2^0(x, \mathbf{y})$ is not zero; the permanent microscopic flow in the fluid domain $Y_f$ is shown in Fig. 3. Different boundary conditions enable for the macroscopic acoustic jet, in general.

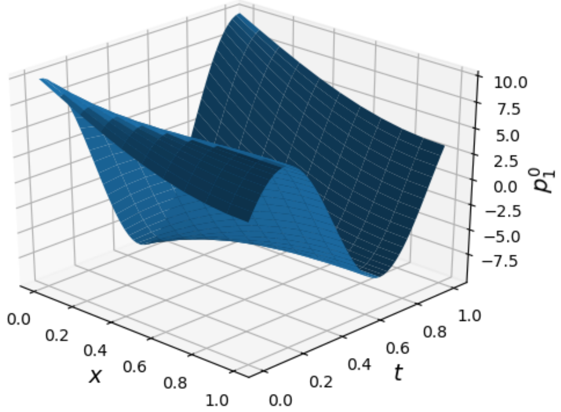


Fig. 1. The space-time distribution of the first order acoustic pressure wave $p_1^0(x,t)$

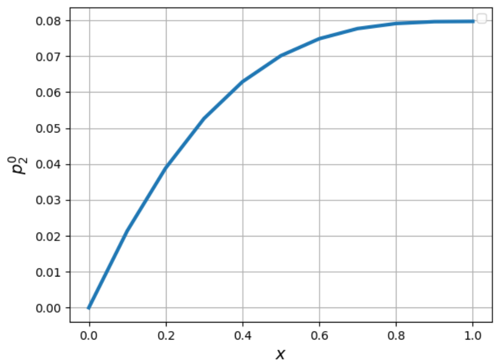


Fig. 2. The distribution of $p_2^0(x)$

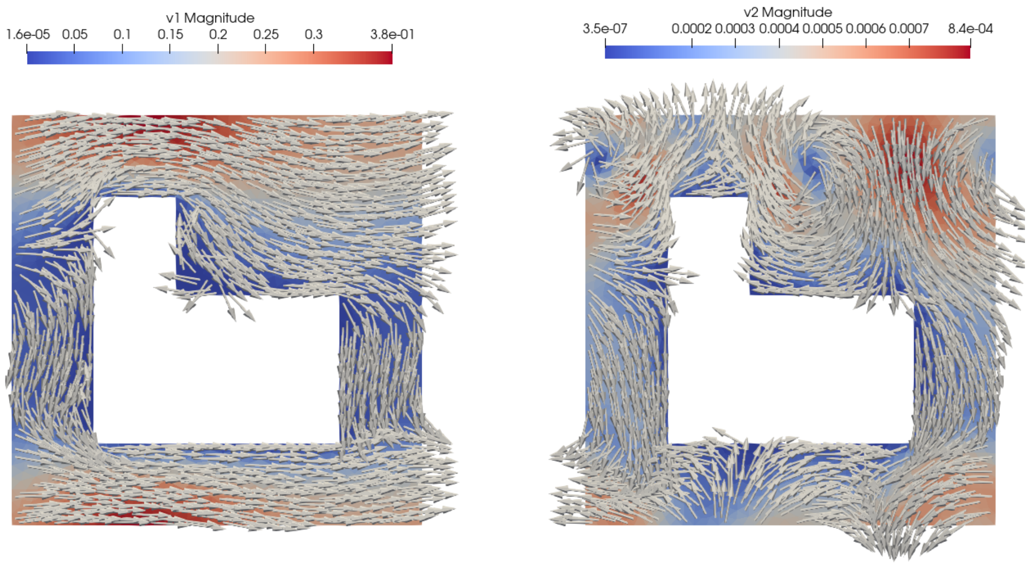


Fig. 3. Reconstruction of the harmonic velocity $\mathbf{v}_1$ at time $t = 0$ (*left*) and permanent $\mathbf{v}_2$ (*right*) in the representative volume $Y_f$ located at the macroscopic position $x = 0$ of the periodic scaffolds


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