FACULTY OF APPLIED SCIENCES
UNIVERSITY
OF WEST BOHEMIA

STUDENT
SCIENTIFIC CONFERENCE 2024

Existence and number of travelling waves in generalised suspension bridge model

Hana Formánková Levá[1], Gabriela Holubová[2]

1 Introduction

We study the existence and number of travelling wave solutions of the problem

$$\begin{cases} u_{tt} + u_{xxxx} + \alpha u^+ - \beta u^- + g(u) = 1, & x \in \mathbb{R},\ t > 0, \\ u(x,t) \to 1/\alpha,\ u_x(x,t) \to 0 \ \text{ for } |x| \to +\infty, \end{cases} \tag{1}$$

where $u = u(x,t)$, $\alpha > 0$, $\beta \geq 0$, $u^\pm = \max\{\pm u, 0\}$ and $g(1/\alpha) = 0$. The problem (1) can be used as a model of an asymmetrically supported bending beam or a generalized model of a suspension bridge.

The existence of travelling waves in suspension bridge model was originally studied in McKenna and Walter (1990), then extended by many others after that. In Holubová and Levá (2023) we proved the existence of infinitely many travelling wave solutions of (1) with wave speed from interval $\left(\sqrt[4]{100\beta/9}, \sqrt[4]{4\alpha}\right)$ under weakened assumptions on the nonlinearity g than in previous literature. We used Mountain Pass Theorem (MPT) and the nonzero weak convergence after a suitable translation. Further, we extended our results in Holubová, Levá and Nečesal (2024) by determining the largest possible range of wave speeds for which the existence of travelling wave can be proved by using MPT.

Next, we would like to address the problem of the number of travelling wave solutions of (1) with fixed wave speed. This is inspired by Champneys and McKenna (1997).

2 Conversion of the problem (1)

For the sake of simplicity, we will now consider only $g \equiv 0$. First, we transform the considered problem (1) by substituting $\alpha u(x,t) = y(\sqrt[4]{\alpha}x - c\sqrt{\alpha}t) = y(s)$. We also introduce the parameter $\xi = \beta/\alpha \geq 0$ and we take $z = z(s) = y(s) - 1$. Thus, the equation in problem (1) can be rewritten as two linear ODEs

$$\begin{cases} z^{(4)} + c^2 z'' + z = 0 & \text{for } z \geq -1, \\ z^{(4)} + c^2 z'' + \xi z = 1 - \xi & \text{for } z \leq -1. \end{cases} \tag{2}$$

We look for special type of solutions, the so called symmetric *one-troughed* waves. These solutions are even and drop under the value -1 just once. We construct them by finding the analytical solutions z_1 and z_2 of both equations in (2), parametrizing the connection point $s = s^*$ (i.e. $z_1(s^*) = z_2(s^*)$) in phase space by $z_1'(s^*) = \theta$ and matching the two solutions and their first three derivatives in s^*.

[1] student of the doctoral degree program Mathematics, e-mail: levah@kma.zcu.cz
[2] Department of Mathematics and NTIS, Faculty of Applied Sciences, e-mail: gabriela@kma.zcu.cz

By eliminating some parameters we reduce the search for one-troughed travelling waves to the search for zeros of a function

$$L_k(c,\xi,\theta) = \frac{1}{2}\left(1+\frac{\kappa_2}{\kappa_1}\right) + \frac{1}{\pi}\arctan\left(\kappa_2 \frac{-\kappa_1^2+\xi+\xi\sqrt{2-c^2}\theta}{\xi\theta\kappa_1^2+\xi\sqrt{2-c^2}+\xi\theta(1-c^2)}\right) - \frac{\kappa_2}{\kappa_1\pi}\arctan\left(\kappa_1 \frac{-\kappa_2^2+\xi+\xi\sqrt{2-c^2}\theta}{\xi\theta\kappa_2^2+\xi\sqrt{2-c^2}+\xi\theta(1-c^2)}\right) - k \tag{3}$$

where $\kappa_{1,2}^2 = (c^2 \mp \sqrt{c^4-4\xi})/2$, $\theta \in (\theta_{min}(c), 0)$ and $k \in \mathbb{N}$.

3 Numerical experiments

In order to find out the number and form of solutions we present some numerical experiments. It seems that for any $c^2 \in (0,2)$ and $\xi = pc^4/4$ with $p \in (0,1)$ there are at most five symmetric one-troughed travelling waves, see Figure 1 on the left. It corresponds with results in Champneys and McKenna (1997).

In contrast with previous results in Holubová and Levá (2023) we are able to find also solutions with more than one local extrema in the interval where z drops below -1, see Figure 1 on the right. We have to note that analytical verification of these numerical results remains an open question.

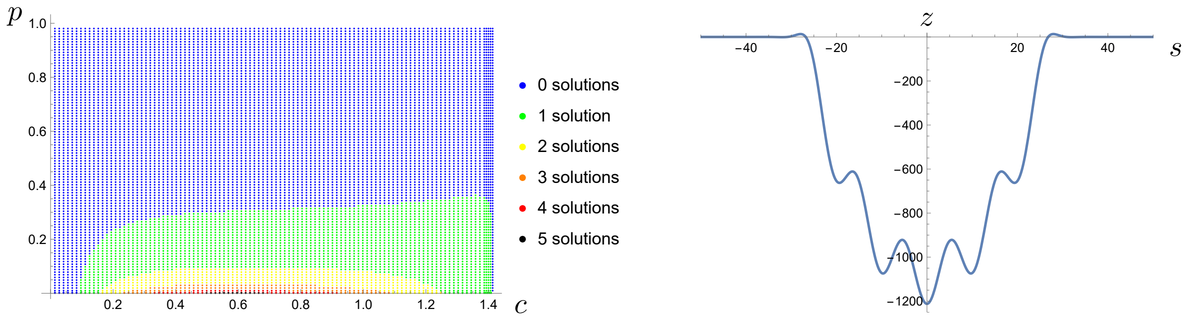

Figure 1: Number of solutions in region with admissible parameters c and p (on the left), one-troughed travelling wave solution with $c = 22\sqrt{2}/51$ and $p = 0.000654528$ (on the right).

Acknowledgement

Both authors were supported by Grant No. 22-18261S of the Grant Agency of the Czech Republic.

References

Holubová, G. and Levá, H. (2023) *Travelling wave solutions of the beam equation with jumping nonlinearity.* Journal of Mathematical Analysis and Applications, 527.2: 127466.

Holubová, G., Levá, H. and Nečesal, P. *Lower bounds for admissible values of the travelling wave speed in asymmetrically supported beam.* Submitted, 2024.

Champneys, A. R. and McKenna, P. J. (1997) On solitary waves of a piecewise linear suspended beam model. *Nonlinearity*, 10(6), 1763.

McKenna, P. J. and Walter, W. (1990) Travelling waves in a suspension bridge. *SIAM Journal of Applied Mathematics*, 50(3), pp. 703–715.