Blending Implicit Shapes Using Smooth Unit Step Functions

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ABSTRACT

In constructive solid geometry (CSG), highly smooth and manageable blending operations are always desirable. In this paper, we propose two novel ways to blend implicitly defined geometric objects using smooth unit step functions. With the proposed techniques, not only is it direct to build an implicit blending operation with controllable blending range, but also the blending operation can be constructed to whatever level of smoothness. In addition, the mathematical descriptions of the techniques are quite simple and elegant.

Keywords

CSG, blending operation, implicit curve, implicit surface.

1 Introduction

Modelling computer graphics objects as implicit functions has received more and more attention in recent years. Unlike parametric geometric objects, an implicit shape in $\mathbb{R}^n$ is represented by a mapping $f : \mathbb{R}^n \to \mathbb{R}$ as the 0-contour of f, or as the set $A = \{\mathbf{P} \in \mathbb{R}^n : f(\mathbf{P}) \geq 0\}$. The implicitly represented shapes have advantages over the parametric shapes in several aspects. First of all, when an object is modelled implicitly, one can directly tell whether a point lies inside or outside the shape and the problem of boundary detection can be easily solved. Secondly, the surface normals are easy to compute. Thirdly, the most commonly used geometric shapes, such as spheres, cylinders, and ellipsoids, take very simple forms. In addition, the value of an implicit function at a point can be used to approximate the signed distance from the point to the surface.

In computer graphics, one of the most extensively used implicit modelling techniques has been the CSG [Ric, MS85, Roc89, BW90, Hof93, PASS95, GP02, HL02]. With this technique, a complex geometric object can be regarded as the result of a series of set-theoretic operations acting on a set of primitive geometric solids. Assume that two objects A, B are represented implicitly by $F_A(\mathbf{P}) \geq 0$ and $F_B(\mathbf{P}) \geq 0$. Then, the union $A \cup B$, the intersection $A \cap B$, and the subtraction $A - B$ of set A and B can be represented by functions $F_{A\cup B}(\mathbf{P})$, $F_{A\cap B}(\mathbf{P})$, and $F_{A-B}(\mathbf{P})$ respectively, which are defined using a binary blending operation $\max(x, y)$ in the following way:

$$F_{A\cup B}(\mathbf{P}) = \max\{F_A(P), F_B(\mathbf{P})\}, \quad (1)$$
$$F_{A\cap B}(\mathbf{P}) = -\max\{-F_A(P), -F_B(\mathbf{P})\}, \quad (2)$$
$$F_{A-B}(\mathbf{P}) = -\max\{-F_1(P), F_2(\mathbf{P})\}. \quad (3)$$

One major problem with the blending operation $\max(x, y)$ is that it is not smooth enough. As a bivariate function, the surface of $z = \max(x, y)$ changes sharply at line $y = x$. The blended shapes based on this function always have a sharp edge at the joint. Many solutions have been proposed to cope with the problem[MS85, Roc89, BW90, Hof93, PASS95, GP02, HL02]. However, two problems still exist. Firstly, while one is able to construct efficient and effective smooth manageable blenders, their mathematical descriptions are becoming more and more complex. They lack mathematical simplicity and elegancy. Secondly, with present approaches, the construction of a blending operation with high degree of smoothness, say, with derivatives up to 3 or 4, is still a tough task.

In this paper, we present two new blending operations for implicitly represented geometric objects. The proposed blenders are not only direct to built, but the

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blending operations built in this way can be highly smooth and manageable. The key idea here is the use of what we have called the smooth unit step functions. The rest of the paper is organized as follows. In section 2, we introduce the concept of smooth unit step functions and their construction. In section 3, two new ways of constructing blending operations are proposed using the smooth unit functions. This is followed by some examples and the conclusion of the paper.

2 The smooth unit step functions and their construction

Definition 1 *Let* $H_0 : \mathbb{R} \to [0,1]$ *be defined by*

$$H_0(t) = \begin{cases} 0, & t < 0; \\ \frac{1}{2}, & t = 0; \\ 1, & t > 0. \end{cases} \tag{4}$$

Then, $H_0(t)$ *is called the Heaviside unit step function.*

Definition 2 *Let* $\mu : \mathbb{R} \to [0,1]$ *be a function satisfying following conditions:*

(1) $\mu(t) = 0$*, when* $t < -1$*;*

(2) $\mu(t) = 1$*, when* $t > 1$*;*

(3) $\mu(t)$ *is continuous and nondecreasing over real line* $\mathbb{R}$*.*

Then, $\mu(t)$ *is called a smooth unit step function.*

2.1 The construction of smooth unit step functions

There are a number of ways to construct smooth unit step functions. In this section, four methods have been proposed with three of them to be constructive. Any of these approaches allow us to construct a unit step function with desired degree of smoothness.

2.1.1 Piecewise polynomial smooth unit step functions

A smooth unit step function can be constructed using piecewise polynomials. Let $H_0(t)$ be the Heaviside unit step function, and let

$$\begin{aligned} f_0(t) &= H_0(t), \qquad (5) \\ f_n(t) &= \frac{t}{n} f_{n-1}(t) + (1 - \frac{t}{n}) f_{n-1}(t-1), \\ & \quad n = 1, 2, 3, \cdots. \end{aligned}$$

Set

$$H_n(t) = f_n(\frac{n(t+1)}{2}), \quad n = 1, 2, 3, \cdots. \tag{6}$$

Then it can be shown directly that all these $H_n(t)$, $n = 1, 2, \ldots$, satisfy the three conditions given in definition 2. Hence, they are smooth unit step functions. Furthermore, it can also be verified that the smooth unit step function $H_n(t)$ is C^{n-1}-smooth. Obviously, any of these smooth unit step functions are piecewise polynomials.

2.1.2 Smooth unit step functions as the integration of uniform B-spline basis functions

Another way to obtain smooth piecewise polynomial unit step functions is to integrate the uniform B-spline basis functions. Let $N_m(t)$ be the m^{th} order uniform B-spline basis function with support $[-1, 1]$. Then we can construct a smooth unit step function by computing the integral defined by:

$$B_m(t) = \frac{1}{\delta} \int_{-\infty}^{t} N_m(s) ds, \tag{7}$$

where

$$\delta = \int_{-1}^{1} N_m(s) ds,$$

$B_m(t)$ is obviously nonnegative monotonic increasing and takes value 0 for $t \leq -1$ and value 1 for $t \geq 1$. Since $N_m(t)$ is C^{m-2}-smooth, $B_m(t)$ is thus a C^{m-1}-smooth function. For instance, the cubic smooth unit step function constructed in this way is:

$$B_3(t) = f(\frac{3(t+1)}{2}), \tag{8}$$

where

$$f(t) = \begin{cases} 0, & t \leq 0; \\ \frac{1}{6}t^3, & 0 < t \leq 1; \\ -\frac{5}{8} + \frac{3}{4}t - \frac{1}{3}(t - \frac{3}{2})^3, & 1 < t \leq 2; \\ 1 - \frac{1}{6}(3-t)^3, & 2 < t \leq 3; \\ 1, & t > 3. \end{cases}$$

which is a C^2-smooth function.

2.1.3 Smooth unit step functions as a special type of rising cutoff functions

Another constructive way of generating smooth unit step functions is to use what have been called "rising cutoff functions", which is introduced by Wickerhauser in [Wic94]. In wavelet theory, rising cutoff functions are used to construct localized trigonometric functions which are then combined into a library of orthonormal bases.

In [Wic94], a rising cutoff function has been defined as a complex function $\mu(t)$ with a real argument t that satisfies the conditions that:

$$|\mu(t)|^2 + |\mu(-t)|^2 = 1 \tag{9}$$

for all $t \in \mathbb{R}$, and that

$$\mu(t) = \begin{cases} 0, & \text{if } t < -1 \\ 1, & \text{if } t > 1. \end{cases} \tag{10}$$

It can be seen that any function satisfying this condition must be of the form

$$\mu(t) = \sin\theta(t)\, e^{i\phi(t)}, \tag{11}$$

where $\theta(t)$ is a real function satisfying

$$\theta(t) + \theta(-t) = \pi/2, \qquad \theta(t) = \begin{cases} 0, & \text{if } t < -1 \\ \frac{\pi}{2}, & \text{if } t > 1\,, \end{cases} \tag{12}$$

and $\phi(t)$ is a real function satisfying

$$\phi(t) = \begin{cases} 2n\pi, & \text{if } t < -1 \\ 2m\pi, & \text{if } t > 1\,. \end{cases} \tag{13}$$

As can be seen from the definition, the rising cutoff function is not necessarily real and monotone. Thus, this type of function does not fully fit our purposes. In this paper, we are only interested in a special type of rising cutoff functions, namely, the smooth unit step functions, any of which is real and non-decreasing. Therefore, this type of rising cutoff functions will always take the form $\mu(t) = \sin(\theta(t))$, where $\theta(t)$ is non-decreasing and satisfies equation (12).

Note that for any smooth unit step function $\mu(t)$, the function defined by $\theta(t) = \frac{\pi}{2}\mu^2(t)$ satisfies equation (12). Therefore, smooth unit step functions can be constructed iteratively. To construct a smooth unit step function, we can first begin with a simple monotonic increasing function $\theta_0(t)$ satisfying the condition (12) with its values in $[0, \frac{\pi}{2}]$. Then for $k = 1, 2, \cdots$, the smooth unit step functions $\mu_k(t)$ can be defined recursively in the following way:

$$\begin{array}{ccc} \mu_k(t) & = & \sin\theta_{k-1}(t) \\ \Uparrow & & \Downarrow \\ \theta_k(t) & = & \dfrac{\pi}{2}\mu_k^2(t) \end{array} \tag{14}$$

The interesting fact is that the degree of smoothness of the unit step function from the procedure will increase with the increase in times of recursion when the initial $\theta_0(t)$ is continuous everywhere and is differentiable except at the points $t = \pm 1$.

Here is an example of a set of smooth unit step functions constructed in this way.

Example 1 *[Wic94] One of the simplest continuous $\theta-$functions can be given by*

$$\theta_0(t) = \begin{cases} 0 & t < -1 \\ \frac{\pi}{4}(1+t)) & -1 \le t \le 1 \\ \frac{\pi}{2} & t > 1. \end{cases} \tag{15}$$

With this initial $\theta-$function, a set of smooth unit step functions can be constructed according to (14).

$$\begin{aligned} R_1(t) &= \sin\theta_0(t) \\ &= \begin{cases} 0 & t < -1 \\ sin(\frac{\pi}{4}(1+t)) & -1 \le t \le 1 \\ 1 & t > 1, \end{cases} \\ \theta_1(t) &= \frac{\pi}{2}R_1^2(t) \\ &= \begin{cases} 0 & t < -1 \\ \frac{\pi}{4}(1+\sin\frac{\pi}{2}t) & -1 \le t \le 1 \\ \frac{\pi}{2} & t > 1 \end{cases} \\ &= \theta_0(\sin\frac{\pi}{2}t), \\ R_2(t) &= \sin\theta_1(t) \\ &= \begin{cases} 0 & t < -1 \\ R_1(\sin\frac{\pi}{2}t) & -1 \le t \le 1 \\ 1 & t > 1, \end{cases} \end{aligned}$$

and in general, we have

$$R_{n+1}(t) = \begin{cases} 0 & t \le -1 \\ R_n(\sin\frac{\pi}{2}t) & -1 < t < 1 \\ 1 & t \ge 1. \end{cases} \tag{16}$$

Obviously, $R_n(t)$, $(n = 1, 2, \cdots)$ constructed above are all non-decreasing. Furthermore, it can be shown that $R_n(t)$ has $2^{n-1} - 1$ vanishing derivatives at $t = 1$ and $t = -1$ for $n = 1, 2, \cdots$.

In practice, both $R_n(t)$ and $R_n^2(t)$ can be used as smooth unit step functions. However, $R_n^2(t)$ is antisymmetric about the point $(0, \frac{1}{2})$, which provides a balanced blending when it is used to combine two implicit functions.

2.1.4 Approximating a smooth unit step function using $\frac{1}{1+e^{-\alpha t}}$

In practice, we can also use following function to represent approximatively a smooth unit step function:

$$S_\alpha(t) = \frac{1}{1+e^{-\alpha t}}, \qquad \alpha > 0. \tag{17}$$

Unlike the smooth unit step functions constructed from the first three methods, this function has a rising range from $-\infty$ to ∞ rather than $[-1, 1]$. However, a smooth unit step function can be well approximated by $S_\alpha(t)$. In fact, for any given small number ϵ, $0 < \epsilon < 1$, select a α such that $\alpha > \ln\frac{1-\epsilon}{\epsilon}$. Then we have

$S_\alpha(t) < \epsilon$ when $t < -1$, and $S_\alpha(t) > 1 - \epsilon$ when $t > 1$.

Obviously, $S_\alpha(t)$ is differentiable at any point and to any order.

2.2 Smooth unit step functions with arbitrary rising range

Definition 3 *Let $\mu(t)$ be a smooth unit step function with rising range $[-1, 1]$, and let ϵ be a positive real number. The real function $\mu(\frac{t}{\epsilon})$, denoted by $\mu_\epsilon(t)$, is called a smooth unit step function with rising range parameter ϵ.*

Obviously, a smooth unit step function with rising range parameter ϵ has a rising range $[-\epsilon, \epsilon]$.

A C^3-continuous smooth piecewise polynomial unit step function constructed from the first method is displayed in figure 1 with different rising parameters. As can be seen later, the rising range parameter can be used to control the size of the transition area between two blending objects.

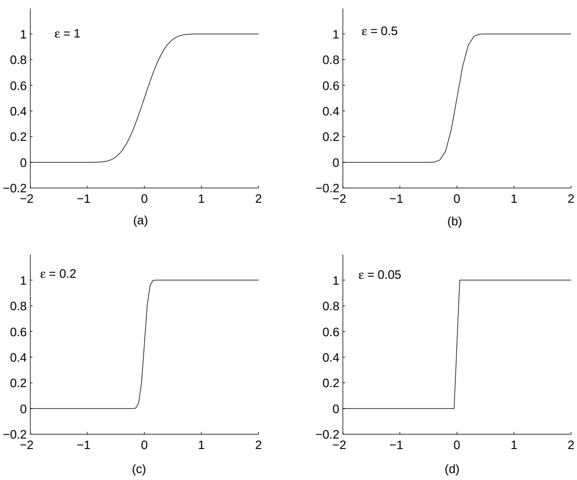


Figure 1: A C^3-continuous smooth unit step function with different rising range parameter ϵ. (a) $\epsilon = 1$;(b) $\epsilon = 0.5$;(c) $\epsilon = 0.2$;(d) $\epsilon = 0.05$.

3 The applications of smooth unit step functions in blending implicit objects

In this section, two techniques, which we called exterior blending and interior blending respectively, are presented to realize a smooth and blending range controllable implicit shape blending. Each of the two techniques is obtained by constructing a highly smoothed bivariate function $f(x, y)$ that well approximates the C^0-smooth function $\max(x, y)$ using smooth unit step functions. They offer different features in blending two implicit objects. With the exterior blending, the blended solid union will generally contain the solids to be blended as its subsets. In contrast, with the interior blending, the blended solid union will always be a subset of the union of the two solids to be blended.

3.1 Exterior blending

In this section, we discuss the exterior blending technique. The basic idea is to approximate the absolute function $|t|$ with a much smoother function $S_{abs}(t)$ using smooth unit step functions.

Let $\delta > 0$ and let $\mu_\epsilon(t)$ be a smooth unit step function with rising parameter ϵ, and let

$$\begin{aligned} f(t) &= \frac{x^2}{2\delta} + \frac{\delta}{2}. \\ g_1(t) &= \mu_\epsilon(t + \delta). \\ g_2(t) &= \mu_\epsilon(t - \delta). \end{aligned} \tag{18}$$

We define

$$\begin{aligned} S_{abs}(t, \delta, \epsilon) &= (g_1(t) + g_2(t) - 1)t \\ &+ g_1(t)(1 - g_2(t))f(t). \end{aligned} \tag{19}$$

Since $g_1(t)$ is a smooth unit step function at $t = -\delta$, and $g_2(t)$ a smooth unit step function at $t = \delta$, $S_{abs}(t, \delta, \epsilon)$ has approximately such properties: $S_{abs}(t, \delta, \epsilon) = -t$ when $t \in (-\infty, -\delta]$; $S_{abs}(t, \delta, \epsilon) = t$ when $t \in [\delta, \infty]$; $S_{abs}(t) = f(t)$ when $t \in (-\delta, \delta)$, where $f(t)$ is a parabola passing both points $(-\delta, \delta)$ and (δ, δ) with derivative -1 at $(-\delta, \delta)$, and derivative 1 at (δ, δ). Thus, $S_{abs}(t, \delta, \epsilon)$ can be regarded as a smooth approximation to the absolute function $s(t) = |t|$, where δ is used as a parameter to control the accuracy of the approximation. Figure 2 shows the shapes of such a function with different δ.

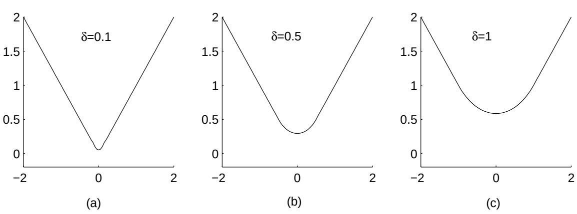


Figure 2: The shapes of $S_{abs}(t, \delta, \epsilon)$ with different δ. (a)$\delta = 0.1$; (b)$\delta = 0.5$; (c)$\delta = 1$.

With function $S_{abs}(t, \delta, \epsilon)$, we define following approximation to $\max(x, y)$:

$$Ext_{max}(x, y, \delta, \epsilon) = \frac{x + y + S_{abs}(x - y, \delta, \epsilon)}{2}. \tag{20}$$

Figure 3 shows the 0-contour of $Ext_{max}(x, y, \delta, \epsilon)$ with $\epsilon = 0.1$ and different values of δ.

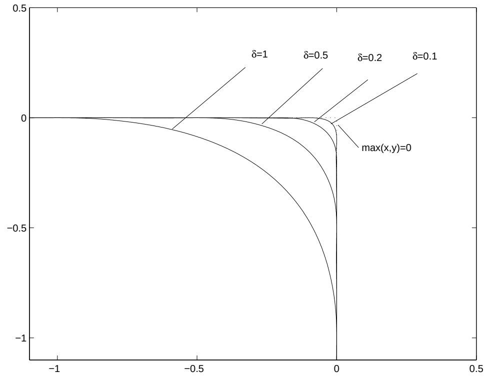


Figure 3: The shapes of $Ext_{max}(x, y, \delta, \epsilon) = 0$ with $\epsilon = 0.05$ and different values of δ.

For two implicit shapes $A : F_A(\mathbf{P}) \geq 0$ and $B : F_B(\mathbf{P}) \geq 0$, we define the set-theoretic operations as follows:

Union:

$$F_{A\cup B}(\mathbf{P}) = Ext_{\max}(F_A(\mathbf{P}), F_B(\mathbf{P}), \delta, \epsilon). \tag{21}$$

Intersection:

$$F_{A\cap B}(\mathbf{P}) = -Ext_{\max}(-F_A(\mathbf{P}), -F_B(\mathbf{P}), \delta, \epsilon). \tag{22}$$

Abstraction:

$$F_{A-B}(\mathbf{P}) = -Ext_{\max}(-F_A(\mathbf{P}), F_B(\mathbf{P}), \delta, \epsilon). \tag{23}$$

Figure 4 - 6 demonstrate how the blending union, the blending intersection, and the blending subtraction can be managed by changing the parameter δ in the blending operations.

The key features of the blending operations defined above are that: (1) The blended union object from operation defined in (21) will contain the objects to be blended as its subsets, unless the rising range parameter is set too big. This can be seen directly from the definitions of $S_{abs}(t)$ and $Ext_{max}(x, y)$. (2) Two completely separated solids can be blended in different blending range by varying the parameters δ and ϵ (see figure 7). (3)The blending operations $F_{A\cup B}$, $F_{A\cap B}$, F_{A-B} can be built to whatever degree of smoothness one wishes if only the smooth unit step function is smooth enough. For example, to define a C^3-continuous blending operation in this way, we need only use a C^3-continuous smooth unit step function in (19) to define S_{abs}. (4) When the implicit shapes $F_A(\mathbf{P})$ and $F_B(\mathbf{P})$ are both polynomials and $S_{abs}(t, \delta, \epsilon)$ used in (20) is defined by the piecewise smooth unit step function, the blended shapes from any blending operations defined in (21) to (23) will be the 0-contours of piecewise polynomials.

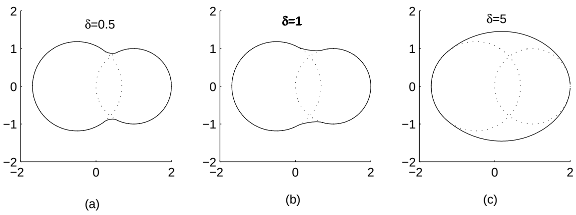


Figure 4: The blending union of two discs using operators defined in (21) with $\epsilon = 0.25$ and different δ: (a) $\delta = 0.5$; (b) $\delta = 1$; (c) $\delta = 5$.

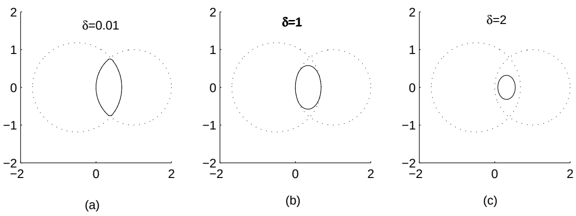


Figure 5: The blending intersection of two discs using the operators defined in (22) with $\epsilon = 0.25$ and different δ: (a) $\delta = 0.01$; (b) $\delta = 1$; (c) $\delta = 2$.

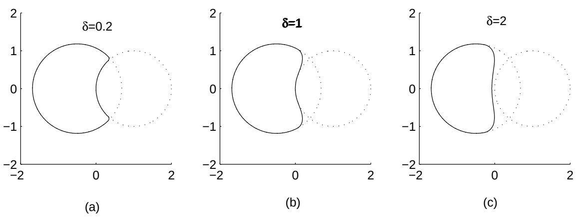


Figure 6: The blending subtraction of two discs using the operators defined in (23) with $\epsilon = 0.25$ and different δ: (a) $\delta = 0.2$; (b) $\delta = 1$; (c) $\delta = 2$.

3.2 Interior blending

We have defined in previous section some blending operations using a highly smooth blending function $Ext_{max}(x, y)$. As have been discussed, the blending union will generally contain the objects to be blended as its subsets. In this section, we are going to define another blending function, $Int_{max}(x, y)$. As will be seen later, the blending union operation derived from

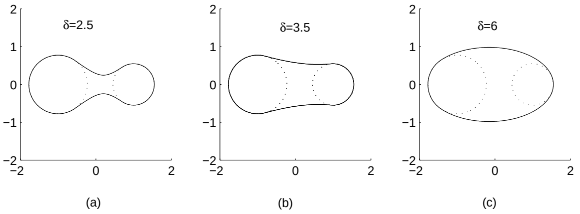


Figure 7: The blending union of two completely separated discs using operators defined in (21) with $\epsilon = 0.25$ and different δ: (a) $\delta = 0.25$; (b) $\delta = 0.35$; (c) $\delta = 6$.

the function $Int_{max}(x, y)$ will always be contained in the union of the objects to be blended. $Int_{max}(x, y)$ is defined as follows:

$$Int_{max}(x, y, \epsilon) = (1 - \mu_\epsilon(y - x))x + \mu_\epsilon(y - x)y. \tag{24}$$

where $\mu_\epsilon(t)$ is a smooth unit step function with rising range parameter ϵ.

Figure 8 shows the 0-contour of $Int_{max}(x, y, \epsilon)$ with different values of ϵ.

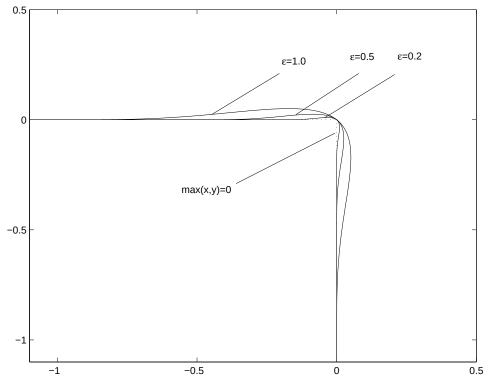


Figure 8: The shapes of $Int_{max}(x, y, \epsilon) = 0$ with different values of ϵ.

For implicit shapes described by functions $A : F_A(\mathbf{P}) \geq 0$ and $B : F_B(\mathbf{P}) \geq 0$, following soft blending operations can be defined using $Int_{max}(x, y)$:

Union:

$$F_{A \cup B}(\mathbf{P}) = Int_{max}(F_A(\mathbf{P}), F_B(\mathbf{P}), \epsilon). \tag{25}$$

Intersection:

$$F_{A \cap B}(\mathbf{P}) = -Int_{max}(x, y)(-F_A(\mathbf{P}), -F_B(\mathbf{P}), \epsilon). \tag{26}$$

Subtraction:

$$F_{A - B}(\mathbf{P}) = -Int_{max}(x, y)(-F_A(\mathbf{P}), F_B(\mathbf{P}), \epsilon). \tag{27}$$

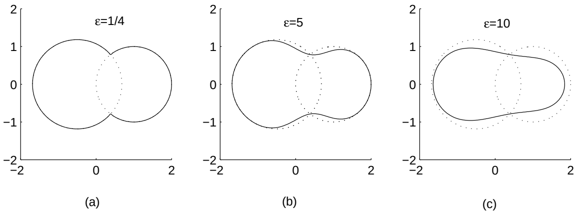


Figure 9: The blending union of two discs using the operators defined in (25) with different rising range parameters ϵ: (a) $\epsilon = 1/4$; (b) $\epsilon = 5$; (c) $\epsilon = 10$.

Figure 9-11 demonstrate how two discs are combined using the three operations.

Unlike the operations derived from function $Ext_{max}(x, y)$, where the blending range is mainly controlled by adjusting parameter δ, the blending range using operations defined in (25)-(27) is mainly controlled by varying the rising range parameter ϵ in the smooth unit step functions: the smaller the parameter ϵ, the larger the surface range of the objects to be blended will be preserved (see Figure 9-11). It also can be observed from the figures that no matter what a rising parameter is used, the joints of the shapes to be blended always lie on their blending. This is obviously implied in the blending function $Int_{max}(x, y)$. Consequently, blending operations defined by $Int_{max}(x, y, \epsilon)$ can only be used to blend intersecting implicit objects.

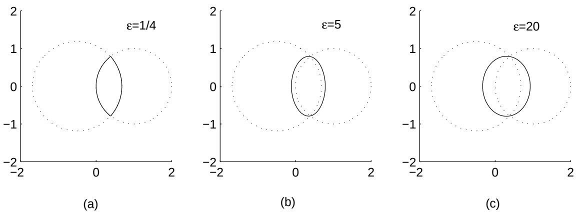


Figure 10: The blending intersection of two discs using the operators defined in (26) with different rising range parameters ϵ: (a) $\epsilon = 1/4$; (b) $\epsilon = 5$; (c) $\epsilon = 20$.

Another feature of the blending technique, as can be seen from the figure 9, is that the blended solid union using operation defined in (25) will always be a subset of the union of the solids to be blended. Contrary to the union, the intersection of the blended object from function (26) always contains the intersection of the solids to be blended. Figure 9- 11 also demonstrate how the transition area can be controlled by adjusting the rising range parameter in the smooth unit step function $\mu_\epsilon(t)$.

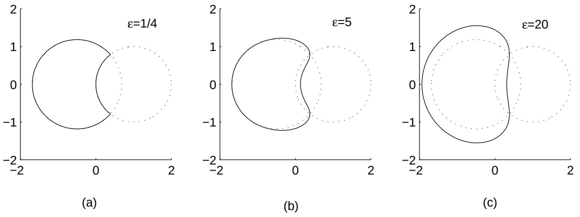


Figure 11: The blending subtraction of one disc from another using the operators defined in (27) with different rising range parameters ϵ: (a) $\epsilon = 1/4$; (b) $\epsilon = 5$; (c) $\epsilon = 20$.

4 Examples and discussions

In this section, we present a few examples of implicitly represented graphical objects to demonstrate the efficiency and the effectiveness of the proposed blending algorithms. All these objects are constructed by blending a set of simple algebraic surfaces. Piecewise cubic polynomial smooth unit step function has been used to define the blending operations. Therefore, the implicit functions corresponding to all these figures are C^2-continuous. Suppose the object to be designed involving implicit surfaces $F_i(\mathbf{P}) = 0$, $i = 0, 1, 2, \cdots, n$. Then the general procedure in producing the object is performed in an iterative way and can be described as follows:

$F(\mathbf{P})$=$F_0(\mathbf{P})$;

for ($i = 1$ to n){

$$F(\mathbf{P}) = blending(F(\mathbf{P}), F_i(\mathbf{P}));$$

}

where $blending(\cdot, \cdot)$ represents a general blending operation.

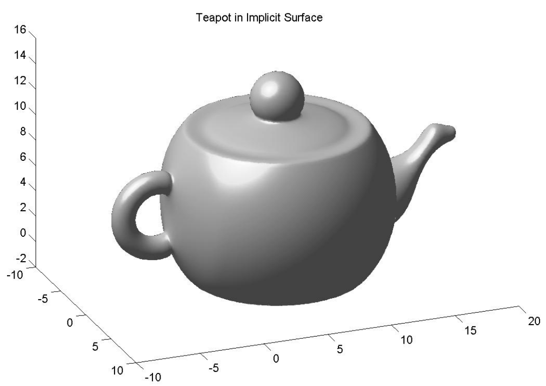


Figure 12: The shape of a teapot is represented as a C^2-smooth function by blending a set of simple algebraic surfaces

Figure 12 presents an implicit teapot constructed using some commonly known algebraic surfaces. The teapot body and lid are built using spheres. The handle is represented by a torus, and the nozzle of the teapot is a blend of three cylinders and a plane using different rising range parameters in the smooth unit step functions. The blending operations used are those derived from bivariate function $Int_{max}(x, y)$.

Figure 13 presents an implicitly represented sea snail. It is constructed using a set of spheres and a plane using blending operations defined by function $Int_{max}(x, y)$. The spheres are evenly located along a 3D spiral curve with increasing radius. They are blended to represent the body of the shell. To obtain a closed surface, the shell body is then blended with a closed half sphere, which is obtained by subtracting a 3D plane from the sphere.

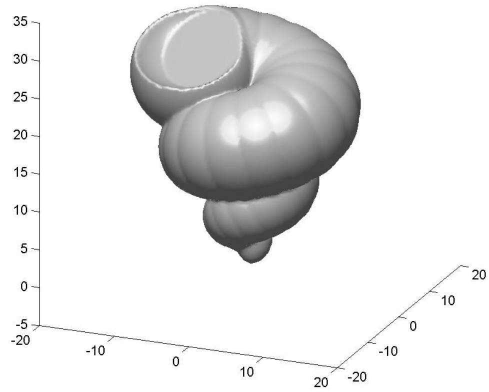


Figure 13: The shape of a sea snail is represented as a C^2-smooth function by blending a set of spheres and a 3D plane.

Figure 14 presents an implicitly represented handwritten number. The implicit shape is obtained by blending a set of equal-sized spheres located along a curve which represents the skeleton of the character. The Blending operation defined in (21) using $Ext_{max}(x, y)$ is used and the parameter δ in function $S_{abs}(t)$ is specified roughly equal to the distance between the center of two adjacent spheres. This example exploits the property of the union blending operation that two completely separated objects can be blended together with expected blending volume. This technique can be used in practice to approximately represent some convolution surfaces.

Figure 15 demonstrates the key difference in blending implicitly represented surfaces. In general, there are no visual differences at all when blending parameter δ and ϵ are very small. However, with the increase in the values of δ and ϵ, the union blending opera-

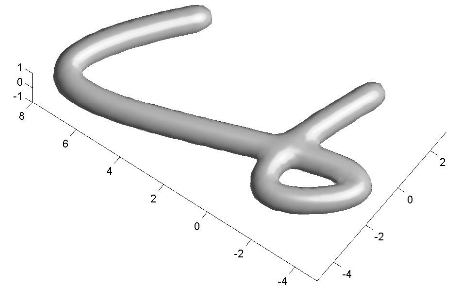
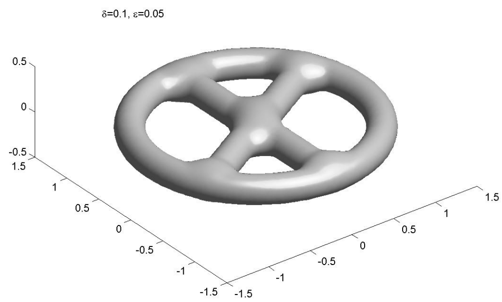
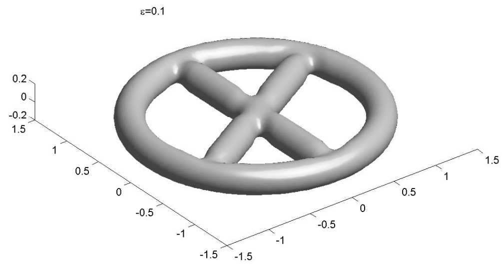


Figure 14: A hand-written character is represented as a C^2-smooth function by blending a set of spheres.

tion defined using Ext_{max} tends to generate a bulge around the surface intersection, while the one defined using Int_{max} tends to contract near the intersection.



(a)



(b)

Figure 15: (a)The union blending operation defined using Ext_{max} tends to generate a bulge around the surface intersection; (b) The union blending operation defined using Int_{max} tends to contract near the surface intersection.

5 conclusion

In this paper, we introduced a new technique to define implicit shape blending operations using smooth unit step functions. Four methods have been given to derive a C^n-continuous smooth unit step function with a given integer n. Two new types of blending operations for implicit objects are proposed. The typical features of the presented techniques are, firstly, that they enable us to build an implicit blender to have whatever degree of smoothness one wishes. Secondly, the blending operations derived from this technique are more flexible in controlling the blending range and smoothness. Furthermore, the mathematical descriptions of the techniques are simple and elegant.

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