Západočeská univerzita v Plzni
Fakultа aplikovaných věd

Postupné vlny v asymetricky podepřeném nosníku

Mgr. Hana Levá

Rigorózní práce k získání
akademického titulu doktor přírodních věd
Studijní program: Matematika a její aplikace

Katedra matematiky
Plzeň, 2023

University of West Bohemia
Faculty of Applied Sciences

FACULTY OF APPLIED SCIENCES
UNIVERSITY
OF WEST BOHEMIA

Travelling Waves in Asymmetrically Supported Beam

Mgr. Hana Levá

Thesis for the award of the degree
of Doctor of Natural Sciences (RNDr.)
The study program: Mathematics and its applications

Department of Mathematics
Pilsen, 2023

Declaration

I hereby declare that, to the best of my knowledge, this thesis is an original report of my own research and that any ideas, techniques or other materials from the work of others are fully acknowledged with the standard referencing practices.

In Pilsen, January 6, 2023 Hana Levá

Acknowledgement

The author would like to thank doc. Ing. Gabriela Holubová, Ph.D. for great guidance and valuable advice. Many thanks belong also to Martin and the author's family for support.

Both author and her supervisor were supported by the Grant Agency of the Czech Republic, Grant No. 22–18261S.

Abstract

This thesis focuses on the nonlinear fourth order partial differential equation which can be used as a model of an asymmetrically supported beam or a generalized model of a suspension bridge. We present an overview of existing results for related problems. Then, the variational approach, in particular Mountain Pass Theorem together with a nonzero weak convergence after suitable translation, is used to prove the existence of a travelling wave solution. We show that it exists under considerably weakened assumptions than those formerly used in literature. On the other hand, it seems that the presence of sign preserving nonlinearities results in a limitation of the possible values of the wave speed. Additionally, we present some numerical experiments in order to find one particular form of classical solutions.

Key words: fourth order nonlinear partial differential equation, travelling wave, suspension bridge model, nonlinear restoring force, asymmetrically supported beam, Mountain Pass Theorem

Abstrakt

V tomto textu se zaměříme na nelineární parciální diferenciální rovnici čtvrtého řádu, která může sloužit jako model asymetricky podepřeného nosníku či jako zobecněný model visutého mostu. Předkládáme shrnutí již existujících výsledků pro související úlohy. K důkazu existence řešení ve tvaru postupné vlny je použit variační přístup, konkrétně věta Mountain Pass Theorem společně s nenulovou slabou konvergencí po vhodné translaci. Ukazujeme, že takové řešení existuje za předpokladů značně oslabených oproti těm dosud využívaným v literatuře. Na druhou stranu se zdá, že připuštěním funkcí zachovávajících znaménko dochází k omezení možných hodnot vlnové rychlosti. Na závěr předkládáme numerické experimenty prováděné za účelem nalezení konkrétního tvaru klasického řešení.

Klíčová slova: nelineární parciální diferenciální rovnice čtvrtého řádu, postupná vlna, model visutého mostu, nelineární vratná síla, asymetricky podepřený nosník, Mountain Pass Theorem

Contents

Preface

In this text, we will deal with nonlinear problems for partial differential equations possesing travelling wave solutions. By travelling wave we mean a function $u(x,t) = U(x - ct) = U(z)$, where $x \in \mathbb{R}$, $t > 0$, constant c corresponds to the wave speed, z is called wave variable and U is a wave profile. The direction of the wave motion depends on the sign of the speed. For $c > 0$ the wave travels to the right, for $c < 0$ it moves to the left. In case that $c = 0$ it is called a stationary wave.

The topic of travelling waves is very broad. Real situations in which travelling waves occur are frequently modelled by first or second order partial differential equations. No less often they can be found as solutions of certain fourth order models and these are the problems we will focus on.

Applications of equations possessing travelling wave solutions can be found in various fields. Typically, these are biology problems such as the population models, morphology problems, pattern formation, neural signal transmission, epidermal wound healing, etc.

Travelling waves also play an important role in physics or mechanics. As an example we can mention the models of phase transitions, the models of fluid flow or even some problems in quantuum mechanics. The vibrations of long structures such as beams or suspension bridges are also a significant phenomenon. Especially in the latter case, the occurrence of solutions of this type is usually unwelcome and information for which system parameters they arise is very useful. Some models of suspension bridges and associated problems will be discussed in Chapter 1.

It is also important to note that we distinguish between localized waves and periodic solutions which contain a chain of waves.

A special case of a localized wave is the so called soliton. It is difficult to find a uniform definition in the literature (see, e.g., [30] or [3]) and it is often characterized by its properties. The most important of these is that it preserves its shape and velocity when interacting with other solitons, except for a phase shift. Solitons can be observed, for example, in shallow waters or narrow channels. We note also that the Korteweg-deVries equation (cf. [30]) was set up in order to have soliton solutions, therefore, it is one of the few mathematical models with a nonlinear partial differential equation that can be solved exactly. In our case, we will generally deal with localized waves.

Now, we define some important notions for travelling wave solutions. The standard procedure is to take a suitable transformation and move from the partial differential equation to the ordinary one. Let U_1^* and U_2^* be two different constant (stationary) solutions of the ordinary differential equation under the study. If $U(z) \to U_1^*$ for $|z| \to +\infty$, U is called a homoclinic solution. Thus, such a solution connects one stationary state with itself. On the other hand, if $U(z) \to U_1^*$ for $z \to -\infty$ and $U(z) \to U_2^*$ for $z \to +\infty$, then the solution is called heteroclinic. In this case, the solution connects two different stationary states. For suspension bridges we will mainly focus on problems with homoclinic solutions.

It is very often impossible to find an exact solution to such problems. However, they can be solved numerically. On the other hand, we can examine the existence of the travelling wave solutions. Topological or variational methods are usually used in the proofs, in particular topological degree theory, fixed point theorems, Saddle Point Theorem or Mountain Pass Theorem. We use the last mentioned in Chapter 2.

Chapter 1

Travelling Waves in Suspension Bridges

1.1 Suspension Bridges

The central topic of this part of the thesis is suspension bridges, their mathematical description and their behaviour under certain conditions. The suspension bridges can be modelled in many ways depending on what types of oscillations and behaviour we are interested in. There are two trends in literature. First one is to make the mathematical model the most realistic. In contrast, the other one is to make it very simple to be able to use analytical methods. We can mention here that there exist several PDE and ODE models, with or without damping. Specifically, we can name single beam model, string beam model, torsional vertical model, etc. For a detailed overview see the paper by P. Drábek, G. Holubová, A. Matas and P. Nečesal [13].

We will discuss in detail the following simple one dimensional nonlinear model presented by A. C. Lazer and P. J. McKenna in [19]. This single beam model is the first mentioned in [13] and it concerns the vertical oscillations of the roadbed of the suspension bridge. In fact, the other models are based on this one. The considered partial differential equation has the form

$$mu_{tt} + EIu_{xxxx} + \delta u_t + bu^+ = W(x) + \varepsilon h(x,t), \qquad (1.1)$$

where $x \in [0, L],\ t > 0$, with initial conditions

$$u(x, 0) = u_0(x) \qquad \text{and} \qquad u_t(x, 0) = u_1(x), \quad x \in [0, L] \tag{1.2}$$

and boundary conditions

$$u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = 0, \quad t > 0, \tag{1.3}$$

where $u = u(x, t)$ describes the deflection of the bridge of the length L from the unloaded state, $u^+(x, t) = \max\{u(x, t), 0\}$ is the positive part of u and $u_0 = u_0(x)$, $u_1 = u_1(x)$ are given real functions of one real variable corresponding with the initial deflection and velocity of the roadbed.

In this case, the main roadbed of the suspension bridge is modelled as one dimensional bending beam fixed at both ends and suspended by cables. Those are modelled as springs with an unilateral elastic force obeying Hooke's law, where $b > 0$ is the stiffness of the springs. Thus the restoring force is nonlinear. There is no force if the cables are compressed, however, if they are stretched, the restoring force is linearly proportional to the displacement of the roadbed.

We now discuss the physical meaning of each term in the equation (1.1). The first term mu_{tt} represents the inertial force with the parameter m indicating the constant mass. The term EIu_{xxxx} denotes the elastic force, where E is the modulus of elasticity in tension and I is the moment of the inertia of the beam cross section. The third term δu_t describes a viscous damping with the coefficient δ. As we mentioned above, the term bu^+ corresponds to the restoring force. The beam deflects by its own weight expressed by the function $W = W(x)$, which denotes the weight per unit length (the action of gravitation force). The function $h = h(x, t)$ involves the action of external forces affecting the beam, such as wind and other natural or artificial influences, and $\varepsilon > 0$.

Sometimes periodic initial conditions can be considered instead of (1.2), i.e.,

$$u(x, t + \tau) = u(x, t), \qquad x \in [0, L],\ t > 0. \tag{1.4}$$

Usually, this is the case where h is also periodic.

From now on, we will take $x \in \mathbb{R}$ and normalize the model. Suspension bridges are long and narrow enough to use a one dimensional infinite beam model. This is convenient for studying travelling wave solutions. We introduce $k = b/m$. For the simplicity, we also take $\delta = 0$ and $\varepsilon = 0$. Thus, the considered model is without damping and without external excitation. On the top of that, we suppose $EI/m = 1$ and $W(x) \equiv m$, i.e., constant loading. Thus $u \equiv 1/k$ is the constant stationary solution of the equation (1.1). Then the model can be written in the form

$$\begin{cases} u_{tt} + u_{xxxx} + ku^+ = 1, \qquad x \in \mathbb{R},\ t > 0, \\ u \to \dfrac{1}{k} \text{ for } |x| \to +\infty, \\ u' \to 0 \text{ for } |x| \to +\infty. \end{cases} \tag{1.5}$$

We have to note here that the model (1.5) is rather simple and do not describe the whole complex behavior of the bridge. However, in a certain way it suffices for modelling the vertical oscillations in reality and a lot of interesting information can be gleaned from it. If this simple model suggests a dangerous behaviour (e.g., large amplitude of the oscillations) it can be inferred that even more complex (and more realistic) model will behave analogously.

Remark 1. The equation in (1.5) can be generalized as

$$u_{tt} + u_{xxxx} + f(u) = 0, \qquad x \in \mathbb{R},\ t > 0. \tag{1.6}$$

In literature, the term with f originating from the restoring force is most often in the form mentioned above, i.e., $f(u) = ku^+ - 1$, or as the so called "smoothed" nonlinearity $f(u) = e^u - 1$ (see, e.g., [4], [9], [12] or [28]). It is convenient for numerical experiments and for engineers in practical applications despite its superlinear growth.

1.2 Overview of the results

In this part of the text we will discuss and summarize several findings from the literature. We will focus on problems with

Figure 1.1: Tacoma Narrows Bridge collapse on November 7, 1940. Retrieved from [24].

travelling wave solutions. Let us emphasize that these are not our results.

The formation of travelling waves was observed, e.g., on the Golden Gate Bridge or the Tacoma Narrows Bridge which is often remembered in this context. Only four months after its opening in 1940, strong wind caused very large oscillations and the bridge collapsed (see Figure 1.1). This has been the cause of many papers to take the mathematical description of the suspension bridges as their subject, starting with an in-depth report by O. H. Amann, T. Kármán and G. B. Woodruff [1].

However, as far as we know, the topic of travelling waves in suspension bridges was not opened until the year 1990 by P. J. McKenna and W. Walter [25]. The object of their research was the model (1.5) from which, by an appropriately chosen substitution (for details see Section 2.1.), they moved to model with ordinary differential equation in the form

$$\begin{cases} z^{(4)} + c^2 z'' + (z+1)^+ - 1 = 0, \\ z, z' \to 0 \quad \text{for } |t| \to +\infty. \end{cases} \tag{1.7}$$

For simplicity, the authors restricted themselves to one special form of solutions and solved (partly numerically) the problem (1.7). The analytical form of the travelling wave solution was found proving the existence of such a solution. We were inspired by their approach and looked for classical solutions analogously (see Section 2.3). However, such approach fails for an arbitrarily small perturbation of the nonlinearity and the entire calculation must be performed again for the problem with new nonlinear term.

McKenna and Walter also showed that the possible wave speeds can be taken from interval $(0, \sqrt{2})$. The solution becomes highly oscillatory for the wave speed going to $\sqrt{2}$. On the contrary, if it goes to zero it seems that the amplitude increases beyond all limits.

We want to remark here that the equation (1.7) is sometimes called the Swift–Hohenberg equation (see, e.g., [29]). On the other hand, if the coeficient of the second derivative is negative, the equation is usually called extended Fisher–Kolmogorov equation. In some sense, the latter is simpler to deal with.

Followed by [8] with the variational approach, the topic has been further explored. In their paper, Y. Chen and P. J. McKenna used the Mountain Pass Theorem and the concentration compactness principle to prove the existence of a travelling wave solution and also the Mountain Pass algorithm developed by Y. S. Choi and McKenna [10] to find some forms of the solution numerically. The stability and other properties were studied and surprising behavior has been observed similar to solitons – two waves emerge almost intact after the collision.

Their work was further enhanced by Chen [7] who considered the so called fast increasing nonlinearitites and gave also the variational proof of the existence result. However, he used quite restrictive assumptions on the nonlinearity, especially the analyticity.

Later on, in [20] by A. C. Lazer and P. J. McKenna the unboundedness of the wave amplitude for the wave speed converging to zero was shown. This was already suggested in [25]. After that, P. Karageorgis and J. Stalker [18] found a lower bound for the amplitude of travelling waves in suspension bridges of McKenna and Walter's type model.

Another paper by Chen and McKenna [9] is also worth mentioning. The authors perform some numerical experiments for the problem with the smoothed nonlinearity $f(u) = e^u - 1$ to find a solution and what is more important, at the same time, they summarize four questions, several of which remain open to the present. One of them is the analytical proof of the existence of more than one solution to the considered problem (with the piecewise linear or the smoothed nonlinearity). The current open questions for generalized problem are listed in the Section 2.4.

The topic of travelling waves has been addressed by a number of other people than those mentioned above. Similar problems have been studied also in [3], [5], [6], [12], [21], [26], [27], [28] (see also references therein).

Chapter 2

Travelling Waves in Asymmetrically Supported Beam

2.1 Problem formulation and preliminaries

In this part of the text, we will focus on the fourth order partial differential equation

$$u_{tt} + u_{xxxx} + \alpha u^+ - \beta u^- + g(u) = 1, \quad x \in \mathbb{R},\ t > 0, \tag{2.1}$$

where $\alpha > 0$ and $\beta \geq 0$ are constants, u^+ and u^- denote the positive and negative parts of u, i.e., $u^\pm = \max\{\pm u, 0\}$ pointwise, $u = u^+ - u^-$ and $g : \mathbb{R} \to \mathbb{R}$ with $g\,(1/\alpha) = 0$. Other properties of g will be specified later. We will study the existence of travelling wave solutions $u = u(x, t)$ of (2.1). This type of equation can be used, for example, as a model of an asymmetrically supported beam or as a generalized model of a suspension bridge.

Since (2.1) has a constant (stationary) solution $u \equiv 1/\alpha$, it is convenient to require the boundary condition

$$u(x, t) \to \frac{1}{\alpha} \ \text{ for } |x| \to +\infty. \tag{2.2}$$

We will look for the homoclinic travelling waves that connect $1/\alpha$ with itself. We also add a condition on the first space derivative of u in the form

$$u_x(x,t) \to 0 \quad \text{for } |x| \to +\infty. \tag{2.3}$$

Remark 2. The jumping (or asymmetric) nonlinearities, i.e., terms $f(u)$ with

$$\lim_{u\to+\infty} \frac{f(u)}{u} = \alpha \quad \text{and} \quad \lim_{u\to-\infty} \frac{f(u)}{u} = \beta$$

or their simplest form $f(u) = \alpha u^+ - \beta u^-$ considered in our problem, were firstly studied by Fučík [15], [16] and Dancer [11] in late seventies and appear also in many applications. They are often used in modelling objects on the interfaces of two different media, we can find them in the models of nonlinearly supported bending beams or floating beams. In particular, they can be used as the above mentioned suspension bridge models or in naval architecture where ships are modelled as vibrating beams or plates with free–end boundary condition.

Now, we suppose that the problem (2.1)–(2.3) has a travelling wave solution. Specifically, we assume it in the form

$$u(x,t) = \frac{1}{a^4} y\left(ax - ca^2 t\right)$$

where $a^4 = \alpha$ and $y \in C^4(\mathbb{R})$. Using this transformation we obtain an ordinary differential equation

$$y^{(4)} + c^2 y'' + y^+ - \xi y^- + g\left(\frac{1}{\alpha} y\right) = 1 \tag{2.4}$$

with parameters $\xi = \beta/\alpha \geq 0$, $c \in \mathbb{R} \setminus \{0\}$. Notice that $|ca|$ corresponds to the wave speed and $y \equiv 1$ is the stationary state (equilibrium). Further, it is convenient to put $z := y - 1$ and translate the equilibrium to zero. Substituting for z and denoting $\tilde{g}(z) = g\left(\frac{1}{\alpha}(z+1)\right)$, (2.4) can be written as

$$z^{(4)} + c^2 z'' + (z+1)^+ - \xi(z+1)^- + \tilde{g}(z) - 1 = 0$$

or, equivalently,

$$z^{(4)} + c^2 z'' + z + \tilde{g}(z) = 0 \quad \text{if } z \geq -1, \tag{2.5}$$

$$z^{(4)} + c^2 z'' + \xi z + \tilde{g}(z) = 1 - \xi \quad \text{if } z \leq -1. \tag{2.6}$$

Thus, solutions $z = z(t)$ of the problem

$$\begin{cases} z^{(4)} + c^2 z'' + (z+1)^+ - \xi(z+1)^- + \tilde{g}(z) - 1 = 0, \\ z, z' \to 0 \quad \text{for } |t| \to +\infty \end{cases} \tag{2.7}$$

correspond to homoclinic travelling wave solutions of the beam equation (2.1) with the conditions (2.2) and (2.3).

Remark 3. We only look for homoclinic travelling wave solutions that connect the value $1/\alpha$ with itself. For $g \equiv 0$, resp. $\tilde{g} \equiv 0$, there is no negative constant solution. However, if we take nontrivial g, the existence of other different constant stationary states depends on the properties of the particular nonlinearity.

Since we are looking for solutions of (2.7) vanishing for $|t| \to +\infty$ and due to the properties of the fundamental systems corresponding to (2.5)–(2.6) with $\tilde{g} \equiv 0$, we obtain trivially the following assertion dealing with possible values of c and the amplitude of the solution z (for details see [25] or [17]).

Lemma 4 ([25])**.** *Let $z = z(t)$ be a nontrivial solution of* (2.7) *with $\tilde{g} \equiv 0$. Then $c^2 \in (0, 2)$ and $z < -1$ for some t.*

For this reason, from now on, we consider $c^2 \in (0, 2)$. Moreover, we introduce a positive constant $C_1 = \frac{4}{4-c^4}$ and require the nonlinearity $\tilde{g}$ to satisfy the following assumptions:

(A1) $\tilde{g} \in C(\mathbb{R})$,

(A2) $\tilde{g}(0) = 0$ and $\tilde{g}'_{\pm}(0) \in \mathbb{R}$,

(A3) $\exists \delta > 0$ and $\nu \in (0, 1)$: $\tilde{g}(z)z \geq -\frac{1-\nu}{C_1} z^2$ for all $|z| \leq \delta$,

(A4) $\lim\limits_{z \to -\infty} \tilde{g}(z) > -\infty$,

(A5) $\tilde{g}$ is convex for $z < 0$ and concave for $z > 0$.

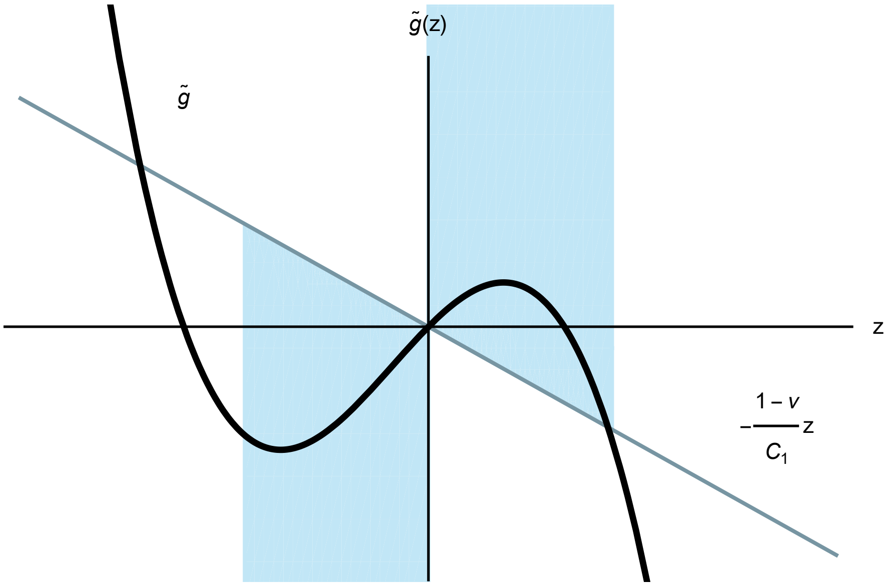

Figure 2.1: Illustration of assumption (A3) for the nonlinearity $\tilde{g}$.

Remark 5. Assumptions (A1) and (A4) imply the existence of $K \geq 0$ such that $\tilde{g}(z) \geq -K$ for all $z \leq 0$. In other words, the nonlinearity $\tilde{g}$ is bounded from below only for negative arguments.

Assumption (A3) also reads as $\tilde{g}(z) \geq -\frac{1-\nu}{C_1} z$ for all $0 \leq z \leq \delta$ and $\tilde{g}(z) \leq -\frac{1-\nu}{C_1} z$ for all $-\delta \leq z \leq 0$. We can see the graphical illustration of this property in Figure 2.1. The colored domain in the δ–neighborhood of the origin is the admissible one for $\tilde{g}$.

Moreover, due to (A5), one–sided derivatives $\tilde{g}'_{\pm}$ exist at every point and $\tilde{g}(z) \leq \tilde{g}'_{+}(0)z$ for $z > 0$, $\tilde{g}(z) \geq \tilde{g}'_{-}(0)z$ for $z < 0$. Then the assumption (A3) can be reduced to the existence of $\delta > 0$ and $\nu \in (0, 1)$ such that $\pm\tilde{g}(\pm\delta)\delta \geq -\frac{1-\nu}{C_1}\delta^2$.

Finally, (A1), (A2) and (A5) together imply that $\tilde{g}$ is Lipschitz continuous on any bounded interval.

To study the existence of solutions of (2.7), we need the corresponding weak and variational formulation. Let us consider the Hilbert space $H = W^{2,2}(\mathbb{R})$ with the standard norm

$$\|z\|_H = \left(\int_{\mathbb{R}} \left((z''(t))^2 + (z'(t))^2 + (z(t))^2 \right) \, \mathrm{d}t \right)^{1/2}.$$

Similarly, we denote $H(a,b) = W^{2,2}(a,b)$.

Definition 6 (**Weak formulation**)**.** A function $z \in H$ is called a weak solution of (2.7) if it satisfies the integral identity

$$\int_{\mathbb{R}} \left(z''(t)\varphi''(t) - c^2 z'(t)\varphi'(t) + (z(t)+1)^+\varphi(t) - \xi(z(t)+1)^-\varphi(t) \right.$$
$$\left. - \varphi(t) \right) \mathrm{d}t + \int_{\mathbb{R}} \tilde{g}(z(t))\varphi(t)\,\mathrm{d}t = 0 \qquad \forall \varphi \in H. \tag{2.8}$$

Now, we will move to the variational formulation of (2.7). Let us consider the functional

$$I(z) = \frac{1}{2} \int_{\mathbb{R}} \Big((z''(t))^2 - c^2 (z'(t))^2 + \left((z(t)+1)^+ \right)^2$$
$$+ \xi \left((z(t)+1)^- \right)^2 - 1 \Big) \, \mathrm{d}t - \int_{\mathbb{R}} z(t)\,\mathrm{d}t + \int_{\mathbb{R}} G(z(t))\,\mathrm{d}t \tag{2.9}$$

with $G(z) = \int_0^z \tilde{g}(w)\,\mathrm{d}w$. Due to the assumptions (A1), (A2), all the integrals in (2.8) and (2.9) converge and I is well defined. Moreover, $I \in C^1(H, \mathbb{R})$, $I(0) = 0$ and the critical points of I correspond to weak solutions of (2.7). Indeed, the Fréchet derivative $I'(z)\varphi$ for z, $\varphi \in H$ coincides with the left–hand side of (2.8).

Remark 7. Notice that the assumptions (A1)–(A5) for $\tilde{g}$ imply the following corresponding properties of G:

(P1) $G \in C^1(\mathbb{R})$,

(P2) $G(0) = 0$,

(P3) $G(z) \geq -\dfrac{1-\nu}{2C_1} z^2$ for all $|z| \leq \delta$,

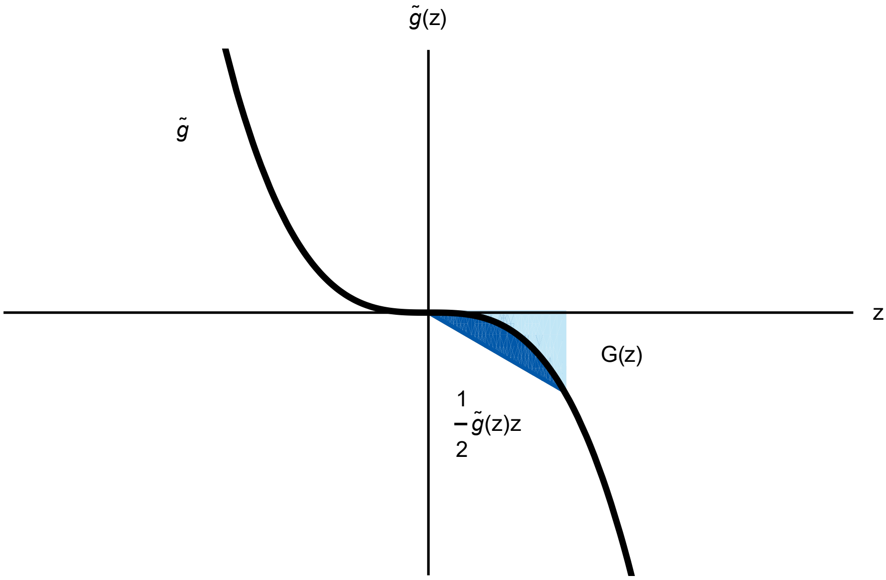

Figure 2.2: Illustration of r.

(P4) $G(z) = \int_0^z \tilde{g}(w)\,\mathrm{d}w = \int_z^0 -\tilde{g}(w)\,\mathrm{d}w \leq -Kz = K|z|$ for any $z \leq 0$,

(P5) $G(z) \geq \frac{1}{2}\tilde{g}(z)z$ for all $z \in \mathbb{R}$.

For the sake of clarity of the text below, we denote $r(z) := 2G(z) - \tilde{g}(z)z \geq 0$. We can find a graphical illustration in Figure 2.2. There is a detail of r in Figure 2.3.

Remark 8. There are several options how to equivalently write I and its Fréchet derivative. We state here two of the variants that will be useful in further text, especially in the proofs. Either, I can

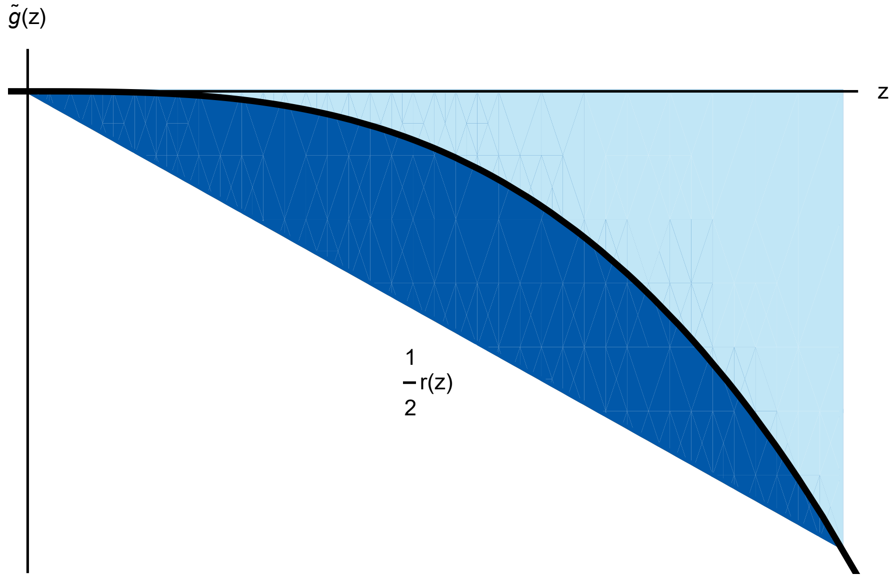

Figure 2.3: Detail of r.

be written as

$$I(z) = \frac{1}{2}\int\limits_{\mathbb{R}} \left((z'')^2 - c^2(z')^2 + z^2\right)\, \mathrm{d}t - \frac{(1-\xi)}{2}\int\limits_{z\leq -1} (z+1)^2\, \mathrm{d}t + \int\limits_{\mathbb{R}} G(z)\,\mathrm{d}t \tag{2.10}$$

with Fréchet derivative for z, $\varphi \in H$ in the form

$$I'(z)\varphi = \int\limits_{\mathbb{R}} \left(z''\varphi'' - c^2 z'\varphi' + z\varphi\right)\, \mathrm{d}t - (1-\xi)\int\limits_{z\leq -1} (z\varphi + \varphi)\, \mathrm{d}t + \int\limits_{\mathbb{R}} \tilde{g}(z)\varphi\,\mathrm{d}t, \tag{2.11}$$

or as

$$I(z) = \frac{1}{2}\int\limits_{\mathbb{R}} \left((z'')^2 - c^2(z')^2\right)\,\mathrm{d}t + \frac{1}{2}\int\limits_{z>-1} z^2\,\mathrm{d}t + \frac{\xi}{2}\int\limits_{z\le -1} z^2\,\mathrm{d}t + \frac{\xi-1}{2}\int\limits_{z\le -1} (2z+1)\,\mathrm{d}t + \int\limits_{\mathbb{R}} G(z)\,\mathrm{d}t \tag{2.12}$$

with Fréchet derivative

$$I'(z)\varphi = \int\limits_{\mathbb{R}} \left(z''\varphi'' - c^2 z'\varphi'\right)\,\mathrm{d}t + \int\limits_{z>-1} z\varphi\,\mathrm{d}t + \int\limits_{z\le -1} (\xi z + \xi - 1)\,\varphi\,\mathrm{d}t + \int\limits_{\mathbb{R}} \tilde{g}(z)\varphi\,\mathrm{d}t. \tag{2.13}$$

As the last preliminary step, we cite and extend an auxiliary lemma concerning relations among various norms. Compared to the paper [8] we have added the inequality (2.14). For the sake of completeness we present here the full proof, last two inequalities are from Lemma 2.2 and 2.3 in [8].

Lemma 9 (cf. [8])**.** *Let $c^2 \in (0,2)$, $C_0 = \frac{3}{2-c^2} > 0$, $C_1 = \frac{4}{4-c^4} > 0$ and let us denote*

$$\|z\|_\infty = \sup_{t\in\mathbb{R}} |z(t)|$$

and

$$\|z\| = \left(\int\limits_{\mathbb{R}} \left((z''(t))^2 - c^2(z'(t))^2 + (z(t))^2\right)\,\mathrm{d}t\right)^{\frac{1}{2}}.$$

Then $\|\cdot\|$ is an equivalent norm on H and the inequalities

$$\|z\|_{L^2}^2 \le C_1 \|z\|^2, \tag{2.14}$$

$$\frac{1}{C_0}\|z\|_H^2 \le \|z\|^2 \le \|z\|_H^2 \tag{2.15}$$

and

$$\|z\|_{\infty} \leq \sqrt{2}\,\|z\|_H \leq \sqrt{2C_0}\,\|z\| \tag{2.16}$$

hold for any $z \in H$.

Proof. First, we show (2.14). Let $\hat{z}(x) = \mathcal{F}(z(t))$ denote the Fourier transform of $z(t)$. Considering $\mathcal{F}(z^{(n)}(t)) = (ix)^n \hat{z}(x)$, where $n \in \mathbb{N}$ is the order of the derivative, $i^2 = -1$, and $\|z\| = \|\hat{z}\|$ (due to Plancherel identity), we can write

$$\|z\|^2 = \int_{\mathbb{R}} (x^4 - c^2x^2 + 1)(\hat{z}(x))^2 \,\mathrm{d}x \geq \min_{x\in\mathbb{R}}(x^4 - c^2x^2 + 1)\,\|\hat{z}\|_{L^2}^2$$
$$= \frac{1}{C_1}\,\|z\|_{L^2}^2 .$$

Since the right inequality $\|z\|_H^2 \geq \|z\|^2$ in (2.15) follows directly from the definition, we will focus on the left inequality and we can argue in a similar way as in the proof of (2.14). In particular, we have

$$\|z\|^2 = \int_{\mathbb{R}} \left(x^4 - c^2x^2 + 1\right)\left(\hat{z}(x)\right)^2 \,\mathrm{d}x$$
$$= \int_{\mathbb{R}} \left[\left(x^4 + x^2 + 1\right) - \left(c^2 + 1\right)x^2\right]\left(\hat{z}(x)\right)^2 \,\mathrm{d}x.$$

From $(x^2 - 1)^2 \geq 0$ we get

$$\|z\|^2 \geq \int_{\mathbb{R}} \left[\left(x^4 + x^2 + 1\right) - \frac{c^2+1}{3}(x^4 + x^2 + 1)\right]\left(\hat{z}(x)\right)^2 \,\mathrm{d}x$$
$$= \frac{2-c^2}{3}\int_{\mathbb{R}} \left(x^4 + x^2 + 1\right)\left(\hat{z}(x)\right)^2 \,\mathrm{d}x = \frac{1}{C_0}\,\|z\|_H^2 .$$

For the inequalities (2.16) the equality

$$z^2(t) = z^2(a) + 2\int_a^t z(x)z'(x)\,\mathrm{d}x \tag{2.17}$$

holds for arbitrary t, $a \in [0,1]$. It follows from the Mean Value Theorem that there is such $a \in (0,1)$ for which

$$z^2(a) = \int_0^1 z^2(t)\,\mathrm{d}t = \|z\|^2_{L^2(0,1)}\,. \tag{2.18}$$

The Cauchy–Schwartz and the Triangle inequalities, (2.17) and (2.18) all together yield

$$\begin{aligned} z^2(t) = z^2(a) + 2\int_a^t z(x)z'(x)\,\mathrm{d}x &\le 2\|z\|^2_{L^2(0,1)} + \|z'\|^2_{L^2(0,1)} \\ &\le 2\int_0^1 \left(|z(x)|^2 + |z'(x)|^2\right)\,\mathrm{d}x \le 2\|z\|^2_{H(0,1)}\,. \end{aligned}$$

Thus

$$\sup_{t\in(0,1)} |z(t)| \le \sqrt{2}\,\|z\|_{H(0,1)}\,. \tag{2.19}$$

Analogously, the relation in (2.19) holds on any interval $(n, n+1)$ with $n \in \mathbb{Z}$. Thus

$$\|z\|_\infty = \sup_{t\in\mathbb{R}} |z(t)| \le \sqrt{2}\,\|z\|_H$$

and together with (2.15) it gives

$$\|z\|_\infty \le \sqrt{2}\,\|z\|_H \le \sqrt{2C_0}\,\|z\|\,.$$

Moreover, it is easy to see that the mapping $\|\cdot\|$ is indeed a norm on H despite it is not stated explicitly in [8]. □

2.2 Main existence result

In this part of text, we present our main result. It reads as follows.

Theorem 10 ([17]). *Let $\tilde{g}$ satisfy assumptions (A1)–(A5). Then for any $\xi \in [0, 9/25)$ and $c^4 \in (100\xi/9, 4)$ the problem* (2.7) *possesses at least one nontrivial weak solution that corresponds to the homoclinic travelling wave solution of* (2.1)–(2.3).

Remark 11. Notice that the assumptions $\xi \in [0, 9/25)$ and $c^4 \in (100\xi/9, 4)$ are equivalent to $c^2 \in (0, 2)$, $\xi \in [0, 9c^4/100)$. The first mentioned expression corresponds more closely to the original problem (2.1)–(2.3), since the parameters α and β are given from the equation (2.1) and therefore ξ is also given. However, the latter will be also used in some cases for better legibility of the text.

Considering given α and β, i.e., fixed $\xi = \beta/\alpha$, Theorem 10 ensures the existence of travelling wave solutions with $\sqrt[4]{\alpha}c \in \left(-\sqrt[4]{4\alpha}, -\sqrt[4]{100\beta/9}\right) \cup \left(\sqrt[4]{100\beta/9}, \sqrt[4]{4\alpha}\right)$. For $c < 0$ it means that the wave is travelling to the left, for positive values it moves to the right.

To prove our statement we use the approach inspired by [8], i.e., the variational formulation of (2.7) and the Mountain Pass Theorem.

Theorem 12 (Mountain Pass Theorem [2], [14])**.** *Let H be a Hilbert space, $I \in C^1(H, \mathbb{R})$ and let there exist $e \in H$ and $r > 0$ such that $\|e\| > r$ and*

$$\inf_{\|z\|=r} I(z) > I(0) \geq I(e). \tag{2.20}$$

Then there exists a sequence $(z_n)_{n=1}^{+\infty} \subset H$ such that

$$I(z_n) \to s, \qquad \|I'(z_n)\| \to 0, \tag{2.21}$$

with s given by

$$s = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \tag{2.22}$$

and $\Gamma = \{\gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = e\}$.

The following series of lemmas shows that the functional I given by (2.9) has the mountain pass geometry, i.e., it satisfies all the assumptions of Theorem 12. The first one gives us the local minimum at 0 and boundedness from below on a neighborhood of the origin. Notice that for now we require the parameter ξ to be less than 1. Other constrains will come up later.

Lemma 13 ([17])**.** *Let* $c^2 \in (0,2)$*,* $\xi \in [0,1)$ *and let* $\tilde{g}$ *satisfy assumptions (A1)–(A3). Then the functional* I *given by* (2.9) *has a local minimum at* 0*. In particular,*

$$\inf_{\|z\|=r} I(z) \geq \omega \tag{2.23}$$

with $r = \min\{\delta,1\}/\sqrt{2C_0} > 0$ *and* $\omega = \min\{\delta^2,1\}\nu/(4C_0) > 0$*.*

Proof. Considering $\|z\| \leq r$ implies $\|z\|_\infty = \sup_{t\in\mathbb{R}} |z(t)| \leq \min\{\delta,1\}$. Thus $\int_{z\leq -1}(z+1)^2\,\mathrm{d}t = 0$ and, applying the property (P3),

$$\int\limits_{\mathbb{R}} G(z)\,\mathrm{d}t \geq -\frac{1-\nu}{2C_1}\int\limits_{\mathbb{R}} |z|^2\,\mathrm{d}t \geq -\frac{1-\nu}{2}\|z\|^2.$$

Hence, using (2.10), we can write

$$\begin{aligned} I(z) &= \frac{1}{2}\|z\|^2 - \frac{1-\xi}{2}\int\limits_{z\leq -1}(z+1)^2\,\mathrm{d}t + \int\limits_{\mathbb{R}} G(z)\,\mathrm{d}t \\ &\geq \frac{1}{2}\|z\|^2 - \frac{1-\nu}{2}\|z\|^2 = \frac{\nu}{2}\|z\|^2. \end{aligned}$$

Since $I(0) = 0$, we can conclude that $I(z)$ has a local minimum at $z = 0$ and (2.23) holds true. □

The following lemma gives us the existence of an element far enough from the origin at which the functional I has a negative value. However, from now on, there is a further limitation of the values of ξ.

Lemma 14 ([17])**.** *Let* $c^2 \in (0,2)$*,* $\xi \in [0, 9c^4/100)$ *and let* $\tilde{g}$ *satisfy assumptions (A1), (A2) and (A4). Then there exists* $e \in H$ *such that* I *given by* (2.9) *satisfies*

$$I(e) < I(0) = 0.$$

Proof. Let us consider a function v in the form

$$v(t) = \begin{cases} -\sin^3 t & \text{for } t \in (0,\pi), \\ 0 & \text{elsewhere.} \end{cases}$$

Obviously, $v \in H$. Let us now denote $z_0(t) = Av(\lambda t)$ for some positive $\lambda \in \mathbb{R}^+$, $A \in \mathbb{R}^+$. Using (2.12), we can write

$$\begin{aligned} I(z_0) &= \frac{1}{2}\int\limits_{\mathbb{R}} ((z_0'')^2 - c^2(z_0')^2)\,\mathrm{d}t + \frac{1}{2}\int\limits_{z_0>-1} z_0^2\,\mathrm{d}t + \frac{\xi}{2}\int\limits_{z_0\le-1} z_0^2\,\mathrm{d}t \\ &\quad + \frac{\xi-1}{2}\int\limits_{z_0\le-1}(2z_0+1)\,\mathrm{d}t + \int\limits_{\mathbb{R}} G(z_0)\,\mathrm{d}t. \end{aligned} \tag{2.24}$$

The particular terms of (2.24) can be calculated and/or estimated in the following way:

- $$\begin{aligned} \int\limits_{\mathbb{R}} ((z_0'')^2 - c^2(z_0')^2)\,\mathrm{d}t &= 9\lambda^4A^2\int\limits_0^{\frac{\pi}{\lambda}} \left(\sin^3(\lambda t) - 2\sin(\lambda t)\cos^2(\lambda t)\right)^2\,\mathrm{d}t \\ &\quad -9c^2\lambda^4A^2\int\limits_0^{\frac{\pi}{\lambda}} \sin^4(\lambda t)\cos^2(\lambda t)\,\mathrm{d}t = -\frac{9}{16}\lambda\pi A^2(c^2-5\lambda^2), \end{aligned}$$

- $$\int\limits_{z_0>-1} z_0^2\,\mathrm{d}t = \int\limits_{0<|z_0|<1} z_0^2\,\mathrm{d}t \le \int\limits_{0<|z_0|} 1\,\mathrm{d}t \le \int\limits_0^{\frac{\pi}{\lambda}} 1\,\mathrm{d}t = \frac{\pi}{\lambda},$$

- $$\xi\int\limits_{z_0\le-1} z_0^2\,\mathrm{d}t \le \xi\int\limits_0^{\frac{\pi}{\lambda}} z_0^2\,\mathrm{d}t = \xi A^2\frac{5\pi}{16\lambda},$$

- $$\begin{aligned} (\xi-1)\int\limits_{z_0\le-1}(2z_0+1)\,\mathrm{d}t &= (1-\xi)\int\limits_{z_0\le-1}(2|z_0|-1)\,\mathrm{d}t \\ &\le (1-\xi)\int\limits_{z_0\le-1} 2|z_0|\,\mathrm{d}t \le (1-\xi)\int\limits_0^{\frac{\pi}{\lambda}} 2A\sin^3(\lambda t)\,\mathrm{d}t = (1-\xi)A\frac{8\pi}{3\lambda}. \end{aligned}$$

Graphical illustration of the second estimate is depicted in Figure 2.4. Finally, using the property (P4), we get

- $$\int\limits_{\mathbb{R}} G(z_0)\,\mathrm{d}t \le K\int\limits_{\mathbb{R}} |z_0|\,\mathrm{d}t = K\int\limits_0^{\frac{\pi}{\lambda}} A\sin^3(\lambda t)\,\mathrm{d}t = KA\frac{4\pi}{3\lambda}.$$

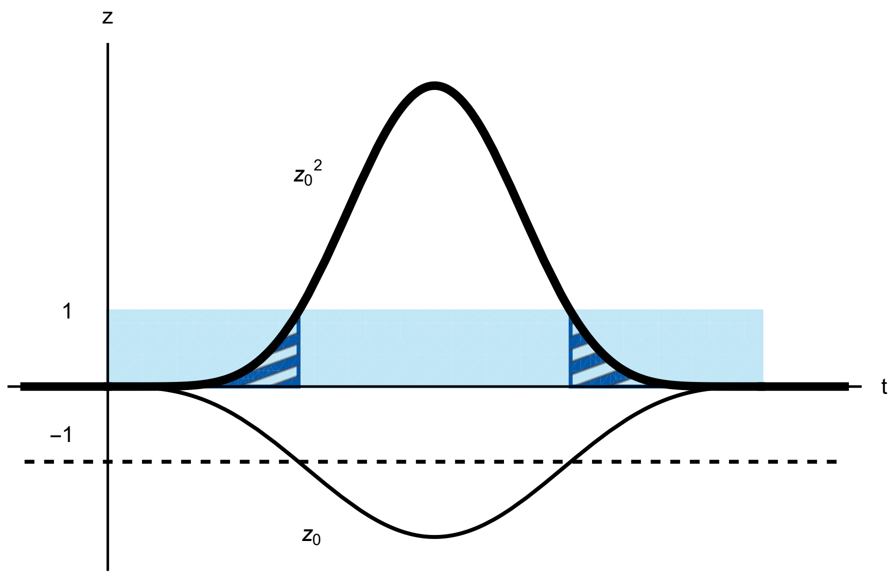

Figure 2.4: Illustration for the estimate of the second term in (2.24).

Now, we are ready to write the required estimate of $I(z_0)$ as

$$I(z_0) \leq -\frac{9}{32}\lambda\pi A^2(c^2-5\lambda^2)+\frac{\pi}{2\lambda}+\xi A^2\frac{5\pi}{32\lambda} +(1-\xi)A\frac{4\pi}{3\lambda}+KA\frac{2\pi}{3\lambda}.$$

The right–hand side of the above inequality is the quadratic function of A. Hence, for $0<\lambda<\sqrt{5}c$ and $0\leq\xi<\frac{9}{5}\lambda^2(c^2-5\lambda^2)$, there exists $A_0>0$ such that $I(z_0)<0$ for any $A>A_0$. Moreover, the largest interval allowed for ξ, namely $\xi\in[0,9c^4/100)$, is obtained for the choice $\lambda_0=\frac{\sqrt{10}}{10}|c|$. Hence, for any $\xi\in[0,9c^4/100)$, the required point $e\in H$ with the property $I(e)<0$ can be chosen as $e=z_0$ with $\lambda=\lambda_0$ and A sufficiently large. □

Remark 15. In the proof of Lemma (14) we first used function

$$v(t)=\begin{cases}-(1+\cos t) & \text{for } t\in(-\pi,\pi),\\ 0 & \text{elsewhere.}\end{cases}$$

Then we get $\xi \in [0, c^4/12)$. In order to extend the interval of admissible values of ξ, we took $v(t) = -\sin^3 t$ for $t \in (0, \pi)$ and zero elsewhere. Notice that function $v(\lambda_0 t)$ with $\lambda_0 = \sqrt{10}|c|/10$ is a solution of the eigenvalue problem

$$\begin{cases} v^{(4)} + c^2 v'' + \mu v = 0 \quad \text{on } (0, a), \\ v(0) = v(a) = v'(0) = v'(a) = 0 \end{cases} \tag{2.25}$$

with eigenvalue $\mu = 9c^4/100$ and $a = \pi/\lambda_0$. This choice of v, μ and a corresponds to the last case where the first eigenfunction of (2.25) does not change its sign. It seems we may be able to further extend the interval for ξ. However, in the time of writing this text the question remains open.

Remark 16. Let $\tilde{g}$ be such that $\int_{\mathbb{R}} G(z)\,\mathrm{d}t \geq 0$ for any $z \in H$. This happens, e.g., if $\tilde{g}(z)z \geq 0$. Then for $\xi \geq 1$ we have trivially $I(z) \geq \frac{1}{2}\|z\|^2$ for any $z \in H$ (cf. (2.10)). Similarly, for $\xi < 1$ we obtain (cf. (2.12))

$$\begin{aligned} I(z) &\geq \frac{1}{2}\int_{\mathbb{R}} ((z'')^2 - c^2(z')^2)\,\mathrm{d}t + \frac{1}{2}\int_{z>-1} z^2\,\mathrm{d}t + \frac{\xi}{2}\int_{z\leq -1} z^2\,\mathrm{d}t \\ &\geq \frac{1}{2}\int_{\mathbb{R}} ((z'')^2 - c^2(z')^2 + \xi z^2)\,\mathrm{d}t. \end{aligned}$$

Using Plancherel identity as in the proof of Lemma 9, we can write

$$\begin{aligned} \int_{\mathbb{R}} ((z'')^2 - c^2(z')^2 + \xi z^2)\,\mathrm{d}t &= \int_{\mathbb{R}} (x^4 - c^2x^2 + \xi)(\hat{z})^2\,\mathrm{d}x \\ &\geq \min_{x\in\mathbb{R}}(x^4 - c^2x^2 + \xi)\|\hat{z}\|^2_{L^2} \end{aligned}$$

and thus

$$I(z) \geq \frac{1}{2}\left(\xi - \frac{c^4}{4}\right)\|z\|^2_{L^2}$$

for any $z \in H$. Hence, we can conclude that if $\int_{\mathbb{R}} G(z)\,\mathrm{d}t \geq 0$ and $\xi > c^4/4$ then the functional I has a global minimum in $z = 0$ and no point $e \in H$ satisfying (2.20) exists.

This suggests that the condition $\xi < 9c^4/100$ from Lemma 14 can really be too strict and it is probably not necessary for (2.20). As we mentioned already in the Remark 15 the open question left is what upper bound for ξ is the necessary and sufficient one.

As we have stated above, Lemmas 13 and 14 imply that for $\xi \in [0, 9c^4/100)$ the functional I has the mountain pass geometry (cf. Theorem 12). That means there exists a sequence $(z_n)_{n=1}^{+\infty}$ satisfying (2.21), (2.22) with $s \geq \min\{\delta^2, 1\}\nu/(4C_0)$. To show the existence of the limit critical point, we need another series of partial lemmas.

The first one concerns the boundedness of $(z_n)_{n=1}^{+\infty}$. Notice that in both of the lemmas we assume $\xi < 1$. Although, the functional I has guaranteed the right geometry for $\xi \in [0, 9c^4/100)$, it does not mean that there cannot be I with ξ out of the interval with required geometry.

Lemma 17 ([17])**.** *Let $c^2 \in (0,2)$, $\xi \in [0,1)$ and let $\tilde{g}$ satisfy assumptions (A1)–(A5). Any sequence $(z_n)_{n=1}^{+\infty} \subset H$ satisfying* (2.21) *with I given by* (2.9) *is bounded.*

Proof. We proceed via contradiction. Let $(z_n)_{n=1}^{+\infty} \subset H$ satisfy $I(z_n) \to s$ and $\|I'(z_n)\| \to 0$ but $\|z_n\| \to +\infty$ for $n \to +\infty$. Using (2.10) and (2.11), we can write

$$I(z_n) = \frac{1}{2}\|z_n\|^2 - \frac{1-\xi}{2} \int\limits_{z_n \leq -1} (z_n+1)^2 \,\mathrm{d}t + \int\limits_{\mathbb{R}} G(z_n)\,\mathrm{d}t$$

and

$$I'(z_n)z_n = \|z_n\|^2 - (1-\xi) \int\limits_{z_n \leq -1} \left(z_n^2 + z_n\right) \,\mathrm{d}t + \int\limits_{\mathbb{R}} \tilde{g}(z_n)z_n\,\mathrm{d}t. \quad (2.26)$$

Since

$$\frac{I(z_n)}{\|z_n\|} \to 0 \qquad \text{and} \qquad \frac{I'(z_n)z_n}{\|z_n\|} \to 0$$

we have

$$\begin{aligned} 0 &= \lim_{n\to+\infty} \frac{2I(z_n) - I'(z_n)z_n}{\|z_n\|} \\ &= \lim_{n\to+\infty} (\xi - 1) \int\limits_{z_n \leq -1} \frac{z_n+1}{\|z_n\|}\,\mathrm{d}t + \lim_{n\to+\infty} \int\limits_{\mathbb{R}} \frac{r(z_n)}{\|z_n\|}\,\mathrm{d}t. \quad (2.27)\end{aligned}$$

Recall that $r(z) = 2G(z) - \tilde{g}(z)z$. Both terms on the right–hand side of (2.27) are nonnegative (cf. Remark 7 and property (P5)), therefore

$$\lim_{n\to+\infty} \int\limits_{z_n\le -1} \frac{|z_n+1|}{\|z_n\|}\,\mathrm{d}t = 0 \quad \text{and} \quad \lim_{n\to+\infty} \int\limits_{\mathbb{R}} \frac{r(z_n)}{\|z_n\|}\,\mathrm{d}t = 0.$$

Furthermore,

$$0 \le \lim_{n\to+\infty} \int\limits_{z_n\le -1} \frac{z_n^2+z_n}{\|z_n\|^2}\,\mathrm{d}t \le \lim_{n\to+\infty} \sqrt{2C_0} \int\limits_{z_n\le -1} \frac{|z_n+1|}{\|z_n\|}\,\mathrm{d}t = 0. \tag{2.28}$$

Hence, (2.26) together with (2.28) imply

$$\begin{aligned} 0 &= \lim_{n\to+\infty} \frac{I'(z_n)z_n}{\|z_n\|^2} = \\ &= 1 + \lim_{n\to+\infty} (\xi - 1) \int\limits_{z_n\le -1} \frac{z_n^2+z_n}{\|z_n\|^2}\,\mathrm{d}t + \lim_{n\to+\infty} \int\limits_{\mathbb{R}} \frac{\tilde{g}(z_n)z_n}{\|z_n\|^2}\,\mathrm{d}t \\ &= 1 + \lim_{n\to+\infty} \int\limits_{\mathbb{R}} \frac{\tilde{g}(z_n)z_n}{\|z_n\|^2}\,\mathrm{d}t. \end{aligned} \tag{2.29}$$

If $\tilde{g}(z)z \ge 0$ on $\mathbb{R}$, we obtain immediately a contradiction. Now, let ε_1 be the positive root of $\tilde{g}(z) = -(1-\nu/2)z/C_1$ and $-\varepsilon_2$ be the negative root of $\tilde{g}(z) = -(1-\nu/2)z/C_1$. If $g(z) > -(1-\nu/2)z/C_1$ for all $z > 0$, we put $\varepsilon_1 = +\infty$ and, similarly, if $g(z) < -(1-\nu/2)z/C_1$ for all $z < 0$, we put $-\varepsilon_2 = -\infty$.

Notice that if this is not the case, then due to the assumptions (A3) and (A5) there exist unique $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ such that $g(z) > -(1-\nu/2)z/C_1$ on $(0, \varepsilon_1)$ (and similarly on $(-\varepsilon_2, 0)$). The graphical illustration of ε_1 and $-\varepsilon_2$ is depicted in Figure 2.5.

Now, we split the last integral in (2.29) into three (possible) parts according to the values of z_n:

$$\int\limits_{\mathbb{R}} \frac{\tilde{g}(z_n)z_n}{\|z_n\|^2}\,\mathrm{d}t = \int\limits_{z_n\le -\varepsilon_2} \frac{\tilde{g}(z_n)z_n}{\|z_n\|^2}\,\mathrm{d}t + \int\limits_{-\varepsilon_2<z_n<\varepsilon_1} \frac{\tilde{g}(z_n)z_n}{\|z_n\|^2}\,\mathrm{d}t + \int\limits_{z_n\ge\varepsilon_1} \frac{\tilde{g}(z_n)z_n}{\|z_n\|^2}\,\mathrm{d}t.$$

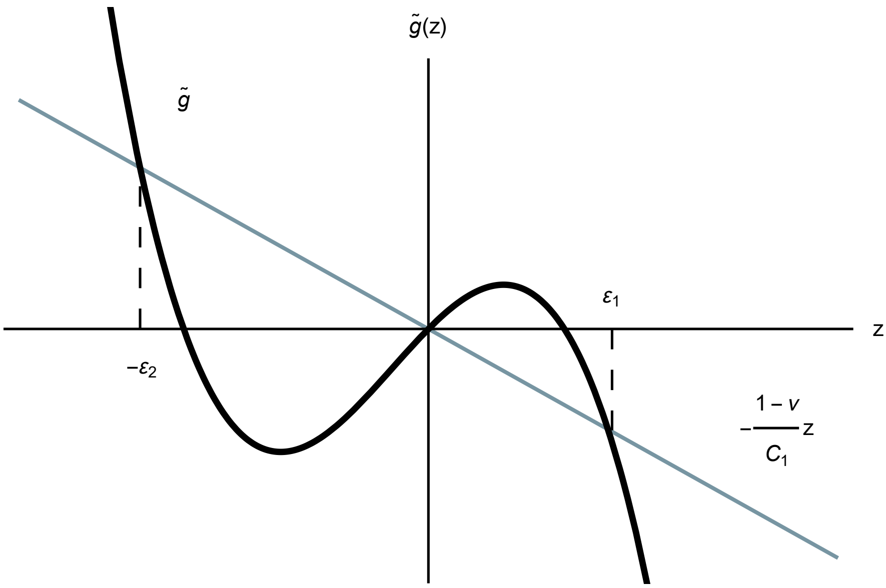

Figure 2.5: Illustration of ε_1 and $-\varepsilon_2$.

For $-\varepsilon_2 < z < \varepsilon_1$, we have $\tilde{g}(z)z > -(1-\nu/2)z^2/C_1$. Hence, the middle integral can be estimated as

$$\int_{-\varepsilon_2<z_n<\varepsilon_1} \frac{\tilde{g}(z_n)z_n}{\|z_n\|^2}\,\mathrm{d}t > \int_{-\varepsilon_2<z_n<\varepsilon_1} \frac{-(1-\nu/2)z_n^2}{C_1\,\|z_n\|^2}\,\mathrm{d}t \geq -1+\frac{\nu}{2}.$$

Further, if $\varepsilon_1 < +\infty$, due to concavity of $\tilde{g}$ on $(0,+\infty)$, there exists $b_{\varepsilon_1} \in (0,\varepsilon_1)$ such that

$$G(z_n) \geq \frac{1}{2}\left(z_n - b_{\varepsilon_1}\right)\tilde{g}(z_n) \quad \text{for any } z_n \geq \varepsilon_1. \tag{2.30}$$

Notice that if $G(\varepsilon_1) < 0$, we can take b_{ε_1} such that the equality $G(\varepsilon_1) = \frac{1}{2}\left(\varepsilon_1 - b_{\varepsilon_1}\right)\tilde{g}(\varepsilon_1)$ holds true. On the other hand, if $G(\varepsilon_1) > 0$ we have to choose b_{ε_1} as the last positive root of $\tilde{g}(z) = 0$. The reader can find a graphical illustration of ε_1 and b_{ε_1} in Figure 2.6.

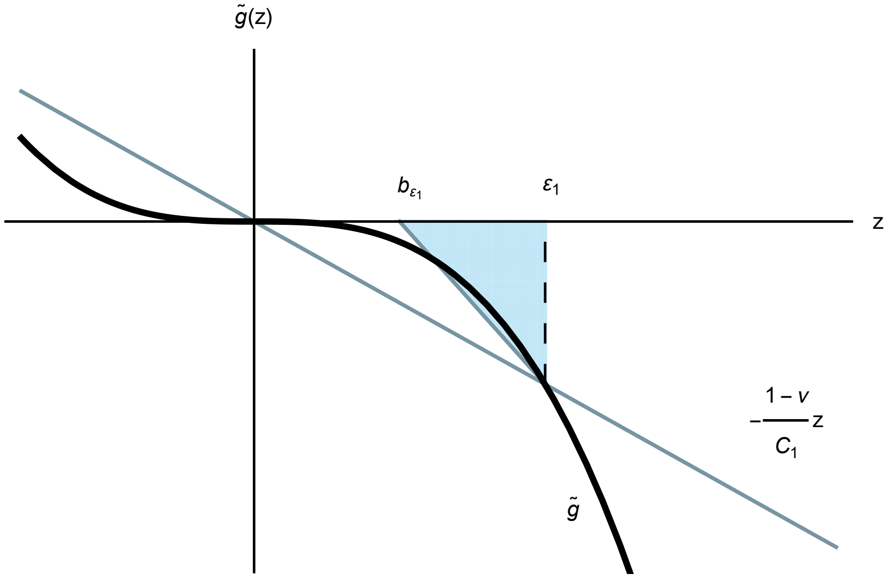

Figure 2.6: Illustration of b_{ε_1}.

The relation (2.30) implies $|\tilde{g}(z_n)| \leq \dfrac{r(z_n)}{b_{\varepsilon_1}}$ for $z_n \geq \varepsilon_1$ and

$$0 \leq \int\limits_{z_n \geq \varepsilon_1} \frac{|\tilde{g}(z_n) z_n|}{\|z_n\|^2} \,\mathrm{d}t \leq \frac{1}{b_{\varepsilon_1}} \int\limits_{z_n \geq \varepsilon_1} \frac{r(z_n) z_n}{\|z_n\|^2} \,\mathrm{d}t$$

$$\leq \frac{\sqrt{2C_0}}{b_{\varepsilon_1}} \int\limits_{z_n \geq \varepsilon_1} \frac{r(z_n)}{\|z_n\|} \,\mathrm{d}t \to 0$$

for $n \to +\infty$. Similarly, we obtain convergence to zero also for the integral over the domain $z_n \leq -\varepsilon_2$. Hence, (2.29) can be continued as

$$0 = 1 + \lim_{n \to +\infty} \int\limits_{\mathbb{R}} \frac{\tilde{g}(z_n) z_n}{\|z_n\|^2} \,\mathrm{d}t = 1 + \lim_{n \to +\infty} \int\limits_{-\varepsilon_2 < z_n < \varepsilon_1} \frac{\tilde{g}(z_n) z_n}{\|z_n\|^2} \,\mathrm{d}t$$

$$\geq 1 - 1 + \frac{\nu}{2} = \frac{\nu}{2},$$

a contradiction. □

Lemma 17 and reflexivity of H imply the existence of a weak limit $z_n \rightharpoonup z \in H$. Since we work on the unbounded domain, we cannot use the standard compact embedding arguments in order to show that the functional I satisfies the Palais–Smale condition, i.e., that the sequence fulfilling (2.21) has a convergent subsequence in the norm of H, and to gain the strong limit being the critical point of the functional I. Instead, we can use either the so called concentration compactness principle introduced by P. Lions [23] or the existence of a nonzero weak limit after a suitable translation (cf. [22]). The latter method is what we present in the proof of the following lemma.

Lemma 18 ([17])**.** *Let $c^2 \in (0,2)$, $\xi \in [0,1)$, let $\tilde{g}$ satisfy assumptions (A1)–(A5) and let $(z_n)_{n=1}^{+\infty} \subset H$ be a sequence satisfying* (2.21) *with I given by* (2.9)*. Then there exists $\tilde{z} \in H$ such that $I'(\tilde{z})\varphi = 0$ for all $\varphi \in H$.*

Proof. The proof will be done in three steps.

Step 1 First of all, we show that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the suprema of $|z_n|$ are bounded away from zero. Indeed, let us assume the contrary, i.e., that there exists a subsequence (denoted again by (z_n)) such that $\|z_n\|_\infty < \min\{\delta, 1\}$ for all $n \in \mathbb{N}$. Then $\{t \in \mathbb{R} : z_n(t) \leq -1\} = \emptyset$ and (2.21) together with the assumption (A3) imply

$$0 = \lim_{n\to+\infty} I'(z_n)z_n = \lim_{n\to+\infty} \left(\|z_n\|^2 + \int_{\mathbb{R}} \tilde{g}(z_n) z_n \,\mathrm{d}t \right)$$

$$\geq \lim_{n\to+\infty} \left(\|z_n\|^2 - \frac{1-\nu}{C_1} \int_{\mathbb{R}} z_n^2 \,\mathrm{d}t \right) \geq \lim_{n\to+\infty} \nu \|z_n\|^2$$

and hence $\|z_n\| \to 0$. Using assumptions (A2) and (A5) (cf. also Remark 5), we obtain

$$\int_{\mathbb{R}} G(z_n)\,\mathrm{d}t = \int_{z_n>0} G(z_n)\,\mathrm{d}t + \int_{z_n<0} G(z_n)\,\mathrm{d}t$$

$$\leq \tilde{g}'_+(0) \int_{z_n>0} \frac{z_n^2}{2}\,\mathrm{d}t + \tilde{g}'_-(0) \int_{z_n<0} \frac{z_n^2}{2}\,\mathrm{d}t \leq \frac{1}{2} M_0 \|z_n\|^2$$

with $M_0 = \max\{\tilde{g}'_+(0), \tilde{g}'_-(0), 0\}C_1 \geq 0$, and thus

$$0 < s = \lim_{n\to+\infty} I(z_n) = \lim_{n\to+\infty} \left(\frac{1}{2}\|z_n\|^2 + \int_{\mathbb{R}} G(z_n)\,\mathrm{d}t \right) \leq \frac{1}{2}(1+M_0) \lim_{n\to+\infty} \|z_n\|^2 = 0,$$

a contradiction. Hence, we can conclude that for all sufficiently large $n \in \mathbb{N}$ there exist $t_n \in \mathbb{R}$ such that

$$|z_n(t_n)| = \sup_{t\in\mathbb{R}} |z_n(t)| \geq \min\{\delta, 1\} > 0. \tag{2.31}$$

Step 2 In spite of (2.31), the weak limit of (z_n) can be zero. For this reason, we translate functions z_n in order to move their suprema into a bounded interval. In particular, let us introduce functions $\tilde{z}_n(t) = z_n(t+t_n)$. Then $|\tilde{z}_n(0)| = |z_n(t_n)| \geq \min\{\delta, 1\} > 0$ for almost all $n \in \mathbb{N}$. The sequence $(\tilde{z}_n)_{n=1}^{+\infty} \subset H$ is again bounded with a subsequence weakly converging to $\tilde{z} \in H$ satisfying $\tilde{z}(0) \neq 0$ and $\tilde{z} \not\equiv 0$ [22].

Step 3 Finally, we want to prove that $I'(\tilde{z}) = 0$. Obviously, $I'(\tilde{z}_n(\cdot))\varphi(\cdot) = I'(z_n(\cdot + t_n))\varphi(\cdot) = I'(z_n(\cdot))\varphi(\cdot - t_n)$ and $\|\varphi(\cdot)\| = \|\varphi(\cdot - t_n)\|$ for all $\varphi \in H$. Hence, $\|I'(\tilde{z}_n)\| = \|I'(z_n)\| \to 0$ and it suffices to prove that $I'(\tilde{z}_n)\varphi \to I'(\tilde{z})\varphi$ for all $\varphi \in H$. The definition of weak convergence gives

$$\int_{\mathbb{R}} \left(\tilde{z}_n''\varphi'' - c^2\tilde{z}_n'\varphi' + \tilde{z}_n\varphi \right)\,\mathrm{d}t \to \int_{\mathbb{R}} \left(\tilde{z}''\varphi'' - c^2\tilde{z}'\varphi' + \tilde{z}\varphi \right)\,\mathrm{d}t$$

for $n \to +\infty$. Now, we have to show that also

$$\int_{\mathbb{R}} \left((1-\xi)(\tilde{z}_n+1)^- + \tilde{g}(\tilde{z}_n) \right) \varphi\,\mathrm{d}t \to \int_{\mathbb{R}} \left((1-\xi)(\tilde{z}+1)^- + \tilde{g}(\tilde{z}) \right) \varphi\,\mathrm{d}t$$

for any $\varphi \in H$ as $n \to +\infty$ (cf. the expression (2.11) for $I'(z)\varphi$). For simplicity, let us denote $h(z) := (1-\xi)(z+1)^- + \tilde{g}(z)$. Since (z_n)

is bounded in H and h is continuous with finite one–sided derivatives $h'_{\pm}(0)$, there exists M_1 such that

$$\|h(\tilde{z}_n) - h(\tilde{z})\|_{L^2(\mathbb{R})} \leq M_1$$

for any $n \in \mathbb{N}$. For arbitrarily small $\varepsilon > 0$ we can choose a compact set $J \subset \mathbb{R}$ such that $\|\varphi\|_{L^2(\mathbb{R}\setminus J)} < \varepsilon/(2M_1)$. It follows from Cauchy–Schwarz inequality that

$$\left| \int\limits_{\mathbb{R}\setminus J} (h(\tilde{z}_n) - h(\tilde{z}))\, \varphi \,\mathrm{d}t \right| \leq \|h(\tilde{z}_n) - h(\tilde{z})\|_{L^2(\mathbb{R}\setminus J)} \, \|\varphi\|_{L^2(\mathbb{R}\setminus J)} \leq \frac{\varepsilon}{2}.$$

Due to Lipschitz continuity of h (cf. Remark 5), there exists a constant L such that $|h(\tilde{z}_n) - h(\tilde{z})| \leq L|\tilde{z}_n - \tilde{z}|$. Let $M_2 := \int_J |\varphi| \,\mathrm{d}t$. Since on any compact set $J \subset \mathbb{R}$, the weak convergence implies the uniform convergence, there exists $n_0 \in \mathbb{N}$ such that $\sup_{t\in J} |\tilde{z}_n - \tilde{z}| \leq \varepsilon/(2LM_2)$ for all $n \geq n_0$. Therefore

$$\left| \int\limits_J (h(\tilde{z}_n) - h(\tilde{z}))\, \varphi \,\mathrm{d}t \right| \leq L \sup_{t\in J} |\tilde{z}_n - \tilde{z}| \int\limits_J |\varphi| \,\mathrm{d}t \leq \frac{\varepsilon}{2}$$

and hence

$$\begin{aligned}
&|I'(\tilde{z}_n)\varphi - I'(\tilde{z})\varphi| \\
&= \left| \int\limits_{\mathbb{R}} \big((1-\xi)(\tilde{z}_n + 1)^- + \tilde{g}(\tilde{z}_n) - (1-\xi)(\tilde{z}+1)^- - \tilde{g}(\tilde{z})\big)\, \varphi \,\mathrm{d}t \right| \\
&\leq \left| \int\limits_{\mathbb{R}\setminus J} (h(\tilde{z}_n) - h(\tilde{z}))\, \varphi \,\mathrm{d}t \right| + \left| \int\limits_J (h(\tilde{z}_n) - h(\tilde{z}))\, \varphi \,\mathrm{d}t \right| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

for all $n \geq n_0$. That is, $I'(\tilde{z}_n)\varphi \to I'(\tilde{z})\varphi$ as $n \to +\infty$ and $I'(\tilde{z})\varphi = 0$ for all $\varphi \in H$. □

Now, we are ready to sum up all the above partial results to prove our main statement.

Proof of Theorem 10. As we have shown above, the weak solutions of (2.7) are equivalent to critical points of functional I given by (2.9). It follows from Lemmas 13 and 14 that I satisfies assumptions of Theorem 12, i.e., I has the mountain pass geometry. Therefore, there exists a sequence $(z_n)_{n=1}^{+\infty}$ satisfying

$$I(z_n) \to s \quad \text{and} \quad \|I'(z_n)\| \to 0$$

with s given by (2.22). Due to Lemma 17, the sequence $(z_n)_{n=1}^{+\infty}$ is bounded and hence weakly convergent in H. Finally, Lemma 18 ensures that after a suitable translation the weak limit is the searched nontrivial critical point of I and hence the homoclinic travelling wave solution of the original equation (2.1) with the conditions (2.2) and (2.3). □

2.3 Possible forms of the classical solutions and some numerical experiments

In this section, inspired by [25], we deal with the analytical expression of the nontrivial classical solution $z \in C^4(\mathbb{R})$ of the problem (2.7) and with associated numerical experiments. For the sake of simplicity, from now on we consider the nonlinearity $\tilde{g} \equiv 0$ and therefore solve the system

$$z^{(4)} + c^2 z'' + z = 0 \quad \text{if } z \geq -1, \tag{2.32}$$

$$z^{(4)} + c^2 z'' + \xi z = 1 - \xi \quad \text{if } z \leq -1. \tag{2.33}$$

There are several ways how to deal with this problem. In our case, we focus on solutions that are even and cross the value -1 just twice as in [25]. That is, $z = z(t)$ will be constructed as the solution of the equation (2.32) on the interval $(-r, r)$ with $r > 0$ being an unknown parameter and $z(r) = z(-r) = -1$. In $t = \pm r$ we joint it smoothly with the solution of the equation (2.33), which vanishes for $t \to \pm\infty$. Moreover, we suppose $\xi \in (0, c^4/4)$. The case $\xi = 0$ is treated in [25].

Lemma 19. *Let $c^2 \in (0,2)$ and $\xi \in (0, c^4/4)$. The general solution of* (2.32) *that satisfies $z \to 0$ for $t \to -\infty$ takes the form*

$$z(t) = \mu e^{\sigma t} \cos(\tau t + \phi),$$

with $\mu \in \mathbb{R}$, $\phi \in \mathbb{R}$ arbitrary and $\sigma = \mathrm{Re}(\lambda)$, $\tau = \mathrm{Im}(\lambda)$ with λ being the root of the associated characteristic equation $\lambda^4 + c^2\lambda^2 + 1 = 0$. The general solution of (2.33) *takes the form*

$$z(t) = \gamma \cos(\kappa_1 t) + \delta \cos(\kappa_2 t) + \frac{1-\xi}{\xi}$$

with $\gamma \in \mathbb{R}$, $\delta \in \mathbb{R}$ arbitrary and $\kappa_1 = \mathrm{Im}(\lambda_1)$, $\kappa_2 = \mathrm{Im}(\lambda_2)$ with λ_1, λ_2 being roots of the associated characteristic equation $\lambda^4 + c^2\lambda^2 + \xi = 0$ in the form

$$\lambda_1 = i\sqrt{-\sqrt{\frac{c^4}{4} - \xi} + \frac{c^2}{2}} \quad \text{and} \quad \lambda_2 = i\sqrt{\sqrt{\frac{c^4}{4} - \xi} + \frac{c^2}{2}}.$$

Remark 20. Since the root of the characteristic equation $\lambda^4 + c^2\lambda^2 + 1 = 0$ associated with (2.32) that lies in the first quadrant of the Gauss plane can be written as $\lambda = e^{i\rho} = \sigma + i\tau = \cos\rho + i\sin\rho$ where $\rho \in (0, \frac{\pi}{2})$, $\sigma > 0$ and $\tau > 0$, we can express the parameter $c^2 \in (0,2)$ as $c^2 = -2\cos(2\rho)$ where $\rho \in (\frac{\pi}{4}, \frac{\pi}{2})$.

Hence, the searched solution can be written as

$$z(t) = \begin{cases} \mu e^{\sigma(t+r)} \cos(\tau(t+r) + \phi) & \text{for } t \le -r, \\ \gamma \cos(\kappa_1 t) + \delta \cos(\kappa_2 t) + \dfrac{1-\xi}{\xi} & \text{for } t \in (-r, r), \\ \mu e^{-\sigma(t-r)} \cos(\tau(t-r) - \phi) & \text{for } t \ge r. \end{cases}$$

Conditions on continuity of z and its derivatives in $t = \pm r$ give us the following system of five equations for five unknown parameters μ, ϕ, γ, δ and r:

$$z(\pm r): \qquad \mu\cos(\phi) = -1$$

$$\gamma\cos(\kappa_1 r) + \delta\cos(\kappa_2 r) + \frac{1-\xi}{\xi} = -1,$$

$$z'(\pm r): \qquad \mu\cos(\phi + \rho) = \kappa_1\gamma\sin(\kappa_1 r) + \kappa_2\delta\sin(\kappa_2 r),$$

$$z''(\pm r): \qquad \mu\cos(\phi + 2\rho) = -\kappa_1^2\gamma\cos(\kappa_1 r) - \kappa_2^2\delta\cos(\kappa_2 r),$$

$$z'''(\pm r): \qquad \mu\cos(\phi + 3\rho) = -\kappa_1^3\gamma\sin(\kappa_1 r) - \kappa_2^3\delta\sin(\kappa_2 r).$$

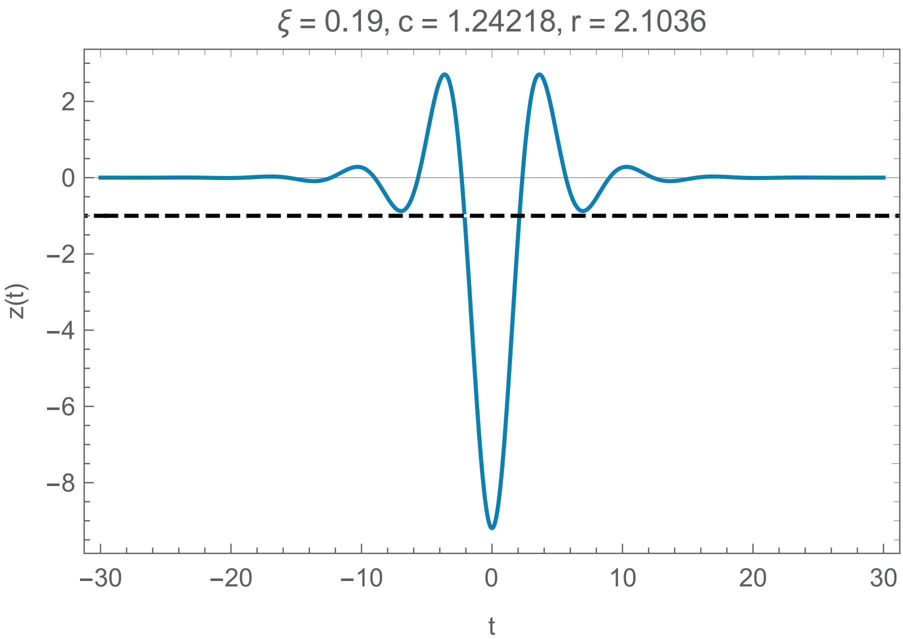

Figure 2.7: Solution of the system (2.32)–(2.33).

It is not possible to solve this system analytically. We perform numerical experiments in MATLAB using a numerical solver `vpasolve()`. The input parameters are ξ, c (which uniquely determine ρ, σ, τ, κ_1 and κ_2, cf. Lemma 19 and Remark 20) and the initial estimate of the solution which we choose as the solution of the system from [25].

Values of parameters μ, ϕ, γ, δ and r (rounded to four decimal places) describing some of the gained solutions are listed in Table 2.1. We can observe that the amplitude (cf. the absolute values of γ, δ and μ) increases considerably for c tending to 0. In Figure 2.7 we can see the solution corresponding to the first row. The other solutions can be found in Appendix 2.4.

We also considered the influence of ξ with fixed c. It seems (cf. results in Table 2.2 for the choice $c = 1.2649$) that as ξ decreases, so does the size of the parameter r which determines the connection points with $z = -1$. A graphical illustration of four selected solutions can be found in Appendix 2.4.

Table 2.1: Values of parameters μ, ϕ, γ, δ and r for different c and ξ describing the solution with enlarging amplitude.

c	ρ	ξ	μ	ϕ	γ	δ	r
1.2422	1.226	0.19	4.8150	1.7800	-10.5707	-2.8861	2.1036
1.0849	1.1	0.11	9.2355	1.6793	-27.8533	-8.1334	2.8468
0.9676	1.029	0.07	14.4016	1.6403	-57.9758	-17.3836	3.4717
0.6741	0.9	0.01	13.1709	1.6468	-203.8950	-40.9035	5.4996
0.2417	0.8	0.0002	56.5095	1.5885	-14383.7979	-3466.8485	18.2793

Table 2.2: Values of parameters μ, ϕ, γ, δ and r for $c = 1,2649$ and different ξ.

ξ	μ	ϕ	γ	δ	r
0.21	4.4561	1.7971	-9.2048	-2.5351	2.0047
0.2	4.2175	1.8102	-9.1157	-2.3878	1.9866
0.15	3.3968	1.8696	-9.6636	-1.8782	1.9046
0.1	2.9134	1.9212	-12.3451	-1.5748	1.8343
0.05	2.5940	1.9666	-21.9187	-1.3719	1.7729
0.01	2.4067	1.9993	-101.6711	-1.2517	1.7290

2.4 Open problems

In Section 2.2 (see Theorem 10) we proved that if the ratio β/α is sufficiently small (less then 9/25) and the nonlinearity g has a controlled slope at $u = 1/\alpha$ and is concave for $u > 1/\alpha$ and convex and bounded from below for $u < 1/\alpha$, the problem (2.1)–(2.3) possesses infinitely many homoclinic travelling wave solutions with arbitrary wave speed in the range $\left(\sqrt[4]{100\beta/9}, \sqrt[4]{4\alpha}\right)$.

Our result corresponds to conclusions obtain in the cited papers (i.e., [8], [7], [25]). However, in comparison with them, we allow the presence of the negative part βu^- in the model and weaken considerably the requirements on the nonlinearity g included in the problem. Specifically, the assumptions on analyticity or differentiability of g were removed. We also allow nonzero slope of g at $1/\alpha$, admit sign preserving functions and simplify proofs of some auxiliary statements. On the other hand, we show that the term βu^- seems to affect the range of wave speeds which the travelling wave is formed with (cf. Remarks 11, 15 and 11).

There are still many open questions left. We will mention three of them which we would like to address in the future. First, in spite of our effort, we have still quite strong requirements on the nonlinearity g (see assumptions (A1)–(A5) in Section 2.1), especially its convexity and concavity properties are restrictive and do not allow to include the smoothed nonlinear term e^u used, e.g., in [29] or [4].

Another one concerns the number of travelling wave solutions and their stability. The computer assisted proof for problem with exponential nonlinearity in [4] show that for given wave speed there exist at least 36 travelling wave solutions. The paper [6] shows that there are at least five different travelling wave solutions of the problem with piecewise nonlinearity for values in a subinterval of the range of wave speeds. It suggests that more than one travelling wave solution for fixed wave speed could appear also in our case. As far as we know, the analytic proof of the existence remains open as well.

The last but not least open question we want to point out is the existence of travelling waves with the speed less than $\sqrt[4]{100\beta/9}$. Remarks 15 and 16 suggest that the range of possible wave speeds could be extended to $\left(\sqrt[4]{4\beta}, \sqrt[4]{4\alpha}\right)$.

Resume

This thesis deals with a fourth order nonlinear partial differential equation, which can be used as a model of an asymmetrically supported beam, and its travelling wave solutions. First, we outline the topic of travelling waves and their occurrence in application problems. Chapter 1 is devoted to suspension bridges and the simple one dimensional model introduced by McKenna and Walter. Then we present and summarize the known results in this field.

Chapter 2 is divided into four sections. In the first one, we present to the reader the problem under our study, important assumptions on the nonlinearity and some auxiliary statements. In the next section, we discuss the formulation and the proof of our main result, which is the existence of a travelling wave solution. The proof is based on variational methods. The third section is devoted to the classical solutions and the numerical experiments performed to find such a solution. We conclude with open questions and problems.

Resumé

Tato práce se zabývá nelineární parciální diferenciální rovnicí čtvrtého řádu, která může sloužit jako model asymetricky podepřeného nosníku, a jejími řešeními ve tvaru postupné vlny. Nejprve v úvodu nastiňujeme problematiku postupných vln a jejich výskyt v aplikačních úlohách. Kapitola 1 je věnována visutým mostům a jednoduchému jednodimenzionálnímu modelu představenému McKennou a Walterem. Dále představujeme a shrnujeme již známé výsledky v této oblasti.

Kapitola 2 je rozdělena do čtyř podkapitol. V první z nich předkládáme čtenáři námi zkoumaný problém, důležité předpoklady na nelinearitu a některá pomocná tvrzení. V následující části se věnujeme formulaci a důkazu našeho hlavního výsledku, kterým je existence řešení ve tvaru postupné vlny. Důkaz je založen na variačních metodách. Třetí podkapitola je věnována hledání klasického řešení a numerickým experimentům prováděným za účelem nalezení takového řešení. Na závěr uvádíme otevřené otázky a problémy.

Appendix A

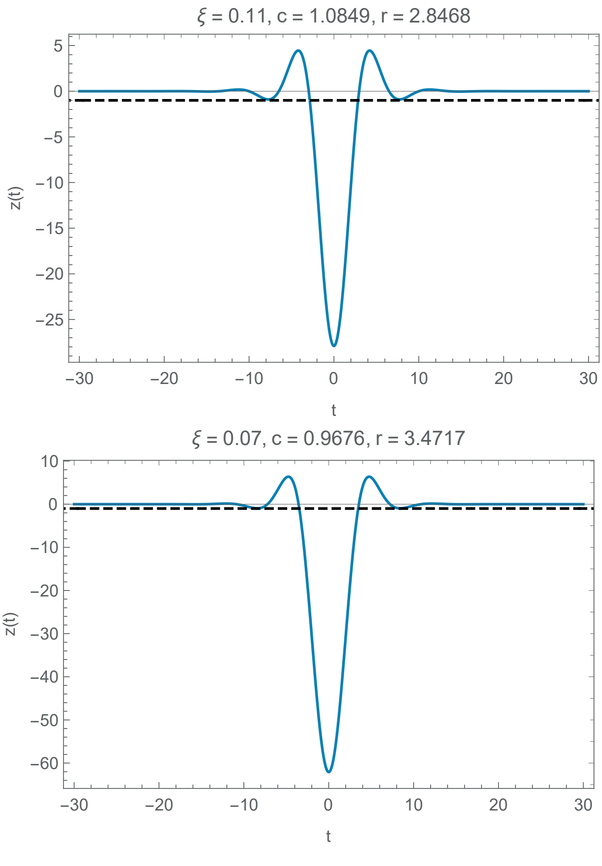

Figure 8: Graphical illustration of increasing amplitude for c tending to 0, the first part.

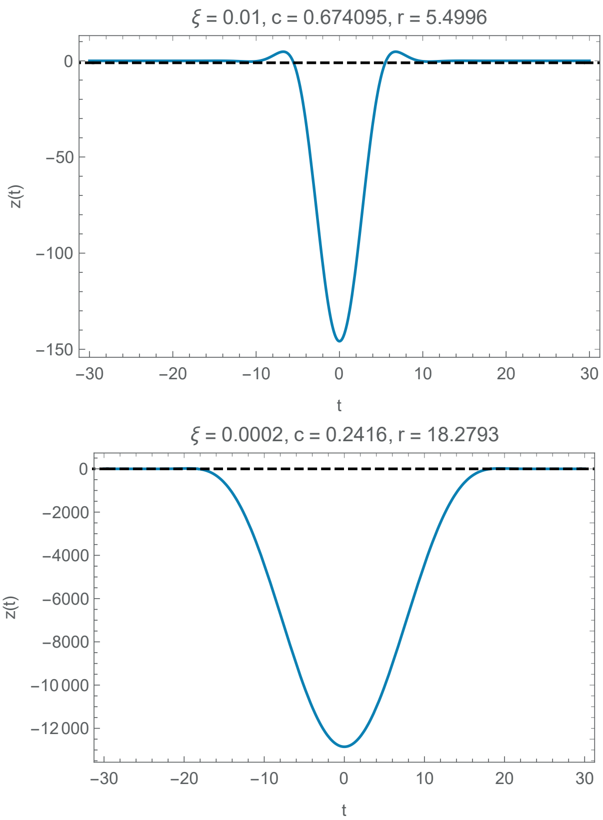

Figure 9: Graphical illustration of increasing amplitude for c tending to 0, the second part.

Appendix B

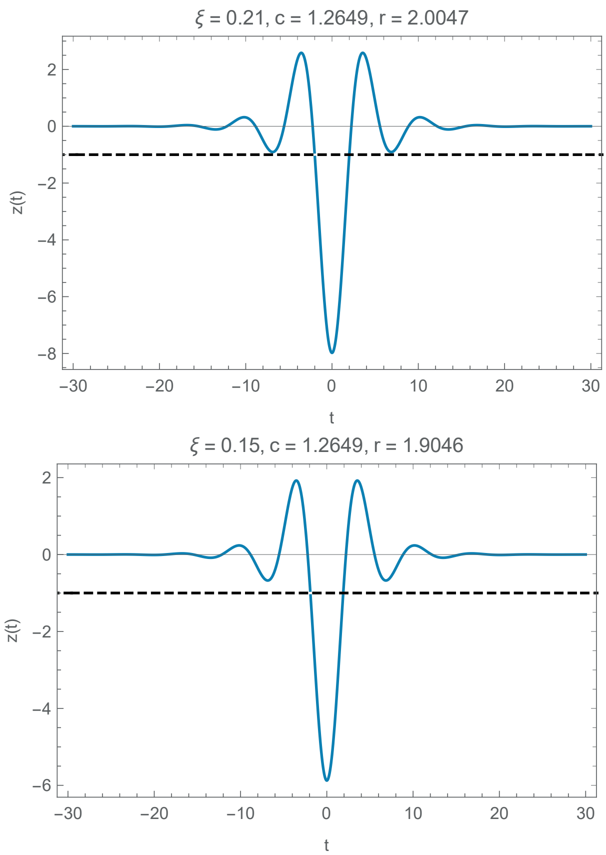

Figure 10: Graphical illustration of the solution for fixed c and changing ξ, the first part.

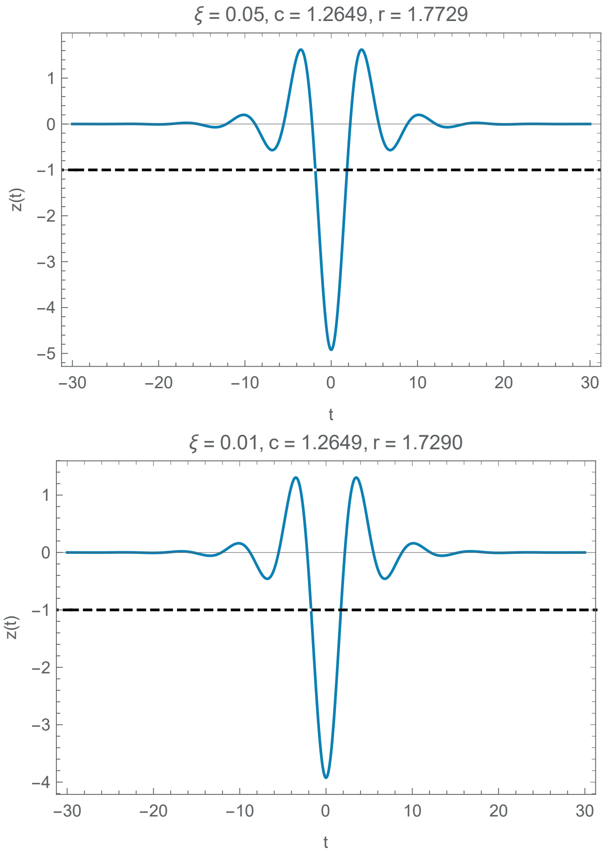

Figure 11: Graphical illustration of the solution for fixed c and changing ξ, the second part

Bibliography

[1] O. H. Amann, T. Kármán, and G. B. Woodruff. The failure of the tacoma narrows bridge. *Federal Works Agency.*, 1941.

[2] A. Ambrosetti and P. H. Rabinowitz. Dual variational methods in critical point theory and applications. *Journal of Functional Analysis*, 14(4):349–381, 1973.

[3] V. Benci and D. Fortunato. Existence of solitons in the nonlinear beam equation. *Journal of Fixed Point Theory and Applications.*, 11(2):1–20, 2011.

[4] B. Breuer, J. Horák, P. J. McKenna, and M. Plum. A computer assisted existence and multiplicity proof for travelling waves in a nonlinearly supported beam. *Journal of Differential Equations*, 224:60–97, 2006.

[5] J. C. Camacho, M. S. Bruzón, J. Ramírez, and M. L. Gandarias. Exact travelling wave solutions of a beam equation. *Journal of Nonlinear Mathematical Physics*, 18(1):33–49, 2011.

[6] A. R. Champneys and P. J. McKenna. On solitary waves of a piecewise-linear suspension bridge model. 1996.

[7] Y. Chen. Traveling wave solutions to beam equation with fast-increasing nonlinear restoring forces. *Applied Mathematics – A Journal of Chinese Universities*, 15(2):156–160, 2000.

[8] Y. Chen and P. J. McKenna. Traveling waves in a nonlinearly suspended beam: Theoretical results and numerical observations. *Journal of differential equations*, 136:325–355, 1997.

[9] Y. Chen and P. J. McKenna. Travelling waves in a nonlinearly suspended beam: Some computational results and four open questions. *Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 355:2175–2184, 1997.

[10] Y. S. Choi and P. J. McKenna. A mountain pass method for the numerical solution of semilinear elliptic problems. *Nonlinear Analysis: Theory, Methods & Applications*, 20(4):417–437, 1993.

[11] E. Dancer. On the dirichlet problem for weakly non-linear elliptic partial differential equations. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 76(4):283–300, 1977.

[12] M. Diaferio and V. Sepe. Smoothed "slack cable" models for large amplitude oscillations of suspension bridges. *Mechanics Based Design of Structures and Machines*, 32(3):363–400, 2004.

[13] P. Drábek, G. Holubová, A. Matas, and P. Nečesal. Nonlinear models of suspension bridges: discussion of the results. *Applications of Mathematics.*, 48(6):497–514, 2003.

[14] P. Drábek and J. Milota. *Methods of Nonlinear Analysis.* Springer Basel, 2013.

[15] S. Fučík. Nonlinear equations with noninvertible linear part. *Czechoslovak Mathematical Journal*, 24(3):467–495, 1974.

[16] S. Fučík. Boundary value problems with jumping nonlinearities. *Časopis pro pěstování matematiky*, 101(1):69–87, 1976.

[17] G. Holubová and H. Levá. Travelling wave solutions of the beam equation with jumping nonlinearity. Submitted.

[18] P. Karageorgis and J. Stalker. A lower bound for the amplitude of traveling waves of suspension bridges. *Nonlinear Analysis: Theory, Methods & Applications*, 75(13):5212–5214, 2012.

[19] A. C. Lazer and P. J. McKenna. Large–amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis. *SIAM Review*, 32:537–578, 1990.

[20] A. C. Lazer and P. J. McKenna. On travelling waves in a suspension bridge model as the wave speed goes to zero. *Nonlinear Analysis: Theory, Methods & Applications*, 74(12):3998–4001, 2011.

[21] T. Li, J. Sun, and T. Wu. Existence of homoclinic solutions for a fourth order differential equation with a parameter. *Applied Mathematics and Computation*, 251:499–506, 2015.

[22] E. H. Lieb and M. Loss. *Analysis.* American Mathematical Society, 2001.

[23] P. L. Lions. The concentration-compactness principle in the calculus of variations. the locally compact case, part 1. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 1(2):109–145, 1984.

[24] P. Long. Tacoma narrows bridge collapses on november 7, 1940. `https://historylink.org/File/5048`. Accessed: October 18, 2022.

[25] P. J. McKenna and W. Walter. Travelling waves in a suspension bridge. *SIAM Journal of Applied Mathematics*, 50(3):703–715, 1990.

[26] L. A. Peletier and W. C. Troy. Multibump periodic travelling waves in suspension bridges. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 128(3):631–659, 1998.

[27] F. Sadyrbaev and I. Yermachenko. Remarks on travel waves problem.

[28] S. Santra and J. Wei. Homoclinic solutions for fourth order traveling wave equations. *SIAM journal on mathematical analysis*, 41(5):2038–2056, 2009.

[29] D. Smets and J. B. van den Berg. Homoclinic solutions for swift–hohenberg and suspension brigde type equations. *Journal of Differential Equations*, 184(1):78–96, 2002.

[30] W. A. Strauss. *Partial Differential Equations, An Introduction Student Solutions Manual.* John Wiley & Sons Inc., New York, 1992.