Applied and Computational Mechanics 7 (2013) 77–86

Semi-analytic solution to planar Helmholtz equation

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Received 21 December 2012; received in revised form 3 May 2013

Abstract

Acoustic solution of interior domains is of great interest. Solving acoustic pressure fields faster with lower computational requirements is demanded. A novel solution technique based on the analytic solution to the Helmholtz equation in rectangular domain is presented. This semi-analytic solution is compared with the finite element method, which is taken as the reference. Results show that presented method is as precise as the finite element method. As the semi-analytic method doesn't require spatial discretization, it can be used for small and very large acoustic problems with the same computational costs.

Keywords: semi-analytic, acoustics, Helmholtz, Galerkin, weighted residuals method

1. Introduction

Numerical methods that are used to simulate acoustic properties inside or outside an acoustic domain are under constant development. Sound, as we hear it, is composed of many tones, that propagate at certain excitation frequencies. These frequencies must not be close to each other. Moreover it is probable that they span from lower over middle to high excitation frequencies. The wide range is a computational difficulty. Many computational methods exist. However limits of their usability restrict them to a limited range of acoustic frequencies. One of the most widely used methods for computing interior problems, the finite element method [14], is generally suitable only for lower excitation frequencies. As the frequency rises and the element size diminishes, pollution error deviates the correct solution. Boundary element method [4] is similar to the finite element method. This method is often used, when exterior or sound radiation problems are to be solved. Usability limits of these two method are given by the number and the size of used elements. The higher the excitation frequency is, the higher is the number of nodes and the smaller the elements become. At some element size numerical errors start to depreciate the solution. Then other methods, as e.g. the statistical energy analysis [13] has to be deployed. However there may be a range of excitation frequencies that is already too high for finite and boundary element method and too low for the statistical energy analysis. To address this "non-solvable" frequency range many extensions to before-mentioned methods were developed. Or new approaches based on some analytic properties were proposed. Approach based on the superposition theory and the integro-modal approach [1] is presented in [6]. Another possibility is to try to solve directly the Helmholtz equation (3). Either by an iterative procedure as in [10] or by the use of variational theory as in [8]. The paper [3] presents improved element free Galerkin method. Approaches that are based on the analytic solution to the Helmholtz equation are also under development. So called Trefftz methods [5] use as approximation functions harmonic

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and evanescent exponential functions that are the solution of the underlying partial differential equation in a rectangular or a block domain. Recently developed wave-based method [11] belongs to the group of Trefftz methods. However the methods based on analytic solution require simpler geometry.

In this text a novel approach based on analytic solution to Helmholtz equation in a rectangular domain (3) is described. Full derivation of the approximation functions and the application on a car-like interior cavity is described.

2. Mathematical description of the acoustic problem

Fluctuation of the acoustic pressure p in two-dimensional space is described in general by wave equation [12]

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2}. \tag{1}$$

In this text time-harmonic acoustic pressure behavior is considered

$$p(x, y, t) = p_0(x, y) \exp(j\omega t), \tag{2}$$

where $j = \sqrt{-1}$, c_0 $[\mathrm{m/s}]$ being the speed of sound in the acoustic fluid of the density ϱ $[\mathrm{kg/m^3}]$ and ω $[\mathrm{s^{-1}}]$ being the radial excitation frequency. The equation (1) is transformed into Helmholtz equation [7]

$$\frac{\partial^2 p_0}{\partial x^2} + \frac{\partial^2 p_0}{\partial y^2} + k^2 p_0 = 0, \tag{3}$$

where the wave number $k = \omega/c_0$. For the sake of notation simplicity, let's denote the acoustic pressure amplitudes $p_0(x, y)$ as $p(x, y)$.

The equation (3) is defined on a bounded region Ω. On the boundary Γ following types of boundary conditions may be applied.

- Normal acoustic velocity $\bar{v}$ on the boundary Γ_v is prescribed by Neumann boundary conditions

$$\frac{j}{\varrho\omega} \frac{\partial p(x, y)}{\partial \vec{n}} = \bar{v} \quad \text{on } \Gamma_v. \tag{4}$$

  $\vec{n}$ being the outward pointing normal of the domain Ω.

- Acoustic pressure $\bar{p}$ on the boundary Γ_p is prescribed by Dirichlet boundary conditions

$$p(x, y) = \bar{p} \quad \text{on } \Gamma_p. \tag{5}$$

The solution of (3) in Cartesian coordinates in rectangular domain Ω is defined according to [7] as

$$p(x, y) = (A \cos \alpha x + B \sin \alpha x) (C \cos \beta y + D \sin \beta y). \tag{6}$$

The constants A, B, C, D and the wave numbers α and β are to be determined from the boundary conditions. Additionally for α and β holds

$$\alpha^2 + \beta^2 = k^2. \tag{7}$$

3. Finite element method

The finite element method, as proposed in [2, 15] operates with elements and nodes, that are defined in the domain Ω. Amplitudes of the acoustic pressure in an element p_f^e are described by polynomial functions (8). The superscript e denotes elements and the subscript f denotes the finite element method. In two dimensional problems the polynomials are usually bilinear functions $g(x^e, y^e)$. In that case p_f^e is defined as

$$p_f^e(x^e, y^e) = \sum_{i=1}^{m_n} p_i^e \cdot g_i(x^e, y^e), \tag{8}$$

where x^e, y^e are element local coordinates and p_i^e are the unknown acoustic pressures in m_n element's nodes. The vector $\boldsymbol{p_f^e} = [p_1^e, \ldots, p_{m_n}^e]^T \cdot e^{j\omega t}$ stores nodal pressures p_i^e of the element's nodes. As stated in [2], for every element mass matrix $\boldsymbol{M_f^e}$, damping matrix $\boldsymbol{B_f^e}$, stiffness matrix $\boldsymbol{K_f^e}$ and the vector of known nodal pressures $\boldsymbol{b_f^e}$ can be derived. Resulting damped element equations of motion are

$$\boldsymbol{M_f^e} \frac{\partial^2 \boldsymbol{p_f^e}}{\partial t^2} + \boldsymbol{B_f^e} \frac{\partial \boldsymbol{p_f^e}}{\partial t} + \boldsymbol{K_f^e} \boldsymbol{p_f^e} = \boldsymbol{b_f^e}. \tag{9}$$

The domain Ω consists of multiple elements. Total number of nodes in Ω is m. Mass, damping and stiffness matrices and right hand side vectors of individual elements are composed together to form global matrices of the whole system. Resulting steady-state acoustic problems are described by

$$\left(-\omega^2 \boldsymbol{M_f} + j\omega \boldsymbol{B_f} + \boldsymbol{K_f}\right) \boldsymbol{p_f} = \boldsymbol{b_f}. \tag{10}$$

The vector $\boldsymbol{p_f} = [p_1, \ldots, p_m]^T$ stores all unknown nodal acoustic pressures in Ω. Matrices in (10) are frequency independent and in case undamped system is being solved damping matrix $\boldsymbol{B_f}$ becomes zero.

The number of used elements and nodes is driven by the highest excitation component of ω. Eight elements per the shortest wavelength $\lambda_{min} = (2\pi c_0)/\omega_{max}$ are said to secure proper and precise solution.

4. Semi-analytic solution

Presented semi-analytic method was derived from the solution (6) and a specific set of boundary conditions applied to rectangular domain Ω.

4.1. Derivation of the basis functions

The approximation of the solution is based on the analytic solution (6). Derivation of the linear combination of the semi-analytic solution requires only Neumann boundary conditions. A specific set of boundary conditions is applied to the rectangular domain Ω with dimensions L_x and L_y, see Fig. 1. Unlike Dirichlet boundary conditions, that have to be continuous in the corners of the domain, Neumann boundary conditions may be discontinuous.

The course of the normal acoustic velocity excitation function $\bar{v}$ along the boundary Γ_v is in Fig. 1. Three of the four domain sides have zero normal acoustic velocity. On the last side a non-zero normal acoustic velocity function is prescribed. Substituted boundary conditions for

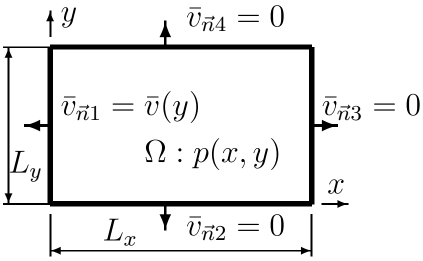

Fig. 1. Normal acoustic velocity boundary conditions for one domain problem

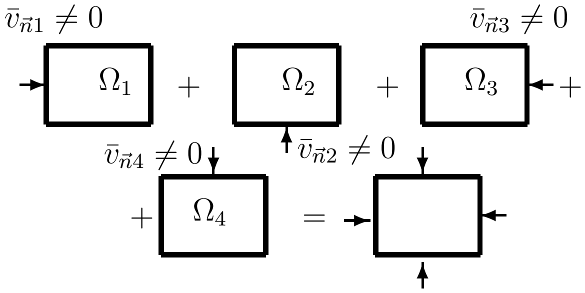

Fig. 2. Schematic procedure of computing rectangular cavities with non-zero boundary conditions on all sides

one domain are as follows:

$$v_{\vec{n}1}(0, y) : \; -\bar{v}(y) = \frac{j\alpha}{\varrho\omega}(-A \sin \alpha 0 + B \cos \alpha 0)(C \cos \beta y + D \sin \beta y), \tag{11}$$

$$v_{\vec{n}2}(x, 0) : \; 0 = \frac{j\beta}{\varrho\omega}(A \cos \alpha x + B \sin \alpha x)(-C \sin \beta 0 + D \cos \beta 0), \tag{12}$$

$$v_{\vec{n}3}(L_x, y) : \; 0 = \frac{j\alpha}{\varrho\omega}(-A \sin \alpha L_x + B \cos \alpha L_x)(C \cos \beta y + D \sin \beta y), \tag{13}$$

$$v_{\vec{n}4}(x, L_y) : \; 0 = \frac{j\beta}{\varrho\omega}(A \cos \alpha x + B \sin \alpha x)(-C \sin \beta L_y + D \cos \beta L_y). \tag{14}$$

In the course of manipulation of equations (11)–(14) it can be assumed, that the space coordinates x, y, and the wavenumbers α and β can be non-zero values. Then the following relations can be determined:

$$D = 0 \quad \text{and} \quad \beta_n = n\frac{\pi}{L_y}, \qquad n = 0, \ldots, n_F. \tag{15}$$

Every wave number β_n is related to α_n by (7). Though α_n can be easily computed. From (13) the equation for A_n

$$A_n = B_n \frac{1}{\tan(\alpha_n L_x)} \tag{16}$$

is obtained. Substituting of (15), (16) and α_n into (6) and combining newly-emerged $B_n \cdot C_n = H_n$ also in (19) the final pressure approximation (17) in one rectangular domain is obtained as

$$\tilde{p}(x, y) = \sum_{n=0}^{n_F} H_n \left(\frac{1}{\tan(\alpha_n L_x)} \cos \alpha_n x + \sin \alpha_n x \right) \cos \beta_n y. \tag{17}$$

In matrix notation,

$$\tilde{p} = \boldsymbol{A}^\star \cdot \boldsymbol{h}^\star, \tag{18}$$

the basis functions are stored in a row vector $\boldsymbol{A}^\star = [A_0, \ldots, A_{n_F}]$ and the unknown coefficients H_n are stored in a column vector $\boldsymbol{h}^\star = [H_0, \ldots, H_{n_F}]^T$. Pressure approximation $\tilde{p}$ for the set of boundary conditions from Fig. 1 is a linear combination for unknown coefficients H_n.

From (11) a relation between the boundary value function $\bar{v}(y)$ and the unknown coefficients H_n is obtained as

$$H_n \alpha_n \cos \beta_n y_m = -\frac{\varrho\omega}{j} \bar{v}(y_m). \tag{19}$$

Unknown H_n from (19) can be obtained e.g. by using the least square method. In that case the boundary function $\bar{v}$ is evaluated at discrete positions $y = y_m$.

The number of basis functions n_F is derived from the requirement that the wavelength corresponding to the highest n is at least half the length of the wavelength in the acoustic fluid excited at the radial frequency ω.

4.2. Extension to more excited sides

Acoustic pressure approximation in (17) can only solve problems with one combination of boundary conditions. However, domain Ω can generally be excited on all sides by non-zero normal acoustic velocities. Acoustic problem is a linear problem of seeking a solution to a partial differential equation (1). For linear problems superposition theorem is valid.

Thus the solution of a rectangular domain excited on all sides is computed as the sum of four properly modified solutions (17). The scheme is in Fig. 2. The non-zero excitation functions can be discontinuous in the corners. As with only one excited side, the approximation for all excited sides can be written in matrix form. A row vector $\boldsymbol{A} = [\boldsymbol{A}_1^\star, \ldots \boldsymbol{A}_4^\star]$ collects all basis functions of all four approximations and the column vector $\boldsymbol{h} = [\boldsymbol{h}_1^\star, \ldots, \boldsymbol{h}_4^\star]^T$ stores all the unknown coefficients.

4.3. Application of the weighted residuals method

The unknown coefficients H_n can be obtained from (19) using least square method. This works for both the problem with single excited side and for the problem with all excited sides. In the latter case it requires computing the coefficients four times, individually for each side.

There may exist an acoustic domain, that is composed of two or more rectangular domains. In that case evaluating four individual solutions represents a problem for the coupling. Independently obtained solutions can not describe properly the continuity conditions on the common boundary between the domains.

Next, let assume an acoustic problem consisting of two rectangular domains. Between the domains there is interface boundary Γ_i. The outer boundary can be excited by both Dirichlet or Neumann boundary conditions.

System of linear equations is a convenient way of solving linear problems. As long as the approximation $\tilde{p}$ from (17) is a linear combination of basis function, Galerkin weighted residuals method described in [9] can be used. Using Galerkin weighted residuals method the unknown coefficient in $\tilde{p}$ are determined from a system of linear equations. In the Galerkin modification of weighted residuals method basis functions in $\tilde{p}$ are used as the weighting functions. The method minimizes the residuals in the domains Ω and on the boundary Γ. The residuals

$$R_\Omega = \frac{\partial^2 \tilde{p}}{\partial x^2} + \frac{\partial^2 \tilde{p}}{\partial y^2} + k^2 \tilde{p} \quad \text{on } \Omega, \tag{20}$$

$$R_{\Gamma_v} = \frac{j}{\varrho\omega} \frac{\partial \tilde{p}}{\partial \vec{n}} - \bar{v} \quad \text{on } \Gamma_v, \tag{21}$$

$$R_{\Gamma_p} = \tilde{p} - \bar{p} \quad \text{on } \Gamma_p \tag{22}$$

are the difference between the exact solution and the approximation. However R_Ω is in every domain identically equal to zero, because the basis functions are solution of the equation (3). Application of Galerkin weighted residuals method leads to the following equation in the case

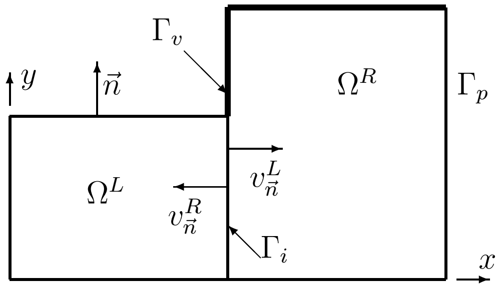

Fig. 3. Scheme of coupling of two domains when the Galerkin weighted residuals method is used

of one domain

$$\underbrace{\int_{\Gamma_p} \frac{j}{\varrho\omega} \frac{\partial \boldsymbol{A}^T}{\partial \vec{n}} \boldsymbol{A} \, \mathrm{d}\Gamma_p}_{\boldsymbol{C_p}} \cdot \boldsymbol{h} + \underbrace{\int_{\Gamma_v} \boldsymbol{A}^T \frac{j}{\varrho\omega} \frac{\partial \boldsymbol{A}}{\partial \vec{n}} \, \mathrm{d}\Gamma_v}_{\boldsymbol{C_v}} \cdot \boldsymbol{h} =$$
$$\underbrace{\int_{\Gamma_p} \frac{j}{\varrho\omega} \frac{\partial \boldsymbol{A}^T}{\partial \vec{n}} \bar{p} \, \mathrm{d}\Gamma_p}_{\boldsymbol{b_p}} + \underbrace{\int_{\Gamma_v} \boldsymbol{A}^T \bar{v} \, \mathrm{d}\Gamma_v}_{\boldsymbol{b_v}} . \tag{23}$$

On the boundary Γ_i both pressure and velocity continuity have to be enforced. Both requirements represent a boundary condition. The pressure continuity is applied to the domain on the "left" side of the interface and the velocity continuity condition to the "right" domain. Let's denote the domains around the boundary Ω^L and Ω^R. The residuals on the interface are

$$R^L_{\Gamma_i} = \tilde{p}^L - \tilde{p}^R, \tag{24}$$
$$R^P_{\Gamma_i} = \frac{j}{\varrho\omega} \frac{\partial \tilde{p}^L}{\partial \vec{n}^L} + \frac{j}{\varrho\omega} \frac{\partial \tilde{p}^R}{\partial \vec{n}^R}. \tag{25}$$

Equations (26) and (27) show the extension of the formulation (23) for a problem with two acoustic domains. The extension adds these members

$$\ldots + \underbrace{\int_{\Gamma_i} \frac{j}{\varrho\omega} \frac{\partial \boldsymbol{A^L}^T}{\partial \vec{n}^L} \boldsymbol{A^L} \, \mathrm{d}\Gamma_i}_{\boldsymbol{C^L_{Bp}}} \cdot \boldsymbol{h^L} + \underbrace{\int_{\Gamma_i} \frac{j}{\varrho\omega} \frac{\partial \boldsymbol{A^L}^T}{\partial \vec{n}^L} \boldsymbol{A^R} \, \mathrm{d}\Gamma_i}_{\boldsymbol{C_{LR}}} \cdot \boldsymbol{h^R} = \ldots, \tag{26}$$

$$\ldots + \underbrace{\int_{\Gamma_i} \frac{j}{\varrho\omega} \frac{\partial \boldsymbol{A^R}^T}{\partial \vec{n}^R} \boldsymbol{A^L} \, \mathrm{d}\Gamma_i}_{\boldsymbol{C_{RL}}} \cdot \boldsymbol{h^L} + \underbrace{\int_{\Gamma_i} \frac{j}{\varrho\omega} \frac{\partial \boldsymbol{A^R}^T}{\partial \vec{n}^R} \boldsymbol{A^R} \, \mathrm{d}\Gamma_i}_{\boldsymbol{C^R_{Bv}}} \cdot \boldsymbol{h^R} = \ldots \tag{27}$$

4.4. *Resulting system of equations*

Assembled matrices in (23) form a system of linear equations. Problem of one domain is solved with

$$[\boldsymbol{C_p} + \boldsymbol{C_v}] \cdot \boldsymbol{h} = \boldsymbol{b_v} + \boldsymbol{b_p}. \tag{28}$$

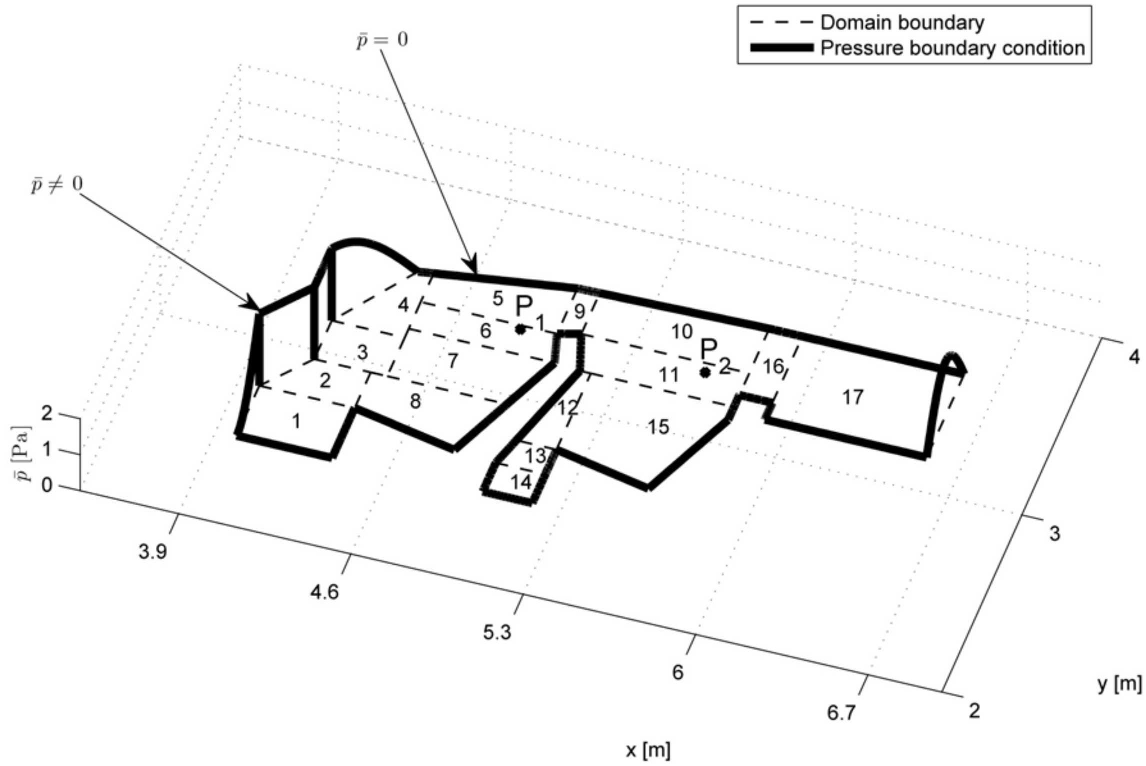

Fig. 4. Course of acoustic pressure boundary conditions. Points $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are depicted as well as the interior boundaries of individual domains

In case more domains are coupled together, the system is extended with the matrices from (26) and (27). In this text only the situation of two coupled domains is shown

$$\begin{bmatrix} \boldsymbol{C}_{\boldsymbol{p}}^{\boldsymbol{L}} + \boldsymbol{C}_{\boldsymbol{v}}^{\boldsymbol{L}} + \boldsymbol{C}_{\boldsymbol{Bp}}^{\boldsymbol{L}} & \boldsymbol{C}_{\boldsymbol{LR}} \\ \boldsymbol{C}_{\boldsymbol{RL}} & \boldsymbol{C}_{\boldsymbol{p}}^{\boldsymbol{R}} + \boldsymbol{C}_{\boldsymbol{v}}^{\boldsymbol{R}} + \boldsymbol{C}_{\boldsymbol{Bv}}^{\boldsymbol{R}} \end{bmatrix} \cdot \begin{pmatrix} \boldsymbol{h}^{\boldsymbol{L}} \\ \boldsymbol{h}^{\boldsymbol{R}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_{\boldsymbol{p}}^{\boldsymbol{L}} + \boldsymbol{b}_{\boldsymbol{v}}^{\boldsymbol{L}} \\ \boldsymbol{b}_{\boldsymbol{p}}^{\boldsymbol{R}} + \boldsymbol{b}_{\boldsymbol{v}}^{\boldsymbol{R}} \end{pmatrix}. \tag{29}$$

Extension of the system (29) for more domains is straightforward. Unlike FEM, derived matrices are frequency dependent and the matrix elements have to be recomputed for every excitation frequency.

5. Application on a car-like acoustic cavity

Proposed acoustic solution was compared to the finite element solution on an example of a car-like interior.

5.1. Acoustic domain and the boundary conditions

The shape, depicted in Fig. 4, represents simplified car interior. Semi-analytic method is designed for the use on convex acoustic domains, that resemble rectangles. To fulfil this demand, the car-like cavity was divided into seventeen nearly rectangular domains, see Fig. 4. The whole system is solved using the Galerkin weighted residuals method applied to (17).

Only acoustic pressure boundary conditions are applied. The course and the location of applied boundary conditions is shown in the Fig. 4. Maximum amplitude of the excitating acoustic pressure is $\bar{p}_{max} = 2$ Pa. The course of the boundary condition function is continuous.

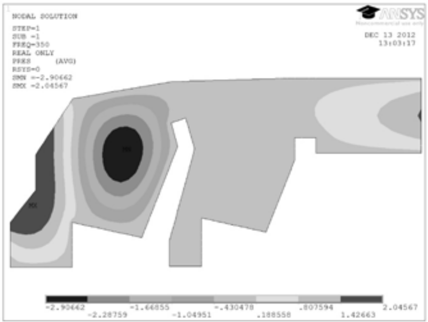

Fig. 5. Acoustic pressure from the finite element analysis. Excitation frequency of $f = 350\,\text{Hz}$

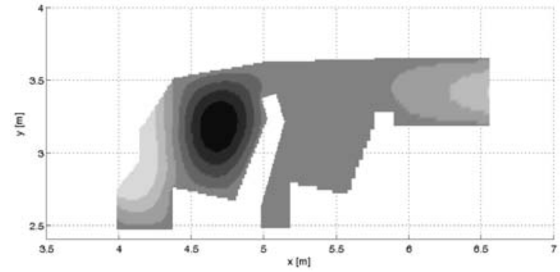

Fig. 6. Acoustic pressure from semi-analytic method. Excitation frequency of $f = 350\,\text{Hz}$

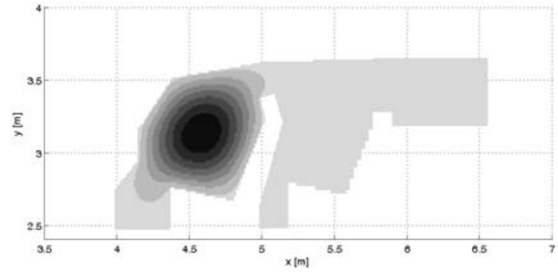

Fig. 7. Acoustic pressure from the semi-analytic method. Excitation frequency of $f = 296\,\text{Hz}$, which is also one of computed system eigenfrequencies

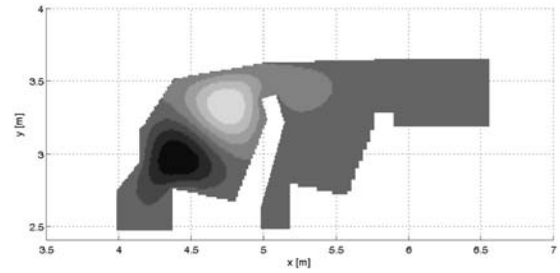

Fig. 8. Acoustic pressure from the semi-analytic method. Excitation frequency of $f = 438\,\text{Hz}$, which is also one of computed system eigenfrequencies

5.2. *Results*

In the finite element analysis an convergence study was carried out. Model with 965 nodes was chosen as the optimal one. For finer meshes the improvement of accuracy was negligible. Matrices of the semi-analytic method are frequency dependent. Maximum number of unknowns H_n of the whole system was 748. This represents 22.5 % less unknowns in comparison with the finite element analysis. Computed transfer functions of both semi-analytic and the finite element methods in the points $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are depicted in Fig. 9 and Fig. 10, respectively. Computation was done in the frequency range of

$$f = [144, \ldots, 474]\,\text{Hz}.$$

The solution of the system excited at $f = 350\,\text{Hz}$ computed by the finite element method is in Fig. 5 and the results of the semi-analytic method are in Fig. 6. Figs. 7 and 8 show the computed acoustic pressure at frequencies of $f = 296\,\text{Hz}$ and $f = 438\,\text{Hz}$, respectively. These frequencies are the eigenfrequencies of the whole system.

The frequency characteristics show the sound pressure level (30). Sound pressure level was computed as follows

$$L_{sp} = 20 \log_{10} \frac{p}{|p_{ref}|}, \tag{30}$$

where $p_{ref} = 20 \cdot 10^{-6}\,\text{Pa}$. As long as the excitation frequency is different from the eigenfrequency of the system, the results exhibit very good agreement with the finite element analysis results, that were taken as reference. Outside of the eigenfrequencies the relative error is lower

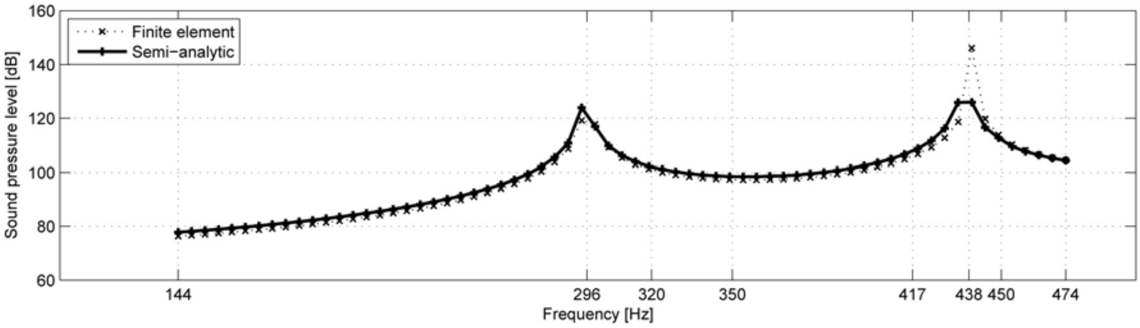

Fig. 9. Transfer functions of the acoustic pressure in the point $\boldsymbol{P}_1$

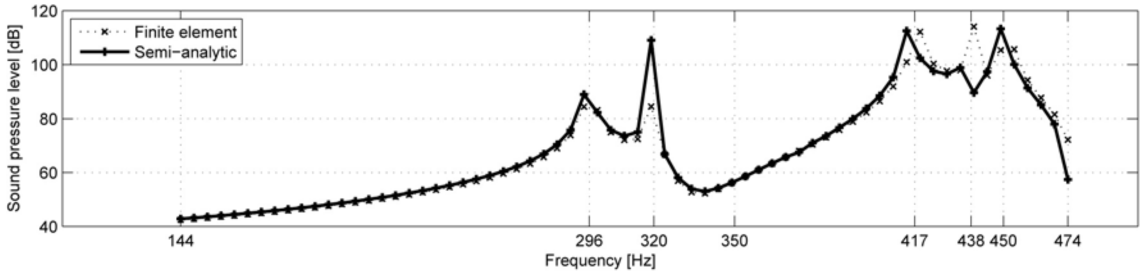

Fig. 10. Transfer functions of the acoustic pressure in the point $\boldsymbol{P}_2$

than three percent. The semi-analytic method predicts the eigenfrequencies of the system very accurately. Pressure fields for excitation frequencies of $f = 296\,\text{Hz}$ in Fig. 7 and $f = 438\,\text{Hz}$ in Fig. 8 are correctly predicted. However these frequencies are the system eigenfrequencies and the amplitudes are deviated from the reference finite element solution. In case the excitation frequency equals any system eigenfrequency, the set of the independent basis functions $\boldsymbol{A}$ becomes linearly dependent. Thus the system of linear equations becomes badly conditioned. The precision of the results of such a system is lowered and the numerical results are deviated from the reference solution.

The bad conditioning could be avoided if damping was introduced. The example has only pressure boundary conditions and thus the system is undamped. When part of the boundary has impedance boundary condition, the system is damped. Then the maximum sound pressure level of a damped system is reduced and the precision improved. Basis functions of a damped semi-analytic method are supposed not to become linearly dependent, when the excitation frequency coincides with the eigenfrequency of the system. Implementation of impedance boundary conditions is the subject of further development.

6. Conclusion

A novel semi-analytic method for solving acoustic problems was suggested. It is based on the exact solution to the Helmholtz equation (3) in a rectangular domain. Acoustic pressure approximation (17) is in the form of linear combination. Derivation of basis functions $\boldsymbol{A}$ as well as the derivation of the system of linear equations was presented. Results of the semi-analytic method were compared with the finite element analysis. A study on a car-like cavity was carried out. The results of the undamped model show very good agreement. Acoustic pressure in points $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ for the excitation frequency that corresponds to the eigenfrequency of the system is a little deviated, however introduction of damping via impedance boundary conditions should restore good correspondence.

The semi-analytic method is based on the analytic solution. With its lower computational requirements it has the potential to solve problems, where other methods such finite element method would require too much computational force.

Acknowledgements

Supported by research project No. P101/12/1306 *Biomechanical modeling of human voice production*.

References

[1] Anyunzoghe, E., Cheng, L., Improved integro-modal approach with pressure distribution assessment and the use of overlapped cavities, Applied Acoustics 63 (11) (2002) 1 233-1 255.
[2] Bathe, K.-J., Finite element procedures in engineering analysis, 1st edition, Prentice-Hall, 1982.
[3] Bouillard, P., Lacroix, V., De Bel, E., A wave-oriented meshless formulation for acoustical and vibro-acoustical applications, Wave Motion 39 (4) (2004) 295-305.
[4] Kirkup, S., The boundary element method in acoustics, 2nd edition, Integrated Sound Software, 2007.
[5] Kita, E., Kamiya, N., Trefftz method: An overview, Advances in Engineering Software 24 (1-3) (1995) 3-12.
[6] Li, Y., Cheng, L., Vibro-acoustic analysis of a rectangular-like cavity with a tilted wall, Applied Acoustics 68 (7) (2007) 739-751.
[7] Merhaut, J., Theory of electro-acoustic devices, Czechoslovak Academy of Sciences, Praha, 1955. (in Czech)
[8] Momani, S., Abuasad, S., Application of He's variational iteration method to Helmholtz equation, Chaos, Solitons & Fractals 27 (5) (2006) 1 119-1 123.
[9] Nellis, G., Klein, S., The Galerkin weighted residual method, Cambridge, 2008, chap. 2.
[10] Otto, K., and Larsson, E., Iterative solution of the Helmholtz equation by a second-order method, UMIACS TR-96-95, (1996), 1-18.
[11] Pluymers, B., Wave based modelling methods for steady-state vibro-acoustics, Ph.D. thesis, Katholieke Universiteit Leuven, Department of Mechanical Engineering, Celestijnenlaan 300 B 3001, Heverlee, Belgium, 2006.
[12] Rienstra, S.-W., Hirschberg, A., An introduction to acoustics, Eindhoven University of Technology, 2004.
[13] Wijker, J., Statistical energy analysis, Random Vibrations in Spacecraft Structures Design, vol. 165 of Solid Mechanics and Its Applications. Springer Netherlands, 2009, pp. 51-320.
[14] Zienkiewics, O., Taylor, R., The finite element method: Fluid dynamics, 5th edition, vol. 3. Butterworth Heinemann, 2000.
[15] Zienkiewics, O., Taylor, R., The finite element method: The basis, 5th edition, vol. 1. Butterworth Heinemann, 2000.