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Front propagation in reaction-diffusion-convection equations with combustion nonlinearity

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1 Introduction

We consider the reaction-diffusion-convection equation

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left[D(v)\left|\frac{\partial v}{\partial x}\right|^{p-2}\frac{\partial v}{\partial x}\right] + \frac{\partial H(v)}{\partial x} + g(v), \quad p > 1. \tag{1}$$

Here $D \in C^1[0,1]$ is a positive diffusion coefficient, $H \in C^2[0,1]$ is a nonlinear convective flux function and the reaction term g is a Lipschitz function satisfying $g(0) = g(1) = 0$, $g(v) = 0$ in $[0,\theta]$ and $g(v) > 0$ in $(\theta, 1)$ for some $\theta \in (0,1)$. This type of nonlinearity g is frequently found in combustion models where θ represents the ignition temperature, see, e.g., Berestycki et al. (1985).

A classical problem associated with reaction-diffusion equations is the existence of traveling wave solutions, i.e., bounded nonconstant solutions that travel without the change of shape at a constant speed. In the absence of convection, there exists a unique admissible wave speed $c = c^*$ and a corresponding wave profile connecting equlibria 0 and 1. The additional convective term might cause the disappearance of this wavefront, as shown in Malaguti and Marcelli (2003) for $p = 2$. We extend the existence and nonexistence results established therein to the more general case $p > 1$.

2 Reduction to a first order problem

The existence of traveling wave solutions $v(x,t) = u(x - ct)$ to (1), where c denotes the unknown speed of propagation, is equivalent to the solvability of the boundary value problem

$$\begin{cases} \left(D(u)|u'|^{p-2}u'\right)' + (c + h(u))\, u' + g(u) = 0, \\ u(-\infty) = 1, \; u(+\infty) = 0. \end{cases} \tag{2}$$

It can be shown that the solution $u = u(\xi)$ is nonincreasing on $\mathbb{R}$ and $u'(\xi) < 0$ for any $\xi \in \mathbb{R}$ such that $0 < u(\xi) < 1$. Thanks to this property, we can proceed similarly as in Enguiça et al. (2013) and transform (2) into a first order boundary value problem

$$\begin{cases} y'(u) = p'\left[(c + h(u))\left(y^+(u)\right)^{\frac{1}{p}} - f(u)\right], \quad u \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{3}$$

where $f(u) = D^{p'-1}(u)g(u)$, $h(u) = \frac{\mathrm{d}}{\mathrm{d}u}H(u)$ and $p' = \frac{p}{p-1}$ is the exponent conjugate. More precisely, *positive* solutions of (3) are uniquely determined by solutions of (2) and vice versa.

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3 Main results

By investigating the first order problem (3), we derive the following results, where h_m denotes the minimum of h over $[0,1]$.

3.1 Existence

Let

$$k = k(p) = \begin{cases} \frac{1}{2^{p'-1}-1} & \text{if } 1 < p < 2, \\ 1 & \text{if } p = 2, \\ p' \left(p' - 1 + \frac{1+p'(p'-1)^{\frac{1}{p'-2}}+(p'-1)^{\frac{p'}{p'-2}}}{\left(1+(p'-1)^{\frac{1}{p'-2}}\right)^{p'}} \right)^{-1} & \text{if } p > 2. \end{cases}$$

This is a continuous function satisfying $\lim_{p\to 1+} k(p) = 0$ and $\lim_{p\to +\infty} k(p) = 2$.

If

$$H(1) \le h_m + \left(k(p) \int_0^1 D^{p'-1}(u) g(u) \, \mathrm{d}u \right)^{\frac{1}{p'}} \tag{4}$$

then there exists a unique $c = c^* > -h_m$ such that the boundary value problem (2) has a unique (up to translation) solution. Moreover, if H satisfies condition (4) with $h_m = 0$, then $c^* > 0$.

We mention that for $p = 2$ our result generalizes the one from Malaguti and Marcelli (2003) by allowing equality to hold in (4).

3.2 Nonexistence

If

$$H(\theta) \ge \theta h_m + \left(p' \int_0^1 D^{p'-1}(u) g(u) \, \mathrm{d}u \right)^{\frac{1}{p'}} \tag{5}$$

then the boundary value problem (2) admits no solutions for any $c > -h_m$.

If $h_m = h(0)$ and strict inequality holds in (5), then (2) has no solution for any $c \in \mathbb{R}$.

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