Západočeská univerzita v Plzni
Fakulta aplikovaných věd

Existence a bifurkace periodických řešení v modelech visutých mostů

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disertační práce
k získání akademického titulu doktor (Ph.D.)
v oboru Aplikovaná matematika

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Katedra matematiky

Plzeň 2021

University of West Bohemia
Faculty of Applied Sciences

Existence and bifurcation of periodic solutions in models of suspension bridges

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Dissertation Thesis
Thesis submitted for the degree of Doctor of Philosophy (Ph.D.)
in the specialization of Applied Mathematics

Supervisor: doc. Ing. Gabriela Holubová, Ph.D.

Department of Mathematics

Pilsen 2021

Declaration

I hereby declare that this thesis is my own work, unless clearly stated otherwise.

Plzeň, ...

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Jakub Janoušek

Acknowledgement

I would like to thank my supervisor, doc. Ing. Gabriela Holubová, Ph.D., for her patience, time and scientific leadership.

Next, I would like to thank all my colleagues from the VP-5 research programme, especially Tomáš Roubal, for many discussions during the length of my study.

And last, but not least, I owe many thanks to my family for heaps of support and positive motivation.

Our work was supported by the Czech Ministry of Education, Youth and Sports (the project LO1506), by the Grant Agency of the Czech Republic (Grant No. 18-03253S) and by the project SGS-2016-003 of the University of West Bohemia.

Abstrakt

Tato práce si klade za cíl shrnout naše výsledky v oblasti modelování visutých mostů, konkrétně se zabývá řešitelností a bifurkacemi periodických řešení v několika vybraných modelech. Text je uspořádán chronologicky podle pořadí, ve kterém jsme se jednotlivými tématy zabývali. Nejprve se soustředíme na jednodimenzionální model s tlumením a přinášíme revizi dosavadního výsledku týkajícího se jednoznačnosti řešení. K tomu využíváme standardních nástrojů, jako je např. Banachova věta o kontrakci, ovšem tentokrát s použitím přesnějšího geometrického zkoumání polohy vlastních čísel odpovídajícího lineárního operátoru. To nám umožnuje dosáhnout značného rozšíření intervalu přípustné tuhosti mostních lan.

Dále zkoumáme model visutého mostu s prostorově proměnnou tuhostí, která přináší možnost, jak do modelu zahrnout informaci o odděleném rozložení připojení lan k mostovce. Zaměřujeme se na kvalitativní a kvantitativní vlastnosti tohoto modelu, které srovnáváme s klasickým modelem s konstatní tuhostí. Zjišťujeme, že pro některá nastavení tuhosti a odpovídajícího profilu rozložení lan dochází v modelu k bifurkacím periodických řešení. V neposlední řadě se také zabýváme významem Fučíkova spektra pro možnost existence tzv. blow-upů.

Poté se věnujeme tématu, které přímo souvisí se zkoumáním vlivu prostorově proměnné tuhosti, a sice hledání postačujících podmínek pro (striktně) inverzní pozitivitu lineárního diferenciálního operátoru čtvrtého řádu, který odpovídá příslušné rovnici nosníku s proměnnou tuhostí. Pomocí rozšíření technik uvedených J. Schröderem v rámci teorie redukce operátorů, ukazujeme, že minima a maxima prostorově proměnné tuhosti mohou značně překročit meze stanovené dřívějšími výsledky.

Závěrem poznamenejme, že tato práce je rozdělena do dvou hlavních úseků. První část slouží jako přehled a kompilace našich výsledků, uvedených v kontextu související literatury, případně předchozích výsledků v dané oblasti. Druhá část je reprezentována přílohou, která zahrnuje tři naše výzkumné články, v nichž jsou k dispozici detailnější informace a také důkazy všech našich tvrzení.

Klíčová slova

Visutý most, nelineární rovnice nosníku, mudely s tlumením, existence slabého řešení, jednoznačnost slabého řešení, skákající nelinearita, proměnný koeficient, bifurkace, rovnice čtvrtého řádu, kladné řešení, inverzní pozitivita

Abstract

This thesis brings an overview of our work concerning solvability and bifurcation in various models of suspension bridges. Our efforts are presented in chronological order. At first, we focus on a one-dimensional damped model of a suspension bridge. We bring a revision of the so far known uniqueness result, by employing standard techniques, such as Banach Contraction Theorem. However, we use more precise geometrical arguments connected to the position of eigenvalues of the corresponding linear operator, and therefore we obtain a significant extension of the allowed interval for the stiffness parameter.

Next, we study a model with a spatially variable stiffness parameter. This is an attempt to take into account the discrete nature of the placement of suspension bridge hangers. We investigate qualitative and quantitative properties of this model, especially in comparison to the standard model with constant stiffness. We also show that bifurcation of periodic solutions occurs for certain combinations of the stiffness parameter and the corresponding hanger placement profile. Additionally, there are also blow-ups to be expected. The existence of those depends on the structure of the so called Fučík spectrum of the corresponding linear operator.

And finally, we search for sufficient conditions for the (strict) inverse-positivity of the linear fourth order operator associated with the one-dimensional beam equation with a spatially variable coefficient. Hence, this topic is very close to our previous work. We incorporate an evolution of techniques, such as the results of operator reduction introduced and developed by J. Schröder, which allows us to show, that the extrema of the coefficient can significantly breach the originally derived bounds.

Let us point out, that this text is divided into two main blocks. The first one serves as the introduction and a brief compilation of our work in the context of related literature and previous results of other authors. Since our main results have been split into three research papers, we add them to this text as appendices, which we consider to be the second main part. There it is possible to find all the proofs and technical details, which were for the sake of brevity omitted from the first part.

Keywords

Suspension bridge, nonlinear beam equation, models with damping, existence of a weak solution, uniqueness of a weak solution, jumping nonlinearity, variable coefficient, bifurcation, fourth order operator, positive solutions, inverse-positivity

Zusammenfassung

Diese Dissertation gibt einen Überblick über unsere Arbeit zur Lösbarkeit und Bifurkation in verschiedenen Modellen von Hängebrücken. Unsere Ergebnisse sind in der chronologischen Reihenfolge dargestellt. Zuerst konzentrieren wir uns auf ein eindimensionales gedämpftes Modell einer Hängebrücke. Wir bringen eine Revision des bisher bekannten Ergebnisses der Eindeutigkeit. Wir verwenden klassische Techniken wie z. B. den Banach-Kontraktionssatz, aber mit genaueren geometrischen Argumenten, die mit der Position der Eigenwerte des entsprechenden linearen Operators verbunden sind. Daher erhalten wir eine signifikante Erweiterung des Intervalls der erlaubten Werte für den Steifigkeitsparameter.

Dann untersuchen wir ein Modell mit einem räumlich variablen Steifigkeitsparameter. Es ist ein Weg, wie die diskrete Art der Anordnung von Hängebrückenhängern zu berücksichtigen. Wir untersuchen qualitative und quantitative Eigenschaften dieses Modells, insbesondere im Vergleich zu dem klassischen Modell mit konstanter Steifigkeit. Für bestimmte Kombinationen des Steifigkeitsparameters und des entsprechenden Profils der Hängerplatzierung, können wir die Bifurkation periodischer Lösungen beobachten. Wir diskutieren auch über die Existenz der "Blow-ups", die mit der Struktur des sogenannten Fučík-Spektrums des entsprechenden linearen Operators verbunden ist.

Und schließlich suchen wir hinreichende Bedingungen für die Inverspositivität des linearen Operators vierter Ordnung, der mit einer eindimensionalen Balkengleichung mit einem räumlich variablen Koeffizienten verbindet ist. Dieses Thema ist sehr nah an unserer bisherigen Arbeit. Wir präsentieren eine Entwicklung von Techniken, die von J. Schröder eingeführt worden. Die Ergebnisse seiner Theorie der Operatorreduktion erlauben uns zu zeigen, dass die Extrema des Koeffizienten viel weiter von Null können sein.

Dieser Text ist in zwei Hauptblöcke geteilt. Der erste Teil dient als Einführung und eine kurze Zusammenstellung unserer Arbeit, die im Kontext der verwandten Literatur und Ergebnisse anderer Autoren präsentiert wird. In dem zweiten Teil stellen wir unsere Forschungsartikeln zur Verfügung, wo man alle Beweise und technischen Details finden kann.

Schlüsselwörter

Hängebrücke, nichtlineare Differentialgleichung, Modelle mit Dämpfung, Existenz der schwachen Lösung, Eindeutigkeit der schwachen Lösung, springende Nonlinearität, variabler Koeffizient, Bifurkation, Operator vierter Ordnung, positive Lösung, Inverspositivität

Contents

Chapter 1

Introduction and model overview

In 1940, the Tacoma Narrows Bridge in the northwestern US collapsed into the Puget Sound after being hit by a slightly above average wind storm. During and after the federal investigation, much effort has been spent into finding an explanation what precisely caused the wild and eventually fatal oscillations of the bridge, since it appeared that the wind itself was not strong enough to cause such a disaster. The federal investigation and following research began with a detailed report (see [2]) written by O. H. Amman, T. von Kármán and G. B. Woodruff. Since then, the research has still been going on, while bringing many approaches, techniques and results in the process. Some branches of research are based on the nonlinearity assumption, i.e., that the bridge's hangers have no restoring force when being compressed. A simple way how to describe the behaviour of such a structure is to consider the bridge as a one-dimensional bending beam with simply supported ends connected to an unmovable object by a set of nonlinear hangers (see Fig. 1). Here, we would like to point out, that the term "hangers" is not universally used and does not appear in a significant part of the cited literature. There are other frequently seen descriptions for this part of the bridge, such as cables and ropes. See the discussion in [21] for further details. The bridge hangers act as linear springs when being streched, however, as mentioned before, when being compressed, they have no restoring force.

In this thesis, we deal with such one-dimensional beam models. Practically all of them can be traced back to the original model, which appeared in the work of A. C. Lazer and P. J. McKenna (see [28]) and has the following form:

$$\begin{aligned} &mu_{tt} + EIu_{xxxx} + bu_t + \kappa u^+ = W(x) + \varepsilon f(x,t), \\ &u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \\ &u(x,t+T) = u(x,t),\ -\infty < t < +\infty,\ x \in (0,L), \end{aligned} \tag{1.1}$$

see e.g. [39]. An overview explaining the meaning of all the coefficients is provided here:

m	mass per unit length
E	Young's modulus
I	moment of inertia of the cross section
b	damping coefficient
κ	cable stiffness
W	weight per unit length
εf	external periodic force
L	length of the center-span of the bridge.

For a more convenient way of examining the model's properties, it is suitable to rescale it by dividing with the mass m and normalizing the length and time period with the corresponding spatial variable x and time variable t, respectively (cf. [39]). Note that we again use the same symbols for rescaled functions, however, the new parameters get a new notation. Specifically, $\alpha^2 := \frac{EIT^2\pi^2}{4mL^4}$, $\beta := \frac{T}{2\pi m}b$ and $k = \frac{T^2}{4\pi^2 m}\kappa$. That is, (1.1) takes the form

$$\begin{aligned} u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + ku^+ &= W(x) + \varepsilon f(x,t), \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) &= u_{xx}(\pi,t) = 0, \\ u(x,t+2\pi) = u(x,t),\ -\infty < t &< +\infty,\ x \in (0,\pi). \end{aligned} \tag{1.2}$$

Here, the displacement $u(x,t)$ of the roadbed is measured as positive in the downward direction. Parameters α^2, β and k represent elastic forces inside the beam, viscous damping and the hangers' stiffness, respectively. The term $W(x)$ stands for the weight per unit length of the roadbed, whereas the external forces affecting the bridge are represented by $\varepsilon f(x,t)$. The already discussed nonlinear behaviour of the hangers is described by the positive part function, where

$$u^+(x,t) := \max\{u(x,t), 0\}.$$

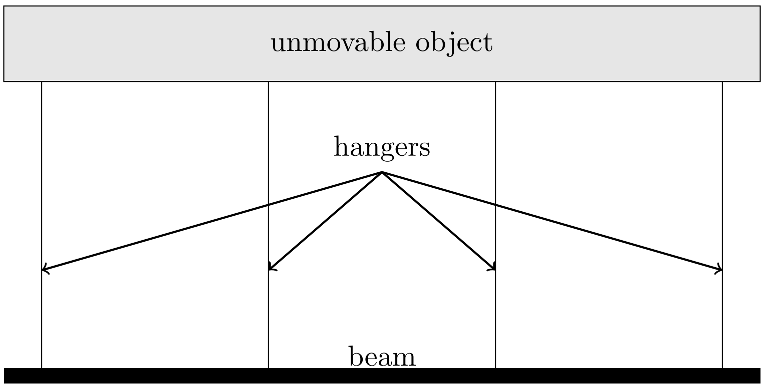

Figure 1.1: Abstract idealization of a suspension bridge.

Although such models are a simplification and omit many real world properties, their behaviour actually shows patterns encountered when observing structures such as the Golden Gate Bridge or, in the past, even Tacoma Narrows Bridge itself. Specifically, we have in mind solutions of large amplitude or multiplicity of solutions, see [10, 13, 16, 23, 28, 29]). This suggests that even simple nonlinear models can reveal some potentially dangerous movement of the roadbed.

Before we continue, let us point out that these models can be relatively simply modified by introducing additional terms, equations (e.g. for more realistic models which consider the bridge hangers connected to a flexible main cable) and/or changing corresponding boundary conditions. For further details, see, e.g., [13, 15, 16] or [28].

However, for the remainder of this text, let us return to simpler one-dimensional models. Before we discuss the results of our work in this field, let us present an overview of standard models and corresponding results in this chapter. This part will also serve not only for better understanding of the basic theory and used mathematical toolset, but also for explaining our general motivation.

From the technical and also historical point of view, the focus on modelling suspension bridges has been split into several branches. We try to follow this trend in making the structure of this text similar, i.e., we treat the damped and non-damped models separately. In addition, in both these branches, further simplifications can be made and, by assuming certain spatial profile of the solution, reduce the corresponding PDE problem to an ODE case.

1.1 PDE models without damping

More than thirty years ago, during the 1980s, A. C. Lazer, P. J. McKenna and W. Walter were studying multiplicity of solutions for various types of equations without damping. These were thought to be a possible tool for modelling suspension bridges (see, eg. [27] or [29]). Their work was followed by Q. H. Choi and T. Jung (see [7]) and also L. Humphreys, who obtained some corresponding numerical results in [23]. These results were extended later by P. Drábek and G. Holubová in [10] by employing global bifurcation theory. Generally, the results of all mentioned authors suggest that the more eigenvalues of the corresponding linear beam operator are crossed by the stiffness parameter k, the more solutions appear.

In order to be more specific, let us concentrate on these PDE models without damping, such as

$$\begin{aligned} u_{tt} + \alpha^2 u_{xxxx} + ku^+ = W(x) + \varepsilon f(x,t) \quad \text{in} \quad \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \times \mathbb{R}, \\ u\left(\pm\tfrac{\pi}{2}, t\right) = u_{xx}\left(\pm\tfrac{\pi}{2}, t\right) = 0, \quad u(x,t) = u(x, t+2\pi). \end{aligned} \tag{1.3}$$

In the above mentioned literature, models of this type were studied and the first results concerning multiplicity of solutions were obtained. For simplification, the right-hand side was considered in a more specific form by putting $W(x) = 1$. However, such simplification is rather realistic, since one expects that the weight per unit length is (more or less) constant for real structures (see e.g. [29]). After adding the symmetry conditions (again, a logical step for the studied structures) and normalizing by changing the variables, the authors of [7, 10, 23] and [29] investigated the following version of (1.3):

$$\begin{aligned} u_{tt} + u_{xxxx} + ku^+ = 1 + \varepsilon f(x,t) \quad \text{in} \quad \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \times \mathbb{R}, \\ u\left(\pm\tfrac{\pi}{2}, t\right) = u_{xx}\left(\pm\tfrac{\pi}{2}, t\right) = 0, \\ u(x,t) = u(-x,t) = u(x,-t) = u(x, t+\pi). \end{aligned} \tag{1.4}$$

First of all, according to McKenna and Walter (see [29]), if $k \in (3, 15)$ then at least two solutions of (1.4) exist. However, for $k \in (-1, 3)$, the problem (1.4) admits a unique solution.

More details were proved by Choi et al. in [7] by a variational reduction method. The authors revealed more information concerning the number and quality of solutions for $k \in (3, 15)$. By their result, under this assumption on k, (1.4) has at least three solutions, two of them being large amplitude ones. The underlying idea that "more solutions appear when k crosses more eigenvalues of the corresponding linear beam operator" was later numerically supported by L. Humphreys in [23]. Again, also the qualitative properties of solutions were discussed, since the author also presented a large-amplitude numerical solution, obtained by a mountain pass algorithm (cf. [8]).

In contrast to previous works, the paper [10] comes with a different approach, studying the problem (1.4) with ε sufficiently small, or with $\varepsilon = 0$, i.e.,

$$\begin{aligned} u_{tt} + u_{xxxx} + ku^+ = 1 \quad \text{in} \quad \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \times \mathbb{R}, \\ u\left(\pm\tfrac{\pi}{2}, t\right) = u_{xx}\left(\pm\tfrac{\pi}{2}, t\right) = 0, \\ u(x,t) = u(-x,t) = u(x,-t) = u(x, t+\pi) \end{aligned} \tag{1.5}$$

with the global bifurcation theory tools in mind. This has brought even more qualitative information about the solution set. Namely, the authors of [10] proved, that each eigenvalue of the corresponding linear problem with odd multiplicity is a point of global bifurcation and there exists a continuum of solutions, which is either unbounded, or reaches another eigenvalue. Moreover, for $k < -1$, the problem (1.5) has no solution, for $k \in (-1, 3)$ it has a unique, positive and stationary solution and for $k \in (3, 15)$ there exist at least two solutions, one of them being positive and stationary and the other one sign changing (see Corollary 2.1 in [10]).

Although there remain some unanswered questions, e.g., whether for any $k > 3$ there exist multiple solutions of (1.5), we decided that it would be more appropriate to concentrate on a model, which would be in some sense more general, however, it would be possible to at least partially use some of the techniques and results from [10] as a background. As an advantage, we would also have some clues, what type of behaviour to expect. That is indeed what we did, since we started the investigation of a generalized model

$$\begin{aligned} u_{tt} + u_{xxxx} + k(x)u^+ &= h(x,t) \quad \text{in } (0,1)\times\mathbb{R}, \\ u(0,t) = u(1,t) = u_{xx}(0,t) &= u_{xx}(1,t) = 0, \\ u(x,t) = u(x,t+2\pi) &= u(x,-t) \end{aligned} \tag{1.6}$$

with non-constant hanger stiffness. This assumption, that is, $k = k(x)$, allows us to reflect the fact, that the bridge hangers are not arbitrarily close to each other, in a relatively simple way. A summary of our results concerning this model can be found in Chapter 3 and [21].

On the other hand, if we would like to obtain more precise results for the price of having a far less general model to work with, we can, under specific circumstances, “reduce” the PDE models into ODE ones. Let us look into this process in more detail.

1.2 ODE models without damping

If we assume that the loading and the external forces have the largest impact in the middle of the whole structure, we can consider the terms of the right-hand side in (1.4) being $W(x) = \cos x$ and $\varepsilon f(x,t) = \varepsilon f(t)\cos x$ and look for no-nodal solutions $u(x,t) = y(t)\cos x$ (see [10] or [28]). These modifications allow to transform (1.4) into an ODE problem

$$\begin{aligned} y'' + y + ky^+ &= 1 + \varepsilon f(t), \\ y(t) = y(-t) &= y(t+\pi). \end{aligned} \tag{1.7}$$

Similar model has been dealt with in more detail in [28] and, again, the authors came to the conclusion that more crossed eigenvalues (by k) of the corresponding linear problem means more solutions of (1.7). Later, by taking ε sufficiently small or even $\varepsilon = 0$ in (1.7) and thus studying the model

$$\begin{aligned} y'' + y + ky^+ &= 1, \\ y(t) = y(-t) &= y(t+\pi), \end{aligned} \tag{1.8}$$

the authors of [10] obtained a strong bifurcation result, which brings detailed information about the set of solutions. In particular, there exists a sequence $\{k_m\}$ where $k_m = 4m^2 - 1,\ m \in \mathbb{N}\cup\{0\}$, such that (1.8) has exactly $2m+1$ solutions whenever $k \in (k_m, k_{m+1})$. Using global bifurcation theorems, the authors also provided a detailed descriprion of solution branches bifurcating from the points $k_m,\ m \geq 1$, which are the negatives of the corresponding linear operator’s eigenvalues (see [10], Theorem 3.1).

What is also an important observation, is the relation between the periodicity interval of solutions and their boundedness, or unboundedness, respectively. Indeed, P. Drábek and P. Nečesal showed in [14] that when one considers the solution of (1.8) not only π-periodic, but generally T-periodic, two behaviour patterns may occur. Specifically, if $T \in (0,\pi)$, the solutions are all uniformly bounded, whereas if $T \geq \pi$ then there exist solutions with an arbitrarily large amplitude. Moreover, there are blow up points if $T > \pi$, that is, in such case, there exist nonstationary solutions with their amplitude approaching infinity.

The appearance of blow up points can be described even more precisely. The authors of [14] found out that this phenomenon in the general T-periodic problem corresponds to the Fučík spectrum of

$$\begin{aligned} y'' + \alpha y^+ - \beta y^- &= 0, \\ y(t) = y(-t) &= y(t+T) \end{aligned} \tag{1.9}$$

such that for a fixed T, the point k is a blow up point if and only if the couple $(k+1, 1)$ belongs to the Fučík spectrum of (1.9). Moreover, as T goes to infinity, the number of blow up points increases. The authors also obtained similar results for small perturbations of 1 on the right-hand side of (1.8), i.e., being in the form $1 + \varepsilon f(t)$ for ε small enough (see [14] for details).

1.3 Damped models

Now, we turn our attention to a one-dimensional model with a viscous damping term. Investigating the properties and multiplicity of solutions for this model is technically more involved, however, this approach is more realistic. We work with a minor modification of (1.2), which has its roots in the work of Lazer and McKenna, who introduced it in [28] and takes the form

$$\begin{aligned} u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + k u^+ &= h(x,t), \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) &= 0, \\ u(x,t+2\pi) = u(x,t),\ -\infty < t < +\infty,\ x &\in (0,\pi), \end{aligned} \tag{1.10}$$

where $\alpha > 0, \beta > 0$ and $k \in \mathbb{R}$. The meaning of all parameters remains the same as in (1.2), however, the right-hand side $h(x,t)$ is considered in a more general form. This model has not been dealt with in that much detail as e.g. the previously mentioned ones without damping, but still some existence results have been obtained.

Let us start with the work of J. M. Alonso and R. Ortega, that employed in some sense a more "out of the ordinary" approach (compared to other cited literature concerning damped models and our work). However, it is interesting from our point of view, since their result has much in common with ours. At the beginning of the 1990s, they studied the global asymptotic stability and uniqueness of a solution of a forced Newtonian system with dissipation (see [1]), i.e.,

$$u''(t) + cu'(t) + Au + \nabla G(u) = p(t), \tag{1.11}$$

where $u : \mathbb{R} \to \mathbb{R}^N$, $c > 0$, A is a symmetric positive semidefinite matrix, $G \in C^2(\mathbb{R}, \mathbb{R}^N)$ and the right-hand side $p \in C(\mathbb{R}, \mathbb{R}^N) \cap L^\infty(\mathbb{R}, \mathbb{R}^N)$. By considering the right-hand side of (1.10) $h(x,t)$ continuous and bounded, using the spatial discretization and the finite difference approach, the authors were able to interpret (1.10) in view of (1.11) and obtained a uniqueness and stability result, which is in the form of a sufficient condition, that is, if $k < \beta^2 + 2\alpha\beta$ then (1.10) has a unique bounded solution that is exponentially asymptotically stable. This specific condition for k partially coincides with our new set of conditions, that can be seen further in this text (Chapter 2), or with more details in [19].

In the same time period, P. Drábek also studied the problem (1.10) (see [9]). His results confirmed the existence of at least one weak solution, even for a more general right-hand side than the one in [1]. Moreover, under additional assumptions, he showed that with sufficiently small external forces, there always exists a solution in some sense near to the equilibrium. This idea was made more clear later in the 1990s, when J. Berkovits, P. Drábek, H. Leinfelder, V. Mustonen and G. Tajčová proved in [4] that for the right-hand side in the form $h(x,t) = W(x) + \varepsilon f(x,t) > 0$, the problem (1.10) becomes linear and it admits a positive nonstationary solution for an arbitrary $k \in \mathbb{R}$.

The work of [9] was followed also by G. Tajčová in [39]. She realized, that if $|k| < \operatorname{dist}(0, \sigma(L))$ then the problem (1.10) has a unique weak solution for an arbitrary right-hand side $h \in L^2(\Omega)$. The generality of the right-hand side makes this result rather strong, however, it also has its drawbacks. Indeed, it unfortunately suggests, that the bridge is "safe enough" if its cables are not really stiff when stretched, which does not often correspond well to reality. Since the result is formulated as a sufficient condition, there is room for a potentially significant improvement. It is not necessary to change the whole approach encountered in [39]. Instead of that, it is possible, with some minor updates, to get new, less strict sufficient conditions. We used the same abstract tools and settings again (cf. [4], [39]), however, by incorporating some new geometric arguments, we were able to extend the "uniqueness interval" for the stiffness parameter k. These new conditions have been obtained in [19] and are summarized in Chapter 2.

But for now, let us complete the initial model overview and briefly mention a simplified ODE model with damping. Since $x \in (0, \pi)$ in (1.10), we now consider the right-hand side of (1.10) in the form $h(x,t) = \sin x + \varepsilon f(t) \sin x$ in order to try to find no-nodal solutions $u(x,t) = y(t) \sin x$. That is, the process to obtain the model simplification as in Section 1.2 is very similar. In the end, we get a damped ODE problem

$$\begin{aligned} y'' + \beta y' + \alpha^2 y + k y^+ &= f(t) \\ y(t) &= y(t + 2\pi). \end{aligned} \tag{1.12}$$

Studying this model was not one of the main topics of our work, however, some information about the existence of a unique solution based on the mutual relation between the values of k and β can be found in Chapter 2.

1.4 Motivation and text structure

To conclude the opening chapter, let us explain our motivation to the reader, since it also has impact on the structure of the following text. At first, our goal was simply to improve the results concerning the damped PDE model that appeared in [39]. The following chapter and [19] are devoted to this topic. However, there is more than that in Chapter 2. We also discuss solvability and/or uniqueness results for simplified models with a specific right-hand side. There are some relatively standard existence results, which appear neither in [19], nor (to the best of our knowledge) in the corresponding literature. Therefore, we present them also with proofs.

Later, we decided to study bifurcations in the sense of [10], but for a generalized model of the (1.6) type, i.e., with the spatially variable stiffness, better reflecting the discrete nature of the hanger placement. Here, we were able to show, that this assumption actually improves the bahaviour of the model. For a detailed discussion, see Chapter 3 and [21].

The study of bifucartions of periodic solutions in a weighted model led us to a more theoretical topic. Establishing the standard bifurcation equation is closely connected to the existence of a positive stationary solution of (1.6). Searching for conditions, which guarantee this existence and positivity brought us to the field of (strictly) inverse-positive operators. Again, we could build our work on previous results, namely those in [11, 12, 26, 38]. Our progress in this field and new, less strict conditions for inverse-positivity of a linear fourth-order operator are available in [20] and in Chapter 4.

Further details and proofs of all main theorems can be found in our research articles [19, 21] and [20]. These are available at the end of this thesis as Appendix A.1, A.2 and A.3, respectively.

Chapter 2

Models with damping

For now, let us concentrate on the one-dimensional model (1.10) with damping. The aim of this chapter is to present our existence and uniqueness results considering a relatively general right-hand side. This was achieved by making some minor improvements of previously introduced techniques. In order to show the results in the right context, we also provide some important auxiliary assertions concerning "basic solvability" of (1.10). We use the same abstract setting as seen in [39] and, for the reader's convenience, we provide its brief summary.

2.1 Abstract formulation

Speaking about the abstract setting, the involvement of the damping term actually has far reaching consequences, specifically for choosing the most suitable function space. Let us begin by denoting $\Omega = (0,\pi) \times (0, 2\pi)$ the considered domain and by $H = L^2(\Omega, \mathbb{R})$ the real Hilbert space equipped with the standard scalar product $\langle u, v\rangle = \int_\Omega uv \,\mathrm{d}x\,\mathrm{d}t$, $u, v \in H$ and the corresponding norm. Further, we denote by $\mathcal{D}$ the set of all smooth functions which satisfy the boundary and periodic conditions from (1.10).

Definition 1. We call a function $u(x,t) \in H$ a *weak solution* of the problem (1.10) if and only if the integral identity

$$\int_\Omega u(v_{tt} + \alpha^2 v_{xxxx} - \beta v_t)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega (h - ku^+)v\,\mathrm{d}x\,\mathrm{d}t \tag{2.1}$$

holds for all $v \in \mathcal{D}$.

However, because of the considered damping, the whole situation becomes slightly more complicated and we need a more general space. Under the term *complexification* of H, we understand the space $H_\mathbb{C} = H + \mathrm{i}H = L^2(\Omega, \mathbb{C})$ with the scalar product $\langle u, v\rangle = \int_\Omega u\bar{v}\,\mathrm{d}x\,\mathrm{d}t$, $u, v \in H_\mathbb{C}$, and the usual norm $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$. The set $\{\mathrm{e}^{\mathrm{i}nt}\sin mx,\ m \in \mathbb{N},\ n \in \mathbb{Z}\}$ forms an orthogonal basis in $H_\mathbb{C}$ ([19, 39]) and hence each function $u(x,t) \in H_\mathbb{C}$ has its representation by the Fourier series

$$u(x,t) = \sum_{n=-\infty}^{+\infty}\sum_{m=1}^{+\infty} u_{mn}\mathrm{e}^{\mathrm{i}nt}\sin mx. \tag{2.2}$$

For obvious reasons, we are investigating real valued solutions, so let us point out, that for real functions $u \in H$ there is $u_{m(-n)} = \bar{u}_{mn}$.

Let us continue with another abstract object, i.e., we have to introduce a proper replacement for the derivatives in our space.

Definition 2. The operator L, such that

$$L:\ \mathrm{dom}(L) \subset H_{\mathbb{C}} \ \to\ H_{\mathbb{C}}, \qquad Lu = \sum_{n=-\infty}^{+\infty} \sum_{m=1}^{+\infty} (\alpha^2 m^4 - n^2 + \mathrm{i}\beta n) u_{mn} \mathrm{e}^{\mathrm{i}nt} \sin mx,$$

where

$$\mathrm{dom}(L) = \Big\{ u \in H_{\mathbb{C}} :\ \sum_{n=-\infty}^{+\infty} \sum_{m=1}^{+\infty} |\alpha^2 m^4 - n^2 + \mathrm{i}\beta n|^2 \, |u_{mn}|^2 < +\infty \Big\}.$$

is called the abstract realization of the linear beam operator

$$u \ \mapsto\ u_{tt} + \alpha^2 u_{xxxx} + \beta u_t$$

with the boundary and periodic conditions from (1.10).

Again, what is important for our interest in real solutions, is the fact, that L maps real-valued functions to real-valued ones, i.e., $u \in \mathrm{dom}(L) \cap H \ \Rightarrow\ Lu \in H$.

Some basic proofs concerning necessary conditions for solvability (which we discuss later in this chapter) require the usage of an adjoint operator L^*. Luckily, it is relatively straightforward to find the formal adjoint for our operator L.

Remark 1. The operator

$$L^*:\ \mathrm{dom}(L^*) \subset H_{\mathbb{C}} \ \to\ H_{\mathbb{C}}, \qquad L^*u = \sum_{n=-\infty}^{+\infty} \sum_{m=1}^{+\infty} (\alpha^2 m^4 - n^2 - \mathrm{i}\beta n) u_{mn} \mathrm{e}^{\mathrm{i}nt} \sin mx.$$

is the adjoint of L on $H_{\mathbb{C}}$. It is also a real operator and its domain is considered in the same sense as for L.

Now it is finally possible to recast the original boundary value problem from (1.10) as an abstract operator equation in $L^2(\Omega, \mathbb{R})$. Hence we get

$$Lu + ku^+ = h. \tag{2.3}$$

The spectrum of L consists only of points $\sigma(L) = \{\lambda_{mn},\ m \in \mathbb{N},\ n \in \mathbb{Z}\}$, with

$$\lambda_{mn} = \alpha^2 m^4 - n^2 + \mathrm{i}\beta n, \quad m \in \mathbb{N},\ n \in \mathbb{Z} \tag{2.4}$$

being the eigenvalues of L, see [4, 19] of [39]. Now, for any real parameter λ, such that $\lambda \notin \sigma(L)$ and for an arbitrary right-hand side $f \in H$, the non-homogeneous equation

$$Lu - \lambda u = f \tag{2.5}$$

has a unique weak solution $u \in H$. Speaking about solutions, let us summarize the last important piece of information in the following lemma (see, e.g., [4, 19] or [39]):

Lemma 3. *The resolvent operator, corresponding to L and denoted by L_λ^{-1}, such that*

$$L_\lambda^{-1}:\ H \to H, \quad L_\lambda^{-1}:\ f \mapsto u$$

is linear, compact and

$$\|L_\lambda^{-1}\| \le \frac{1}{\mathrm{dist}(\lambda, \sigma(L))}. \tag{2.6}$$

After this preliminary discussion with establishing the operator equation (2.3), and before discussing the improved uniqueness condition from [19], let us, for now, turn our attention to some auxiliary results. Indeed, we can get elementary information about the damped model (1.10), such as conditions for its solvability (either necessary, or sufficient). Note that the existence of at least one weak solution for positive values of the stiffness parameter k is a natural requirement. Fortunately, we are able to prove the existence result for an arbitrary L^2 right-hand side with $k > -\alpha^2$. As a side quest, we can reduce (1.10) in view of Section 1.2 into a periodic ODE problem and thus extract more detailed information (see Section 2.3 below). But let us start with the original damped model (1.10).

2.2 Basic solvability - PDE case

Our first task is to establish the necessary condition for the solvability of the equation from (1.10). With our abstract setting in mind, we may safely continue with the operator formulation of (1.10). Also, we use the fact, that the normed first eigenfunction of L, which corresponds to the smallest real eigenvalue $\lambda_{10} = \alpha^2$ and is denoted by v_{10}, is just the sine half-wave, i.e., $v_{10} = \sin x$ on $(0, \pi)$.

Lemma 4. *Let $u(x,t)$ be a weak solution of (1.10) and let $\int_\Omega h\, v_{10} > 0$. Then $k > -\alpha^2$.*

Proof. Since the problem (1.10) can be viewed as (2.3) in $H_{\mathbb{C}}$, then also

$$\left(\left(Lu + ku^+\right), v_{10}\right) = (h, v_{10})$$

holds and hence

$$(Lu, v_{10}) + k\left(u^+, v_{10}\right) = (h, v_{10})\,.$$

Now, we can continue by employing the adjoint operator L^* (see Remark 1). Note that for sufficiently differentiable functions, L^* has the form $L^*u = u_{tt} + \alpha^2 u_{xxxx} - \beta u_t$. Thus we get

$$(u, L^* v_{10}) + k\left(u^+, v_{10}\right) = (h, v_{10})\,.$$

Since $L^* v_{10} = \alpha^2 v_{10}$ and $u = u^+ - u^-$, we obtain

$$\alpha^2\left(u^+ - u^-, v_{10}\right) + k\left(u^+, v_{10}\right) = (h, v_{10})\,,$$

which reads

$$\left(\alpha^2 + k\right)\left(u^+, v_{10}\right) = (h, v_{10}) + \alpha^2\left(u^-, v_{10}\right).$$

The eigenfunction $v_{10} = \sin x$ is strictly positive for $x \in (0, \pi)$ and thus the inequalities

$$\left(u^+, v_{10}\right) \geq 0 \quad \text{and} \quad \left(u^-, v_{10}\right) \geq 0$$

hold. Hence, under the assumption, that $\int_\Omega h\, v_{10} > 0$, we have

$$k > -\alpha^2.$$

□

Remark 2. The positivity assumption on the integral $\int_\Omega h\, v_{10}$ is quite natural, since it is satisfied for all in the literature considered right-hand sides of the form $h(x,t) = W(x) + \varepsilon f(x,t)$ with ε being small enough (see e.g. [10]). Our assumption means that the external force $h(x,t)$ consisting of the roadbed's weight combined with some external perturbation can be even negative for some sufficiently small parts of the roadbed, which actually allows the perturbation $\varepsilon f(x,t)$ to be quite large.

After dealing with the necessary condition for the solvability of (1.10), it seems logical to proceed with a sufficient solvability condition. Before we formulate this condition, we present the following technical lemma (for more information and the proof, see [4], Lemma 3.3 and Remark 3.1), which will find its use in the proof of the main assertion.

Lemma 5. *Let $k > -\alpha^2$. If $h \in L^2(\Omega)$ does not depend on the time variable, then also the weak solution $u = u(x)$ is time independent and the estimate*

$$\|u\| \leq \frac{1}{\min\{\alpha^2, \alpha^2 + k\}} \|h\| \tag{2.7}$$

holds.

Remark 3. The proof of Lemma 3.3 in [4] is done for a more general nonlinearity $g(\cdot)$, for which the estimate

$$|g(\xi)| \leq c_1 + c_2|\xi|, \quad c_1, c_2 > 0 \tag{2.8}$$

holds for any $\xi \in \mathbb{R}$, and under additional assumption, that g, after the abstract formulation of the problem, leads to a monotone operator. Note that in case of $g(u) = ku^+$, the estimate (2.8) holds for an arbitrary $k \in \mathbb{R}$, but the monotonicity assumption is valid only for values $k > 0$. However, if we consider the nonlinearity in the proof of Lemma 3.3 ([4]) specifically in the form

$$g(u) = ku^+, \quad k < 0,$$

we can complete the proof without the corresponding nonlinear operator being monotone and obtain the estimate from Lemma 5 also for $k \in \left(-\alpha^2, 0\right]$.

Remark 4. Since $h \equiv 0$ is time-independent, we have $h \equiv 0 \Rightarrow u \equiv 0$ for any $k > -\alpha^2$, i.e., in such case, $Lu + ku^+ = 0$ has only a trivial solution.

Following these preparatory observations, we can show the most important assertion of this section. In its proof, we use the Leray-Schauder degree theory. The proof naturally splits into rather standard blocks, i.e., we set the correct operator formulation in the form "identity plus a compact mapping", and, after that, we construct an admissible homotopy, which makes a connection to a mapping with a known degree.

Theorem 6. *Let $k > -\alpha^2$ and $h \in L^2(\Omega)$ be arbitrary. Then (1.10) admits at least one weak solution.*

Proof. We start from the inverse operator formulation of (1.10)

$$u = L_0^{-1}(h - ku^+)$$

and put every nonzero term on the left-hand side. Note that $L_0^{-1}(h - ku^+)$ is a composition of a bounded map $(\cdot)^+$ and a compact operator L_0^{-1}. Hence we have

$$u - L_0^{-1}(h - ku^+) = o. \tag{2.9}$$

Now we construct a homotopy which connects the mapping in (2.9) to the identity mapping and is in the following form

$$H(t, u) = u - t\left(L_0^{-1}(h - ku^+)\right), \quad t \in [0, 1]. \tag{2.10}$$

If there exists a ball $B(o, r)$ in $L^2(\Omega)$, such that the considered homotopy is admissible, then the values of the degree are

$$\deg\big(I - L_0^{-1}(h - k(\cdot)^+), B(o, r), o\big) = \deg\big(H(1, u), B(o, r), o\big) =$$

$$= \deg\big(H(0, u), B(o, r), o\big) = \deg\big(I, B(o, r), o\big) = 1 \neq 0$$

and therefore, we have at least one weak solution of (1.10). So, the main task is the admissibility check, i.e., showing that there really exists this sufficiently large ball $B(o,r)$, where

$$H(t,u) \neq o$$

for all $u = \|r\|$ and all $t \in [0,1]$. Let us assume by contradiction, that there exist sequences $(u_n), (t_n)$, such that

$$\|u_n\| \to +\infty, \quad t_n \to t \in [0,1]$$

and for all $n \in \mathbb{N}$ we have

$$H(t_n, u_n) = u_n + t_n k L_0^{-1}(u_n^+) - t_n L_0^{-1}(h) = o.$$

Assuming $\|u_n\| \neq 0$ for all $n \in \mathbb{N}$, we get

$$\frac{u_n}{\|u_n\|} + t_n k \frac{L_0^{-1}(u_n^+)}{\|u_n\|} - t_n \frac{L_0^{-1}(h)}{\|u_n\|} = o,$$

which is, by linearity of L_0^{-1}, equal to

$$\frac{u_n}{\|u_n\|} + t_n k L_0^{-1}\left(\frac{u_n^+}{\|u_n\|}\right) - t_n L_0^{-1}\left(\frac{h}{\|u_n\|}\right) = o.$$

Let us denote $v_n := \dfrac{u_n}{\|u_n\|}$. Then

$$v_n + t_n k L_0^{-1}\left(v_n^+\right) - t_n L_0^{-1}\left(\frac{h}{\|u_n\|}\right) = o. \tag{2.11}$$

For $n \to +\infty$, the term $t_n L_0^{-1}\left(\frac{h}{\|u_n\|}\right)$ goes to the zero element. Since $L^2(\Omega)$ is a Hilbert (and hence a reflexive Banach) space, we are by Eberlein-Shmulyan's Theorem able to find a weakly convergent subsequence in every bounded sequence. The sequences (v_n) and v_n^+ are both bounded and thus there exist elements $v \in L^2(\Omega)$ and $y \in L^2(\Omega)$ such that

$$v_n \rightharpoonup v \quad \text{and} \quad v_n^+ \rightharpoonup y.$$

Note that we may pass to a subsequence if necessary. The compactness of L_0^{-1} implies the existence of an element $z \in L^2(\Omega)$ which is a strong limit of the sequence $\left(L_0^{-1}\left(v_n^+\right)\right)$, i.e.,

$$v_n^+ \rightharpoonup y \quad \Rightarrow \quad L_0^{-1}\left(v_n^+\right) \to z.$$

By going back to (2.11) and proceeding to corresponding limits, we obtain the equality

$$v = -tkz. \tag{2.12}$$

Let us recall that v is a weak limit of (v_n) and tkz is a strong limit of the product $t_n k L_0^{-1}\left(v_n^+\right)$. But since v is equal to $-tkz$, it is necessarily not only weak, but also a strong limit of (v_n). Hence (2.12) is the strong limit case of (2.11) for $n \to +\infty$. We have

$$v_n \to v \quad \Rightarrow \quad v_n^+ \to v^+$$

and thus

$$L_0^{-1}\left(v_n^+\right) \to L_0^{-1}\left(v^+\right), \tag{2.13}$$

since the operator $(\cdot)^+$ is continuous. Hence, $z = L_0^{-1}\left(v^+\right)$ and the limit version of (2.11) has the form

$$v + tkL_0^{-1}\left(v^+\right) = o, \tag{2.14}$$

or

$$Lv + tkv^+ = o. \tag{2.15}$$

For every $n \in \mathbb{N}$, the norm $\|v_n\| = \dfrac{u_n}{\|u_n\|} = 1$. The strong limit of such sequence has the same norm, i.e., $\|v\| = 1$. But for $k > -\alpha^2$, the problem (2.15) has only a trivial solution (Lemma 5, Remark 4), which is a contradiction. Hence the considered homotopy is admissible and (1.10) has at least one weak solution. □

2.3 Solvability - ODE case

In the previous chapter, we talked about the simplified ODE models. They are obviouly not as accurate and realistic as the PDE ones, however, can bring useful insights and suggest what phenomena to look for in more complex models. So, let us now take a side step in some sense, and investigate an ODE version of (1.10). If we consider the right-hand side of (1.10) in the form $h(x,t) = \sin x + \varepsilon f(t) \sin x$, try to find no-nodal solutions $u(x,t) = y(t) \sin x$ and put this information into (1.10), we get similar model simplification as in Section 1.2, that is, a damped ODE problem

$$\begin{aligned} y'' + \beta y' + \alpha^2 y + k y^+ &= f(t) \\ y(t) &= y(t + 2\pi), \end{aligned} \tag{2.16}$$

where $y^+ := \max\{y(t), 0\}$ and $f(t)$ is generally an L^2 function. Notice that for this problem, we can get the same results as in the PDE case, however, due to the simplified nature of it, we are able to obtain much more.

Let us very briefly go through the abstract setting for dealing with this problem. It is actually a reduction of the setting from Section 2.1 and hence it contains many similarities. Because of that, we take the liberty to use the same notation for the corresponding objects, e.g., the linear part of the equation is again represented by a linear operator denoted by L etc. Since the damping is present, we generally work in complex spaces, however, similarly to the PDE situation, we use the real Hilbert space (this time $H = L^2((0, 2\pi), \mathbb{R})$) as a starting point for a suitable complexification. This space is equipped with the standard scalar product and the corresponding norm. Again, we denote by $\mathcal{D}$ the set of all smooth functions which satisfy the periodic condition from (2.16).

Definition 7. We call a function $y(t) \in H$ a *weak solution* of the problem (2.16) if and only if the integral identity

$$\int_0^{2\pi} y(v'' - \beta v' + \alpha^2 v)\, \mathrm{d}t = \int_0^{2\pi} (f - k y^+) v\, \mathrm{d}t \tag{2.17}$$

holds for all $v \in \mathcal{D}$.

In order to continue the same way as before, let us introduce the complexification $H_{\mathbb{C}} = H + \mathrm{i}H = L^2((0, 2\pi), \mathbb{C})$ of H with the scalar product $\langle y, v \rangle = \int_0^{2\pi} y \bar{v}\, \mathrm{d}t$, $y, v \in H_{\mathbb{C}}$, and the usual norm $\|y\| = \langle y, y \rangle^{\frac{1}{2}}$. The set $\{\mathrm{e}^{\mathrm{i}nt},\ n \in \mathbb{Z}\}$ forms an orthogonal basis in $H_{\mathbb{C}}$ and thus each function $y(t) \in H_{\mathbb{C}}$ can be represented by the Fourier series

$$y(t) = \sum_{n=-\infty}^{+\infty} y_n \mathrm{e}^{\mathrm{i}nt}, \tag{2.18}$$

where for real functions $y \in H$ we have $y_{-n} = \bar{y}_n$. Next, we define the "abstract derivatives" through the following linear operator.

Definition 8. The operator L, such that

$$L: \operatorname{dom}(L) \subset H_{\mathbb{C}} \to H_{\mathbb{C}}, \qquad Ly = \sum_{n=-\infty}^{+\infty} (\alpha^2 - n^2 + \mathrm{i}\beta n) y_n \mathrm{e}^{\mathrm{i}nt},$$

where

$$\operatorname{dom}(L) = \Big\{ y \in H_{\mathbb{C}} : \sum_{n=-\infty}^{+\infty} |\alpha^2 - n^2 + \mathrm{i}\beta n|^2 \, |y_n|^2 < +\infty \Big\},$$

is called the *abstract realization* of the second order differential operator $y \mapsto y'' + \beta y' + \alpha^2 y$ (together with the associated periodic condition).

Again, L is a real operator, i.e., $y \in \operatorname{dom}(L) \cap H \Rightarrow Ly \in H$. And finally, there also exists an adjoint operator L^* such that

$$L^* : \operatorname{dom}(L^*) \subset H_{\mathbb{C}} \to H_{\mathbb{C}}, \qquad L^* y = \sum_{n=-\infty}^{+\infty} (\alpha^2 - n^2 - \mathrm{i}\beta n) y_n \mathrm{e}^{\mathrm{i}nt}.$$

It is also a real operator (cf. Section 2.1) and its domain is considered in the same sense as it has been done for the operator L.

Remark 5. As in the PDE case considered in Section 2.1, the spectrum $\sigma(L)$ of the operator L consists only of eigenvalues, which now have the form $\lambda_n = \alpha^2 - n^2 + \mathrm{i}\beta n$. Notice that $\lambda_n = \lambda_{1n}$ from the PDE case. Further, for all $\lambda \notin \sigma(L)$, the corresponding resolvent operator L_λ^{-1}, $L_\lambda^{-1} : H \to H$, is linear, compact and the estimate (2.6) holds.

Now, with the slightly revisited setting, we are able to bring the solvability conditions for the problem (2.16). Note that this time, the first normed eigenfunction of L, denoted by v_0, corresponds to the only real eigenvalue $\lambda_0 = \alpha^2$ and is in the form $v_0 \equiv 1$. Concerning the text structure, we keep the ordering from Section 2.2 and start with the necessary condition.

Lemma 9. *Let $y(t)$ be a weak solution of (2.16) and let $\int_0^{2\pi} f \, dt > 0$. Then $k > -\alpha^2$.*

Proof. The proof is similar to the PDE case (cf. Lemma 4). We start with the operator formulation and obtain the equation

$$(Ly, v_0) + k\left(y^+, v_0\right) = (f, v_0).$$

Again, as in the PDE case, we proceed with application of the adjoint operator L^*, which, for sufficiently differentiable functions, can be considered in the form $L^* y = y'' - \beta y' + \alpha^2 y$. Hence we obtain (cf. the proof of Lemma 4)

$$(y, L^* v_0) + k\left(y^+, v_0\right) = (f, v_0)$$

and consequently (by the decomposition of u and using the equality $L^* v_0 = \alpha^2 v_0$)

$$\left(\alpha^2 + k\right)\left(y^+, v_0\right) = (f, v_0) + \alpha^2\left(y^-, v_0\right).$$

Since $v_0 = 1$ and thus

$$\left(y^+, v_0\right) \geq 0 \quad \text{and} \quad \left(y^-, v_0\right) \geq 0,$$

we get $k > -\alpha^2$ whenever $\int_0^{2\pi} f\, v_0 \, \mathrm{d}t > 0$, i.e., whenever the mean of the function f is positive, since $v_0 \equiv 1$. □

Next on our list is the sufficient solvability condition. We aim to use the Theorem 6-type of proof, i.e., using the degree theory. That is why we again need a technical lemma in the sense of Lemma 5. This time, however, the assertion is more advanced, since it is combined with the standard regularity result and incorporates a much weaker assumption on the right-hand side, as only continuity is needed.

Theorem 10. *Let $f(t) \in C^0\big([0, 2\pi]\big)$. Then $y(t) \in C^2\big((0, 2\pi)\big) \cap C^0\big([0, 2\pi]\big)$ and*

$$\|y\| \leq \frac{1 + \sqrt{1 + 4\frac{\gamma}{\beta^2}}}{2\gamma} \|f\|, \tag{2.19}$$

where

$$\gamma = \begin{cases} \alpha^2 & \text{for } k \geq 0, \\ \alpha^2 + k & \text{for } k \in \left(-\alpha^2, 0\right). \end{cases} \tag{2.20}$$

Proof. Due to standard regularity arguments, the weak solution $y(t)$ is also the classical one in the class $C^2\big((0,2\pi)\big) \cap C^0\big([0,2\pi]\big)$. Next, if we multiply the equation from (2.16) by y' and integrate with respect to t over the interval of periodicity, we get

$$\int_0^{2\pi} y''y' + \beta \int_0^{2\pi} (y')^2 + \alpha^2 \int_0^{2\pi} \frac{1}{2}\,(y^2)' + k \int_0^{2\pi} y^+ y' = \int_0^{2\pi} f y'$$

If we use the information, that

$$\int_0^{2\pi} y''y' = \frac{1}{2}\Big[(y')\Big]_0^{2\pi} = 0, \qquad \int_0^{2\pi} (y^2)' = \Big[y^2\Big]_0^{2\pi} = 0 \tag{2.21}$$

and, finally, that also

$$\int_0^{2\pi} y^+ y' = 0, \tag{2.22}$$

we obtain

$$\int_0^{2\pi} (y')^2 = \frac{1}{\beta} \int_0^{2\pi} f y'.$$

Hölder inequality guarantees, that the estimate

$$\|y'\|^2 \le \frac{1}{\beta}\|f\|\|y'\|$$

holds and thus

$$\|y'\| \le \frac{1}{\beta}\|f\|. \tag{2.23}$$

Next, we multiply the equation from (2.16) by y and integrate with respect to t over $[0,2\pi]$ again, i.e.,

$$\int_0^{2\pi} y''y + \beta \int_0^{2\pi} y'y + \alpha^2 \int_0^{2\pi} y^2 + k \int_0^{2\pi} y^+ y = \int_0^{2\pi} f y$$

By applying similar arguments, we obtain

$$-\int_0^{2\pi} (y')^2 + \frac{\beta}{2} \int_0^{2\pi} \frac{1}{2}\,(y^2)' + \alpha^2 \int_0^{2\pi} y^2 + k \int_0^{2\pi} y^+ y = \int_0^{2\pi} f y \tag{2.24}$$

It follows from (2.21) and (2.22), that (2.24) actually reads

$$-\int_0^{2\pi} (y')^2 + \alpha^2 \int_0^{2\pi} y^2 + k \int_0^{2\pi} y^+ y = \int_0^{2\pi} f y \tag{2.25}$$

Further, for $k \ge 0$, we have $k\int_0^{2\pi} y^+ y \ge 0$ and thus we can reformulate (2.25) as the following inequality.

$$\alpha^2 \int_0^{2\pi} y^2 \le \int_0^{2\pi} f y + \int_0^{2\pi} (y')^2$$

After employing the Hölder inequality, it can be seen that

$$\alpha^2 \|y\|^2 \le \|f\|\|y\| + \|y'\|^2.$$

Now we can finally use the inequality (2.23) and obtain the final relation

$$\alpha^2 \|y\|^2 \le \|f\|\|y\| + \frac{1}{\beta^2}\|f\|^2,$$

which can be viewed as a quadratic inequality with respect to $\|y\|$. This finally yields

$$\|y\| \le \frac{1 + \sqrt{1 + 4\frac{\alpha^2}{\beta^2}}}{2\alpha^2}\|f\|.$$

For $k \in \left(-\alpha^2, 0\right)$, the inequalities

$$\alpha^2 \int_0^{2\pi} y^2 + k \int_0^{2\pi} y^+ y \geq \alpha^2 \int_0^{2\pi} y^2 + k \int_0^{2\pi} y^2$$

and

$$\left(\alpha^2 + k\right) \int_0^{2\pi} y^2 \leq \int_0^{2\pi} f y + \int_0^{2\pi} \left(y'\right)^2$$

hold. Using the same arguments, we obtain a slightly changed quadratic inequality, i.e.,

$$\left(\alpha^2 + k\right) \|y\|^2 \leq \|f\| \|y\| + \frac{1}{\beta^2} \|f\|^2.$$

Hence we get that for $k \in \left(-\alpha^2, 0\right)$ the estimate

$$\|y\| \leq \frac{1 + \sqrt{1 + 4\frac{\alpha^2 + k}{\beta^2}}}{2\left(\alpha^2 + k\right)} \|f\|$$

holds. □

Remark 6. Obviously, $f \equiv 0$ is a continuous function. Hence, we get $f \equiv 0 \Rightarrow y \equiv 0$ whenever $k > -\alpha^2$. That is, for such values of k, the equation $Ly + ky^+ = 0$ has only a trivial solution.

Finally, let us provide the ODE equivalent of Theorem 6. The proof is technically the same and thus we only point out the main parts of it.

Theorem 11. *Let $k > -\alpha^2$ and $f \in L^2(0, 2\pi)$ be arbitrary. Then (2.16) admits at least one weak solution.*

Proof. Again, we use the Leray-Schauder degree theory and start with an operator formulation

$$y - L_0^{-1}(f - ky^+) = o$$

The considered homotopy has the same form as in the previous section, i.e.,

$$H(t, y) = y - t\left(L_0^{-1}(f - ky^+)\right),$$

so we only check the admissibility of it for a sufficiently large ball $B(o, r)$ in $L^2(0, 2\pi)$. This is again done by contradiction (cf. the proof of Theorem 6), i.e., we work with sequences (y_n), (t_n), such that $\|y_n\| \to +\infty$ and $t_n \to t \in [0, 1]$. For all $n \in \mathbb{N}$, we assume that

$$\frac{y_n}{\|y_n\|} + t_n k L_0^{-1}\left(\frac{y_n^+}{\|y_n\|}\right) - t_n L_0^{-1}\left(\frac{h}{\|y_n\|}\right) = o.$$

In the end, after denoting $w_n := \dfrac{y_n}{\|y_n\|}$ and $w := \lim_{n \to +\infty} w_n$, we once more obtain the "limit" equality

$$w + tkL_0^{-1}\left(w^+\right) = o, \tag{2.26}$$

or

$$Lw + tkw^+ = o. \tag{2.27}$$

For every $n \in \mathbb{N}$, the norm $\|w_n\| = \dfrac{y_n}{\|y_n\|} = 1$. The strong limit of such sequence has the same norm, i.e., $\|w\| = 1$. But for $k > -\alpha^2$, the problem (2.27) has only a trivial solution (see the estimates in Theorem 10 and Remark 6), which is a contradiction. Therefore, the considered homotopy is admissible and thus (2.16) has a nontrivial weak solution for every $k > -\alpha^2$. □

Let us end this section with a bonus paragraph, where we can fully utilize the simplified ODE nature of (2.16). In contrast with Section 2.2, we can relatively easily obtain information about positivity of the solution in this case. Namely, if we consider the right-hand side in a specific form, we may formulate the following necessary and sufficient condition for the existence of a positive solution with respect to the sign properties of the right hand side.

Lemma 12. *There exists a strictly positive solution $y > 0$ of (2.16) with $f(t) = 1 + \varepsilon \sin t$ if and only if $\varepsilon^2 \leq \frac{(\alpha^2-1+k)^2+\beta^2}{(k+\alpha^2)^2}$.*

Proof. The linear problem corresponding to (2.16) has a solution in the form

$$y_{\mathrm{Lin}}(t) = A + B\sin(t+\varphi),$$

where

$$A = \frac{1}{\alpha^2+k}, \; k > -\alpha^2 \quad \text{and} \quad |B| = \frac{\varepsilon}{\sqrt{(\alpha^2-1+k)^2+\beta^2}}.$$

Next, for $k \neq -\alpha^2 + 1$,

$$\varphi = \arctan\left(-\frac{\beta}{\alpha^2-1+k}\right).$$

Notice that $k = -\alpha^2 + 1$ implies $\cos\varphi = 0$, i.e., this case corresponds to the shift $\varphi = \frac{\pi}{2}$. Now, if $\varepsilon^2 \leq \dfrac{(\alpha^2-1+k)^2+\beta^2}{(k+\alpha^2)^2}$, then for all t in the periodicity interval we have $y_{\mathrm{Lin}}(t) > 0$. The positivity of a linear solution implies

$$y_{\mathrm{Lin}}^{+} = y_{\mathrm{Lin}},$$

hence such a solution satisfies also problem (2.16).

On the other hand, if there exists a positive solution y of (2.16), then it satisfies the corresponding linear problem, since

$$y^{+} = y = y_{\mathrm{Lin}}.$$

But that implies $\varepsilon^2 \leq \dfrac{(\alpha^2-1+k)^2+\beta^2}{(k+\alpha^2)^2}$. □

For the visual interpretation of the condition on k and ε from Lemma 12 for $\alpha = 1$ and certain values of β, see Figure 2.1.

Remark 7. Let $f(t) = 1 + \varepsilon \sin t$ and $|\varepsilon| < 1$. Then the right-hand side of (2.16) is strictly positive. However, the positivity of it does not guarantee the existence of a positive weak solution. Indeed, if e.g. $k = \beta = \frac{1}{2}$ and $\alpha^2 = 1$, then there exists a positive weak solution if and only if $|\varepsilon| < \dfrac{\sqrt{2}}{3} < 1$. This means that there are values of ε such that $\dfrac{\sqrt{2}}{3} < \varepsilon < 1$ or $-\dfrac{\sqrt{2}}{3} > \varepsilon > -1$, for which $f(t) > 0$, however, the corresponding solution $y(t)$ changes sign. On the other hand, if e.g. $k = \beta = 3$ and $\alpha^2 = 1$, there exists a positive weak solution for every $|\varepsilon| < \dfrac{3\sqrt{2}}{4}$, which means, that there are values of ε, with the absolute value $1 < |\varepsilon| < \dfrac{3\sqrt{2}}{4}$, such that the corresponding solution $y(t)$ is positive, even though $f(t)$ changes sign.

2.4 Uniqueness for a general right hand side

At last, let us deal with the uniqueness of a weak solution. Since we tried to improve the previous results from [39] by using the (in the corresponding literature) standardized abstract setting (see Section 2.1), however with some fine-tuned discussion concerning the model parameters, it was

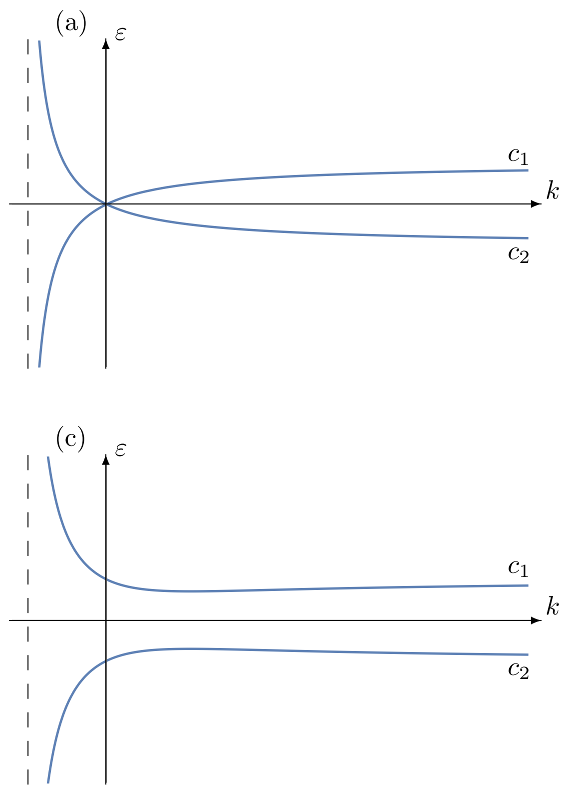

Figure 2.1: The area between the curves c_1 and c_2 represents points $[k, \varepsilon]$ which correspond to the existence of a positive solution for a specific right hand side $f(t) = 1 + \varepsilon \sin t$. In all cases, $\alpha = 1$. The picture (a) corresponds to the value $\beta = 0$, whereas in (b), $\beta = 0.2$, in (c), $\beta = 1$ and in the picture (d), $\beta = 2$. The dashed line represents the limit value $k = -\alpha^2 = -1$.

crucial to interpret the eigenvalues (2.4) geometrically in the complex plane. Fortunately, it was possible, since they could be viewed as intersections of parabolas p_m and lines l_n parallel to the real axis (see Fig. 2.2 for illustration), where

$$p_m = \left\{ (x, y) : \; x = \alpha^2 m^4 - \frac{y^2}{\beta^2} \right\}, \quad m \in \mathbb{N}$$

and

$$l_n = \{(x, y) : \; y = \beta n, \}, \quad n \in \mathbb{Z}.$$

This geometric interpretation makes the manipulation with the eigenvalues much more "visual" and thus it is possible to work with the concept of distance of parameter λ from the spectrum of L more easily. Just based on this visualisation, our first improvement is increasing the readability of the uniqueness condition from [39]. Indeed, it is now easier to verify for specific parameter values. The key step is to determine the type of ordering relation between the parameters α and β and when does it guarantee the smallest real eigenvalue λ_{10} to be the closest one to the point of origin, i.e., when

$$\operatorname{dist}(0, \sigma(L)) = |\lambda_{10}| = \alpha^2. \tag{2.28}$$

In other words, the open disc $D_0 = \{z \in \mathbb{C} : \; |z| < \alpha^2\}$ does not contain any other eigenvalue λ_{mn}. Since these eigenvalues can be identified as parabola-line intersections, it is sufficient to check if the first parabola p_1, or the first pair of lines, that is, l_1, l_{-1}, is outside D_0. With this approach, we can observe the following behaviour (see Fig. 2.3 for illustration and [19] for more information).

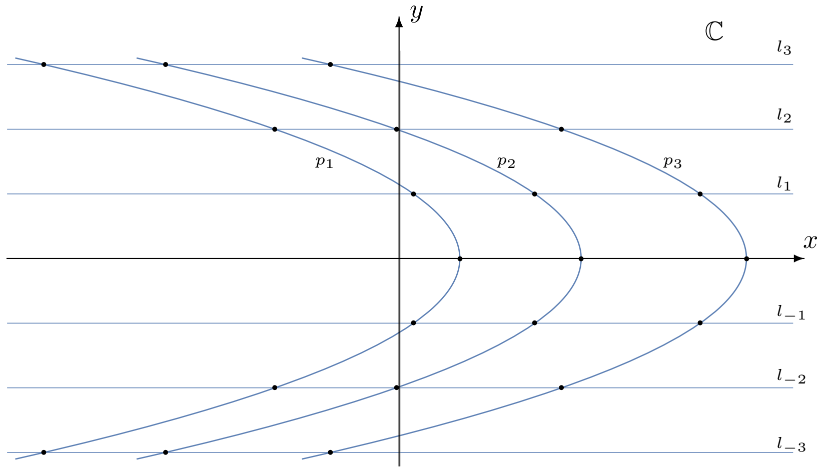

Figure 2.2: Illustration of the eigenvalues λ_{mn} in the complex plane.

Remark 8. The mutual position of the parabolas, lines and D_0 has these consequences:

1. If $\beta \geq \alpha^2$, then no horizontal line l_n, $n \in \mathbb{Z}$, intersects D_0.
2. If $\beta \geq \sqrt{2}\alpha$, then no parabola p_m, $m \in \mathbb{N}$, intersects D_0.
3. If $\alpha > 1$ and $\sqrt{2\alpha^2 - 1} \leq \beta < \sqrt{2}\alpha$, then the only parabola intersecting D_0 is p_1, but $\lambda_{1n} \notin D_0$ for all $n \in \mathbb{Z}$.

Hence, if one of these relations holds then (2.28) is true, we obtain (in view of the result from [39]) uniqueness of a weak solution for any $k \in \big(-\alpha^2, \alpha^2\big)$. Now, let us summarize all discussed facts.

Proposition 13 ([19])**.** *Let $\beta \geq \alpha^2$ for $\alpha < 1$ and $\beta \geq \sqrt{2\alpha^2 - 1}$ for $\alpha \geq 1$. Then the problem (1.10) has a unique weak solution $u \in H$ for an arbitrary right-hand side $h \in H$ whenever $k \in (-\alpha^2, \alpha^2)$.*

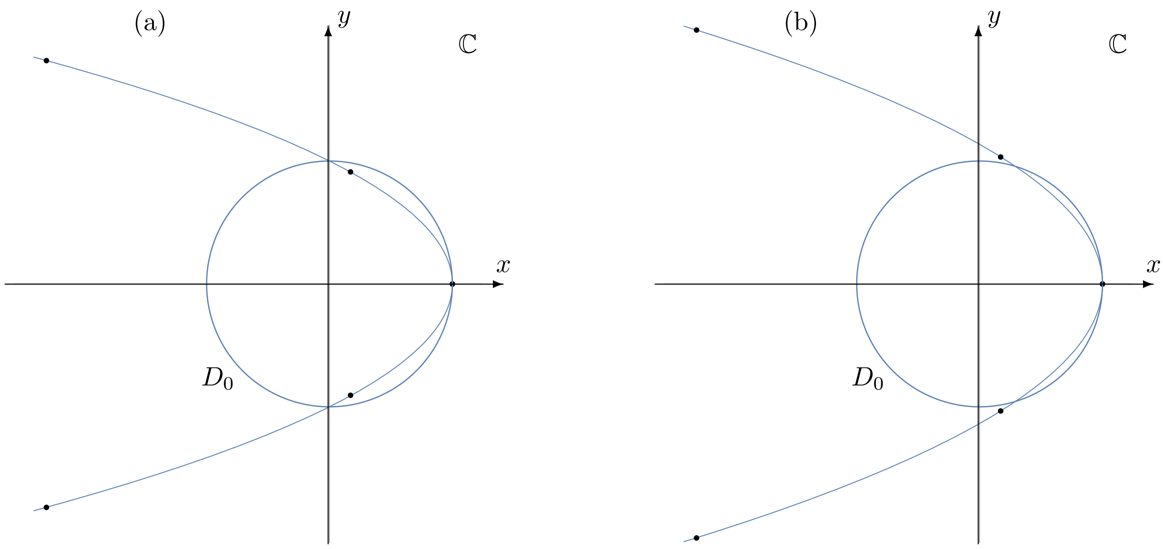

Figure 2.3: Spectrum $\sigma(L)$ and disc D_0 for (a) $\alpha = 1.1$, $\beta = 1.1$, (b) $\alpha = 1.1$, $\beta = 1.25$. Here, $\sigma(L) \cap D_0 = \{\lambda_{1\pm1}\}$ in (a), whereas $\sigma(L) \cap D_0 = \emptyset$ in (b).

In this place, we would like to point out the fact, that Proposition 13 allows just the more straightforward usability of the original condition. So far, it is not a "real" improvement - quite the opposite, it is in fact weaker. The main issue is that it makes the chain of implications, which "enable" the general uniqueness condition from [39], longer.

Hence, if we want to achieve a more noticeable improvement of the previous uninequeness condition, it is desirable to modify the operator equation (2.3). In analysis, there is often a standard approach of adding and subtracting the same term, which, if handled correctly, brings a new perspective for the same problem. Indeed, that is why we introduced an ε-shift to the operator equation, i.e., (2.3) now has the form

$$Lu - \varepsilon u + \varepsilon u + ku^+ = h$$

and allows us to consider an equivalent equation, which, however, works with an ε-shifted operator, that is,

$$(L - \varepsilon I)u = -(\varepsilon u + ku^+) + h. \tag{2.29}$$

Actually, the main reason for introducing this ε-shift stands out the most, if we look at Figs. 2.2 and 2.3. Checking the distance between the origin and $\sigma(L)$ is simply too restrictive and thanks to the parabolical shape of p_m, it would be much better to measure the distance between $\sigma(L)$ and some other point ε on the real axis, especially for $\varepsilon < 0$. Some limited improvement is possible also for $\varepsilon > 0$, however this side of zero is not much interesting from the suspension bridge point of view.

Thus, by considering ε not to be an eigenvalue of L and using the decomposition $\varepsilon u = \varepsilon u^+ - \varepsilon u^-$ on the right hand side of (2.29) (note that this decomposition comes as a natural step, since the equation already contained a positive-part term ku^+), we get a fixed point formulation

$$u = L_\varepsilon^{-1}\left(-(k+\varepsilon)u^+ + \varepsilon u^- + h\right), \tag{2.30}$$

where L_ε^{-1} denotes the resolvent operator $(L - \varepsilon I)^{-1}$. Next, in view of [39], we again employ Banach Contraction Theorem together with the estimate

$$\|(k+\varepsilon)(v^+ - u^+) - \varepsilon(v^- - u^-)\| \leq \max\{|k+\varepsilon|,\, |\varepsilon|\}\|v - u\|$$

(see [19]) and obtain that if

$$\max\{|k+\varepsilon|,\, |\varepsilon|\} < \operatorname{dist}(\varepsilon, \sigma(L)), \tag{2.31}$$

then the operator $L_\varepsilon^{-1}\left(-(k+\varepsilon)(\cdot)^+ + \varepsilon(\cdot)^- + h\right)$ is contractive.

Apparently, the value $|k+\varepsilon|$ expresses the distance between ε and $-k$ and $|\varepsilon|$ the distance between ε and the origin. For now, if we keep the "distance language", the inequality (2.31) actually reads

$$\operatorname{dist}(\varepsilon, 0) < \operatorname{dist}(\varepsilon, \sigma(L)) \quad \wedge \quad \operatorname{dist}(\varepsilon, -k) < \operatorname{dist}(\varepsilon, \sigma(L)).$$

In order to have both inequalities under control, it is optimal to consider $k = -2\varepsilon$, which implies $|k+\varepsilon| = |\varepsilon|$. In that case, if we find the maximal positive values $\varepsilon_m, \varepsilon_M$ such that $\operatorname{dist}(\varepsilon, 0) < \operatorname{dist}(\varepsilon, \sigma(L))$ holds for any $\varepsilon \in (-\varepsilon_M, \varepsilon_m)$ then $\operatorname{dist}(\varepsilon, -k) < \operatorname{dist}(\varepsilon, \sigma(L))$ holds for any $k \in (-2\varepsilon_m, 2\varepsilon_M)$. This actually means, that the more we can shift ε, the larger interval of uniqueness for values of the stiffness parameter k we get.

However, finding the values $\varepsilon_m, \varepsilon_M$ is not necessarily simple. The first way how to deal with this problem is to find their safe (however, not necessarily optimal) estimates, as it can be seen in the following existence and uniqueness theorem, which brings the above suggested process into action.

Theorem 14 ([19])**.** *Let $\varepsilon_M > 0$ and $\varepsilon_m > 0$ be the maximal real numbers for which*

$$\left\{z \in \mathbb{C} : \ (|z - \varepsilon_m| < \varepsilon_m) \vee (|z + \varepsilon_M| < \varepsilon_M)\right\} \cap \sigma(L) = \emptyset. \tag{2.32}$$

Then the problem (1.10) has a unique weak solution $u \in H$ for an arbitrary right-hand side $h \in H$ whenever $k \in (-2\varepsilon_m, 2\varepsilon_M)$. Moreover, the following estimates hold:

$$\varepsilon_M \geq \tilde{\varepsilon}_M = \begin{cases} \dfrac{2\alpha\beta + \beta^2}{2} & \text{for } \beta \geq 2(1-\alpha), \\ \beta & \text{for } \beta < 2(1-\alpha), \end{cases} \tag{2.33}$$

and

$$\varepsilon_m \geq \tilde{\varepsilon}_m = \begin{cases} \dfrac{\alpha^2}{2} & \text{for } \beta \geq \min\left\{\alpha, \frac{\alpha^2}{2}\right\}, \\ \dfrac{2\alpha\beta - \beta^2}{2} & \text{for } \beta \leq \min\{\alpha, 2(\alpha-1)\}, \\ \beta & \text{for } \alpha < 2 \text{ and } 2(\alpha-1) \leq \beta \leq \frac{\alpha^2}{2}. \end{cases} \tag{2.34}$$

For the proof, see [19], however, its main idea has already been discussed in the previous paragraph. If we go back to the visual interpretation of λ_{mn} (see Fig. 2.2), i.e., if we have in mind, that the eigenvalues are in fact intersections of p_m, $m \in \mathbb{N}$, and l_n, $n \in \mathbb{Z}$, we realize, that the safe lower bounds $\tilde{\varepsilon}_m$, $\tilde{\varepsilon}_M$ for ε_m, ε_M are in fact the radii of discs, which are positioned such that their centers are on the real line, they are touching the origin and, either the first parabola p_1 (see Figure 2.4), or the first pair of lines l_{-1}, l_1.

Roughly speaking, the estimates depend only on the mutual position (directly determined by α and β) of p_1 and $l_{\pm 1}$. It is sufficient (with respect to the nature of the eigenvalues λ_{mn}) to check whether the first parabola or the first pair of lines pass through the corresponding discs (see Figs. 2.4 and 2.5 for illustration).

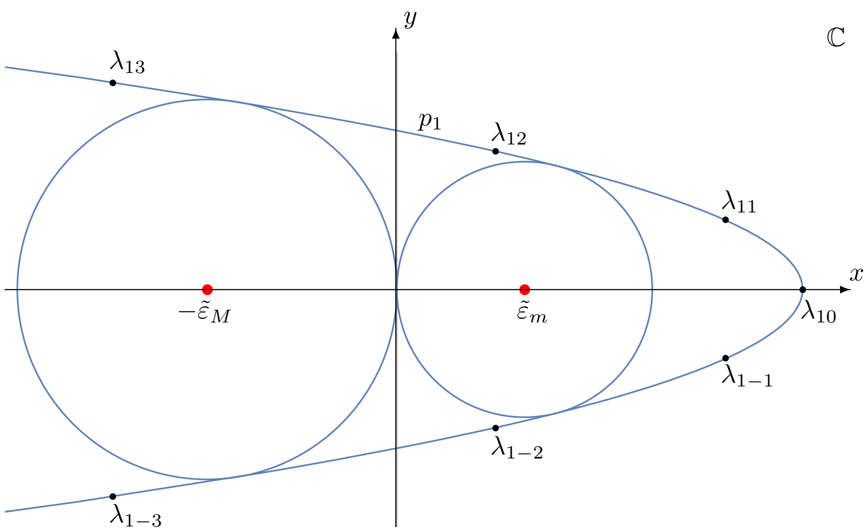

Figure 2.4: The values $\tilde{\varepsilon}_m$, $\tilde{\varepsilon}_M$ for $2(1-\alpha) \leq \beta \leq \alpha$ and the corresponding "safe" discs, where none of the eigenvalues may appear.

Although these estimates for $\varepsilon_m, \varepsilon_M$ based on this discussion actually bring a significant improvement, they are geometrically quite basic, since we questioned neither the position of other parabolas, nor other pairs of lines. Hence, since the room for improvement is generally still large, let us discuss our possibilities.

Before we suggest another approach in order to deal with this issue, let us briefly mention the possible ways of improving the estimates $\tilde{\varepsilon}_m$ and $\tilde{\varepsilon}_M$. By a closer inspection of the parabolas p_1 and p_2, we find out that for $\alpha^2 < 1$ and $\beta < \frac{1-\alpha^2}{(4+\sqrt{15})\alpha}$ we may construct a disc touching the origin and p_2 with no eigenvalues lying on p_1 in its interior. In this case, we can improve the estimate, which is then in the form $\varepsilon_M \geq \bar{\varepsilon}_M = \frac{8\alpha\beta+\beta^2}{2}$.

So, if one would ask, whether it is possible to generate more precise estimates, the general answer would be "yes", however, for the price of getting more and more complicated conditions on α and β, since it is necessary to take into account more parabolas and pairs of lines.

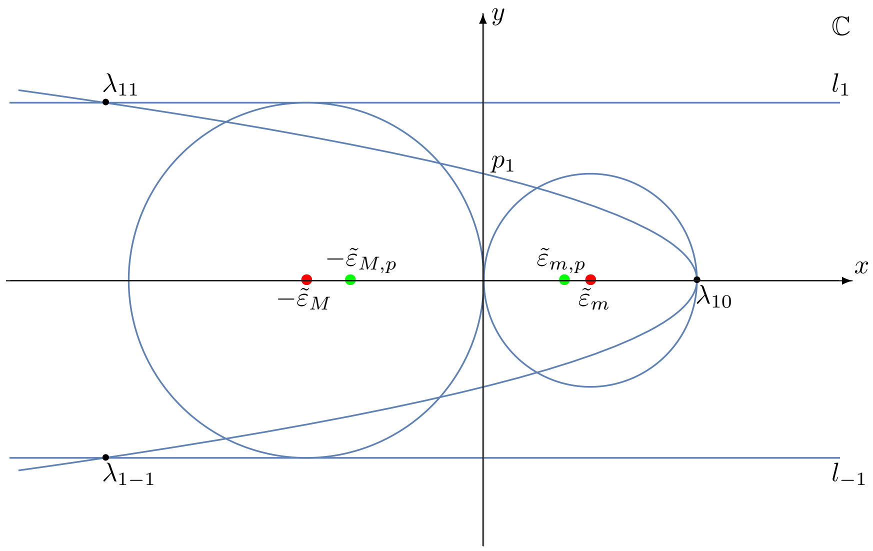

Figure 2.5: The values $\tilde{\varepsilon}_m$, $\tilde{\varepsilon}_M$ and the corresponding "safe" discs, where none of the eigenvalues may appear, however, now for the case $\frac{\alpha^2}{2} \leq \beta < 2(1-\alpha)$, when the first pair of lines gains importance. Here, $\tilde{\varepsilon}_{M,p}$ and $\tilde{\varepsilon}_{m,p}$ stand for the more restrictive estimates, which would have been obtained by checking the position of the first parabola p_1.

There is, however, another possibility, that is, to avoid the process of finding some "near optimal" estimates and instead of that to compute the optimal values $\varepsilon_m, \varepsilon_M$ directly via an algorithm working for specific given values α and β. The suggested algorithm is discussed in [19] and can be described in four steps.

Remark 9. The value ε_M can be computed by following this procedure:

1. Put $\lambda_{\text{opt}} = \lambda_{1n_0}$ with $n_0 = \lfloor \alpha + 1 \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part of a real number. (Notice that λ_{1n_0} is the closest eigenvalue to the imaginary axis with a negative real part on p_1.)

2. Find an open disc D with the center on the real axis, whose boundary is going through an eigenvalue λ_{opt} and the origin, i.e., $D = \{z \in \mathbb{C};\ |z + \varepsilon_D| < \varepsilon_D\}$ with

$$\varepsilon_D = \frac{|\lambda_{\text{opt}}|^2}{2\,|\text{Re}\,(\lambda_{\text{opt}})|}.$$

3. If there is no other $\lambda_{mn} \in D$, put $\varepsilon_M = \varepsilon_D$ and quit. In the other case find indexes $M = \max\{m : \lambda_{mn} \in D\}$ and $N = \min\{n : \lambda_{Mn} \in D\}$, i.e., find the indexes of such an eigenvalue with a negative real part inside D, which is the closest one to the imaginary axis and lies on the parabola, whose branches are the furthest from the real axis.

4. Put $\lambda_{\text{opt}} = \lambda_{MN}$ and go back to Step 2.

The same algorithm (only with modifications corresponding to the fact, that the constructed discs find themselves on the other half-plane in $\mathbb{C}$) can be applied in the case of ε_m.

So far, we discussed several techniques, which lead to significant improvements of the uniqueness result from [39]. Now, let us show a pair of examples, which serve as an overview of all tools presented in [19] and also in this chapter.

Example 1 ([19])**.** Let $\alpha \geq 2$, $q \geq \sqrt{2}$ and put $\beta = q\alpha$. Then the assumptions of Proposition 13 are satisfied and the problem (1.10) has a unique weak solution for all $k \in (-\alpha^2, \alpha^2)$. If we employ Theorem 14 with its estimates (2.33), (2.34), we get

$$\tilde{\varepsilon}_M = \frac{2\alpha\beta + \beta^2}{2}, \quad \tilde{\varepsilon}_m = \frac{\alpha^2}{2}$$

and thus obtain a much larger uniqueness interval

$$k \in \Big(-\alpha^2, \ \big(q^2 + 2q\big)\, \alpha^2\Big),$$

i.e., the positive part of the interval, which is more interesting from the physical point of view, is $(q^2 + 2q)$-times larger than the original conditions allow.

Example 2 ([19])**.** Let $s \in \mathbb{N}$ be arbitrary and put $\alpha = s$, $\beta = \frac{1}{s}$. Here, the refining of previous results, i.e., Proposition 13, gives no information about solvability of (1.10), since $\alpha^2 \geq 1$ and $\beta \leq 1 \leq \sqrt{2\alpha^2 - 1}$. However, since $s \in \mathbb{N}$, $\lambda_{1s} = 0 + \mathrm{i}$, and the open disc

$$D = \{z \in \mathbb{C};\ |z| < 1\} \tag{2.35}$$

contains no other eigenvalue λ_{mn} (i.e., $\min_{m\in\mathbb{N},n\in\mathbb{Z}} |\lambda_{mn}| = |\lambda_{1s}| = 1$), the original general result from [39] guarantees the existence and uniqueness of a weak solution of (1.10) for an arbitrary right-hand side $h \in H$ whenever $k \in (-1, 1)$.

By applying Theorem 14, the interval $(-1, 1)$ can be enlarged. It is easy to see, that for $s = 1$ the estimates (2.33), (2.34) yield $k \in \big(-\alpha^2,\ 2\alpha\beta + \beta^2\big) = (-1,\ 3)$, and for $s \geq 2$ we obtain

$$k \in \big(-2\alpha\beta + \beta^2,\ 2\alpha\beta + \beta^2\big) = \left(-2 + \frac{1}{s^2},\ 2 + \frac{1}{s^2}\right). \tag{2.36}$$

Note that these uniqueness intervals are twice as large as the original one. Moreover, e.g., for $s = 1$, the closest eigenvalue to the imaginary axis with a negative real part on p_1 is $\lambda_{12} = -3+2\mathrm{i}$ and the disc D, whose boundary is passing through it, contains no other eigenvalue in its interior. Hence, using our algorithm, we get $\varepsilon_M = \varepsilon_D = \frac{|\lambda_{12}|^2}{2|\mathrm{Re}\,(\lambda_{12})|} = \frac{13}{6}$ and the uniqueness result holds for any $k \in \left(-1, \frac{13}{3}\right)$.

Although it may seem a bit adventurous to directly compare results for non-damped and damped models, we would like to find a good enough confirmation of the general expectation, that the damping term should guarantee a more stable behaviour, than the similar non-damped model has (see also [25]).

Remark 10. When we look at Example 2 with using the estimates for $s = 1$ and at the result [10] for (1.4) and (1.5), we get the same uniqueness interval $(-1, 3)$. If we utilize our algorithm from Remark 9, the uniqueness interval is larger: $(-1, 13/3)$. This really suggests, that adding the damping term into the model may extend its uniqueness behaviour to a larger interval. For better illustration, we may consider $\alpha = 1$ and $\beta > 1$. By employing our estimates $\tilde{\varepsilon}_m$, $\tilde{\varepsilon}_M$ from Theorem 14, we get uniqueness for any $k \in \big(-1, 2\beta + \beta^2\big)$, where $2\beta + \beta^2 > 3$. That yields another direct comparison, since, as we discussed e.g. in Chapter 1, for $3 < k < 15$ without damping, there are more solutions guaranteed (see also [10]). However, with sufficient damping, we get a unique solution even for some $k > 3$.

To conclude this chapter, we should also point out, that our results from [19] are fully applicable also in the ODE case, which was investigated in Section 2.3.

Remark 11. Obviously, the extended uniqueness result from Theorem 14 holds also for the problem (2.16). Moreover, the fact that the eigenvalues $\lambda_n = \lambda_{1n}$ lie on no other parabolas than p_1 (cf. Fig. 2.2), means, that the estimates $\tilde{\varepsilon}_m$ and $\tilde{\varepsilon}_M$ are more accurate (see the discussion between Theorem 14 and Remark 9). For some appropriate setting of the bridge's parameters, it may be even possible to find the precise values $\varepsilon_m, \varepsilon_M$. Also, the algorithm from Remark 9 for optimal values of $\varepsilon_m, \varepsilon_M$ is easier to go through, and we may realistically expect to compute ε_m and ε_M in reasonable time, since the number of possible eigenvalues in the disc D is lower.

Chapter 3

Weighted non-damped PDE model

In Chapter 1, we discussed the possibility of introducing a reasonably straightforward generalization of (1.2), which would deal more realistically with the problem that the bridge hangers actually should not be viewed as a "continuous force" acting on the roadbed (cf. (1.6)). Since, in reality, they are placed with some fixed distance between them, one of the simplest ways to reflect this is introducing a *density* (or technically speaking *weight*) function $r(x)$ into the basic "constant-stiffness" PDE model. The restoring force is supposed to attain its maximum where the hangers are connected to the roadbed. On the other hand, it should be considerably weaker in between.

Hence, in this chapter, we provide the results published in [21], where we study a suitably modified version of a standard ([28, 29]) one-dimensional nonlinear beam model of a suspension bridge, i.e.,

$$\begin{aligned} &u_{tt} + u_{xxxx} + k\, r(x) u^+ = h(x,t) \quad \text{in } (0,1) \times \mathbb{R}, \\ &u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \\ &u(x,t) = u(x,t+2\pi) = u(x,-t). \end{aligned} \tag{3.1}$$

The restoring force of the bridge hangers, which is considered nonlinear, is represented by the term $k\, r(x) u^+$. The stiffness constant is denoted by k and the placement density by $r(x)$. In almost all cases, we consider $r(x)$ to be a continuous function on $(0,1)$, such that $0 < r(x) \leq 1$ *almost everywhere* in $(0,1)$. This is meant in the standard sense, i.e., the subset of $(0,1)$, where r violates this asssumption, is of zero Lebesgue measure. There are a few exceptions in this chapter, where the weight is considered as a more general function, however, this is always explicitly noted to avoid confusion.

Let us briefly summarize the work done for the constant density case, i.e., $r(x) \equiv 1$, which we somewhat vaguely called a "continuous force" of the hangers. In this field, there are many results concerning multiplicity of periodic solutions, which served as our motivation: see e.g. [28, 29] and also [7], [23]. These works provide an excellent example of the problem setting and consequently also the application of various abstract tools. Their common narrative is reaching a conclusion, that the more eigenvalues of the corresponding linear beam operator are crossed by the hanger stiffness k, the more solutions appear. These articles were followed by [10] and [14], which took a different approach of this problem and utilized a global bifurcation framework. Specifically, [10] was a major source of inspiration for us, as we tried to keep the same "abstract setting $\to$ bifurcation equation" structure, only repurposed for the variable density $r(x)$.

Speaking about the non-constant hanger placement density, it is in some sense a negative (however expected) fact, that establishing a standard bifurcation scheme is a more difficult task when compared to the constant density case. It is necessary to employ some additional results from [11] and [12], which is documented in more detail in Section 3.2 and of course in [21]. For illustrations of specific hanger distributions, we are mainly interested in weight functions which are similar to a high even power of the cosine function, since, in our opinion, such a density function is simple enough and sufficiently resembles the discrete placement of the bridge hangers (see Fig. 3.1).

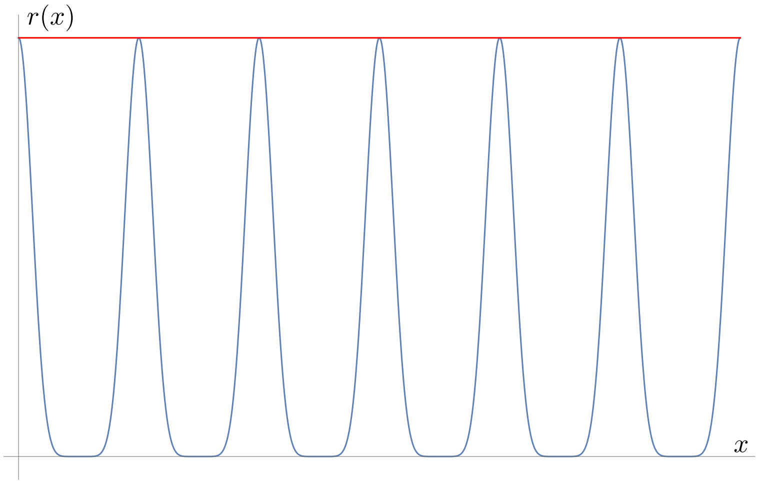

Figure 3.1: The maxima of the cosine-type curve (blue), represent the connections of the bridge hangers and the road-bed. For illustration, we choose $r(x) = \cos^6(x)$. The red line represents the constant density $r(x) \equiv 1$, which omits any spatial difference in stiffness.

When thinking about applications in suspension bridges, we should have in mind, that from the purely mechanical point of view, only positive stiffness k makes sense. However, we deal with the problem without this limitation, since it is mathematically interesting also for k being negative and in many cases, we would have to artificially limit our proofs and would lose important comparison to previous constant density results.

Warning. In [21], we use the letter b for the stiffness parameter. This is due to historical reasons, since we considered our paper as a continuation of [10], which also uses this notation. However, for the notation consistency in this text, we keep the letter k for denoting the stiffness, as in the previous chapters.

Now, the motivation and background for our work with spatially variable stiffness is clear and we can present all the details concerning the operator setting of (3.1).

3.1 Weighted space and eigenvalues

The hanger placement density represented by a weight function comes with a challenge concerning the abstract setting of the problem. We have to find the right balance of difficulty, i.e., we are asking, whether it is better to consider a weighted space with a less complicated abstraction of the beam operator, or a standard L^2 space for the price of a not-so-straightforward abstract realization of the operator.

Actually, there are less obstacles on the former path, i.e., working in a weighted space. This is mainly because the use of weighted spaces for fourth-order problems has been at least partially documented in the work of C. P. Gupta with J. Mawhin [18] and S. A. Janczewsky [24]. Moreover, working in the weighted space is also beneficial in the sense, that structurally, we proceed very similarly as in Chapter 2 when formulating the problem as an operator equation.

Let us denote the considerd domain Ω by

$$\Omega = (0, 1) \times (0, 2\pi)$$

and let $h/\sqrt{r}$ be in $L^2(\Omega)$. As foreshadowed earlier, we consider the weighted space $L^2_r(\Omega) :=$

$L^2(\Omega, r(x))$ with the inner product

$$(u,v)_r = \int_\Omega r(x)u(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

and the corresponding norm $\|u\|_r = \sqrt{(u,u)_r}$. This also has the impact that $h/\sqrt{r} \in L^2(\Omega)$ means $h/r \in L^2_r(\Omega)$.

If we want to establish the notion of a weak solution, we have to consider a space $H \subset L^2_r(\Omega)$ to be a subspace of functions in $L^2_r(\Omega)$ being even in the time variable. Further, let $\mathcal{D}$ stand for all C^∞-functions $\psi : [0,1] \times \mathbb{R} \to \mathbb{R}$ which satisfy the boundary and periodic conditions from (3.1).

Definition 15. A function $u : (0,1) \times \mathbb{R} \to \mathbb{R}$ is called *a weak solution* of the problem (3.1) if and only if

$$\int_\Omega u(x,t)(\psi_{tt}(x,t) + \psi_{xxxx}(x,t))\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \big(h(x,t) - k\, r(x)u^+(x,t)\big)\,\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

for all $\psi \in \mathcal{D}$, and the restriction of u belongs to H. Here, again, u^+ denotes the positive part of u. Let us add that u^- stands for the negative part of u and $u = u^+ - u^-$.

As the last detail for now, we point out that if $u \in H$, both u^+ and u^- are also elements of H. Now, if we want to investigate solvability and bifurcations in (3.1), it is not surprising, that we have to understand the structure of the spectrum of the operator that represents the linear part of the equation in (3.1). This time, the introduced density function forces us to deal with the weighted spectrum of a linear beam operator. That is, we consider the eigenvalue problem with a weight function r in the form

$$\begin{aligned} &u_{tt} + u_{xxxx} = \lambda\, r(x)u \quad \text{in } (0,1)\times\mathbb{R}, \\ &u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \\ &u(x,t) = u(x,t+2\pi) = u(x,-t). \end{aligned} \tag{3.2}$$

At this point, we show in [21], that this problem can be investigated in the view of the so-called *regular Sturmian systems*. Indeed, we can take advantage of the results from [18] and [24]. In order to make it possible, we start with a rather standard approach, that is, employing the separation of variables. Considering the solution in the separated form $u(x,t) = X(x)T(t)$, there exists $\mu \in \mathbb{R}$ such that

$$\begin{aligned} T'' + \mu T &= 0 \\ T(t) = T(t+2\pi) &= T(-t), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} X^{\mathrm{IV}} - \mu X &= \lambda r(x) X \\ X(0) = X(1) = X''(0) &= X''(1) = 0. \end{aligned} \tag{3.4}$$

It is easy to see, that (3.3) has a nontrivial solution if and only if $\mu = n^2$, $n \in \mathbb{N} \cup \{0\}$ and $T_n(t) = \cos nt$. Then, for any fixed n, the problem (3.4) has the properties of a regular Sturmian system with a weight function r. The implications of this procedure with the involvement of [18] and [24] can be summarized in the following lemma.

Lemma 16 ([21])**.** *All the eigenvalues of the problem* (3.2) *are real and form an infinite sequence*

$$(\lambda_{m,n})_{m,n=0}^{+\infty}$$

with the following properties.

1. *For any* $m, n \in \mathbb{N} \cup \{0\}$, $\lambda_{m,n} \neq 0$.

2. $\lim\limits_{m \to +\infty} \lambda_{m,n} = +\infty$, $\lim\limits_{n \to +\infty} \lambda_{m,n} = -\infty$.

3. *For any fixed* $n \in \mathbb{N} \cup \{0\}$, *all the eigenvalues, for which* $\lambda_{m,n} > -n^2$, *are simple, i.e.,* $\lambda_{m_1,n} \neq \lambda_{m_2,n}$ *whenever* $m_1 \neq m_2$.

4. *All the eigenvalues* $\lambda_{m,0}$ *are positive.*

5. *For any* $m, n \in \mathbb{N} \cup \{0\}$
$$|\lambda_{m,n}| \geq \left|(m+1)^4\pi^4 - n^2\right|.$$

6. *For any fixed* $m \in \mathbb{N} \cup \{0\}$,
$$\lambda_{m,n_1} \geq \lambda_{m,n_2}, \quad \text{whenever} \;\; n_1 < n_2.$$

The eigenfunctions corresponding to $\lambda_{m,n}$ *take the form*
$$\varphi_{m,n}(x,t) = X_m(x,n)\cos nt$$
with $X_m(x,n)$ *being a nontrivial solution of the ODE problem*
$$\begin{gathered} X^{IV} - n^2 X = \lambda_{m,n}\, r(x) X, \\ X(0) = X(1) = X''(0) = X''(1) = 0. \end{gathered} \tag{3.5}$$
All the eigenfunctions $\varphi_{m,n}(x,t)$ *form a complete orthogonal system on* Ω *with the weight* $r(x)$, *i.e.,*
$$\int_\Omega r(x)\varphi_{m,n}(x,t)\varphi_{k,l}(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0, \quad \text{whenever} \;\; m \neq k \text{ or } n \neq l.$$
The eigenfunction $\varphi_{0,0}(x,t) = X_0(x,0)$ *is strictly positive in* Ω. *Moreover, if* $r(x) > 0$ *on* $(0,1)$, *all the functions* $X_m(x,n)$ *corresponding to* $\lambda_{m,n} > -n^2$ *have exactly* m *zero points in* $(0,1)$.

If we would like to extract the most interesting fact, it would most likely be the way of shifting the weighted eigenvalues. That is, if $r(x)$ decreases, the nonzero eigenvalues *do not shift closer to zero.* In other words, for positive weights such that $r(x) < 1$ almost everywhere in $(0,1)$, we can expect the eigenvalues to shift further from zero. As for the constant "maximal" weight $r(x) \equiv 1$, the eigenvalues take the form $\lambda_{m,n} = (m+1)^4\pi^4 - n^2$, $m, n \in \mathbb{N} \cup \{0\}$, they are simple and the corresponding eigenfuctions are in the product form $\varphi_{m,n} = \sin(m+1)\pi x\, \cos nt$.

This type of results found in Lemma 16, i.e., infinite sequence of real eigenvalues without a finite cluster point and the corresponding r-orthogonality of the eigenfunctions, hold also for $r \in L^1(0,1)$ (this is due to J. Mawhin, see [18]). Unfortunately, in such a case it is not possible to utilize the results from [24], which means that we lose any information concerning the shift of the weighted eigenvalues.

Let us proceed similarly to the setting in Chapter 2, that is, utilizing Fourier series for definitions of all needed abstract objects. According to Lemma 16, the eigenfunctions $\varphi_{m,n}$ form an r-orthogonal basis in H and therefore any function $u \in H$ can be expanded into
$$u(x,t) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} u_{m,n}\varphi_{m,n}$$
with the coefficients
$$u_{m,n} = \frac{(u, \varphi_{m,n})_r}{(\varphi_{m,n}, \varphi_{m,n})_r}.$$

Definition 17. We call $L : \mathrm{dom}(L) \subset H \to H$,

$$Lu = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \lambda_{m,n} u_{m,n} \varphi_{m,n}$$

with

$$\mathrm{dom}(L) = \left\{ u \in H;\ \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \lambda_{m,n}^2 u_{m,n}^2 < \infty \right\},$$

the abstract realization of the beam operator $\frac{1}{r(x)}(\partial_{tt} + \partial_{xxxx})$ on H.

Remark 12. Note that L is a linear, closed, densely defined symmetric operator. Its weighted spectrum consists of real points

$$\sigma_r(L) = \{\lambda_{m,n}\}_{m,n=0}^{+\infty}$$

given by Lemma 16. Its resolvent $(L - \lambda I)^{-1}$ with $\lambda \notin \sigma_r(L)$ is a compact normal operator on H and its norm is given by

$$\|L_\lambda^{-1}\|_r = \frac{1}{\min\limits_{m,n \in \mathbb{N}_0} |\lambda_{m,n} - \lambda|} = \frac{1}{\mathrm{dist}\,(\lambda, \sigma_r(L))}. \tag{3.6}$$

Before we proceed further, we would like to point out again, that especially the choice of the weighted space $L_r^2(\Omega)$ allowed us to build the abstract formulation in a similar way as in the previous chapter. Hence, $u \in H$ is a weak solution of the problem (3.1) whenever it solves the abstract equation

$$Lu + k\, u^+ = g \tag{3.7}$$

with $g = h/r \in H$.

3.2 Stationary solution

Now, we have enough information about the weighted spectrum and the corresponding eigenfunctions. This allows us to prove two auxiliary existence and/or uniqueness assertions, however with some limitations for either the values of k, or for the right-hand side, since for an arbitrary right-hand side $g = h/r \in H$ combined with any $k \in \mathbb{R}$, the existence of a weak solution of (3.1) is not generally guaranteed. Let us begin with showing the existence and uniqueness result for values of stiffness around $k = 0$. This assertion bears technical similarity to Theorem 14, as its proof also utilizes an appropriately shifted operator and Banach Contraction Theorem.

Proposition 18 ([21])**.** *Let $\lambda_q < 0 < \lambda_p$ be such that $\sigma_r(L) \cap [\lambda_q, \lambda_p] = \{\lambda_q, \lambda_p\}$, and let $g = h/r \in H$ be arbitrary. Then the problem* (3.1) *has a unique weak solution for any $k \in (-\lambda_p, -\lambda_q)$.*

This actually gives us information that for (3.1) there is a bounded, generally asymmetric interval of non-resonance around zero. Also, notice that λ_p is the smallest positive eigenvalue of L, and λ_q is the largest negative eigenvalue of L. Moreover, we can quantify the statement of Proposition 18 more precisely. It can be computed, that for $r(x) \equiv 1$, there is $\lambda_q = \lambda_{0,10} = \pi^4 - 100$ and $\lambda_p = \lambda_{0,9} = \pi^4 - 81$. Using the fifth property in Lemma 16 (i.e., the shift of the eigenvalues for $r(x) \leq 1$), we find out, that the problem (3.1) has a unique weak solution for any $k \in (81 - \pi^4, 100 - \pi^4)$ (see Remark 5 in [21]).

Next, let us restrict ourselves to investigating (3.1) with a positive right-hand side h. It allows us to obtain the necessary condition for the solvability of (3.1) almost for free, since the proof can be done by keeping the structure from Lemma 4, that is, passing to suitable scalar products (now with the weight r) and thus considering the equation

$$(Lu, \varphi_{0,0})_r + k(u^+, \varphi_{0,0})_r = (g, \varphi_{0,0})_r,$$

using the symmetry of L with the fact, that $L\varphi_{0,0} = \lambda_{0,0}\,\varphi_{0,0}$ and also that $u = u^+ - u^-$, we obtain

$$(\lambda_{0,0} + k)(u^+, \varphi_{0,0})_r = (g, \varphi_{0,0})_r + \lambda_{0,0}(u^-, \varphi_{0,0})_r.$$

Realizing that $\varphi_{0,0}$ is strictly positive in Ω, $\lambda_{0,0} > 0$ and checking the sign properties of all other functions involved, we get the following condition.

Proposition 19 ([21])**.** *Let $g = h/r \in H$, $h(x,t) \geq 0$ a.e. in Ω, $h(x,t) \not\equiv 0$, and $u \in H$ be a weak solution of* (3.1)*. Then necessarily $k > -\lambda_{0,0}$ and $u \not\leq 0$.*

The detailed proof of Proposition 19 (see [21, Proposition 6 and Remark 7]) suggests that this assertion is valid also for a more general right-hand side, which satisfies the integral inequality $\int_\Omega h\,\varphi_{0,0} > 0$. However, it is not possible to verify this assumption for a non-constant weight function $r(x)$, because in such a case, the exact form of $\varphi_{0,0}$ is unknown. Hence, for now, it makes much more sense to use a more restrictive positivity setting for h.

Another fact worth pointing out is, that the two assertions presented in this section hold also for a more general setting, where the weight r is considered to be an L^1 function on $(0,1)$. However, in that case, it is impossible to apply the results of Janczewsky from [24], as they require the continuity of r. Also, when studying stationary solutions of (3.1), it is necessary to use some results from [11] and [12], which do not hold for L^1 weights.

Reminder. From now to the end of Chapter 3, the weight function r is always considered continous.

Our goal is to build a standard bifurcation equation (as in [10]) with the application of Rabinowitz Global Bifurcation Theorem (see e.g. [32]) in mind. For that, we need to know, under which circumstances (3.1) has a *positive stationary solution*. Hence, let us turn our attention to a time-independent right-hand side h and introduce some necessary notation.

Definition 20. Let $y = y(x)$ be a continuous function on $[0,1]$.

1. Let us denote $y_{\min} := \min_{x\in[0,1]} y(x)$ and $y_{\max} := \max_{x\in[0,1]} y(x)$.
2. We say that y is *strictly positive* on $(0,1)$, if it satisfies $y(x) > 0$ for any $x \in (0,1)$ with $y'(0) > 0$ and $y'(1) < 0$.

Definition 21. By c_0, let us denote the value $c_0 := 4\kappa_0^4$, with κ_0 being the smallest positive solution of the equation $\tan\kappa = \tanh\kappa$.

Note that $\kappa_0 \approx 3.9266$ and $c_0 \approx 950.8843$.

The stationary solution of (3.1) (denoted by u_{st}) must solve the stationary problem

$$\begin{aligned} &u^{(4)} + k\,r(x)u^+ = h(x) \quad \text{in } (0,1),\\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{3.8}$$

We employ Lemma 16 in order to find out, that eigenvalues of the stationary eigenvalue problem

$$\begin{aligned} &u^{(4)} = \lambda\,r(x)u \quad \text{in } (0,1),\\ &u(0) = u(1) = u''(0) = u''(1) = 0 \end{aligned} \tag{3.9}$$

are equal to the weighted eigenvalues $\lambda_{m,0}$ of L. Not only that, they are also simple, positive and form an increasing sequence going to infinity. Also, we have $\lambda_{m,0} \geq (m+1)^4\pi^4$ for any $m \in \mathbb{N} \cup \{0\}$. With this knowledge, let us summarize the situation regarding the stationary solution.

Proposition 22 ([21])**.** *Let $h = h(x) \in C([0,1])$. Then for any $k > -\lambda_{0,0}$ the problem* (3.1) *has a unique classical stationary solution $u_{\rm st} = u_{\rm st}(x) \in C^4([0,1])$. Moreover, if $h(x) \geq 0$, $h(x) \not\equiv 0$ on $(0,1)$, then there exists λ_M (depending on h and r) such that u is strictly positive whenever $k \in (-\lambda_{0,0}, \lambda_M]$, where*

$$-\lambda_{0,0} \leq \min\left\{k_m,\, -\pi^4\right\}, \qquad \lambda_M \geq \min\left\{k_{M1},\, k_{M2}\right\} \tag{3.10}$$

with

$$\begin{aligned} k_m &= -\frac{4\pi^2}{\int_0^1 r(x)\,\mathrm{d}x}, \\ k_{M1} &= c_0 + \frac{h_{\min}}{h_{\max}} 2\pi\sqrt{\pi^4 + c_0}, \\ k_{M2} &= c_0 + 2\pi\frac{h_{\min}}{h_{\max}}\left(\pi\frac{h_{\min}}{h_{\max}} r_{\min} + \sqrt{c_0 r_{\min} + \left(\frac{h_{\min}}{h_{\max}}\right)^2 \pi^2 r_{\min}^2 + \pi^4}\right). \end{aligned} \tag{3.11}$$

The idea of the proof is as follows. Combining the techniques used for the non-stationary case, i.e., Banach Contraction and Proposition 19, we realize, that for a non-negative right-hand side, the inequality $k > -\lambda_{0,0}$ is actually the necessary and sufficient condition for weak solvability of (3.8). The standard regularity arguments then imply that for $h \in C([0,1])$ there is $u_{\rm st} \in C^4([0,1])$.

The strict positivity of $u_{\rm st}$ is a result of the information from [11], [12] and [41] and the fact, that a strictly positive stationary solution has to solve also the linear problem

$$\begin{aligned} &u^{(4)} + k\, r(x) u = h(x) \quad \text{in } (0,1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{3.12}$$

As for the interval of positivity $(-\lambda_{0,0}, \lambda_M]$, the existence of the finite upper bound is a consequence of [41, Corollary 2.1]. The estimates (3.10), (3.11) come from applying certain inverse-positivity conditions from [11] and [12]. Their rather complicated structure is a result of checking that inequalities

$$k \leq c_0 + \frac{h_{\min}}{h_{\max}} 2\pi\sqrt{\pi^4 + c_0}$$

and

$$k \leq c_0 + \frac{h_{\min}}{h_{\max}} 2\pi\sqrt{\pi^4 + k r_{\min}}$$

hold at the same time. For the complete proof, see [21, Proposition 8].

If we look closely at the value k_m in (3.11), we realize, that it is actually the estimate of the principal weighted eigenvalue $\lambda_{0,0}$. For $r(x) \equiv 1$, we have $\lambda_{0,0} = \pi^4$. On the other hand, when $\int_0^1 r(x)\,\mathrm{d}x$ tends to zero, the eigenvalue $\lambda_{0,0}$ goes to infinity. Concerning the upper bound, for $r(x) \equiv 1$, we get $\lambda_M = c_0$. In particular, for an arbitrary $h(x) \geq 0$, $u_{\rm st}$ is strictly positive whenever the stiffness k is between $-\pi^4$ and c_0. In practice, it means, that for any $k > c_0$, we can find a right-hand side $h \geq 0$ such that the stationary solution changes sign (cf. [38]). For illustration, see Fig. 3.2(a) and 3.2(b).

However, what is an important implication of Proposition 22, is the fact, that the positivity interval for k is enlarged by the influence of a non-constant weight r, which is demonstrated in Fig. 3.2(c) and 3.2(d)). There is only a minor drawback, that is, the amplitude of $u_{\rm st}$ is slightly larger than in the constant r case. For comparison, see Fig. 3.2(b) and 3.2(c).

The last observation in correspondence with Proposition 22 is connected to a constant right-hand side. According to [29], for $r(x) \equiv 1$ and $h(x) \equiv 1$, $u_{\rm st}$ is strictly positive *for any*

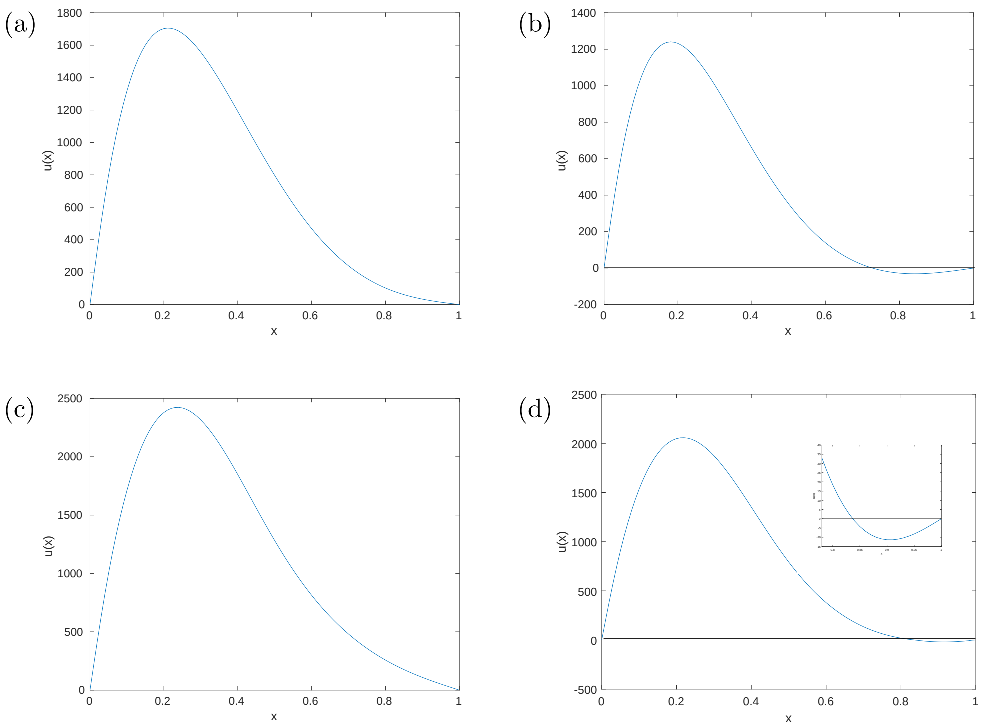

Figure 3.2: Stationary solutions of (3.8) for the constant stifness $k = 800$ (a) and $k = 1500$ (b), and also for the nonconstant stifness $k = 1500\,r(x)$ (c) and $k = 2500\,r(x)$ (d), where $r(x) = \cos^4(2\pi x)$. In all cases $h(x)$ is a positive, piecewise constant function. In particular, $h(x) = 10^8$ for $x \in [0, 0.05]$ and $h(x) = 10^4$ for $x \in (0.05, 1]$. Note that the solution $u(x)$ in (b) and (d) changes sign (see also the zoomed picture in (d) corresponding to the interval $(0.8, 1)$ on the x-axis).

$k > -\pi^4$. This shows us that the estimates k_{M1}, k_{M2} are not optimal and can be improved. However, extending the bounds of inverse-positivity is not easy in general, as the reader can see in Chapter 4.

Before we definitively turn our attention to bifurcations, let us look more closely at the aforementioned paper [41] by M. Ulm, where a much more general setting is used. Specifically, [41] guarantees the existence of an h, r-independent bound $\lambda_U \geq c_0$, such that for any $k \in (-\lambda_{0,0}, \lambda_U]$ and any nonzero right-hand side $h(x) \geq 0$, the corresponding stationary solution is strictly positive. For a given h and r, the positivity interval can be stretched to the estimated value λ_M, however, one loses the "versatility" of λ_U. So if hanger density or external forces change, it is necessary to provide new estimates for λ_M using (3.11).

3.3 Global bifurcation

Finally, let us go through the last few steps on the path to reformulate (3.7) correctly as a bifurcation equation. So, we again consider the right-hand side $h \not\equiv 0$ to be time-independent and u_{st} to be a strictly positive stationary solution of (3.1) and thus also (3.8). Crucially, we represent u as a perturbation of the stationary solution (a trick encountered in [10]), i.e., $u := u_{\text{st}} + w$, where w is also a function from H. Naturally, we also incorporate the standard decomposition $u^+ = u + u^-$, so that (3.7) can be reformulated as

$$L(u_{\text{st}} + w) + k(u_{\text{st}} + w) + k(u_{\text{st}} + w)^- = g.$$

The positivity of $u_{\rm st}$ ensures that $Lu_{\rm st} + ku_{\rm st} = g$, which leaves us with

$$Lw + kw + k(u_{\rm st} + w)^- = 0. \tag{3.13}$$

After that, it only remains to apply the inverse operator L^{-1} on both sides of (3.13), which yields

$$w + kL^{-1}w + kL^{-1}(u_{\rm st} + w)^- = 0. \tag{3.14}$$

So far, it is clear, that there is the identity mapping and the compact operator L^{-1} in (3.14), however, it is necessary to check, whether the term $kL^{-1}(u_{\rm st} + w)^-$ has a correct behaviour. Fortunately, this is possible thanks to the structure of the proofs in [10, Lemma 2.3, Lemma 2.4]. Hence, let us define the set $E := (-\lambda_{0,0}, \lambda_M) \times H$ and present all the details.

Lemma 23 ([21]). *The operator $N : E \to H$ defined by $N(k, w) := kL^{-1}(u_{\rm st} + w)^-$ is compact. Moreover, given any compact subinterval J of $(-\lambda_{0,0}, \lambda_M)$, the limit*

$$\lim_{\|w\|\to 0} \frac{N(k, w)}{\|w\|} = 0$$

is uniform with respect to $k \in J$.

Now, we have successfully verified, that (3.14) is a proper bifurcation scheme and that it is suitable for employing Rabinowitz Global Bifurcation Theorem.

Theorem 24 ([21]). *Every $k = -\lambda_{m,n} \in (-\lambda_{0,0}, \lambda_M) \cap \sigma_r(-L)$, where $\lambda_{m,n}$ has an odd multiplicity, is a point of global bifurcation of (3.14). That is, there exists a continuum of solutions $C_{m,n}$ in $\overline{E}$, $(-\lambda_{m,n}, 0) \in C_{m,n}$, such that at least one of the following properties holds:*

1. *$C_{m,n}$ is not a compact set in E,*
2. *$C_{m,n}$ contains an odd number of points $(-\lambda, 0) \in E$, where $\lambda \neq \lambda_{m,n}$ is an eigenvalue of L of odd multiplicity.*

Moreover,

$$\mathrm{proj}_{\mathbb{R}} C_{m,n} \subset (-\lambda_{0,0}, -\lambda_p] \cup [-\lambda_q, +\infty), \tag{3.15}$$

where $\mathrm{proj}_{\mathbb{R}} C_{m,n} := \{k \in \mathbb{R};\ (k, w) \in C_{m,n}\}$ *and λ_p, λ_q are the smallest positive and the largest negative eigenvalues of L.*

In addition, for $\lambda_{m,n}$ simple, $C_{m,n}$ consists of two subcontinua $C^+_{m,n}$, $C^-_{m,n}$ bifurcating from the point $(-\lambda_{m,n}, 0)$ in the directions of the corresponding eigenfunctions $\varphi_{m,n}$, and $-\varphi_{m,n}$, respectively, such that

$$C^+_{m,n} \cap C^-_{m,n} \cap B_\varrho(-\lambda_{m,n}, 0) = \{(-\lambda_{m,n}, 0)\} \quad \textit{and} \quad C^\pm_{m,n} \cap \partial B_\varrho(-\lambda_{m,n}, 0) \neq \emptyset$$

for sufficiently small $\varrho > 0$.

Note that the bounds in relation (3.15) are due to Proposition 18 and 19. Next, let us discuss the presence of eigenvalues in $(-\lambda_{0,0}, \lambda_M)$. For $r \equiv 1$, this can be documented relatively easily (see [21, Remark 12]). Here, $\lambda_M = c_0$ and the interval contains, e.g., the points $-\lambda_{0,1}, -\lambda_{0,2}, ..., -\lambda_{0,32}$. On the other hand, the set $(-\lambda_{0,0}, c_0)$ contains zero, is relatively small and bounded. Therefore, it can contain at most one value $-\lambda_{m,n_0}$ for any sufficiently large m. Here, $n_0 = \lfloor (m+1)^2\pi^2 \rfloor$, or $n_0 = \lceil (m+1)^2\pi^2 \rceil$. Speaking about "sufficiently large", already for $m = 7$, the distance between $(m+1)^4\pi^4 - \lfloor (m+1)^2\pi^2 \rfloor$ and $(m+1)^4\pi^4 - \lceil (m+1)^2\pi^2 \rceil$ is greater than the length of $(-\lambda_{0,0}, c_0)$. The problem with variable density $r \not\equiv 1$ is, that we do not have the count of weighted eigenvalues in $(-\lambda_{0,0}, \lambda_M)$ under precise control, because not only the eigenvalues, but also the interval bounds shift away from zero. Hence, we cannot specify, whether there are as many eigenvalues as for $r \equiv 1$, or significantly less.

Notice that unlike in [10], we cannot be sure, if $w \in C^1(\Omega)$ and therefore we do not have the information about the possible linear behaviour of the solution branches close to the bifurcation points $(-\lambda_{m,n}, 0)$ (see [21, Remark 13]).

3.4 Bifurcation from infinity and the Fučík spectrum

As the last piece of the puzzle in this bifurcation section, let us concentrate on bifurcation from infinity. For this task, we use the so called Fučík spectrum (see, e.g., [22]).

Definition 25. The set of pairs $(\alpha, \beta) \in \mathbb{R}^2$, denoted by $\Sigma(L)$, such that $Lu = \alpha u^+ - \beta u^-$ has a nontrivial solution u, is called the *Fučík spectrum of the operator* L.

To obtain a necessary condition for a bifurcation from infinity, we can use some technical ideas from the proof of Theorem 6. That is, considering a sequence of solutions w_n, dividing (3.14) by the norm of w_n, using regularity of u_{st}, compactness of L^{-1} and continuity of $(\cdot)^-$, we may pass from weak limits to strong limits and realize, that the existence of the bifurcation is equivalent to the existence of a non-trivial solution (denoted by v) of

$$Lv + k_0 v^+ = 0. \tag{3.16}$$

Therefore, the pair $(-k_0, 0)$ belongs to the Fučík spectrum. All the details are available in the following assertion.

Proposition 26 ([21])**.** *If a bifurcation from infinity of* (3.14) *occurs in* E, *i.e., if there exists a sequence* $(k_n, w_n) \subset E$ *such that* (3.14) *holds with* $(k, w) = (k_n, w_n)$ *for any* $n \in \mathbb{N}$, *and* $k_n \to k_0$, $\|w_n\| \to \infty$, *then necessarily* $(-k_0, 0) \in \Sigma(L)$.

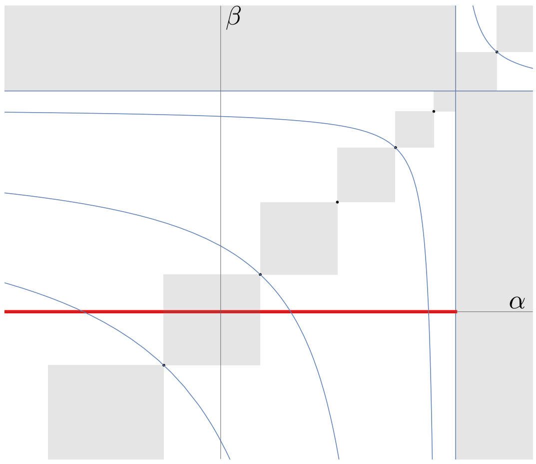

Figure 3.3: Known parts of the Fučík spectrum for L with $r(x) \equiv 1$. The red line marks the set of pairs $(-k, 0)$ in the half-plane $\alpha < \lambda_{0,0}$. The intersections of the Fučík curves with the red line correspond to possible blow-up values, whereas, in the grey inadmissible areas, none of the curves may appear.

When we stay close to the line $\alpha = \beta$, the Fučík spectrum $\Sigma(L)$ of L consists of a finite number of decreasing curves, which cross at the points (λ, λ), $\lambda \in \sigma_r(L)$ and are symmetric with respect to the diagonal $\alpha = \beta$. The so called trivial part of $\Sigma(L)$ is the cross $(\alpha - \lambda_{0,0})(\beta - \lambda_{0,0}) = 0$. In contrast, no parts of $\Sigma(L)$ are located in the squares between two consecutive eigenvalues and in the area $(\alpha - \lambda_{0,0})(\beta - \lambda_{0,0}) < 0$ (see [3] and [22] for further details and Fig. 3.3 for illustration). Hence, $(-k_0, 0) \in \Sigma(L)$ implies $k_0 > -\lambda_{0,0}$ which corresponds well to Proposition 19.

However, when thinking globally, there is a major problem. The complete global description is known neither for $\Sigma(L)$, nor at least for all its intersections with the line $\beta = 0$. Hence, the all possible blow-up values of the stiffness k remain hidden even for the constant weight $r(x) \equiv 1$. On, the other hand, at least some partial analytical and numerical results can be found in [30] and references therein. The lack of the complete description of $\Sigma(L)$ together with multiplicities of the eigenvalues means that we cannot determine the topological degree of the operator $u \mapsto u - L^{-1}(\alpha u^+ - \beta u^-)$ between the Fučík curves. The consequence of this is that for now, it is impossible to confirm the existence of bifurcations from infinity. Also, we are not able to confirm the general existence of a weak solution of (3.1), when h depends on time.

In conclusion, let us discuss the results shown in this chapter. When we look at Lemma 16, Proposition 18, 19 and Theorem 24, we see, that the introduced variable hanger placement density function r eventually improves the behaviour of the model, since, the shift of the eigenvalues away from zero generates a larger uniqueness interval for k and postpones the appearance of additional solutions, potentionally dangerous for the suspension bridge. Moreover, variable r keeps the model's qualitative bifurcation properties. As examined in Section 3.3 and 3.4, the bifurcations from the stationary solution and from infinity still occur.

On the other hand, not everything is so positive and straightforward. We would like to point out, that with non-constant r, we lose the information about regularity of the bifurcating solutions. As opposed to [10], we cannot use the corresponding embedding results for anisotropic Sobolev spaces (cf. [4] and/or [42]), since, as far as we have found, none of them are available for our weighted space setting. Also, the information concerning existence and specially blow-ups of solutions is incomplete, which is largely the consequence of only local description of the Fučík spectrum of the corresponding linear beam operator.

However, not all the problems from this chapter remain a mystery. When trying to implement the standard global bifurcation theory, we realized, that this process relies on the existence of a positive stationary solution under positive constant loading. For non-constant r, its existence is not guaranteed for an arbitrary value of k. When we used the results from [11, 12] and [41], we touched the topic of strictly inverse-positive operators. In that moment, we decided to try and improve the estimates for $-\lambda_{0,0}$ and especially for λ_M, which would become the main narrative of the following chapter.

Chapter 4

Strictly inverse-positive operators

Investigating the bifurcation properties of a weighted model of a suspension bridge led us to studying the stationary problem, which has the form (cf. (3.12))

$$
\begin{aligned}
&u^{(\mathrm{iv})} + c(x)u = h(x), \\
&u(0) = u(1) = u''(0) = u''(1) = 0,
\end{aligned} \tag{4.1}
$$

where $c, h \in C[0,1]$ and $h \geq 0,\ h \not\equiv 0$.

For the purpose of further investigation and discussion, let us consider a differential operator L of the fourth order in the form

$$
\begin{aligned}
&L_c : X \subset C^4[0,1] \to C[0,1], \\
&L_c u(x) := u^{(\mathrm{iv})}(x) + c(x)u(x),
\end{aligned} \tag{4.2}
$$

where $X = \{u \in C^4[0,1] : u(0) = u(1) = u''(0) = u''(1) = 0\}$. Note that now it is possible to formulate (4.1) as

$$
L_c u = h. \tag{4.3}
$$

4.1 Method of reduction

Since we are interested in conditions for extremal values of $c = c(x)$, which would guarantee the positivity of the solution u, we delve in the topic of the so-called strictly inverse-positive (SIP for short) operators.

Definition 27. We say that L_c is *strictly inverse positive* on X if any solution $u \in X$ of (4.3) with an arbitrary nonnegative nontrivial right-hand side $h \in C[0,1]$ is *strictly positive*, i.e., $u > 0$ in $(0,1)$ and $u'(0) > 0$, $u'(1) < 0$.

The history of this field stretches back to the 1960s, when Johann Schröder published the pioneering articles [34, 35, 36] (in German). Here he discussed the so called *method of reduction*, i.e., a decomposition of a general fourth order differential operator into two second order operators (see also [37] and [38]). Then he used this theory to derive bounds for either constant or non-constant coefficient c, which guarantee the SIP property of L_c. It is worth mentioning, that J. Schröder also discussed this topic from the engineering point of view (see e.g. [33]). The broad coverage and volume of his work meant that he established himself as a founder of this field.

There are many authors, who followed Schröder's steps later. The constant coefficient case was covered by B. Kawohl and G. Sweers in [26]. These authors basically re-interpreted and explained the meaning of Schröder's approach in more detail. On the other hand, M. Ulm in [41], obtained the same bounds, however by using a different, more "brute-force" based technique. This is by no means a pejorative description, as Ulm's result is a nice verification of previous efforts by Schröder and/or Kawohl & Sweers. But let us turn our attention to the non-constant coefficient case. Here, many papers, e.g. [5], or more recently, [6], [11], or [12], brought a significant progress, since they showed that the extrema of $c(x)$ can in fact cross the bounds earlier obtained by Schröder's followers of by himself. Our contribution to this topic is available in [20].

Now, let us briefly discuss the outcome of the method of reduction. The reason for it is the appearance of certain functions, which will be used as a tool for determining the SIP property for L_c. This summary (in *much* more detail) can be found e.g. in [38]). The key part of the process is to express the reduction as $P(L_c) = A - B$, where B is a linear positive operator and A is, roughly speaking, a second order differential inverse-positive operator. If this is possible, then $P(L_c)$ is stricly inverse-positive ([38, Proposition 1.4]) and the same holds for L_c itself ([38, p. 96]). Indeed, by applying a suitably chosen integral operator P on $L_c u(x)$, where

$$PU(x) := \int_0^1 G(x,s)U(s)ds, \quad G(x,s) = \begin{cases} \psi(x)\varphi(s) & \text{for } 0 \le x \le s \le 1, \\ \psi(s)\varphi(x) & \text{for } 0 \le s \le x \le 1, \end{cases} \tag{4.4}$$

one obtains the required problem structure, since it is possible to verify that there exist operators A, B with the aforementioned properties, such that $PL_c u(x) = Au(x) - Bu(x)$ (see, [38, p. 102] for details). Note that using the final results of Schröder's reduction means, that the explicit form of these reduction operators A, B is actually irrelevant for us. Instead of that, it suffices to check that the considered functions φ, ψ fulfill several conditions, which are discussed in the following assertion.

Proposition 28. *[38, Proposition 4.3] If there exist functions $\varphi, \psi \in C^4[0,1]$ such that $\varphi, \psi > 0$ in $(0,1)$, with*

$$\varphi(1) = \psi(0) = 0, \quad \varphi''(1) \ge 0 \ \text{ and } \ \psi''(0) \ge 0, \tag{4.5}$$

$$\varphi'(1) < 0, \quad \psi'(0) > 0, \tag{4.6}$$

$$\varphi^{(\mathrm{iv})} + c\varphi \le 0, \quad \psi^{(\mathrm{iv})} + c\psi \le 0 \ \text{ in } [0,1] \tag{4.7}$$

and

$$p := \varphi\psi' - \varphi'\psi \ge 0 \tag{4.8}$$

together with a function $z \in X$ such that $z \ge 0$ and $L_c z \ge 0$, $L_c z \not\equiv 0$ then L_c is strictly inverse-positive.

Let us note that [38] uses different notation for order relations, however, our notation is compatible with the ordering used by Schröder. For further details, see Chapter 1 in [38]. Concerning functions φ, ψ, Schröder used specific ones, which appear as solutions of problems

$$\begin{aligned} &\varphi^{(\mathrm{iv})} + k\varphi = 0, \\ &\varphi(0) = \varphi(1) = \varphi''(1) = 0, \quad \varphi''(0) = -1 \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\psi^{(\mathrm{iv})} + k\psi = 0, \\ &\psi(0) = \psi(1) = \psi''(0) = 0, \quad \psi''(1) = -1. \end{aligned} \tag{4.10}$$

As for the function z appearing in Proposition 28, it suffices to take $z = \sin \pi x$, which combined with φ, ψ yields a relatively well-known sufficient condition, i.e., if $-\pi^4 < c(x) < c_0$, then L is strictly inverse-positive (see [38, Proposition 4.4]). Just to recall, let us note, that $c_0 = 4\kappa^4$, with κ being the smallest positive solution of $\tan \kappa = \tanh \kappa$ (see Definition 21 and, e.g., [26, 38, 41]).

The SIP interval bounds are in fact certain eigenvalues, however, both have a different origin. The lower threshold value $-\pi^4$ is the opposite of the first eigenvalue of L_0, that is, when $c \equiv 0$. For simplicity, let us denote $L := L_0$.

On the other hand, the upper "c_0-type" threshold appears as the eigenvalue of the problem

$$\begin{aligned} &u^{(\text{iv})} + \lambda u = 0, \\ &u(0) = u'(0) = u''(0) = 0, \quad u(1) = 0, \end{aligned} \tag{4.11}$$

for which the corresponding eigenfunction is positive. In this chapter, we understand such an eigenvalue to be the *principal* one.

The reasoning behind the meaning of these SIP bounds is relatively complex. Schröder's book [38] provides both technical and physical arguments, that is, the discussion concerning the positivity of the Green function associated with (4.1) in combination with observations of the behaviour of a bending beam with the considered boundary conditions. Perhaps even more straightforward summary of these discussions can be found in [26]. Since we consider the weight function $r \not\equiv 1$, we extend this concept and search for the c_0-type boundary in a similar way, however, quite different in one important aspect. Our main idea is, that instead of classical eigenvalues, we view the SIP bounds as weighted eigenvalues of corresponding problems.

4.2 Beyond the value c_0

Since our work is connected to the suspension bridges modelling, let us interpret the variable coefficient c in a specific form, i.e., $c(x) = kr(x) \geq 0$ with $k \geq 0$, $0 \leq r(x) \leq 1$, $x \in [0,1]$, which reflects the idea of having a "stiffness constant" k times a certain variable function profile r. This way, it is also easier to compare the extrema of c to corresponding weighted eigenvalues. Also, let us point out, that in this section, we concentrate purely on the upper bound in the case of a positive, or positive semidefinite weight, i.e., the assumption $c(x) \geq 0$ means that $c(x)$ may be equal to zero on a subinterval of $(0,1)$. The negative definite and indefinite weights shall be discussed in the following section, since they correspond to the lower bound of the SIP interval.

If we look back at Schröder's functions (4.9) and (4.10), it turns out, that in order to investigate the possible shift of the upper bound of the SIP interval, we have to work with auxiliary (3,1) and (1,3) conjugate boundary value problems (see, e.g., [6] and [43]) and the corresponding operators. Let us define spaces

$$\begin{aligned} X^{3,1} &= \{u \in C^4[0,1] : u(0) = u'(0) = u''(0) = u(1) = 0\}, \\ X^{1,3} &= \{u \in C^4[0,1] : u(0) = u(1) = u'(1) = u''(1) = 0\}, \end{aligned}$$

and consider the standard fourth order differential operators $L^{3,1} : X^{3,1} \to C[0,1]$ and $L^{1,3} : X^{1,3} \to C[0,1]$ defined by

$$L^{3,1}u = L^{1,3}u = -u^{(\text{iv})}.$$

Before proceeding to work with the weighted eigenvalues of these operators, let us begin with a definition.

Definition 29. We say that λ is a *weighted eigenvalue* of $L^{3,1}$ with the weight function r if there exists a nontrivial solution $u \in X^{3,1}$ of

$$L^{3,1}u = \lambda r u.$$

Moreover, we call it the *principal eigenvalue*, if at least one corresponding eigenfunction is positive in $(0,1)$.

Now, it is necessary to verify the existence of a principal eigenvalue of both these conjugate problems and discuss any possible similarities between them. Actually, it is possible to show that not only they exist, but they are the same, as our following lemma suggests.

Lemma 30 ([20])**.** *Both the operators $L^{3,1}$, $L^{1,3}$ have the same principal eigenvalue with respect to the weight function r, that is, $\Lambda_r = \Lambda_r'$, which is positive and it is the smallest eigenvalue in absolute value. Moreover, it satisfies*

$$\Lambda_r \geq \Lambda_1 = c_0$$

with the equality only for $r \equiv 1$.

The connection with the (3,1) and (1,3) conjugate problems becomes even more obvious, when we introduce modifications of Schröder's functions. Indeed, let φ, ψ be now the solutions of the problems (cf. (4.9), (4.10))

$$\begin{aligned} &\varphi^{(\mathrm{iv})} + kr(x)\varphi = 0, \\ &\varphi(0) = \varphi(1) = \varphi''(1) = 0, \quad \varphi''(0) = -1, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &\psi^{(\mathrm{iv})} + kr(x)\psi = 0, \\ &\psi(0) = \psi(1) = \psi''(0) = 0, \quad \psi''(1) = -1, \end{aligned} \tag{4.13}$$

respectively.

Note that due to symmetry reasons (also cf. Lemma 30), it suffices to deal only with the eigenvalue problem for $\psi(x)$. Nevertheless, for $k \geq 0$, both problems (4.12), (4.13) possess a unique solution. This can be verified by reformulating them as uniquely solvable boundary value problems with a nontrivial right-hand side. In the case of ψ, when we put $\psi = \psi_h + h(x)$ with $h(x) = \frac{1}{6}(x - x^3)$, then (4.13) is equivalent to $L\psi_h + kr(x)\psi_h = -kr(x)h(x)$. This problem is uniquely solvable for any $k > -\lambda_1$, which is the first eigenvalue of L. The function φ can be dealt with the same way. As a consequence, both problems (4.12), (4.13) can be viewed as initial value problems with unique solutions, which depend continuously on k. For more details, see [20].

Next, in the two following assertions, we check that the modified functions φ, ψ satisfy the assumptions from Proposition 28. See [20] for detailed proofs and the influence of the above suggested continuous dependence on k.

Lemma 31 ([20])**.** *Let Λ_r be the principal eigenvalue of $L^{3,1}$ and $L^{1,3}$, respectively, with the (semidefinite) weight r. Then for any $k \in [0, \Lambda_r]$, both the solutions of* (4.12), (4.13) *satisfy $\varphi,\ \psi > 0$ in $(0,1)$. Moreover, $\psi'(0) > 0$, $\varphi'(1) < 0$ for $k \in [0, \Lambda_r)$, and $\psi'(0) = \varphi'(1) = 0$ for $k = \Lambda_r$.*

Lemma 32 ([20])**.** *Let $k \in [0, \Lambda_r]$ be arbitrary and $\varphi,\ \psi$ be the positive solutions of* (4.12), (4.13)*, respectively. Then $p := \varphi\psi' - \varphi'\psi$ is positive in $(0,1)$ as well, and $p(0) = p'(0) = p(1) = p'(1) = 0$.*

Even in the case of these modified functions and weighted eigenvalues, it is possible to take $z(x) = \sin \pi x$ and conclude this section with a theorem dealing with the upper bound of the SIP interval.

Theorem 33 ([20])**.** *Let Λ_r be the principal eigenvalue of $L^{3,1}$ (and $L^{1,3}$, respectively) with the weight function r. Then L_c with $c(x) = k\,r(x)$ is strictly inverse-positive whenever $0 \leq k < \Lambda_r$.*

4.3 Lower bound of the SIP interval

When explaining the SIP bounds earlier, we discussed one of the original results of Schröder, that is, the lower bound corresponding to the opposite value of the first eigenvalue of L. Not only we expand this concept by (again) introducing the weighted eigenvalues, but also by shifting the operator whenever the coefficient function c is indefinite. That is, when c changes sign in $[0,1]$. Thus we consider

$$c(x) = kr^+(x) - lr^-(x) \tag{4.14}$$

with k, $l \geq 0$ and $r^\pm$ being the weight functions (or *profiles*), i.e., $0 \leq r^\pm \leq 1$. Notice that if $r^+r^- \equiv 0$, we have $kr^+ = c^+$ (the positive part of c) and $lr^- = c^-$ (the negative part of c).

If the function c is negative or negative semidefinite (i.e., $k = 0$), we can consider the lower bound of the SIP interval to be the opposite of the first weighted eigenvalue of L. However, if both $k > 0$ and $l > 0$, we deal with the shifted eigenvalue problem of the type

$$L_q u = \lambda r u.$$

The operator L_q is considered precisely in the sense of (4.2). The weight r in this eigenvalue problem is considered semidefinite. We explain its role in decomposing the indefinite coefficient in a short while. All the weighted eigenvalues of this problem are real and positive (see Section 2 in [20]), and for the first one, we proved (with the help of useful insights from [40]) the following assertion.

Lemma 34 ([20]). *The first (weighted) eigenvalue of L_q with the semidefinite weight r — denoted by $\lambda_{q,r}$ — satisfies*

$$\lambda_{q,r} \geq \lambda_{0,r} \geq \lambda_{0,1} = \pi^4$$

with equalities only for $r \equiv 1$ and $q \equiv 0$.

Note that other spectral properties of L_q are available in [17, 18] and [40].

At last, let us fully turn our attention to the indefinite coefficient c. Using the decomposition (4.14), the operator equation (4.3) is equivalent to

$$L_{kr^+} u = lr^- u + h. \tag{4.15}$$

Now, it is possible to interpret the lower bound of the SIP interval as the opposite of the eigenvalue of the shifted operator L_{kr^+} with respect to the weight function r^- (cf. Lemma 34).

Theorem 35 ([20]). *Let $0 \leq k < \Lambda_{r^+}$. Then L_c with $c = kr^+ - lr^-$ is strictly inverse positive whenever $0 \leq l < \lambda_{kr^+,r^-}$.*

When proving this theorem, we were inspired by the approach in [12]. The main idea is, that as long as L_{kr^+} remains SIP, it is possible to construct monotone increasing successive iterations in the form

$$L_{kr^+} u_{n+1} = lr^- u_n + h, \quad u_0 \equiv 0$$

and basically prove the convergence of such iterations under a specific condition. This condition appears to be none other than $0 \leq l < \lambda_{kr^+,r^-}$. For all the steps of the proof in detail, see Theorem 2 in [20].

4.4 Extended SIP interval

We investigated the lower und upper bound of the SIP interval separately, and we showed, how the weighted eigenvalues of corresponding operators can be incorporated in order to go beyond the SIP bounds discussed in the literature so far. Now, we combine Theorem 33 and Theorem 35 to provide an overview, which summarizes the SIP criteria for L_c with a generally indefinite coefficient c.

Corollary 36 ([20])**.** *The operator L_c is strictly inverse positive for any $c \in C([0,1])$ satisfying*

$$-\lambda_{c^+,r^-} < c(x) < \Lambda_{r^+},$$

where

- $c^{\pm}(x)$ *are the positive and negative parts of* $c(x)$*,*
- $r^{\pm}(x)$ *are their "profiles", i.e.,* $r^{\pm}(x) = c^{\pm}(x)/\max_{[0,1]}(\pm c(x))$ *(or* $r^{\pm} \equiv 0$*, if* $\max_{[0,1]}(\pm c(x)) = 0$*),*
- λ_{c^+,r^-} *is the first eigenvalue of* L_{c^+} *with respect to the weight* r^-*,*
- Λ_{r^+} *is the principal eigenvalue of* $L^{3,1}$ *(and* $L^{1,3}$*) with respect to the weight* r^+*.*

In theory, the introduction of weighted eigenvalues brings an improvement (see Lemma 30 and Lemma 34), which, however, cannot be quantified independently on the profile of c. That is why, in the rest of this chapter, we provide estimates for these eigenvalues.

4.5 Eigenvalue estimates

Fortunately, there are tools, which allow us to approximately compute the shift of the corresponding weighted eigenvalues (and thus the SIP bounds) for a particular choice of c. Originally, we discovered this theory in articles by J. R. L. Webb and K. Q. Lan (see [43, 44]) in connection with conjugate problems, however, it can be utilized for any eigenvalue problem, which is suitable for reformulating as a linear Hammerstein equation in the form

$$\lambda u(x) = Tu(x) := \int_0^1 k(x,y)g(y)u(y)\,\mathrm{d}y, \tag{4.16}$$

where the integrand has to satisfy the following three assumptions (cf. [43, 44]).

1. The integral kernel k is measurable and for every $\xi \in [0,1] : \lim_{x\to\xi} |k(x,y) - k(\xi,y)| = 0$ for a. e. $y \in [0,1]$.
2. There exist an interval $[a,b] \subset [0,1]$, a function $\Phi \in L^\infty[0,1]$ and a constant $d \in (0,1]$ such that
$$k(x,y) \le \Phi(y) \quad \text{for } x \in [0,1] \text{ and a.e. } y \in [0,1],$$
$$k(x,y) \ge d\Phi(y) \quad \text{for } x \in [a,b] \text{ and a.e. } y \in [0,1].$$
3. The weight function $g \ge 0$ a. e., $g\Phi \in L^1[0,1]$ and $\int_a^b \Phi(s)g(s)\,\mathrm{d}s > 0$.

We strongly recommend the reader to check the discussion in [31, 43, 44]. Note that the theory behind the estimates of weighted eigenvalues transcends the topics discussed in this thesis. In the following paragraph, we only provide a very brief sketch of facts and tools, which Webb and Lan used in order to prove the existence of the principal eigenvalue of (4.16) and to show the possibility to compute upper and lower estimates for its reciprocal value. They utilized the rather standard Krein-Rutman theory. This requires to consider the following cones of continuous functions in a Banach space Y (especially when $Y = C[0,1]$), i.e.,

$$P = \{u \in C[0,1] : u \ge 0\},$$

$$\tilde{P} = \left\{ u \in P : \min_{x\in[a,b]} u(x) \ge d \max_{x\in[0,1]} u \right\},$$

where a, b and d come from the three initial assumptions $1. - 3.$

Note that the set $K \subset Y$ is called a *cone* if it is convex, for any $a \geq 0$ we have $aK \subset K$ and $K \cap -K = \{o\}$. To show, that the eigenfunction corresponding to the principal eigenvalue of (4.16) is strictly positive, the authors of [44] worked in more specific cones in Y. The cone is said to be *reproducing* or *generating*, if $Y = \{x - y : x, y \in K\}$. This is often denoted by $Y = K - K$. The possibility to decompose a continuous function $u(x)$ such that $u = u^+ - u^-$ shows that the cone P is reproducing in the Banach space of continuous functions. And finally, a cone K is considered *total*, if Y is the norm closure of $\{x - y : x, y \in K\}$.

Webb and Lan proved, that if the assumptions $1. - 3.$ hold then T from (4.16) is a bounded linear compact operator, maps P into $\tilde{P}$ and its *spectral radius* $r(T) := \lim_{n\to+\infty} \|T^n\|^{1/n} > 0$. Here, the norm $\|T\|$ is the standard linear operator norm, i.e., there exists $C \in \mathbb{R}$ such that

$$\|Tu\|_Y \leq C\|u\|_Y$$

for every $u \in Y$, with $\|L\| = \sup_{u\in Y} \frac{\|Lu\|_Y}{\|u\|_Y}$. The strict positivity of $r(T)$ guarantees the existence of the nontrivial positive eigenfunction corresponding to $r(T)$, which is the largest eigenvalue. The last important piece of information is that the weighted eigenvalues of the integral operator T are in fact reciprocals of the eigenvalues of corresponding differential operators (cf. [44]). This will become more obvious when we introduce Green functions for our problems and from the estimates, see e.g. Lemma 37.

Let us return back to the weighted eigenvalues of $L^{3,1}, L^{1,3}$ and L. Their approximate position on the real axis can be measured using these "Webb-type" estimates from [43] and [44]. The considered integral kernels for the general operator, which we denoted above by T are nothing else than Green functions associated to given problems. The generality of the introduced framework makes it possible to find such estimates for either the conjugate problems, or the one with symmetric boundary conditions.

Firstly, let us start with the $(3, 1)$ conjugate problem, since, in the above mentioned literature, the estimating tools are usually used for this particular case (especially in [43]). Let us present the Green function corresponding to the $(3, 1)$ conjugate problem (cf. [43]), i.e.,

$$G_{3,1}(x, y) = \begin{cases} \frac{x^3(1-y)^3-(x-y)^3}{6} & x \geq y, \\ \frac{x^3(1-y)^3}{6} & x < y \end{cases} \tag{4.17}$$

and the form of the estimates in the following lemma.

Lemma 37 (Webb, Lan, [43, 44])**.** *Let $G_{3,1}$ be the Green function of $L^{3,1}$ given by* (4.17)*. Then $m \leq \Lambda_r \leq M$, where*

$$m = \left(\sup_{0\leq x\leq 1} \int_0^1 G_{3,1}(x, y) r(y)\, dy \right)^{-1},$$

$$M = \inf_{0\leq a<b\leq 1} \left(\inf_{a\leq x\leq b} \int_a^b G_{3,1}(x, y) r(y)\, dx \right)^{-1}.$$

We show Lemma 37 in a specific form, but again, let us point out, that it holds not only for $L^{3,1}$ with $G_{3,1}$, but also for $L^{1,3}$ and L, for which there also are suitable Green functions. Indeed, obviously $G_{1,3}(x, y) = G_{3,1}(1 - x, 1 - y)$. The Green function corresponding to L with symmetric boundary conditions takes the form (see e.g. [26])

$$G_{\text{sym}}(x, y) = \begin{cases} \frac{1}{6} x(1 - y)(1 - x^2 - (1 - y)^2) & \text{for } x \leq y, \\ \frac{1}{6} y(1 - x)(1 - y^2 - (1 - x)^2) & \text{otherwise} \end{cases}$$

where $0 \le x, y \le 1$. It can be verified (see [43] and [20]), that all the presented Green functions satisfy the assumptions 1. − 3. when considering any semidefinite continuous weight function r.

The estimates from Webb and Lan are actually not the only ones, since there exist also more precise, iterated estimates, which are built upon them. This improvement was suggested in [45] by Bo Yang. He considered functions θ_0, σ_0 to be a priori bounds for the eigenfunction u corresponding to the appropriate eigenvalue, e.g., Λ_r. More specifically, we need

$$\sigma_0 \le \frac{u}{\|u\|_{C[0,1]}} \le \theta_0.$$

Then it is possible to define sequences σ_n and θ_n, $n \in \mathbb{N}$, by

$$\theta_{n+1}(x) = \int_0^1 G_{3,1}(x,y) r(y) \theta_n(y)\,\mathrm{d}y, \quad \sigma_{n+1}(x) = \int_0^1 G_{3,1}(x,y) r(y) \sigma_n(y)\,\mathrm{d}y,$$

and threshold values

$$m_n := \left(\sup_{0 \le x \le 1} \theta_n(x) \right)^{-\frac{1}{n}} \quad \text{and} \quad M_n := \left(\sup_{0 \le x \le 1} \sigma_n(x) \right)^{-\frac{1}{n}}.$$

The next lemma from Yang then utilizes these values a provides a more sophisticated estimate for (in this case) Λ_r.

Lemma 38 (Bo Yang, [45])**.** *Let θ_n, σ_n, m_n and M_n be defined as above. Then for each $n \in \mathbb{N}$, we have $m_n \le \Lambda_r \le M_n$.*

In [45], Yang provided partially optimized bounds, or "initial guesses" σ_0, θ_0, however, we did not have to use those and the reason for it is straightforward. Since we aim to enlarge the SIP interval, we investigate, whether the weighted eigenvalues are further from zero, than their non-weighted counterparts. That means, we are not particularly interested in values M and M_n, instead, our goal is to compute m and especially m_n. To compute Yang's iterations θ_n for m_n, we only need the initial value θ_0, which is the a priori upper estimate for $u/\|u\|_{C[0,1]}$. It turns out, that the initial guess $\theta_0 \equiv 1$ is a good enough option, from the computational point of view. Obviously, in this case m_1 coincides with m given by Lemma 37.

Now, let us present three different types of weights, which we used for computing the either Webb-Lan based, or Yang based estimates of both Λ_r and $\lambda_{0,r}$. Each type has been chosen for a different reason, which we discuss separately.

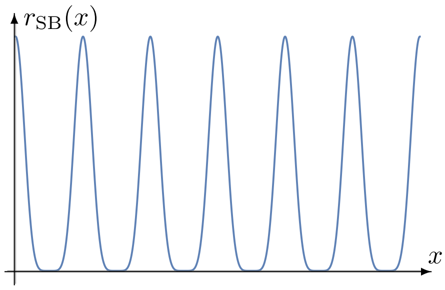

Figure 4.1: Here, we show the function $r_{\mathrm{SB}}(x) = \cos^6(6\pi x)$, which illustrates the placement of hangers of a suspension bridge.

Example 3. Our first example of a weight function, $r_{\mathrm{SB}}(x) = \cos^6(6\pi x)$, should resemble hanger placement density of a suspension bridge, i.e., the maxima correspond to the hanger attachment to the road, whereas the minima correspond to the places precisely in between them.

Example 4. Our next example is the "hill" function

$$r_{\mathrm{H}}(x) = \begin{cases} -16(x-0.5)^2+1 & \text{for } x \in \left[\frac{1}{4}, \frac{3}{4}\right]. \\ 0 & \text{otherwise} \end{cases}$$

We chose this weight function to demonstrate, which types of weights are the most problematic and correspond to smaller gains in terms of the shift of SIP interval bounds. The reason for it is that the position of $\lambda_{0,r}$ depends on the product $ru\sin\pi x$ and thus the behaviour of r near 0.5 affects it the most. Hence, every function, which attains its maximum around the centre of $[0,1]$ and is close to zero otherwise, leads to relatively small improvements. Also, the eigenvalue Λ_r is affected in a similar way, since the product of the corresponding conjugate eigenfunctions has similar properties as $ru\sin\pi x$ (for details, see our Remark 3 in [20]).

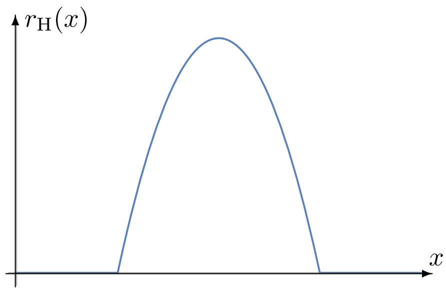

Figure 4.2: Here, we show the function $r_{\mathrm{H}}(x)$, which illustrates the type of function corresponding to a less significant improvement of SIP bounds.

Example 5. The last example is the pair of "half-parabolas"

$$r_{\mathrm{RP}}(x) = \begin{cases} 4(x-0.5)^2 & \text{for } x \in \left[\frac{1}{2}, 1\right], \\ 0 & \text{otherwise,} \end{cases} \qquad r_{\mathrm{LP}}(x) = r_{\mathrm{RP}}(1-x).$$

This pair of functions is interesting, since it provides a relatively big shift of SIP bounds. Moreover, the lower estimates m or m_n (this time only Λ_r estimates, *not* the $\lambda_{0,r}$ ones) for $r_{\mathrm{RP}}(x)$ are also applicable in the $r_{\mathrm{LP}}(x)$ case. Indeed, since $r_{\mathrm{LP}}(x) = r_{\mathrm{RP}}(1-x)$ and $G_{3,1}(x,y) = G_{1,3}(1-x,1-y)$, then also

$$\int_0^1 G_{3,1}(x,y) r_{\mathrm{RP}}(y)\ \mathrm{d}y = \int_0^1 G_{1,3}(x,y) r_{\mathrm{LP}}(y)\ \mathrm{d}y,$$

i.e, m_n for r_{RP} and $L^{3,1}$ are the same as m_n for r_{LP} and $L^{1,3}$. Here, we can utilize the fact, that both conjugate problems for the same weight have the same principal eigenvalue (see Lemma 30). Therefore, it is possible to compute lower bounds m_n of Λ_r for $L^{3,1}$ with both r_{RP} and r_{LP} and choose the value further from zero. This process is actually possible for any assymetrical weight r, since we can compute the m_n values for both $r(x)$ and $r(1-x)$ and use the larger ones.

4.6 Comparison with previous results

At last, let us provide a summary of computed estimates for the weighted eigenvalues Λ_r and $\lambda_{0,r}$ for all introduced example weights. We used *Wolfram Mathematica* for all computations. The tables can be also found in [20].

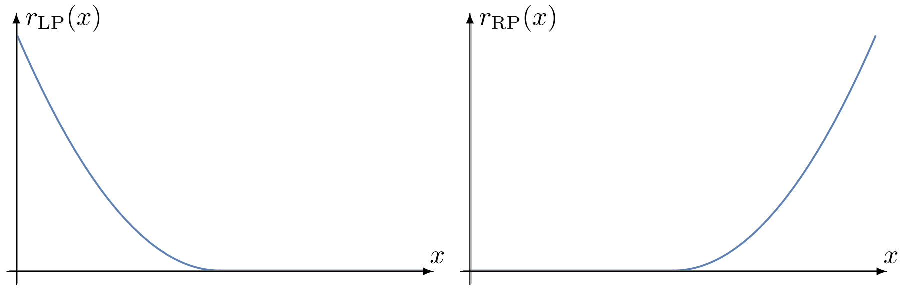

Figure 4.3: Here, we show the functions $r_{\rm LP}(x)$ and $r_{\rm RP}(x)$, which illustrate the symmetric couple of weights with a strong improvement of SIP bounds.

In the first table, we show the Webb-Lan type estimates.

weight	m	M
1	227.56	2783.13
$r_{\rm SB}(x)$	741.79	8881.62
$r_{\rm H}(x)$	502.53	4415.23
$r_{\rm LP}(x)$	1109.19	114749.00
$r_{\rm RP}(x)$	11590.90	29153.50

The estimate m for $r \equiv 1$, is quite far from the non-weighted eigenvalue $\Lambda_1 = c_0 \approx 950.884$. If this trend holds, it suggests, that for non-constant weights r, the weigthed eigenvalues Λ_r could be far from their computed lower estimate as well. However, the more positive information is, that we can already see, that m for $r_{\rm LP}$ and especially for $r_{\rm RP}$ are significantly beyond the value c_0.

Hence, let us use the Yang estimates. The next table illustrates the significant improvement of lower bounds of Λ_r given by iterations m_n. In this table, we show the first value m_n such that m_n is significantly larger than c_0 and the number of the iteration, which it corresponds to.

weight	n	m_n
$r_{\rm SB}$	4	1950.29
$r_{\rm H}$	4	1050.31
$r_{\rm LP}$	3	6339.80
$r_{\rm RP}$	1	11590.90

In view of the discussion in Example 5, note that in fact we have $\Lambda_r > 11590.9$ for both $r_{\rm LP}$ and $r_{\rm RP}$.

Finally, let us provide the m_n estimates for the other eigenvalue, $\lambda_{0,r}$. Again, to better demonstrate the shift, we recall the non-weighted eigenvalue, i.e., for $r \equiv 1$, $\lambda_{0,1} = \pi^4 \approx 97,409$.

weight	m_1	m_2	m_3
1	76.80	83.14	87.45
$r_{\rm SB}$	248.55	267.66	295.80
$r_{\rm H}$	154.37	156.70	169.96
$r_{\rm LP}$	808.09	1052.14	1129.67
$r_{\rm RP}$	808.09	1156.14	1220.50

Here, the first lower estimate m_1 is the same for both half-parabolas. This is not an error, but an expected fact, since both $G_{\rm sym}$ and θ_0 are symmetric. However, that does not hold for the next iterations of θ_n, which is the reason why the further iterations of m_n do not coincide.

In the end, let us compare the shift of boundary values of the SIP interval with some previously achieved results. The original bounds given by Schröder, that is $-\pi^4$ and c_0, have already been broken in [11] and [12]. The new bounds therein mean a significant improvement under certain circumstances, however, they omit any details of the profile of $c(x)$ and, for the upper bound of the SIP interval, depend also on the right-hand side of (4.3). To be more specific, they depend on the ratio $\frac{\min h(x)}{\max h(x)}$ over $x \in [0,1]$. The best-case scenario here is a constant right-hand side h. But even in that case, our new estimates bring a significant improvement over [11] and [12].

In the first comparison table, let us show the estimates of Λ_r. Note that for a constant right-hand side, [11] provides the estimate $\Lambda_r \geq m_{\text{old}} = c_0 + 2\pi^3 \approx 1012.89$ regardless of the profile of r.

weight	m_{old}	m_{new}
r_{SB}	1012.89	1950.29
r_{H}		1050.31
r_{LP}		6339.80
r_{RP}		11590.90

To conclude this overview, let us concentrate on the lower bound for $\lambda_{0,r}$. The estimate based on [12] reads

$$\lambda_{0,r} \geq m_{\text{old}} = \frac{4\pi^2}{\int_0^1 r(x)\,\mathrm{d}x}.$$

The results are provided in the following table.

weight	m_{old}	m_{new}
1	39.48	87.45
r_{SB}	126.33	295.80
r_{H}	118.44	169.96
r_{LP}	236.87	1129.67
r_{RP}	236.87	1220.50

Note that the tested subjects $r_{\text{SB}}, r_{\text{H}}, r_{\text{LP}}, r_{\text{RP}}$ can not only be considered as examples of semidefinite weights by themselves, but also as profiles of positive, or negative parts of an indefinite coefficient $c(x)$ (cf. Theorem 35 and Corollary 36).

The work of Webb, Lan and Yang allowed us to demonstrate, that the influence of weighted eigenvalues ensures possibly a several times larger interval of strict inverse-positivity valid for an arbitrary continuous right-hand side and brings a significant improvement not only over Schröder's original bounds, but also over improved ones from a recently published literature (e.g., [11, 12]).

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- G. Holubová and J. Janoušek. One-dimensional model of a suspension bridge: revision of uniqueness results. *Appl. Math. Lett.*, 71:6–13, 2017

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- G. Holubová and J. Janoušek. Extending the threshold values for inverse-positivity of a linear fourth order operator. *Positivity*, Jun 2021

Author's active participations on conferences and seminars

- *Mathematical models of suspension bridges*, Meeting of Ph.D. Students of Mathematical Analysis and Differential Equations 2016, Mathematical Institute, Czech Academy of Sciences, Prague, Czech Republic, January 25-28, 2016

- *Mathematical models of suspension bridges*, XXX Seminar in Differential Equations, Ostrov u Tisé, Czech Republic, 2016

- *Suspension bridges in one dimension: Revision of previous uniqueness results*, International Conference on Differential & Difference Equations and Applications 2017, Amadora, Portugal, 2017

- *Damped models of suspension bridges: Existence and uniqueness results*, XXXI Seminar in Differential Equations, Velehrad, Czech Republic, 2018

- *Existence and uniqueness results for damped models of suspension bridges*, Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems. Flagstaff, Arizona, USA, 2018

- *Existence, uniqueness and bifurcation results for various models of suspension bridges*, Nonlinear analysis seminar under Mobility 3.0, ZČU, Pilsen, Czech Republic, 2019

- *Bifurcation of periodic solutions in a nonlinear model of a suspension bridge with variable cable stiffness*, Czech-Georgian Workshop on Boundary Value Problems, Brno, Czech Republic, 2019

- *Bifurcation results for a nonlinear model of a suspension bridge with variable stiffness*, Equadiff 2019, Leiden, The Netherlands, 2019

- *Weighted eigenvalues, conjugate problems and the interval of inverse-positivity of a linear fourth order operator*, Analysis seminar, ZČU, Pilsen, Czech Republic, 2021

Co-author's statement

The co-author Gabriela Holubová of the following papers

- [19] G. Holubová and J. Janoušek. One-dimensional model of a suspension bridge: revision of uniqueness results. *Appl. Math. Lett.*, 71:6–13, 2017

- [21] G. Holubová and J. Janoušek. Suspension bridges with non-constant stiffness: bifurcation of periodic solutions. *Z. Angew. Math. Phys.*, 71(6):187, 2020

- [20] G. Holubová and J. Janoušek. Extending the threshold values for inverse-positivity of a linear fourth order operator. *Positivity*, Jun 2021

confirms, that J. Janoušek's contribution to research related to each of these papers is approximately 50 %.

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Gabriela Holubová

Appendices

Appendix A.1

[19] G. Holubová and J. Janoušek. One-dimensional model of a suspension bridge: revision of uniqueness results. *Appl. Math. Lett.*, 71:6–13, 2017

Applied Mathematics Letters 71 (2017) 6–13

Contents lists available at ScienceDirect

Applied Mathematics Letters

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One-dimensional model of a suspension bridge: Revision of uniqueness results

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A R T I C L E I N F O

Article history:
Received 1 February 2017
Received in revised form 14 March 2017
Accepted 14 March 2017
Available online 19 March 2017

Keywords:
Suspension bridge
Beam equation
Unique weak solution

A B S T R A C T

This paper brings a revision of the so far known uniqueness result for a one-dimensional damped model of a suspension bridge. Using standard techniques, however with finer arguments, we provide a significant improvement and extension of the allowed interval for the stiffness parameter.

1. Introduction

We consider a nonlinear one-dimensional model of a suspension bridge introduced by Lazer and McKenna [1] and studied later in many papers (e.g., [2–8]):

$$\begin{aligned}&mu_{tt} + EIu_{xxxx} + bu_t + \kappa u^+ = h(x,t),\\&u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0,\\&u(x,t+2\pi) = u(x,t),\ -\infty < t < +\infty,\ x \in (0,L),\end{aligned} \tag{1}$$

or its rescaled form, respectively,

$$\begin{aligned}&u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + ku^+ = h(x,t),\\&u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0,\\&u(x,t+2\pi) = u(x,t),\ -\infty < t < +\infty,\ x \in (0,\pi).\end{aligned} \tag{2}$$

This model represents the bridge as a damped beam with simply supported ends, subject to a periodic external force and to the nonlinear restoring force of cables hanging on a solid frame. The displacement $u(x,t)$ is measured as positive in the downward direction and the cables are taken as one-sided springs

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http://dx.doi.org/10.1016/j.aml.2017.03.011
0893-9659/

obeying Hooke's law, with a restoring force proportional to the displacement if they are stretched, and with no restoring force if they are compressed. We recall that $u^+(x,t) = \max\{0, u(x,t)\}$ is the positive part of $u(x,t)$ and k (or κ, respectively) can be interpreted as the stiffness of the cables. The meaning of other parameters can be found, e.g., in [3]. Evidently, only $\alpha > 0$, $\beta > 0$ and $k > 0$ make sense from the physical point of view, however, for the sake of generality, we will deal with $k \in \mathbb{R}$ throughout the text.

The aim of this paper is to revise the original result of [9], which says that for sufficiently small $|k|$, the problem (2) admits a unique solution for any right-hand side. Using the same techniques, however with finer arguments, we provide a significant improvement and extension of the allowed values of k. This means that even for a more pronounced asymmetry, the system possesses a unique solution for any loading and no bifurcations can occur.

2. Abstract setting

Let us denote by $\Omega = (0,\pi) \times (0,2\pi)$ the considered domain and by $H = L^2(\Omega,\mathbb{R})$ the real Hilbert space equipped with the standard scalar product and the corresponding norm. Further, we denote by $\mathcal{D}$ the set of all smooth functions which satisfy the boundary and periodic conditions from (2). We call a function $u(x,t) \in H$ a *weak solution* of the problem (2) if and only if the integral identity

$$\int_\Omega u(v_{tt} + \alpha^2 v_{xxxx} - \beta v_t)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega (h - ku^+)v\,\mathrm{d}x\,\mathrm{d}t \tag{3}$$

holds for all $v \in \mathcal{D}$.

Now, let us consider the complexification $H_\mathbb{C} = H + \mathrm{i}H = L^2(\Omega,\mathbb{C})$ of H with the scalar product $\langle u,v\rangle = \int_\Omega u\bar{v}\,\mathrm{d}x\,\mathrm{d}t$, $u,v \in H_\mathbb{C}$, and the usual norm $\|u\| = \langle u,u\rangle^{\frac{1}{2}}$. Since the set $\{\mathrm{e}^{\mathrm{i}nt}\sin mx,\ m \in \mathbb{N},\ n \in \mathbb{Z}\}$ forms an orthogonal basis in $H_\mathbb{C}$, each function $u(x,t) \in H_\mathbb{C}$ has its representation by the Fourier series

$$u(x,t) = \sum_{n=-\infty}^{+\infty}\sum_{m=1}^{+\infty} u_{mn}\mathrm{e}^{\mathrm{i}nt}\sin mx. \tag{4}$$

Notice that for real functions $u \in H$ there is $u_{m(-n)} = \bar{u}_{mn}$.

Finally, we denote by L the abstract realization of the linear beam operator

$$u \mapsto u_{tt} + \alpha^2 u_{xxxx} + \beta u_t$$

with the given boundary and periodic conditions, i.e.,

$$L:\ \mathrm{dom}(L) \subset H_\mathbb{C} \to H_\mathbb{C}, \qquad Lu = \sum_{n=-\infty}^{+\infty}\sum_{m=1}^{+\infty}(\alpha^2 m^4 - n^2 + \mathrm{i}\beta n)u_{mn}\mathrm{e}^{\mathrm{i}nt}\sin mx,$$

where

$$\mathrm{dom}(L) = \Big\{u \in H_\mathbb{C} : \sum_{n=-\infty}^{+\infty}\sum_{m=1}^{+\infty}\left|\alpha^2 m^4 - n^2 + \mathrm{i}\beta n\right|^2 |u_{mn}|^2 < +\infty\Big\}.$$

Notice that L is a real operator in the sense that

$$u \in \mathrm{dom}(L) \cap H \Rightarrow Lu \in H.$$

Using this setting, the original boundary value problem (2) can be formulated in the abstract way as

$$Lu = -ku^+ + h. \tag{5}$$

The spectrum of the operator L consists only of the point spectrum $\sigma(L) = \{\lambda_{mn},\ m \in \mathbb{N},\ n \in \mathbb{Z}\}$ with

$$\lambda_{mn} = \alpha^2 m^4 - n^2 + \mathrm{i}\beta n, \quad m \in \mathbb{N}, n \in \mathbb{Z} \tag{6}$$

being the eigenvalues of L. Considering an arbitrary $\lambda \in \mathbb{R},\ \lambda \notin \sigma(L)$, the non-homogeneous equation

$$Lu - \lambda u = f \tag{7}$$

has a unique weak solution $u \in H$ for an arbitrary right-hand side $f \in H$. Moreover, the corresponding resolvent operator L_λ^{-1},

$$L_\lambda^{-1} : H \to H, \quad L_\lambda^{-1} : f \mapsto u$$

is linear, compact and its norm can be estimated as

$$\|L_\lambda^{-1}\| \le \frac{1}{\mathrm{dist}(\lambda, \sigma(L))} \tag{8}$$

(see, e.g., [3] or [9]).

Remark 1. If we denote $x = \mathrm{Re}\,(\lambda_{mn})$ and $y = \mathrm{Im}\,(\lambda_{mn})$, then the eigenvalues λ_{mn} of L can be interpreted as intersections of parabolas

$$p_m = \left\{(x, y) :\ x = \alpha^2 m^4 - \frac{y^2}{\beta^2}\right\}, \quad m \in \mathbb{N},$$

with horizontal lines

$$l_n = \{(x, y) :\ y = \beta n\}, \quad n \in \mathbb{Z}$$

(see Fig. 1 for illustration and recall that $\alpha > 0,\ \beta > 0$).

3. Revision of previous results

In this section, we recall shortly results of [9] and provide a more detailed analysis of the uniqueness condition for the solvability of (5) obtained therein.

As $\alpha > 0,\ \beta > 0$ and hence zero is not an eigenvalue of L, the problem (5) can be rewritten into an equivalent form

$$u = L_0^{-1}(-ku^+ + h). \tag{9}$$

To obtain the existence of a unique solution $u \in H$, one can apply the Banach contraction theorem. Since for any $u, v \in H$ we have

$$\begin{aligned}\|L_0^{-1}(-ku^+ + f) - L_0^{-1}(-kv^+ + f)\| &= \|L_0^{-1}(k(v^+ - u^+))\| \ \le\ \|L_0^{-1}\||k|\|v^+ - u^+\| \\ &\le \frac{|k|}{\mathrm{dist}(0, \sigma(L))}\|u - v\|,\end{aligned}$$

the operator $L_0^{-1}(-k(\cdot)^+ + f)$ is contractive under the condition $\frac{|k|}{\mathrm{dist}(0,\sigma(L))} < 1$, or, equivalently, under

$$|k| < \mathrm{dist}(0, \sigma(L)).$$

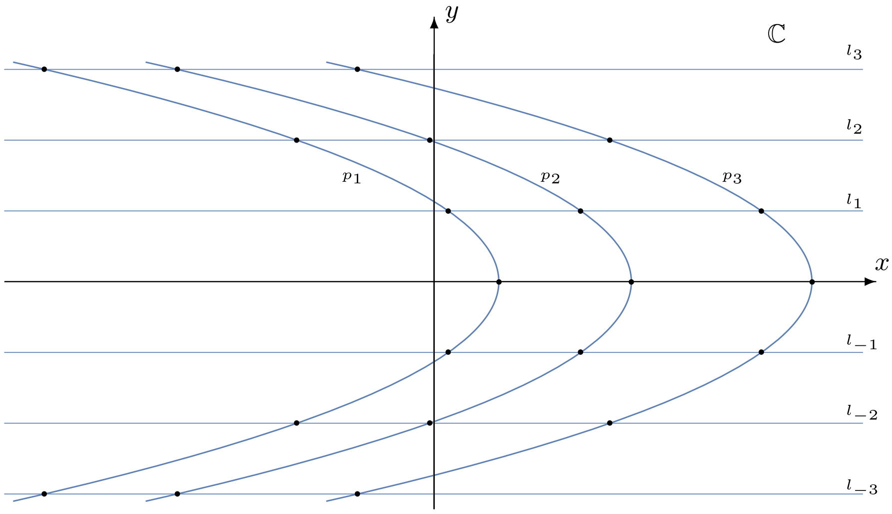

Fig. 1. Illustration of the eigenvalues λ_{mn} in the complex plane.

Since $\operatorname{dist}(0, \sigma(L)) = \min_{m\in\mathbb{N},n\in\mathbb{Z}}|\lambda_{mn}|$, we obtain that the sufficient condition for the unique solvability of (5) (and hence of (2)) reads as

$$|k| < \min_{m\in\mathbb{N},n\in\mathbb{Z}} |\lambda_{mn}|, \tag{10}$$

where

$$\min_{m\in\mathbb{N},n\in\mathbb{Z}} |\lambda_{mn}| = \min_{m\in\mathbb{N},\, n\in\mathbb{Z}} \sqrt{(\alpha^2 m^4 - n^2)^2 + (\beta n)^2}$$

(cf [9]). Notice that there are only finitely many candidates for the eigenvalue with the minimal modulus: it is either the smallest real eigenvalue $\lambda_{mn} = \lambda_{10} = \alpha^2$, or possibly some complex eigenvalue lying even closer to the origin, i.e., λ_{mn} inside an open disc $D_0 = \{z \in \mathbb{C}; |z| < \alpha^2\}$. Taking in mind that $\lambda_{mn} \in p_m \cap l_n$ (cf. Remark 1), we can do, for example, the following observations:

- If $\beta \geq \alpha^2$, then no horizontal line l_n, $n \in \mathbb{Z}$, intersects D_0.
- If $\beta \geq \sqrt{2}\alpha$, then no parabola p_m, $m \in \mathbb{N}$, intersects D_0.
- If $\alpha > 1$ and $\sqrt{2\alpha^2 - 1} \leq \beta < \sqrt{2}\alpha$, then the only parabola intersecting D_0 is p_1, but $\lambda_{1n} \notin D_0$ for all $n \in \mathbb{Z}$.

Hence, in all these cases, $\sigma(L) \cap D_0 = \emptyset$, thus $\min_{m\in\mathbb{N},n\in\mathbb{Z}}|\lambda_{mn}| = \alpha^2$ and we can state the following refining of the original result of [9].

Proposition 1. *Let $\beta \geq \alpha^2$ for $\alpha < 1$ and $\beta \geq \sqrt{2\alpha^2 - 1}$ for $\alpha \geq 1$. Then the problem (2) has a unique weak solution $u \in H$ for an arbitrary right-hand side $h \in H$ whenever $k \in (-\alpha^2, \alpha^2)$.*

4. Main result

Proposition 1 as well as the general condition (10) for the existence and uniqueness of the solution of (2) can be quite restrictive. We can improve them if we consider an ε-shift in Eq. (5):

$$Lu + \varepsilon u - \varepsilon u = -ku^+ + h,$$

or equivalently (using the decomposition $\varepsilon u = \varepsilon u^+ - \varepsilon u^-$)

$$(L - \varepsilon I)u = -(k+\varepsilon)u^+ + \varepsilon u^- + h. \tag{11}$$

Considering $\varepsilon \in \mathbb{R}$ not to be an eigenvalue of the operator L and applying the resolvent operator L_ε^{-1}, we can rewrite Eq. (11) into the form

$$u = L_\varepsilon^{-1}(-(k+\varepsilon)u^+ + \varepsilon u^- + h). \tag{12}$$

Now, we are ready to formulate our main result.

Theorem 2. *Let $\varepsilon_M > 0$ and $\varepsilon_m > 0$ be the maximal real numbers for which*

$$\{z \in \mathbb{C};\ (|z - \varepsilon_m| < \varepsilon_m) \vee (|z + \varepsilon_M| < \varepsilon_M)\} \cap \sigma(L) = \emptyset. \tag{13}$$

Then the problem (2) has a unique weak solution $u \in H$ for an arbitrary right-hand side $h \in H$ whenever $k \in (-2\varepsilon_m, 2\varepsilon_M)$. Moreover, the following estimates hold:

$$\varepsilon_M \geq \tilde{\varepsilon}_M = \begin{cases} \dfrac{2\alpha\beta + \beta^2}{2} & \text{for} \quad \beta \geq 2(1-\alpha), \\ \beta & \text{for} \quad \beta < 2(1-\alpha), \end{cases} \tag{14}$$

and

$$\varepsilon_m \geq \tilde{\varepsilon}_m = \begin{cases} \dfrac{\alpha^2}{2} & \text{for} \quad \beta \geq \min\left\{\alpha,\ \dfrac{\alpha^2}{2}\right\}, \\ \dfrac{2\alpha\beta - \beta^2}{2} & \text{for} \quad \beta \leq \min\{\alpha,\ 2(\alpha-1)\}, \\ \beta & \text{for} \quad \alpha < 2 \ \text{and} \ 2(\alpha-1) \leq \beta \leq \dfrac{\alpha^2}{2}. \end{cases} \tag{15}$$

Proof. We come from the formulation (12) with $\varepsilon \in \mathbb{R}$, $\varepsilon \notin \sigma(L)$, and use again the Banach contraction theorem. For arbitrary $u, v \in H$, we have

$$\begin{aligned} &\|L_\varepsilon^{-1}(-(k+\varepsilon)u^+ + \varepsilon u^- + h) - L_\varepsilon^{-1}(-(k+\varepsilon)v^+ + \varepsilon v^- + h)\| \\ &= \|L_\varepsilon^{-1}((k+\varepsilon)(v^+ - u^+) - \varepsilon(v^- - u^-))\| \ \leq \ \|L_\varepsilon^{-1}\|\|(k+\varepsilon)(v^+ - u^+) - \varepsilon(v^- - u^-)\|. \end{aligned} \tag{16}$$

If we use the inequality

$$\|(k+\varepsilon)(v^+ - u^+) - \varepsilon(v^- - u^-)\| \leq \max\{|k+\varepsilon|,\ |\varepsilon|\}\|v - u\|$$

and the relation (8), we get the estimate

$$\|L_\varepsilon^{-1}(-(k+\varepsilon)u^+ + \varepsilon u^- + h) - L_\varepsilon^{-1}(-(k+\varepsilon)v^+ + \varepsilon v^- + h)\| \leq \frac{\max\{|k+\varepsilon|,\ |\varepsilon|\}}{\operatorname{dist}(\varepsilon, \sigma(L))}\|v - u\|.$$

Hence, the operator $L_\varepsilon^{-1}(-(k+\varepsilon)(\cdot)^+ + \varepsilon(\cdot)^- + h)$ is contractive if

$$\max\{|k+\varepsilon|,\ |\varepsilon|\} < \operatorname{dist}(\varepsilon, \sigma(L)). \tag{17}$$

That is, (17) is the sufficient condition which guarantees the existence of a unique solution of (12) (and hence of (2)). It reads as

$$\operatorname{dist}(\varepsilon, 0) < \operatorname{dist}(\varepsilon, \sigma(L)) \quad \wedge \quad \operatorname{dist}(\varepsilon, -k) < \operatorname{dist}(\varepsilon, \sigma(L)). \tag{18}$$

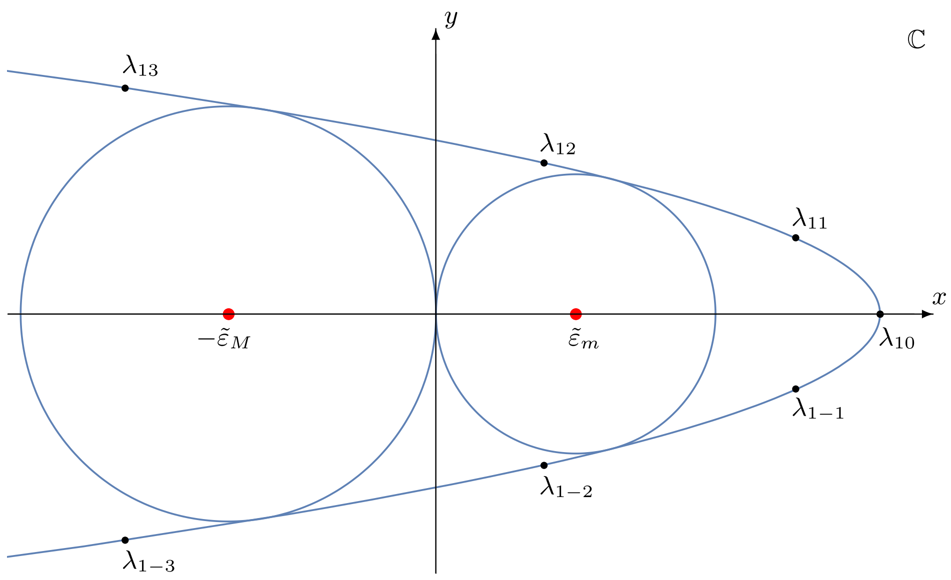

Fig. 2. The values $\tilde{\varepsilon}_m$, $\tilde{\varepsilon}_M$ and the corresponding "safe" discs, where none of the eigenvalues may appear.

Thus, if we find the maximal values $\varepsilon_m > 0$, $\varepsilon_M > 0$ such that the first condition in (18) is satisfied for any $\varepsilon \in (-\varepsilon_M, \varepsilon_m)$, i.e., (13) holds true, then the latter condition in (18) is satisfied for any $k \in (-2\varepsilon_m, 2\varepsilon_M)$.

Considering the fact that the eigenvalues λ_{mn} lie at intersections of parabolas p_m, $m \in \mathbb{N}$, and horizontal lines l_n, $n \in \mathbb{Z}$ (see Remark 1), we can determine the "safe" lower bounds $\tilde{\varepsilon}_m$, $\tilde{\varepsilon}_M$ for ε_m, ε_M as the radii of discs with centers on the real line and touching the origin as well as the first parabola p_1 (see Fig. 2), or the first pair of lines l_{-1}, l_1. Depending on α, β (and hence on the mutual position of p_1 and $l_{\pm 1}$), we obtain the estimates (14), (15) and the required assertions.

5. Final remarks and examples

Remark 2. Notice that $\varepsilon_M = \tilde{\varepsilon}_M$ if $\beta = \frac{n^2-\alpha^2}{\alpha}$ for some $n \in \mathbb{N}$ or $\beta = 1 - \alpha^2 m^4$ (i.e., there exists an eigenvalue λ_{1n} or λ_{m1} coinciding with the corresponding tangent point, cf. Fig. 2). Similarly for ε_m. If this is not the case, the allowed interval for k can be further enlarged.

For example, if $\alpha^2 < 1$ and $\beta < \frac{1-\alpha^2}{(4+\sqrt{15})\alpha}$, then the disc which touches the origin from the left and the second parabola p_2 contains no eigenvalues located on p_1. Hence, in this case, we can estimate

$$\varepsilon_M \geq \bar{\varepsilon}_M = \frac{8\alpha\beta + \beta^2}{2}.$$

Instead of improving the estimates $\tilde{\varepsilon}_m$ and $\tilde{\varepsilon}_M$, we can gain the optimal values ε_m, ε_M in a similar way as in Section 3. Namely, we can find some suitable candidates for the "first" eigenvalues in the left and right half-planes, construct the discs passing through these eigenvalues and decrease them until they contain no eigenvalue in their interiors. In particular, we can proceed via the following algorithm.

1. Put $\lambda_{\mathrm{opt}} = \lambda_{1n_0}$ with $n_0 = \lfloor \alpha + 1 \rfloor$. (Here, $\lfloor \cdot \rfloor$ denotes the integer part of a real number, and λ_{1n_0} is the "first" eigenvalue with a negative real part on the parabola p_1.)
2. Find an open disc D with the center on the real axis, whose boundary circle is going through an eigenvalue λ_{opt} and the origin, i.e., $D = \{z \in \mathbb{C};\ |z + \varepsilon_D| < \varepsilon_D\}$ with

$$\varepsilon_D = \frac{|\lambda_{\mathrm{opt}}|^2}{2\,|\mathrm{Re}\,(\lambda_{\mathrm{opt}})|}.$$

3. If there are no eigenvalues inside D, put $\varepsilon_M = \varepsilon_D$ and quit, in the other case find indexes $M = \max\{m : \lambda_{mn} \in D\}$ and $N = \min\{n : \lambda_{Mn} \in D\}$, i.e., find the indexes of the "first" eigenvalue with the negative real part on the "highest" parabola inside D.
4. Put $\lambda_{\text{opt}} = \lambda_{MN}$ and go back to Step 2.

In the case of ε_m we proceed similarly, starting from the "first" eigenvalue with the positive real part.

Remark 3. It is worth mentioning that the condition $k < \beta^2 + 2\alpha\beta = 2\tilde{\varepsilon}_M$ for $= \beta \geq 2(1-\alpha)$ agrees with a result presented by Alonso and Ortega in [2], however, they obtained it by a different approach using spatial discretization of the problem (2) and its reduction to a finite system of ODEs.

The following examples demonstrate the rate of improvement given by the estimates (14) and (15).

Example 1. Let $\alpha \geq 2$, $q \geq \sqrt{2}$ and put $\beta = q\alpha$. Then the assumptions of Proposition 1 are satisfied and the problem (2) has a unique weak solution for all $k \in (-\alpha^2, \alpha^2)$. If we employ Theorem 2 with its estimates (14), (15), we get

$$\tilde{\varepsilon}_M = \frac{2\alpha\beta + \beta^2}{2}, \quad \tilde{\varepsilon}_m = \frac{\alpha^2}{2}$$

and thus obtain a much larger uniqueness interval

$$k \in \left(-\alpha^2,\, \left(q^2 + 2q\right)\alpha^2\right),$$

i.e., the positive part of the interval, which is more interesting from the physical point of view, is $(q^2 + 2q)$-times larger than the original conditions allow.

Example 2. Let $s \in \mathbb{N}$ be arbitrary and put $\alpha = s$, $\beta = \frac{1}{s}$. Proposition 1 gives no information about solvability of (2), since $\alpha^2 \geq 1$ and $\beta \leq 1 \leq \sqrt{2\alpha^2 - 1}$. However, the sufficient condition (10) (cf. the original result in [9]) in its generality allows us to state that the existence and uniqueness of the weak solution of (2) is guaranteed for an arbitrary right-hand side $h \in H$ whenever $k \in (-1, 1)$.

Indeed, since $s \in \mathbb{N}$, $\lambda_{1s} = 0 + \mathrm{i}$, and the open disc $D = \{z \in \mathbb{C};\ |z| < 1\}$ contains no other eigenvalue λ_{mn}. Hence, $\min_{m\in\mathbb{N}, n\in\mathbb{Z}} |\lambda_{mn}| = |\lambda_{1s}| = 1$.

However, the interval $(-1, 1)$ can be enlarged by applying Theorem 2. It is easy to see, that for $s = 1$ the estimates (14), (15) provide us

$$k \in \left(-\alpha^2,\, 2\alpha\beta + \beta^2\right) = (-1,\, 3)\,,$$

and for $s \geq 2$ we get

$$k \in \left(-2\alpha\beta + \beta^2,\, 2\alpha\beta + \beta^2\right) = \left(-2 + \frac{1}{s^2},\, 2 + \frac{1}{s^2}\right).$$

In all cases, these "safe" uniqueness intervals are twice as large as the original one. Moreover, e.g., for $s = 1$, the "first" eigenvalue with a negative real part is $\lambda_{12} = -3 + 2\mathrm{i}$ and the disc D passing through contains no other eigenvalue in its interior (cf. the algorithm in Remark 2). Hence, $\varepsilon_M = \varepsilon_D = |\lambda_{12}|^2 / (2\,|\mathrm{Re}\,(\lambda_{12})|) = 13/6$ and the uniqueness result holds for any $k \in (-1, 13/3)$.

Acknowledgments

Both authors were partially supported by the project LO1506 of the Czech Ministry of Education, Youth and Sports. The first author was also supported by the Grant Agency of the Czech Republic, Grant No. 13-00863S, the second author was supported by the project SGS-2016-003 of the University of West Bohemia.

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Appendix A.2

[21] G. Holubová and J. Janoušek. Suspension bridges with non-constant stiffness: bifurcation of periodic solutions. *Z. Angew. Math. Phys.*, 71(6):187, 2020

Z. Angew. Math. Phys. (2020) 71:187
© 2020 Springer Nature Switzerland AG
0044-2275/20/060001-11
published online October 16, 2020
https://doi.org/10.1007/s00033-020-01415-4

Zeitschrift für angewandte Mathematik und Physik ZAMP

Suspension bridges with non-constant stiffness: bifurcation of periodic solutions

Gabriela Holubová and Jakub Janoušek

Abstract. We consider a modified version of a suspension bridge model with a spatially variable stiffness parameter to reflect the discrete nature of the placement of the bridge hangers. We study the qualitative and quantitative properties of this model and compare the cases of constant and non-constant coefficients. In particular, we show that for certain values of the stiffness parameter, the bifurcation occurs. Moreover, we can expect also the appearance of blowups, whose existence is closely connected with the so-called Fučík spectrum of the corresponding linear operator.

Mathematics Subject Classification. 35B10, 35B32, 70K30.

Keywords. Suspension bridge, Jumping nonlinearity, Variable coefficient, Bifurcation.

1. Introduction

We study a modified version of a standard [12,13] one-dimensional nonlinear beam model of a suspension bridge

$$\begin{aligned} u_{tt} + u_{xxxx} + b\,r(x)u^+ = h(x,t) \quad \text{in } (0,1)\times\mathbb{R}, \\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \\ u(x,t) = u(x,t+2\pi) = u(x,-t). \end{aligned} \tag{1}$$

Here, the term $b\,r(x)u^+$ represents the nonlinear restoring force due to the bridge hangers (sometimes called suspender cables in the literature) with the constant stiffness b and a variable hanger placement density $r(x)$. Unless stated otherwise, we consider $r(x)$ to be a continuous function on $(0,1)$ such that $0 < r(x) \le 1$ almost everywhere in $(0,1)$.

For $r(x) \equiv 1$, there are several results concerning multiplicity of periodic solutions: see [12,13] and also [2,10] as an example of the problem setting and the application of various tools leading to a conclusion, that when the hanger stiffness b crosses eigenvalues of the corresponding linear beam operator, more solutions appear. These works were followed by [4] and [7], which approached this problem from a different perspective through utilizing a global bifurcation framework.

In this paper, we use some of the basic ideas appearing in [4] and implement them in (1) with a generally non-constant function $r(x) \not\equiv 1$. The reason for introducing the density (or weight) function $r(x)$ is to interpret more realistically the fact that the hangers are actually not a uniformly distributed force acting on the roadbed, but they are "distinctly distributed." That is, the restoring force should attain its maximum where the hangers are connected to the roadbed, whereas being considerably weaker in between. Such phenomenon can be described, e.g., with $r(x)$ being a high even power of the cosine function.

We show that making the stiffness parameter spatially variable actually improves the behavior of the considered model while not changing its qualitative properties (see the results in Sects. 3 and 5). Bifurcation phenomena (from the stationary solution as well as from infinity) are still observable. But

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since the variable coefficient shifts the eigenvalues of the corresponding linear beam operator away from zero, it brings more "room" (with respect to b) for the existence of a unique solution and postpones the appearance of additional solutions. Their existence can result in the buckling phenomenon, possibly dangerous for the bridge structure. (Let us note that the destructive torsional oscillations of the infamous Tacoma Narrows Bridge appeared as a sudden change in behavior.)

On the other hand, the non-constant weight function brings certain difficulties and disables to obtain a more detailed description of the solution set. First of all, the implementation of the bifurcation theory relies on the existence of a positive stationary solution under a positive constant loading. Unlike the constant coefficient case, its existence is not guaranteed for any b (cf. Sect. 4). Moreover, several fundamental questions concerning existence and blowups of solutions stay open, since answering them would require detailed knowledge of the influence of the asymmetric nonlinearity, which is closely connected with the so-called Fučík spectrum of the corresponding linear beam operator (see Sect. 5).

Finally, let us note that for applications in real-world mechanics, only positive values of the parameter b are relevant. In this paper, however, we omit this limitation, since from the mathematical point of view, the problem (and the comparison of constant and non-constant coefficient cases) is interesting also for b being negative. Similarly, a more realistic model should contain non-unit values of the bridge parameters (length, mass, etc.). However, their consideration does not affect our results qualitatively.

2. Preliminaries and operator setting

Let us denote by Ω the domain $(0,1)\times(0,2\pi)$ and let $h/\sqrt{r}\in L^2(\Omega)$. We consider the weighted space $L^2_r(\Omega) := L^2(\Omega, r(x))$ with the inner product $(u,v)_r = \int_\Omega r(x)u(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t$ and the norm $\|u\|_r = \sqrt{(u,u)_r}$. Notice that $h/\sqrt{r}\in L^2(\Omega)$ means $h/r\in L^2_r(\Omega)$.

Further, let $H\subset L^2_r(\Omega)$ be a subspace of functions in $L^2_r(\Omega)$ being even in the time variable and let $\mathcal{D}$ stand for all C^∞-functions $\psi : [0,1]\times\mathbb{R}\to\mathbb{R}$ satisfying the conditions from (1). A function $u:(0,1)\times\mathbb{R}\to\mathbb{R}$ is then called *a weak solution* of the problem (1) if and only if

$$\int_\Omega u(x,t)(\psi_{tt}(x,t)+\psi_{xxxx}(x,t))\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \left(h(x,t) - b\,r(x)u^+(x,t)\right)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

for all $\psi\in\mathcal{D}$, and the restriction of u belongs to H. Here, u^+ denotes the positive part of u. Similarly, u^- stands for the negative part of u and $u = u^+ - u^-$. Moreover, both u^+ and u^- are elements of H.

The solvability of (1) is connected to the weighted spectrum of a linear beam operator. Hence, we first consider the following eigenvalue problem with a weight function r, i.e.,

$$\begin{aligned} u_{tt} + u_{xxxx} &= \lambda\, r(x)u \quad \text{in } (0,1)\times\mathbb{R},\\ u(0,t) = u(1,t) &= u_{xx}(0,t) = u_{xx}(1,t) = 0,\\ u(x,t) &= u(x,t+2\pi) = u(x,-t). \end{aligned} \tag{2}$$

Lemma 1. *All the eigenvalues of the problem (2) are real and form an infinite sequence*

$$(\lambda_{m,n})_{m,n=0}^{+\infty}$$

with the following properties.

(i) *For any $m,n\in\mathbb{N}\cup\{0\}$, $\lambda_{m,n}\neq 0$.*
(ii) $\lim_{m\to+\infty}\lambda_{m,n} = +\infty$, $\lim_{n\to+\infty}\lambda_{m,n} = -\infty$.
(iii) *For any fixed $n\in\mathbb{N}\cup\{0\}$, all the eigenvalues, for which $\lambda_{m,n} > -n^2$, are simple, i.e., $\lambda_{m_1,n}\neq\lambda_{m_2,n}$ whenever $m_1\neq m_2$.*
(iv) *All the eigenvalues $\lambda_{m,0}$ are positive.*

(v) *For any* $m, n \in \mathbb{N} \cup \{0\}$

$$|\lambda_{m,n}| \geq \left|(m+1)^4\pi^4 - n^2\right|.$$

(vi) *For any fixed* $m \in \mathbb{N} \cup \{0\}$*,*

$$\lambda_{m,n_1} \geq \lambda_{m,n_2}, \quad \textit{whenever } n_1 < n_2.$$

The eigenfunctions corresponding to $\lambda_{m,n}$ *take the form*

$$\varphi_{m,n}(x,t) = X_m(x,n)\cos nt$$

with $X_m(x,n)$ *being a non-trivial solution of the ODE problem*

$$\begin{aligned} X^{IV} - n^2 X &= \lambda_{m,n}\, r(x) X, \\ X(0) = X(1) &= X''(0) = X''(1) = 0. \end{aligned} \tag{3}$$

All the eigenfunctions $\varphi_{m,n}(x,t)$ *form a complete orthogonal system on* Ω *with the weight* $r(x)$*, i.e.,*

$$\int_\Omega r(x)\varphi_{m,n}(x,t)\varphi_{k,l}(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0, \quad \textit{whenever } m \neq k \textit{ or } n \neq l.$$

The eigenfunction $\varphi_{0,0}(x,t) = X_0(x,0)$ *is strictly positive in* Ω*. Moreover, if* $r(x) > 0$ *on* $(0,1)$*, all the functions* $X_m(x,n)$ *corresponding to* $\lambda_{m,n} > -n^2$ *have exactly* m *zero points in* $(0,1)$*.*

Proof. Using the separation of variables, i.e., assuming that $u(x,t) = X(x)T(t)$, we obtain that there exists a constant μ such that the problem (2) is equivalent to the couple of ODE problems

$$\begin{aligned} T'' + \mu T &= 0, \\ T(t) = T(t+2\pi) &= T(-t), \end{aligned} \tag{4}$$

and

$$\begin{aligned} X^{IV} - \mu X &= \lambda\, r(x) X, \\ X(0) = X(1) &= X''(0) = X''(1) = 0. \end{aligned} \tag{5}$$

Problem (4) has a non-trivial solution if and only if $\mu = n^2$, $n \in \mathbb{N} \cup \{0\}$, with $T_n(t) = \cos nt$. Hence, for a given $n \in \mathbb{N} \cup \{0\}$, problem (5) reads as (3), which is a *regular Sturmian system* with a weight function $r(x)$.

All the statements of Lemma 1 now follow directly from [8,11]. Namely, [11, Theorem 1] implies that if $r(x)$ decreases, the nonzero eigenvalues do not decrease in absolute value. That is, if $r_1(x) \geq r_2(x)$ are two weight functions, then $|\lambda_{m,n}(r_1)| \leq |\lambda_{m,n}(r_2)|$. This implies

$$|\lambda_{m,n}| \geq \left|(m+1)^4\pi^4 - n^2\right|$$

for any $m, n \in \mathbb{N} \cup \{0\}$, since $\bar{\lambda}_{m,n} = (m+1)^4\pi^4 - n^2$, $m \in \mathbb{N} \cup \{0\}$ are the eigenvalues of (3) with the constant weight function $\bar{r}(x) \equiv 1 \geq r(x)$.

Similarly, if n increases, the eigenvalues do not increase.

Since the systems $(X_m(x))_{m=0}^{+\infty}$ and $(\cos nt)_{n=0}^{+\infty}$ are complete in the corresponding one-dimensional subspaces (cf. [8]), their product $\varphi_{m,n}(x,t) = X_m(x)\cos nt$ forms a complete system in H as well and (2) possess no other eigenfunction in non-separated form. □

Remark 2. For $r(x) \equiv 1$, all the eigenvalues $\lambda_{m,n} = (m+1)^4\pi^4 - n^2$, $m, n \in \mathbb{N} \cup \{0\}$, are simple and the corresponding eigenfuctions take the form $\varphi_{m,n} = \sin(m+1)\pi x\, \cos nt$.

Remark 3. Some results of this type (e.g., real eigenvalues forming an infinite sequence with no finite cluster point, r-orthogonality of corresponding eigenfunctions, etc.) hold also for $r \in L^1(0,1)$ (see [8]). Notice, however, that in this case one cannot apply the results of [11], i.e., the monotonicity of the eigenvalues is not guaranteed.

Since the eigenfunctions $\varphi_{m,n}$ form an r-orthogonal basis in H (cf. Lemma 1), any function $u \in H$ can be expanded into Fourier series

$$u(x,t) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} u_{m,n}\varphi_{m,n}$$

with the coefficients

$$u_{m,n} = \frac{(u,\varphi_{m,n})_r}{(\varphi_{m,n},\varphi_{m,n})_r}.$$

Let us define the operator $L : \mathrm{Dom}(L) \subset H \to H$ by

$$Lu = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \lambda_{m,n}u_{m,n}\varphi_{m,n},$$

with

$$\mathrm{Dom}(L) = \left\{u \in H;\ \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \lambda_{m,n}^2 u_{m,n}^2 < \infty\right\}.$$

Let us note that L can be understood as the abstract realization of the beam operator $\frac{1}{r(x)}(\partial_{tt}+\partial_{xxxx})$ on H. Moreover, L is a linear, closed, densely defined symmetric operator with a real point (weighted) spectrum

$$\sigma_r(L) = \{\lambda_{m,n}\}_{m,n=0}^{+\infty}$$

given by Lemma 1. Its resolvent $(L-\lambda I)^{-1}$ with $\lambda \notin \sigma_r(L)$ is a compact normal operator on H and its norm is given by

$$\left\|(L-\lambda I)^{-1}\right\|_r = \frac{1}{\min\limits_{m,n\in\mathbb{N}_0} |\lambda_{m,n}-\lambda|} = \frac{1}{\mathrm{dist}\,(\lambda,\sigma_r(L))}. \tag{6}$$

Hence, $u \in H$ is a weak solution of the problem (1) whenever it solves the abstract equation

$$Lu + b\,u^+ = g \tag{7}$$

with $g = h/r \in H$.

3. Existence and uniqueness

When considering an arbitrary right-hand side $g = h/r \in H$ and any $b \in \mathbb{R}$, the existence of a weak solution of (1) is not guaranteed in general. Therefore, let us first restrict ourselves to certain values of b and/or less general right-hand sides. First of all, we state the existence and uniqueness result on a bounded, generally asymmetric interval of non-resonance around $b = 0$.

Proposition 4. *Let $\lambda_q < 0 < \lambda_p$ be such that $\sigma_r(L)\cap[\lambda_q,\lambda_p] = \{\lambda_q,\lambda_p\}$, and let $g = h/r \in H$ be arbitrary. Then the problem (1) has a unique weak solution for any $b \in (-\lambda_p,-\lambda_q)$.*

Proof. Let us put $\varepsilon := \frac{1}{2}(\lambda_p+\lambda_q)$ (notice that $\varepsilon \notin \sigma_r\,(L)$) and take an ε-shifted modification of the abstract form (7)

$$(L-\varepsilon I)u = g - bu^+ - \varepsilon u,$$

or, equivalently, using $u = u^+ - u^-$,

$$(L-\varepsilon I)u = g - (b+\varepsilon)u^+ + \varepsilon u^-.$$

Since $\varepsilon \notin \sigma_r(L)$, the operator $(L - \varepsilon I)$ is invertible and we can write

$$u = (L - \varepsilon I)^{-1}(g - (b + \varepsilon)u^+ + \varepsilon u^-).$$

Let us denote the right-hand side by $T(u)$, i.e., $T: H \to H$, $T(u) = (L-\varepsilon I)^{-1}(g-(b+\varepsilon)u^+ + \varepsilon u^-)$. Using the expression (6) for the norm of the resolvent $(L - \varepsilon I)^{-1}$ and knowing that $\operatorname{dist}(\varepsilon, \sigma_r(L)) = \frac{1}{2}(\lambda_p - \lambda_q)$, we obtain

$$\begin{aligned}\|T(u) - T(v)\|_r &= \|(L - \varepsilon I)^{-1}((b + \varepsilon)(v^+ - u^+) - \varepsilon(v^- - u^-))\|_r \\ &\le \|(L - \varepsilon I)^{-1}\|_r \, \|((b + \varepsilon)(v^+ - u^+) - \varepsilon(v^- - u^-))\|_r \\ &\le 2(\lambda_p - \lambda_q)^{-1} \max\{|b + \varepsilon|, |\varepsilon|\} \, \|(v - u)\|_r.\end{aligned}$$

Hence, T is contractive for $b \in (-\lambda_p, -\lambda_q)$ and the existence of a unique solution of (7) is guaranteed under this condition. □

Remark 5. (i) Notice that λ_p is the smallest positive eigenvalue of L, and λ_q is the largest negative eigenvalue of L.

(ii) Using the monotonicity of the eigenvalues (cf. Lemma 1, prop. (v)) and the fact that for $r(x) \equiv 1$ we have $\lambda_q = \lambda_{0,10} = \pi^4 - 100$ and $\lambda_p = \lambda_{0,9} = \pi^4 - 81$, we can conclude that the problem (1) has a unique weak solution for any $b \in (81 - \pi^4, 100 - \pi^4)$.

In what follows, we will mostly deal with a positive and/or time-independent right-hand side h. Let us start with the necessary condition for the solvability of (1).

Proposition 6. *Let $g = h/r \in H$, $h(x,t) \ge 0$ a.e. in Ω, $h(x,t) \not\equiv 0$, and $u \in H$ be a weak solution of (1). Then necessarily $b > -\lambda_{0,0}$ and $u \not\le 0$.*

Proof. It follows from (7) that

$$(Lu, \varphi_{0,0})_r + b(u^+, \varphi_{0,0})_r = (g, \varphi_{0,0})_r.$$

If we use the fact that L is symmetric, $L\varphi_{0,0} = \lambda_{0,0}\,\varphi_{0,0}$ and the decomposition $u = u^+ - u^-$, we can transform the above relation into the form

$$(\lambda_{0,0} + b)(u^+, \varphi_{0,0})_r = (g, \varphi_{0,0})_r + \lambda_{0,0}(u^-, \varphi_{0,0})_r.$$

Since $\varphi_{0,0}$ is strictly positive in Ω, $\lambda_{0,0} > 0$ and g, r, u^+, u^- are nonnegative functions, we see that necessarily $u^+ \not\equiv 0$ and

$$b > -\lambda_{0,0}.$$

□

Remark 7. In fact, Proposition 6 is valid also for a (more general) right-hand side such that $\int_\Omega h\,\varphi_{0,0} > 0$. Unfortunately, the exact form of $\varphi_{0,0}$ is not known for a non-constant $r(x)$, which makes this assumption practically unverifiable. Hence, we use a more restrictive, but, on the other hand, also more reasonable setting for h.

It is also possible to obtain partial existence and uniqueness results for a more general setting with $r \in L^1(0,1)$. However, one cannot apply the results of [5,6,11], which require the continuity of r. Hence, any results from Sect. 4 onward are not guaranteed for L^1 weights.

4. Stationary solution

For now, let us consider a time-independent right-hand side h. For a continuous function $y = y(x)$ on $[0,1]$ let us denote

$$y_{\min} := \min_{x\in[0,1]} y(x) \quad \text{and} \quad y_{\max} := \max_{x\in[0,1]} y(x).$$

Further, we say that y is *strictly positive* on $(0,1)$, if it satisfies

$$y(x) > 0 \quad \text{for any } x \in (0,1) \quad \text{and} \quad y'(0) > 0,\ y'(1) < 0.$$

Set $c_0 := 4k_0^4$ with k_0 being the smallest positive solution of the equation $\tan k = \tanh k$ (i.e., $k_0 \approx 3.9266$ and $c_0 \approx 950.8843$).

Proposition 8. *Let $h = h(x) \in C([0,1])$. Then for any $b > -\lambda_{0,0}$ the problem (1) has a unique classical stationary solution $u_b = u_b(x) \in C^4([0,1])$. Moreover, if $h(x) \geq 0$, $h(x) \not\equiv 0$ on $(0,1)$, then there exists λ_M (depending on h and r) such that u is strictly positive whenever $b \in (-\lambda_{0,0}, \lambda_M]$, where*

$$-\lambda_{0,0} \leq \min\left\{b_m,\ -\pi^4\right\}, \qquad \lambda_M \geq \min\left\{b_{M1},\ b_{M2}\right\} \tag{8}$$

with

$$\begin{aligned} b_m &= -\frac{4\pi^2}{\int_0^1 r(x)\,\mathrm{d}x}, \\ b_{M1} &= c_0 + \frac{h_{\min}}{h_{\max}} 2\pi\sqrt{\pi^4 + c_0}, \\ b_{M2} &= c_0 + 2\pi\frac{h_{\min}}{h_{\max}}\left(\pi\frac{h_{\min}}{h_{\max}} r_{\min} + \sqrt{c_0 r_{\min} + \left(\frac{h_{\min}}{h_{\max}}\right)^2 \pi^2 r_{\min}^2 + \pi^4}\right). \end{aligned} \tag{9}$$

Proof. If u_b is a stationary solution of (1), then it must solve the stationary problem

$$\begin{aligned} u^{(4)} + b\, r(x) u^+ &= h(x) \quad \text{in } (0,1), \\ u(0) = u(1) = u''(0) &= u''(1) = 0. \end{aligned} \tag{10}$$

The eigenvalues of the stationary weighted eigenvalue problem

$$\begin{aligned} u^{(4)} &= \lambda\, r(x) u \quad \text{in } (0,1), \\ u(0) = u(1) &= u''(0) = u''(1) = 0, \end{aligned} \tag{11}$$

coincide with the weighted eigenvalues $\lambda_{m,0}$ of L, i.e., they are simple, positive and form a sequence

$$0 < \lambda_{0,0} < \lambda_{1,0} < \cdots \to +\infty$$

with the property $\lambda_{m,0} \geq (m+1)^4\pi^4$ for any $m \in \mathbb{N} \cup \{0\}$.

Using similar arguments as in the PDE case above (e.g., the Banach contraction principle), we can see that $b > -\lambda_{0,0}$ is the sufficient condition for the existence of a unique (weak) solution u_b of (10). For $h \in C([0,1])$, the standard regularity arguments imply that $u_b \in C^4([0,1])$.

If we consider a nonnegative right-hand side $h = h(x) \geq 0$, $h(x) \not\equiv 0$, then $b > -\lambda_{0,0}$ is also the necessary condition for solvability of (10) (cf. Proposition 6). Moreover, we can exploit results of [5,6] and [17] to obtain the strict positivity of u_b. Indeed, if u_b is a positive solution of (10), then it also solves the linear problem

$$\begin{aligned} u^{(4)} + b\, r(x) u &= h(x) \quad \text{in } (0,1), \\ u(0) = u(1) = u''(0) &= u''(1) = 0. \end{aligned} \tag{12}$$

Since the weight function $r(x)$ does not change sign, we can exploit [17, Corollary 2.1] and obtain the existence of λ_M such that u_b is strictly positive for any $b \in (-\lambda_{0,0}, \lambda_M]$.

The estimates (8), (9) for $-\lambda_{0,0}$ and λ_M, respectively, are obtained by employing conditions from [5, Theorem 4] and [6, Theorem 4]. Note that in view of [5, Remark 6], we have to guarantee that $b \leq c_0 + \frac{h_{\min}}{h_{\max}} 2\pi\sqrt{\pi^4 + c_0}$ and $b \leq c_0 + \frac{h_{\min}}{h_{\max}} 2\pi\sqrt{\pi^4 + b r_{\min}}$ at the same time. The bound from the first inequality is denoted by b_{M1}, and the largest value which satisfies the latter inequality is denoted by b_{M2} (see (8), (9)). □

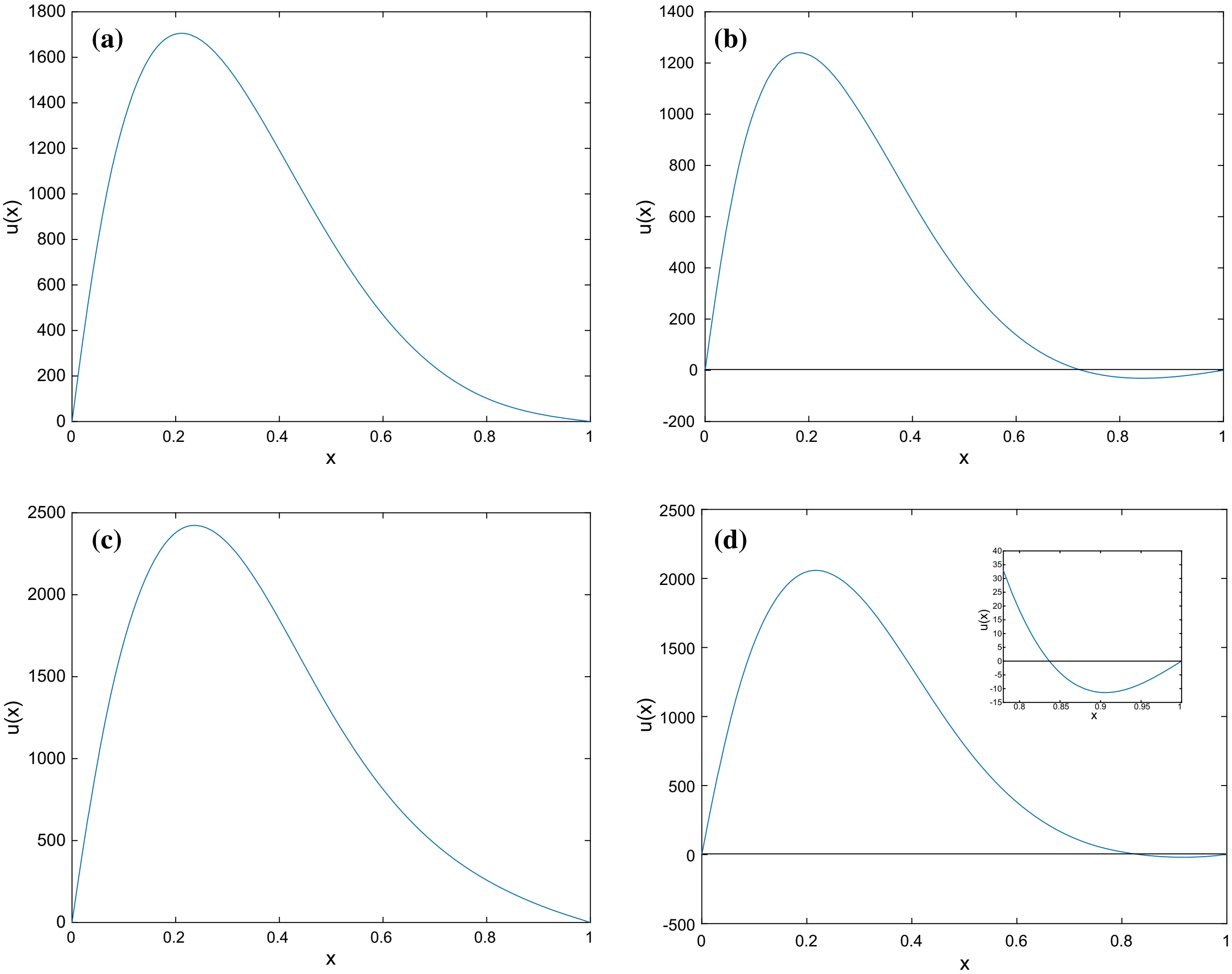

Fig. 1. Stationary solutions of (10) for the constant stiffness $b = 800$ (a) and $b = 1500$ (b), and also for the non-constant stiffness $b = 1500\,r(x)$ (c) and $b = 2500\,r(x)$ (d), where $r(x) = \cos^4(2\pi x)$. In all cases, $h(x)$ is a positive, piecewise constant function. In particular, $h(x) = 10^8$ for $x \in [0, 0.05]$ and $h(x) = 10^4$ for $x \in (0.05, 1]$. Note that the solution $u(x)$ in (b) and (d) changes sign (see also the zoomed picture in (d) corresponding to the interval $(0.8, 1)$ on the x-axis)

Remark 9. (i) Relation (8) provides the estimate of the principal weighted eigenvalue $\lambda_{0,0}$. Obviously, $\lambda_{0,0} = \pi^4$ for $r(x) \equiv 1$, and $\lambda_{0,0}$ approaches infinity when $\int_0^1 r(x)\,\mathrm{d}x$ tends to zero.

(ii) In fact, [17, Corollary 2.1] ensures the existence of a unique "universal" bound $\lambda_U \geq c_0$ such that u_b is strictly positive for any $b \in (-\lambda_{0,0}, \lambda_U]$ and any right-hand side $h(x) \geq 0$, $h \not\equiv 0$. For a particular given h and r, the positivity interval can be enlarged up to the value λ_M estimated as in (8), (9).

(iii) For the constant weight $r(x) \equiv 1$, we have $\lambda_{0,0} = \pi^4$ and $\lambda_U = c_0$. That is, for an arbitrary $h(x) \geq 0$, u_b is strictly positive whenever $-\pi^4 < b \leq c_0$. For $b > c_0$, we can find $h \geq 0$ such that the solution of (10) changes sign (for illustration, see Fig. 1a, b, and cf. [16]). As Lemma 8 shows, the positivity interval for b is stretched by introducing a non-constant weight $r(x)$ (see Fig. 1c, d). However, the non-constant weight results in a slightly larger amplitude of u_b (compare Fig. 1b and c).

(iv) For the constant weight $r(x) \equiv 1$ and constant right-hand side $h(x) \equiv 1$, u_b is strictly positive even for any $b > -\pi^4$ (see [13]). This means that the estimates b_{M1}, b_{M2} are not optimal. Unfortunately, for non-constant weights and constant right-hand sides, no extension is known.

5. Bifurcation results

Let $h \not\equiv 0$ be a nonnegative time-independent right-hand side and u_b a strictly positive stationary solution of (1) (or (10), respectively). Let us put $u := u_b + w$ with $w \in H$. Using the equality $u^+ = u + u^-$, the equation (7) can be written in the equivalent form

$$L(u_b + w) + b(u_b + w) + b(u_b + w)^- = g.$$

Moreover, since u_b is positive, we have $Lu_b + bu_b = g$ and we end up with

$$Lw + bw + b(u_b + w)^- = 0. \tag{13}$$

Applying L^{-1} on both sides of (13) we obtain

$$w + bL^{-1}w + bL^{-1}(u_b + w)^- = 0. \tag{14}$$

Let us denote $E := (-\lambda_{0,0}, \lambda_M) \times H$. Using completely the same arguments as in [4, Lemma 2.3, Lemma 2.4], we can prove the following statement.

Lemma 10. *The operator $N : E \to H$ defined by $N(b, w) := bL^{-1}(u_b + w)^-$ is compact. Moreover, given any compact subinterval J of $(-\lambda_{0,0}, \lambda_M)$, the limit*

$$\lim_{\|w\|\to 0} \frac{N(b, w)}{\|w\|} = 0$$

is uniform with respect to $b \in J$.

Now, we are ready to formulate the main bifurcation result.

Theorem 11. *Every $b = -\lambda_{m,n} \in (-\lambda_{0,0}, \lambda_M) \cap \sigma_r(-L)$, where $\lambda_{m,n}$ has an odd multiplicity, is a point of global bifurcation of (14). That is, there exists a continuum of solutions $C_{m,n}$ in $\overline{E}$, $(-\lambda_{m,n}, 0) \in C_{m,n}$, such that at least one of the following properties holds:*

(i) *$C_{m,n}$ is not a compact set in E,*
(ii) *$C_{m,n}$ contains an odd number of points $(-\lambda, 0) \in E$, where $\lambda \neq \lambda_{m,n}$ is an eigenvalue of L of odd multiplicity.*

Moreover,

$$\mathrm{proj}_{\mathbb{R}} C_{m,n} \subset (-\lambda_{0,0}, -\lambda_p] \cup [-\lambda_q, +\infty), \tag{15}$$

where $\mathrm{proj}_{\mathbb{R}} C_{m,n} := \{b \in \mathbb{R};\ (b, w) \in C_{m,n}\}$ and λ_p, λ_q are the smallest positive and the largest negative eigenvalues of L.

In addition, for $\lambda_{m,n}$ simple, $C_{m,n}$ consists of two subcontinua $C^+_{m,n}$, $C^-_{m,n}$ bifurcating from the point $(-\lambda_{m,n}, 0)$ in the directions of the corresponding eigenfunctions $\varphi_{m,n}$, and $-\varphi_{m,n}$, respectively, such that

$$C^+_{m,n} \cap C^-_{m,n} \cap B_\varrho(-\lambda_{m,n}, 0) = \{(-\lambda_{m,n}, 0)\} \quad \text{and} \quad C^\pm_{m,n} \cap \partial B_\varrho(-\lambda_{m,n}, 0) \neq \emptyset$$

for sufficiently small $\varrho > 0$.

Proof. Due to Proposition 8 and Lemma 10, the operator equation (14) represents the classical bifurcation scheme in E, i.e.,

$$\left(I + bL^{-1} + N(b)\right) w = 0,$$

where I is the identity mapping, bL^{-1} is a linear compact operator (see Sect. 2) and $N(b)$ has the appropriate limit behavior (see Lemma 10). As a consequence of that, we may employ the Rabinowitz theorem [15] directly (see also, e.g., [3]) and the result follows. Finally, relation (15) is a consequence of Proposition 4 and Proposition 6. □

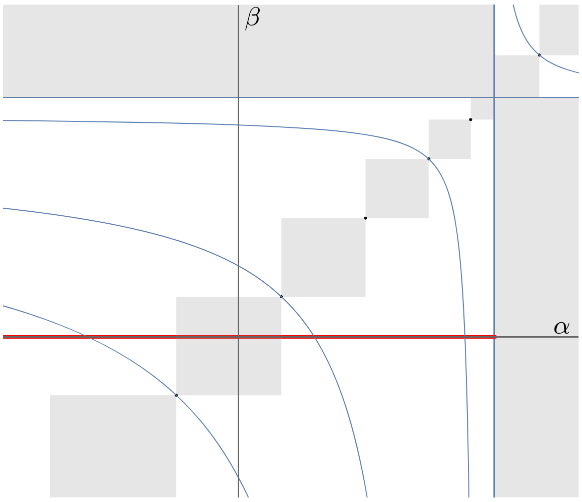

FIG. 2. Known parts of the Fučík spectrum for L with $r(x) \equiv 1$. The red line marks the set $-b < \lambda_{0,0}$. The intersections of the Fučík curves with the red line correspond to possible blowup values, whereas, in the grey inadmissible areas, none of the curves may appear

Remark 12. The set $(-\lambda_{0,0}, \lambda_M) \cap \sigma_r(-L)$ is indeed non-empty for $r \equiv 1$, $\lambda_M = c_0$, as it contains, e.g., the points $-\lambda_{0,1}, -\lambda_{0,2}, ..., -\lambda_{0,32}$. Moreover, since the interval $(-\lambda_{0,0}, c_0)$ is bounded, relatively "small" and contains zero, it can contain at most one value $-\lambda_{m,n_0}$ for any sufficiently large m. Here, either $n_0 = \lfloor (m+1)^2\pi^2 \rfloor$, or $n_0 = \lceil (m+1)^2\pi^2 \rceil$. Notice that already for $m = 7$, the distance between $(m+1)^4\pi^4 - \lfloor (m+1)^2\pi^2 \rfloor$ and $(m+1)^4\pi^4 - \lceil (m+1)^2\pi^2 \rceil$ is greater than the length of $(-\lambda_{0,0}, c_0)$. When $r \not\equiv 1$, the situation gets more complicated, since not only the eigenvalues, but also the interval bounds are shifted away from zero, and hence, the number of elements in $(-\lambda_{0,0}, \lambda_M) \cap \sigma_r(-L)$ cannot be specified.

Remark 13. Notice that if $w \in C^1(\Omega)$, then for any $\lambda_{m,n}$ simple there exists $s = s(\lambda_{m,n})$ such that $(b, w) \in C_{m,n} \cap B_s(-\lambda_{m,n}, 0)$ implies $b = -\lambda_{m,n}$ and $w = c\varphi_{m,n}$ with some $c \in \mathbb{R}$ small enough. Indeed, since u_{b_k} is strictly positive, every $(b, w) \in C_{m,n} \cap B_s(-\lambda_{m,n}, 0)$ with s small enough satisfies $(u_b + w)^- = 0$ in Ω and (14) reduces to a linear eigenvalue problem.

For the next property, we need the so-called Fučík spectrum $\Sigma(L)$ of the operator L, which is defined as

$$\Sigma(L) = \{(\alpha, \beta) \in \mathbb{R}^2 : \ Lu = \alpha u^+ - \beta u^- \text{ has a non-trivial solution } u\}.$$

Proposition 14. *If a bifurcation from infinity of (14) occurs in E, i.e., if there exists a sequence $(b_n, w_n) \subset E$ such that (14) holds with $(b, w) = (b_n, w_n)$ for any $n \in \mathbb{N}$, and $b_n \to b_0$, $\|w_n\| \to \infty$, then necessarily $(-b_0, 0) \in \Sigma(L)$.*

Proof. Dividing (14) with $b = b_n$, $w = w_n$ by $\|w_n\|$ and denoting $v_n = w_n/\|w_n\|$, we obtain

$$v_n + b_n L^{-1} v_n + b_n L^{-1} \left(\frac{u_{b_n}}{\|w_n\|} + v_n \right)^- = 0.$$

Since $u_{b_n} \in C^4([0, 1])$, we have $u_{b_n}/\|w_n\| \to 0$. Boundedness of $(v_n) \subset H$ implies $v_n \rightharpoonup v$ in H, and due to compactness of L^{-1} and continuity of $(\cdot)^-$, we obtain $v_n \to v$ in H and

$$v + b_0 L^{-1} v + b_0 L^{-1} v^- = 0$$

which is equivalent to

$$Lv + b_0 v^+ = 0, \quad \|v\| = 1, \tag{16}$$

i.e., v is a non-trivial solution of (16), and therefore $(-b_0, 0) \in \Sigma(L)$. □

Remark 15. Locally, the Fučík spectrum $\Sigma(L)$ of the beam operator L consists of a finite number of decreasing curves crossing at (λ, λ), $\lambda \in \sigma_r(L)$. These curves are symmetric with respect to the diagonal $\alpha = \beta$, the trivial part of $\Sigma(L)$ is the cross $(\alpha - \lambda_{0,0})(\beta - \lambda_{0,0}) = 0$ and no parts of $\Sigma(L)$ are located in the squares between two consecutive eigenvalues and in the area $(\alpha - \lambda_{0,0})(\beta - \lambda_{0,0}) < 0$ (see [1,9] and Fig. 2 for illustration). Hence, $(-b_0, 0) \in \Sigma(L)$ implies $b_0 > -\lambda_{0,0}$ which is in correspondence with Proposition 6.

Unfortunately, the complete description of the structure of $\Sigma(L)$ (or at least its intersection with the line $\beta = 0$) and hence the knowledge of possible blowup values of the stiffness b is not known even for the case of the constant weight $r(x) \equiv 1$. Some partial analytical and numerical results can be found in [14] (see also references therein).

Remark 16. The complete description of $\Sigma(L)$ (including multiplicities of the eigenvalues) would enable to determine the topological degree of the operator $u \mapsto u - L^{-1}(\alpha u^+ - \beta u^-)$ in between the Fučík curves and thus to confirm or disprove the existence of bifurcations from infinity as well as to answer the existence questions concerning the problem (1) with a time-dependent right-hand side h.

Acknowledgements

Both authors were supported by the Grant Agency of the Czech Republic, Grant No. 18-03253S. The second author was also supported by the project LO1506 of the Czech Ministry of Education, Youth and Sports.

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(Received: February 24, 2020; revised: September 3, 2020; accepted: October 5, 2020)

Appendix A.3

[20] G. Holubová and J. Janoušek. Extending the threshold values for inverse-positivity of a linear fourth order operator. *Positivity*, Jun 2021

Positivity
https://doi.org/10.1007/s11117-021-00841-6

Positivity

Extending the threshold values for inverse-positivity of a linear fourth order operator

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Received: 6 December 2020 / Accepted: 21 May 2021

Abstract
In this paper, we study sufficient conditions for the (strict) inverse-positivity of the linear fourth order operator corresponding to the one-dimensional beam equation with a spatially variable coefficient. We use a modification of results obtained by the operator reduction technique introduced by Schröder and show that the extrema of the coefficient can go beyond the originally derived bounds significantly.

Keywords Fourth order operator · Positive solutions · Inverse-positivity

Mathematics Subject Classification 34L40 · 34B05 · 34B30 · 34L15

1 Introduction

We study the existence of positive solutions of the linear problem

$$\begin{aligned} &u^{(\text{iv})} + c(x)u = h(x),\\ &u(0) = u(1) = u''(0) = u''(1) = 0, \end{aligned} \tag{1}$$

where $c = c(x)$, $h = h(x)$ are continuous functions on $[0, 1]$ and $h(x) \geq 0$, $h(x) \not\equiv 0$, $x \in [0, 1]$. Our work is partially motivated by problems which arise during the study of nonlinear models of suspension bridges (see, e.g., [6] for an overview of such models), however, with spatially variable stiffness $c(x)$. Specifically, when analyzing bifurcations of periodic solutions in these models, it is necessary for the linear part of the bridge equation to have a strictly positive stationary solution (cf. [3,9]). For mod-

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els which consider the homogeneous hanger[1] placement density, i.e, $c(x) \equiv c > 0$, and a positive constant load, McKenna and Walter proved an important result in [12] regarding positivity of the stationary solution. However, when considering a spatially non-homogeneous hanger stiffness $c(x)$ that better reflects the naturally discrete distribution of the suspension bridge hangers, the positivity of the corresponding stationary solution is not guaranteed in general. To improve results known so far in this field we revive more than half a century old tools of the inverse positive operator theory.

We set

$$X = \{u \in C^4[0,1] : u(0) = u(1) = u''(0) = u''(1) = 0\}$$

and consider a differential operator $L_c : X \to C[0,1]$ defined by

$$L_c u(x) := u^{(\mathrm{iv})}(x) + c(x)u(x). \tag{2}$$

Using this notation, (1) can be written as

$$L_c u = h. \tag{3}$$

Definition 1 We say that L_c is *strictly inverse positive* (SIP for short) on X if any solution $u \in X$ of (3) with an arbitrary nonnegative nontrivial right-hand side $h \in C[0,1]$ is *strictly positive*, i.e., $u > 0$ in $(0,1)$ and $u'(0) > 0$, $u'(1) < 0$.

In the 1960s, J. Schröder was among the first authors to investigate the SIP property by using the method of reduction, i.e., decomposing a general fourth order differential operator into two operators of the second order (see, e.g., [13] and [14]). He also derived bounds for either constant or non-constant coefficient c, which guarantee the SIP property of L_c. Many authors continued his work later. The constant coefficient case was covered in detail by [10] and also [16], where the same bounds were obtained, however, by using a different technique. Concerning the non-constant coefficient case, many papers, e.g. [1], or more recently, [2,4], or [5], demonstrated that the extrema of $c(x)$ can in fact cross the bounds which were earlier obtained by Schröder and his followers.

In this paper, we use the results of Schröder's operator reduction, and, by applying it more precisely for the needs of non-constant coefficients, we are able to extend the SIP interval, i.e., allowing the maximum and/or minimum of c to reach larger values. First of all, let us cite an important result, which serves as a starting point for our improvements.

Proposition 1 [14, Proposition 4.3] *If there exist functions* $\varphi, \psi \in C^4[0,1]$ *such that* $\varphi, \psi > 0$ *in* $(0,1)$, *with*

$$\varphi(1) = \psi(0) = 0, \quad \varphi''(1) \ge 0 \ \ \textit{and} \ \ \psi''(0) \ge 0, \tag{4}$$

$$\varphi'(1) < 0, \quad \psi'(0) > 0, \tag{5}$$

[1] In the pioneering paper [11] and following literature, the authors use also the term "cable-stays".

$$\varphi^{(\mathrm{iv})} + c\varphi \leq 0, \quad \psi^{(\mathrm{iv})} + c\psi \leq 0 \ \ in\ [0, 1] \tag{6}$$

and

$$p := \varphi\psi' - \varphi'\psi \geq 0 \tag{7}$$

together with a function $z \in X$ such that $z \geq 0$ and $L_c z \geq 0$, $L_c z \not\equiv 0$ then L_c is strictly inverse-positive.[2]

The functions ψ, φ chosen by Schröder are positive solutions of problems

$$\begin{aligned} &\psi^{(\mathrm{iv})} + k\psi = 0, \\ &\psi(0) = \psi(1) = \psi''(0) = 0, \quad \psi''(1) = -1, \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\varphi^{(\mathrm{iv})} + k\varphi = 0, \\ &\varphi(0) = \varphi(1) = \varphi''(1) = 0, \quad \varphi''(0) = -1, \end{aligned} \tag{9}$$

with constant coefficient k. Combined with $z = \sin \pi x$, Proposition 1 implies that L_c is strictly inverse-positive for $-\pi^4 < c(x) < c_0$ (i.e., the original Schröder's result, see [14, Proposition 4.4]). In fact, it is possible to show that the upper inequality need not be strict, as Schröder did also in [14] by more refined, local usage of the method of reduction.

The threshold values have different meanings. The lower bound $-\pi^4$ is the opposite value to the first eigenvalue of L_0 and the upper bound c_0 is given by $c_0 = 4\kappa^4$ with κ being the smallest positive solution of $\tan \kappa = \tanh \kappa$. Moreover, c_0 coincides with the value λ for which the following (3,1) and (1,3) conjugate problems

$$\begin{aligned} &u^{(\mathrm{iv})} + \lambda u = 0, \\ &u(0) = u'(0) = u''(0) = 0, \quad u(1) = 0, \end{aligned} \tag{10}$$

and

$$\begin{aligned} &u^{(\mathrm{iv})} + \lambda u = 0, \\ &u(0) = 0, \quad u(1) = u'(1) = u''(1) = 0, \end{aligned} \tag{11}$$

possess positive solutions. See [10], [14,16] for the technical (positivity of the Green function associated with (1)) and also physical (the behaviour of a bending beam with the considered boundary conditions) arguments.

The weakness of this approach is the usage of "classical" eigenvalues for determining these bounds for non-constant coefficients. That is why we update Schröder's procedure and treat the threshold values in a more precise way, using weighted eigenvalues where possible. The main goal is to show that both of the original bounds, $-\pi^4$

[2] Let us note that [14] uses different notation for order relations.

and c_0, can be crossed if the coefficient c is variable. We treat constant sign coeffients (i.e., positive or negative definite ones) as well as coefficients changing sign or vanishing on a set of non-zero measure (i.e., the indefinite and semidefinite ones). For more details, see Theorems 1, 2 and Corollary 1. For simplicity, we split checking the assumptions of Proposition 1 into several lemmas in Sect. 3. The improvement rate of obtained results is demonstrated in a series of computations in Sect. 5.

2 Preliminaries

Throughout the text, we consider $c = c(x)$, $h = h(x)$, $r = r(x)$ and $q = q(x)$ to be continuous functions on $[0, 1]$. Moreover, we assume

1. $h \geq 0, h \not\equiv 0$,
2. $q \geq 0$,
3. $0 \leq r \leq 1, r \not\equiv 0$,

for $x \in [0, 1]$. That is, both h and r can vanish on subintervals of $[0, 1]$, but they are positive on a set of non-zero measure.

We consider a differential operator $L_q : X \to C[0, 1]$ in the sense of (2), and, for the sake of brevity, if $q \equiv 0$, we write L instead of L_0.

As usual, we say that λ is a *(weighted) eigenvalue* of L_q with the weight function r if there exists a nontrivial solution $u \in X$ of

$$L_q u = \lambda r u.$$

We speak about the *principal eigenvalue*, if (at least one) corresponding eigenfunction is positive in $(0, 1)$. Due to non-negativity of q and symmetry of L_q (with respect to the scalar product in the space $W^{2,2}(0, 1) \cap W_0^{1,2}(0, 1)$, cf. [5] and Sect. 4), all its weighted eigenvalues are real and positive, and for the first one, the following statement holds.

Lemma 1 *The first (weighted) eigenvalue of L_q with the (semidefinite) weight r — denoted by $\lambda_{q,r}$ — satisfies*

$$\lambda_{q,r} \geq \lambda_{0,r} \geq \lambda_{0,1} = \pi^4$$

with equalities only for $r \equiv 1$ and $q \equiv 0$.

Proof The first inequality $\lambda_{q,r} \geq \lambda_{0,r}$ follows directly from the fact that

$$\lambda_{q,r} = \inf \left\{ \int_0^1 ((u'')^2 + qu^2);\ \int_0^1 ru^2 = 1,\ u \in W^{2,2}(0, 1) \cap W_0^{1,2}(0, 1) \right\}.$$

For any semidefinite weight r, the first eigenvalue $\lambda_{0,r}$ of L is simple and the principal one with the positive eigenfunction $u \in X$ (cf., e.g., [15]). For $r \equiv 1$, we have $\lambda_{0,1} = \pi^4$ and $u(x) = \sin \pi x$. Multiplying $Lu = u^{(\text{iv})} = \lambda_{0,r} r u$ by $\sin \pi x$ and integrating over (0, 1), we get

$$\lambda_{0,r} = \pi^4 \frac{\int_0^1 u \sin \pi x}{\int_0^1 r u \sin \pi x} \geq \pi^4 \tag{12}$$

with the equality only for $r \equiv 1$. □

For other spectral properties of L_q (or L, respectively), see [7,8,15]. For more information concerning dependence of $\lambda_{0,r}$ on the weight function r, see Sect. 5.

As a consequence we obtain that the problem $L_q u + kru = h$ is uniquely solvable for any $k > -\lambda_{q,r}$.

To deal with the upper bound of the SIP interval, we need to work with operators corresponding to auxiliary (3,1) and (1,3) conjugate boundary value problems (see, e.g., [2] and [18].) Let us put

$$X^{3,1} = \{u \in C^4[0,1] : u(0) = u'(0) = u''(0) = u(1) = 0\},$$
$$X^{1,3} = \{u \in C^4[0,1] : u(0) = u(1) = u'(1) = u''(1) = 0\},$$

and consider $L^{3,1} : X^{3,1} \to C[0,1]$ and $L^{1,3} : X^{1,3} \to C[0,1]$ defined by

$$L^{3,1}u = L^{1,3}u = -u^{(\mathrm{iv})}.$$

The following statement concerns the principal eigenvalues of $L^{3,1}$ and $L^{1,3}$. Let us denote these by Λ_r and Λ'_r, respectively.

Lemma 2 *Both the operators $L^{3,1}$, $L^{1,3}$ have the same principal eigenvalue with respect to the weight function r, that is, $\Lambda_r = \Lambda'_r$, which is positive and it is the smallest eigenvalue in absolute value. Moreover, it satisfies*

$$\Lambda_r \geq \Lambda_1 = c_0$$

with the equality only for $r \equiv 1$.

Proof The existence of the principal eigenvalue Λ_r of $L^{3,1}$ and the corresponding positive eigenfunction $u \in X^{3,1}$ is proved in [17], together with the property that Λ_r is the inverse of the spectral radius of $L^{3,1}$, i.e., it is positive and the smallest eigenvalue in absolute value.

Similarly, we obtain Λ'_r as the principal eigenvalue of $L^{1,3}$ with the positive eigenfunction $v \in X^{1,3}$. Multiplying the equality $L^{3,1}u = -u^{(\mathrm{iv})} = \Lambda_r ru$ by v and integrating over (0, 1) results in

$$(\Lambda'_r - \Lambda_r) \int_0^1 ruv = 0$$

and hence Λ'_r and Λ_r coincide.

Now, let $w \in X^{1,3}$ denote the positive eigenfunction of $L^{1,3}$ corresponding to Λ_1 with $r(x) \equiv 1$, i.e., $L^{1,3}w = \Lambda_1 w$. Notice that this problem coincides with (11) and $\Lambda_1 = c_0$. Multiplying $L^{3,1}u = \Lambda_r r u$ by w, we obtain the estimate

$$\Lambda_r = c_0 \frac{\int_0^1 uw}{\int_0^1 ruw} \geq c_0 \tag{13}$$

with the equality only for $r \equiv 1$. □

For more information concerning dependence of Λ_r on the weight function r, see Sect. 5.

3 Positive semidefinite case

To investigate the SIP property of L_c, we start with the positive semidefinite case, i.e., through this section, we consider

$$c(x) = kr(x) \geq 0 \quad \text{with } k \geq 0 \text{ and } 0 \leq r \leq 1.$$

Thus, if r attains the value 1, the parameter k coincides with the maximum value of c on [0, 1].

Now, let φ, ψ be the solutions of the boundary value problems

$$\begin{aligned} &\varphi^{(\mathrm{iv})} + kr\varphi = 0, \\ &\varphi(0) = \varphi(1) = \varphi''(1) = 0, \quad \varphi''(0) = -1, \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\psi^{(\mathrm{iv})} + kr\psi = 0, \\ &\psi(0) = \psi(1) = \psi''(0) = 0, \quad \psi''(1) = -1, \end{aligned} \tag{15}$$

respectively. Notice that for $k \geq 0$ both problems (14), (15) are uniquely solvable. Indeed, putting $\psi = \psi_h + h$ with $h(x) = \frac{1}{6}(x - x^3)$, (15) is equivalent to $L\psi_h + kr\psi_h = -krh$ which has a unique solution for any $k \geq 0$. (In fact, we have the unique solvability for $k > -\lambda_{0,r}$.) Similarly for (14). Hence, both problems (14), (15) can be reformulated as initial value problems with unique solutions φ and ψ that depend continuously on the parameter k.

Lemma 3 *Let Λ_r be the principal eigenvalue of $L^{3,1}$ and $L^{1,3}$, respectively, with the (semidefinite) weight r. Then for any $k \in [0, \Lambda_r]$, both the solutions of (14), (15) satisfy $\varphi,\ \psi > 0$ in $(0, 1)$. Moreover, $\psi'(0) > 0$, $\varphi'(1) < 0$ for $k \in [0, \Lambda_r)$, and $\psi'(0) = \varphi'(1) = 0$ for $k = \Lambda_r$.*

Proof For $k = 0$, we have trivially $\psi = h = \frac{1}{6}(x - x^3)$, which is strictly positive in $(0, 1)$. Since ψ depends continuously on k, we can assume that there exists $k_0 > 0$

and $x_0 \in [0, 1]$ such that ψ is strictly positive for any $k \in [0, k_0)$ and ψ solving (15) with $k = k_0$ satisfies $\psi(x_0) = \psi'(x_0) = 0$. Multiplying (15) by ψ and integrating by parts over $(0, x_0)$, we obtain

$$\int_0^{x_0} (\psi'')^2 + k_0 \int_0^{x_0} r\psi^2 = 0.$$

Hence, $x_0 = 0$ and ψ coincides with a positive eigenfunction of $L^{3,1}$ corresponding to $\lambda = k_0 = \Lambda_r$. The analogous result holds for the solution φ of (14). □

Lemma 4 *Let $k \in [0, \Lambda_r]$ be arbitrary and φ, ψ be the positive solutions of (14), (15), respectively. Then $p := \varphi\psi' - \varphi'\psi$ is positive in $(0, 1)$ as well, and $p(0) = p'(0) = p(1) = p'(1) = 0$.*

Proof The boundary values of φ, ψ imply directly $p(0) = p(1) = 0$, and since

$$p' = \varphi\psi'' - \varphi''\psi,$$

we have also $p'(0) = p'(1) = 0$. Further, multiplying (15) by φ and integrating over the interval $(0, x)$, $x \in (0, 1]$, we obtain the identity

$$\varphi'\psi'' - \varphi''\psi' = \psi'''\varphi - \psi\varphi''' + \psi'(0). \tag{16}$$

Hence,

$$p'' = \varphi'\psi'' + \varphi\psi''' - \varphi'''\psi - \varphi''\psi' = 2(\psi'''\varphi - \psi\varphi''') + \psi'(0). \tag{17}$$

Thus, $p''(0) = p''(1) = \psi'(0)$. For $k \in [0, \Lambda_r)$, $\psi'(0) > 0$ and hence p is positive close to the end points of $[0, 1]$.

For $k = \Lambda_r$ being the principal eigenvalue of $L^{3,1}$ and $L^{1,3}$, $\psi'(0) = 0$. Differentiating (17) we obtain

$$p''' = 2(\psi'''\varphi' - \psi'\varphi''')$$

and $p'''(0) = \psi'''(0)\varphi'(0) > 0$, $p'''(1) = -\psi'(1)\varphi'''(1) < 0$. (Notice that signs of $\psi', \psi''', \varphi', \varphi'''$ in the end points follow from the positivity of ψ, φ and the boundary conditions.) Hence, p is positive close to the end points of $[0, 1]$ also in the case $k = \Lambda_r$.

For $k = 0$, we have trivially $\varphi = \frac{1}{6}x(x-1)(x-2)$, $\psi = \frac{1}{6}(x - x^3)$ and hence $p = \frac{1}{12}x^2(x-1)^2 > 0$ in $(0, 1)$. To prove that $p > 0$ in $(0, 1)$ for any $k \in [0, \Lambda_r]$, we argue via contradiction. Let us assume that p changes sign for some $k \in (0, \Lambda_r]$. Since φ, ψ depend continuously on k, there must exist values $k_0 \in (0, \Lambda_r]$ and $x_0 \in (0, 1)$ such that $p(x_0) = p'(x_0) = 0$ and $p''(x_0) \geq 0$.

Assumption $p(x_0) = 0$ implies $\psi'(x_0) = \frac{\psi(x_0)}{\varphi(x_0)}\varphi'(x_0)$, similarly, $p'(x_0) = 0$ implies $\psi''(x_0) = \frac{\psi(x_0)}{\varphi(x_0)}\varphi''(x_0)$. Hence, $\varphi'\psi'' - \varphi''\psi'|_{x=x_0} = 0$, the identity (16) results in $\psi'''\varphi - \psi\varphi'''|_{x=x_0} = -\psi'(0)$ and

$$p''(x_0) = -\psi'(0).$$

For $k \in [0, \Lambda_r)$ this means $p''(x_0) < 0$, a contradiction. For $k = \Lambda_r$, we have $p''(x_0) = 0$. But (17) implies $\psi'''(x_0) = \frac{\psi(x_0)}{\varphi(x_0)}\varphi'''(x_0)$, and since ψ, φ solve the same equation, $\psi \equiv \frac{\psi(x_0)}{\varphi(x_0)}\varphi$ for all $x \in [0, 1]$. But this is a contradiction with $\psi''(0) = 0 \neq -1 = \varphi''(0)$. □

Now, we are ready to employ Proposition 1.

Theorem 1 *Let Λ_r be the principal eigenvalue of $L^{3,1}$ (and $L^{1,3}$, respectively) with the weight function r. Then L_c with $c(x) = k\,r(x)$ is strictly inverse-positive whenever $0 \leq k < \Lambda_r$.*

Proof Obviously, the functions φ, ψ defined as unique solutions of (14), (15), respectively, satisfy the assumptions (4) and (6) of Proposition 1. Moreover, due to Lemma 3 and Lemma 4, the assumptions (5), (7) are satisfied as well for any $k \in [0, \Lambda_r)$. Similarly as in [14], we can take $z(x) = \sin \pi x$ as the function $z \in X$ satisfying $z \geq 0$, $L_c z \geq 0$, $L_c z \not\equiv 0$. □

4 Negative semidefinite and indefinite cases

Now, we can extend our considerations to the cases of non-positive coefficients c and to coefficients c changing sign in $[0, 1]$. That is, we consider

$$c(x) = kr^+(x) - lr^-(x)$$

with $k, l \geq 0$ and $r^\pm$ being the weight functions, i.e., $0 \leq r^\pm \leq 1$ and $r^\pm \not\equiv 0$. Notice that if $r^+r^- \equiv 0$, we have $kr^+ = c^+$ (the positive part of c) and $lr^- = c^-$ (the negative part of c). Using this notation, (3) is equivalent to

$$L_{kr^+}u = lr^-u + h. \tag{18}$$

Theorem 2 *Let $0 \leq k < \Lambda_{r^+}$. Then L_c with $c = kr^+ - lr^-$ is strictly inverse positive whenever $0 \leq l < \lambda_{kr^+,r^-}$.*

Proof Obviously, (18) has a unique solution for any h whenever $l < \lambda_{kr^+,r^-}$ (cf. Lemma 1). In order to show that λ_{kr^+,r^-} is also the threshold value for positivity of the solution, we use successive iterations. Let us define

$$L_{kr^+}u_{n+1} = lr^-u_n + h, \;\text{ with } u_0 \equiv 0 \tag{19}$$

(cf. [5]). Note that the sequence (u_n) is monotone increasing. Indeed, the assumption $0 \leq k < \Lambda_{r^+}$ implies that L_{kr^+} is SIP, and $L_{kr^+}(u_{n+1} - u_n) = lr^-(u_n - u_{n-1})$, i.e., for any $n \in \mathbb{N}$, $u_n > u_{n-1}$ implies $u_{n+1} > u_n$ and $u_1 > u_0 \equiv 0$.

To prove the convergence of (u_n), we have to pass to weak formulation. Let us denote $H := W^{2,2}(0,1) \cap W_0^{1,2}(0,1)$ and recast (18) as

$$\int_0^1 u''v'' + \int_0^1 kr^+uv = \int_0^1 lr^-uv + \int_0^1 hv, \tag{20}$$

for $u \in H$ and any $v \in H$. Since kr^+ is nonnegative, the left-hand side of (20) has the property of the scalar product in H. By introducing the operator $S_{r^-} : H \to H$, $(S_{r^-}u, v) = \int_0^1 r^-uv$, and $h^* \in H$, $(h^*, v) = \int_0^1 hv$, we can write (20) as

$$u = lS_{r^-}u + h^*.$$

Similarly, (19) can be written as

$$u_{n+1} = lS_{r^-}u_n + h^*, \; u_0 = 0. \tag{21}$$

According to [5], if $l < \|S_{r^-}\|^{-1}$, the sequence $(u_n) \subset H$ is bounded. Its convergence to a strictly positive weak solution $u \in H$ is ensured by its monotonicity and the compactness of S_{r^-}. Standard regularity argument for ODEs guarantees that u is also a strictly positive solution of (18) in X. For further details, see the proof of Theorem 4 in [5].

Since S_{r^-} is a normal bounded operator, we have $\|S_{r^-}\| = \mu$ with μ being the largest eigenvalue of S_{r^-}. That is $S_{r^-}u = \mu u$, or equivalently,

$$\int_0^1 r^-uv = \mu\left(\int_0^1 u''v'' + \int_0^1 kr^+uv\right)$$

for any $v \in H$. Thus, $1/\mu$ coincides with the minimal eigenvalue of L_{kr^+} with the weight r^-, i.e., $\|S_{r^-}\|^{-1} = \lambda_{kr^+,r^-}$. □

Using Theorems 1 and 2, the SIP criteria for L_c with generally indefinite coefficient c can be also summarized as follows.

Corollary 1 *The operator L_c is strictly inverse positive for any $c \in C([0,1])$ satisfying*

$$-\lambda_{c^+,r^-} < c(x) < \Lambda_{r^+},$$

where

- *$c^\pm(x)$ are the positive and negative parts of $c(x)$,*
- *$r^\pm(x)$ are their "profiles", i.e., $r^\pm(x) = c^\pm(x)/\max_{[0,1]}(\pm c(x))$ (or $r^\pm \equiv 0$, if $\max_{[0,1]}(\pm c(x)) = 0$),*
- *λ_{c^+,r^-} is the first eigenvalue of L_{c^+} with respect to the weight r^-,*
- *Λ_{r^+} is the principal eigenvalue of $L^{3,1}$ (and $L^{1,3}$) with respect to the weight r^+.*

5 Eigenvalue estimates

Dependence of the principal eigenvalue on the semidefinite weight function r can be partially quantified using estimates introduced in [17,18] and [19], which incorporate the Green function corresponding to the given linear operator.

Estimates of Λ_r. Let us first consider the operator $L^{3,1}$ and its principal eigenvalue Λ_r. Let $G_{3,1}(x, y)$ denote the corresponding Green function, i.e.,

$$L^{3,1}u = h \quad \Leftrightarrow \quad u(x) = \int_0^1 G_{3,1}(x, y)h(y)\mathrm{d}y.$$

We have (cf. [17, Theorem 3.1.] or [19])

$$G_{3,1}(x, y) = \begin{cases} \frac{x^3(1-y)^3-(x-y)^3}{6} & x \geq y, \\ \frac{x^3(1-y)^3}{6} & x < y, \end{cases} \tag{22}$$

with $0 \leq x, y \leq 1$. The following lemma is a direct consequence of results in [18] (or [19], respectively).

Lemma 5 (Webb and Lan [18]) *Let $G_{3,1}$ be the Green function of $L^{3,1}$ given by (22). Then $m \leq \Lambda_r \leq M$, where*

$$m = \left(\sup_{0\leq x\leq 1} \int_0^1 G_{3,1}(x, y)r(y)\mathrm{d}y \right)^{-1},$$

$$M = \inf_{0\leq a<b\leq 1} \left(\inf_{a\leq x\leq b} \int_a^b G_{3,1}(x, y)r(y)\mathrm{d}y \right)^{-1}.$$

Remark 1 For simplicity, we present Lemma 5 in a specific form for $L^{3,1}$, but it can be formulated for a more general class of operators with appropriate integral kernels, i.e., analogue statements hold also for $L^{1,3}$ and L itself (see [18] for details).

Notice that the lower bound m suggests how far beyond c_0 the SIP property holds when the weight function is not constant and we are not able to determine the exact value of Λ_r. For illustration, we use four different non-constant weight functions. The first one, $r_{\mathrm{SB}}(x) = \cos^6(6\pi x)$ should resemble hanger placement density of a suspension bridge. The other ones should represent various types of positive semidefinite weights. These are the "hill" function

$$r_{\mathrm{H}}(x) = \begin{cases} -16(x-0.5)^2 + 1 & \text{for } x \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ 0 & \text{otherwise} \end{cases}$$

and the pair of "half-parabolas"

$$r_{\mathrm{RP}}(x) = \begin{cases} 4(x-0.5)^2 & \text{for } x \in \left[\frac{1}{2}, 1\right], \\ 0 & \text{otherwise,} \end{cases} \qquad r_{\mathrm{LP}}(x) = r_{\mathrm{RP}}(1-x).$$

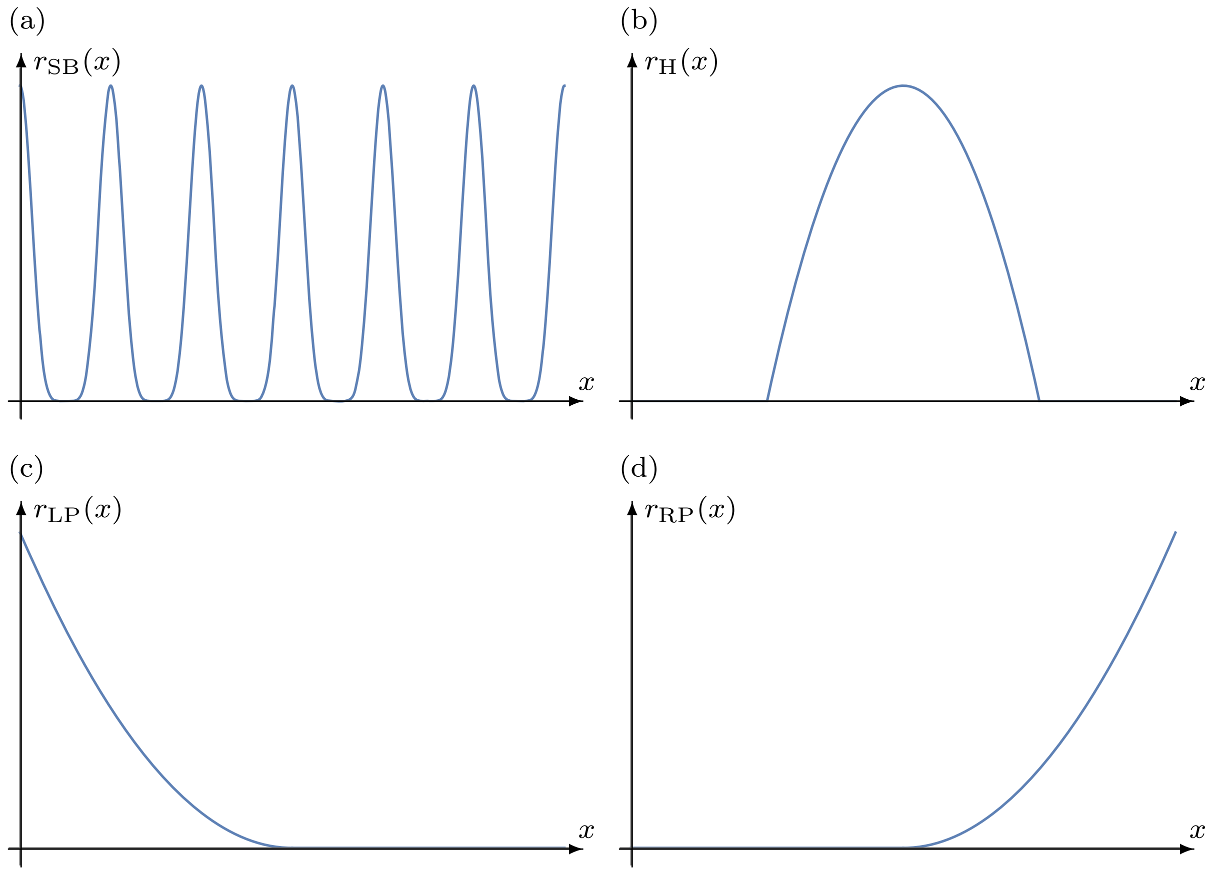

Fig. 1 Overview of used weight functions. In the picture (**a**), we show the function $r_{SB}(x) = \cos^6(6\pi x)$, which should illustrate the placement of hangers of a suspension bridge. In pictures (**b–d**), we present examples of semidefinite weights

See Fig. 1 for illustration. The following table reveals the corresponding values m and M computed in *Mathematica*.

Weight	m	M
1	227.56	2783.13
$r_{SB}(x)$	741.79	8881.62
$r_H(x)$	502.53	4415.23
$r_{LP}(x)$	1109.19	114,749.00
$r_{RP}(x)$	11,590.90	29,153.50

Notice that for $r \equiv 1$, the value m is quite far from the real eigenvalue $\Lambda_1 = c_0 \approx 950.884$. This gives us hope that for non-constant weights r, the real values of Λ_r could be much higher than their lower bounds as well. On the other hand, we can see that m for r_{LP} and especially for r_{RP} cross the value c_0 significantly.

Another improvement was suggested in [19] by an iteration technique. Let us consider functions θ_0, σ_0 being a priori bounds for the eigenfunction u corresponding to Λ_r, i.e., $\sigma_0 \leq u/\,\|u\|_{C[0,1]} \leq \theta_0$. Then we can define sequences (σ_n) and (θ_n) by

$$\theta_{n+1}(x) = \int_0^1 G_{3,1}(x, y) r(y) \theta_n(y) \mathrm{d}y, \quad \sigma_{n+1}(x) = \int_0^1 G_{3,1}(x, y) r(y) \sigma_n(y) \mathrm{d}y,$$

and values

$$m_n := \left(\sup_{0\le x\le 1} \theta_n(x)\right)^{-\frac{1}{n}} \quad \text{and} \quad M_n := \left(\sup_{0\le x\le 1} \sigma_n(x)\right)^{-\frac{1}{n}}.$$

The next assertion yields improved estimates for Λ_r.

Lemma 6 (Yang [19]) *Let θ_n, σ_n, m_n and M_n be defined as above. Then for each $n \in \mathbb{N}$, we have $m_n \le \Lambda_r \le M_n$.*

Obviously, we can choose $\theta_0 \equiv 1$ and then m_1 coincides with m given by Lemma 5. Finer estimates for both θ_0 and σ_0 are introduced in [19].

The following table illustrates the significant improvement of lower bounds of Λ_r given by iterations m_n for the same weight functions as considered above. For illustration, we show the first value m_n such that m_n is significantly larger than c_0.

Weight	n	m_n
r_{SB}	4	1950.29
r_H	4	1050.31
r_{LP}	3	6339.80
r_{RP}	1	11,590.90

Remark 2 In fact, we have $\Lambda_r > 11{,}590.90$ for both r_{LP} and r_{RP}. Indeed, since $r_{LP}(x) = r_{RP}(1-x)$ and $G_{3,1}(x, y) = G_{1,3}(1-x, 1-y)$, then also

$$\int_0^1 G_{3,1}(x, y) r_{RP}(y)\, \mathrm{d}y = \int_0^1 G_{1,3}(x, y) r_{LP}(y)\, \mathrm{d}y,$$

i.e, m_n for r_{RP} and $L^{3,1}$ are the same as m_n for r_{LP} and $L^{1,3}$. Since both conjugate problems for the same weight have the same principal eigenvalue (see Lemma 2), it is possible to compute lower bounds m_n of Λ_r for $L^{3,1}$ with both r_{RP} and r_{LP} and choose the better estimate. Similarly, for any assymetrical weight r, we may compute the m_n-bounds for both $r(x)$ and $r(1-x)$ and use the better ones.

Estimates of $\lambda_{0,r}$. Let us now consider the symmetric operator L on X and its eigenvalue $\lambda_{0,r}$. The corresponding Green function takes the form (see, e.g., [10], Section 6)

$$G_{\mathrm{sym}}(x, y) = \begin{cases} \frac{1}{6} x(1-y)(1-x^2-(1-y)^2) & \text{for } x \le y, \\ \frac{1}{6} y(1-x)(1-y^2-(1-x)^2) & \text{otherwise} \end{cases}$$

with $0 \le x, y \le 1$. We can easily verify that G_{sym} together with any semidefinite weight function r satisfy again the assumptions (C_1)–(C_3) of [17] and/or [18]. Hence,

$\lambda_{0,r}$ is the principal (weighted) eigenvalue of L and we apply Yang's iteration technique and analogy of Lemma 6 to obtain the lower (and upper) bounds of $\lambda_{0,r}$ with respect to r.

We again consider the same illustrative weight functions and using *Mathematica*, we get the following values. Recall that for $r \equiv 1$, $\lambda_{0,1} = \pi^4 \approx 97.409$.

Weight	m_1	m_2	m_3
1	76.80	83.14	87.45
r_{SB}	248.55	267.66	295.80
r_{H}	154.37	156.70	169.96
r_{LP}	808.09	1052.14	1129.67
r_{RP}	808.09	1156.14	1220.50

Note that the first lower bound m_1 is the same for both half-parabolas. This is to be expected due to the symmetry of G_{sym} and θ_0. However, θ_1 is not symmetric and so the higher iterations of m_n do not coincide.

Remark 3 The value of $\lambda_{0,r}$ is governed by the product $ru \sin \pi x$ (see (12)) and thus the behaviour of r near the centre of [0, 1] affects it the most. Hence, functions that are "concentrated" around 0.5 (see, e.g., r_{H}) lead to smaller improvements, than, e.g., semidefinite parabolas like r_{RP}. As for the conjugate problem, Λ_r is affected in a similar way, that is, by the product of the corresponding conjugate eigenfunctions uw (see (13)). Note that the product uw, its first and second derivatives vanish at $x = 0$ (or $x = 1$) and hence the behaviour of r is also less important close to the boundary of [0, 1].

Comparison with previous results. The SIP bounds crossing the original Schröder's values $-\pi^4$ and c_0 can already be found in [4] and [5]. However, estimates therein omit any details concerning the profile of $c(x)$ and—in the case of the upper bound—depend also on the extremal values of the right-hand side, specifically on the ratio $\frac{\min h(x)}{\max h(x)}$ for $x \in [0, 1]$. Even considering the best-case scenario, i.e., a constant h, our new estimates bring a significant improvement over [4] and [5].

According to Corollary 1, the SIP property is guaranteed for $-\lambda_{c^+,r^-} < c < \Lambda_{r^+}$ with $c(x) = c^+(x) - c^-(x) = kr^+(x) - lr^-(x)$. Let us first focus on the upper bound Λ_{r^+} and consider r^+ to have the same form as the four examples of weights above (cf. Fig. 1). For a constant right-hand side, [4] provides the estimate $\Lambda_{r^+} \geq m_{\mathrm{old}} = c_0 + 2\pi^3 \approx 1012.89$ regardless of the profile of r^+. The comparison with our new estimates is in the following table.

Weight	m_{old}	m_{new}
r_{SB}	1012.89	1950.29
r_{H}		1050.31
r_{LP}		6339.80
r_{RP}		11,590.90

Concerning the lower bound, we have $\lambda_{c^+,r^-} \geq \lambda_{0,r^-}$ (cf. Lemma 1). Again, let us consider the same four examples of weights, this time standing for the profile of r^-. The estimate based on [5] reads

$$\lambda_{0,r^-} \geq m_{\text{old}} = \frac{4\pi^2}{\int_0^1 r^-(x)\,\mathrm{d}x}.$$

Again, we provide the following table for comparison with our new results.

Weight	m_{old}	m_{new}
1	39.48	87.45
r_{SB}	126.33	295.80
r_{H}	118.44	169.96
r_{LP}	236.87	1129.67
r_{RP}	236.87	1220.50

In short, the new estimates sometimes bring several times larger interval of strict inverse-positivity valid for an arbitrary continuous right-hand side.

Acknowledgements Both authors were supported by the Grant Agency of the Czech Republic, Grant No. 18-03253S. The second author was also supported by the project LO1506 of the Czech Ministry of Education, Youth and Sports.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed.

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