University of West Bohemia
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Bachelor Thesis

Mathematical Models of Value at Risk

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Declaration

I hereby declare that this bachelor thesis is completely my own work and that I used only the cited sources.

Pilsen, day ________________ ______________________
signature

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Preface

The aim of the thesis is to study mathematical methods of Value at Risk. At first we describe the basic concepts of Value at Risk. Then we focus specifically on three methods of computation Value at Risk, namely Historical simulation, Analytical method and Monte Carlo simulation. We describe the basic principals of these methods and show examples. Subsequently we compare these methods by selected criteria. Finally we examine the simulation Monte Carlo in more details and we conduct several simulations. Most of our computations are performed in the MATLAB® environment and processed in MS Excel® 2003.

Keywords: Value at Risk, Monte Carlo simulation, Historical simulation, Analytical method

Alžběta Holá May 27, 2013

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Chapter 1

Introduction

Glyn A. Holton describes in his work History of Value-at-Risk: 1922-1998 [1] the origin of the name Value at Risk (VaR). Several similar names were used during the 1990's. For example:

- **dollars-at-risk (DaR)**
- **capital-at-risk (CaR)**
- **income-at-risk (IaR)**
- **earnings-at-risk (EaR)**

It seemed that users liked the "-at-risk" moniker, but were uncomfortable labeling exactly what was "at-risk". The "dollars" label of DaR was too provincial for use in many countries.

The "capital" label of CaR seemed too application-specific. Some applications of VaR-such as VaR limits-were unrelated to capital. The "income" and "earnings" labels of IaR and EaR had accounting connotations unrelated to market risk.

Software vendor Wall Street Systems went so far as to call its software "money-at-risk". It is perhaps the uncertainty of the label "value" that made "value-at-risk" attractive. Also, its use in the RiskMetrics Technical Document added to its appeal. By 1996, other names were falling out of use.

In 1985 was suggested the name "value-at-risk" originated within JP Morgan.

During the 1990's, Value-at-Risk (VaR) was widely adopted for measuring market risk in trading portfolios. Its origins can be traced back as far as 1922 to capital requirements the New York Stock Exchange imposed on member firms. VaR also has roots in portfolio theory and a crude VaR measure published in 1945.

There are lots of ways how to access to computing Value at Risk. We can focus on Conditional Value at Risk (CVaR), marginal VaR, incremental VaR and component VaR, Gaussian VaR. About these problems we can read for more for example in [2] or [3].

In Chapter 2 we describe methods of computation of Value at Risk, specifically three most often used ones. These are historical simulation, analytical method known also as variance and covariance method and Monte Carlo simulation. We show computation examples of mentioned methods. Finally we provide a comparison of these methods.

In Chapter 3 we focus on the Monte Carlo simulation. We describe the simulation process on an example.

Chapter 4 is closely related to Chapter 3. We show different simulation experiments in this chapter such as accuracy experiment 4.1 or simulation time experiment 4.2.

Value at Risk is a well known problem, that can be studied from many sources. To name just a few [4] and [5] written by Manfredo and Leuthold,different author is Giorgio Szegö who wrote the book: Risk Measures for the 21st Century [6] or papers from the Wharton School of the University of Pennsylvania [7], [8].

Chapter 2

Value at Risk method (VaR)

Value at Risk (VaR) as we know it today originate in 90's of the 20th century when world banks and other financial institutions started to use VaR for measuring exchange rate risk. Then it started to spread also to non-financial corporations to other market risks. Today it is one of the most widely used method for measuring market risk portfolio.

In this chapter we follow book written by John C. Hull, *Option, Futures, and other Derivatives* [9] , Thesis [10] and Bachelor Thesis [11]. More information can be found also in[12], Section IV.I.

2.1 Definition of VaR

John C. Hull defines Value at Risk in the statement "We are X percent certain that we will not lose more than V dollars in the next N days."

The variable V is here the Value at Risk. It is a function of two variables:

- N : **The time horizon** - It is a period of time over which VaR is measured. It is traditionally measured in trading days rather than calender days. In practise, analysts most frequently set $N = 1$, because there is not enough data to estimate the behaviour of market variables over longer period of time.

  If when the changes in the value of the portfolio on successive days have independent identical normal distributions with mean zero, We can use assume

$$N\text{-day VaR} = 1\text{-day VaR} \cdot \sqrt{N} \tag{2.1}$$

- X : **The confidence level** - In this work we will denote it by α. Most frequently used confidence levels are 95% or 99%.[1]

Value at risk is the maximum amount of money that may be lost on a portfolio over a given period of time, with a given level of confidence.

[1]Note that in the literature the notation is not consistent and sometimes the confidence level is considered $(1 - \alpha)$, i.e. 95% level corresponds to $\alpha = 0.05$

As we said VaR is often measured at 95% confidence level and time horizon of one day. 95% confidence means that there is a 95% chance of the loss on the portfolio being lower than the VaR calculated. Then we have the following exact definition.

Value at risk is the maximum amount of money that may be lost on a portfolio in 24 hours, with 95% confidence.

Then we can say that the interpretation of VaR is for example: $1 million with these parameters indicates, that the loss of the portfolio the following day with a 95% probability will not exceeding $1 million.

You can see the graphic interpretation of VaR in the following three pictures.
In Figure 2.1 the black line at -1.645 means 5% Value at Risk is 1.645. The left area under the red line represents 5% of the total area under the curve. The right area under the blue line represents 95% of the total area under the curve. The curve represents a hypothetical Profit and Lost probability density function. Normal distribution with parameters $\mu = 0, \sigma = 1$. See Appendix A.1.

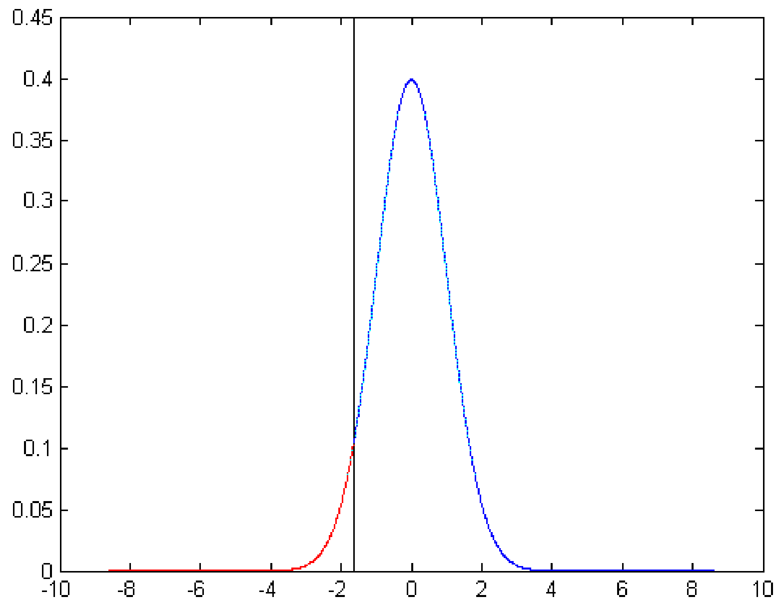

Figure 2.1: 5%VaR diagram for Normal distribution

In Figure 2.2 the black line at -6.314 means 5% Value at Risk is 6.314. The left area under the red line represents 5% of the total area under the curve. The right area under the blue line represents 95% of the total area under the curve. The curve represents a hypothetical Profit and Lost probability density function. Student's *t*- distribution with one degree of freedom. See Appendix A.2.

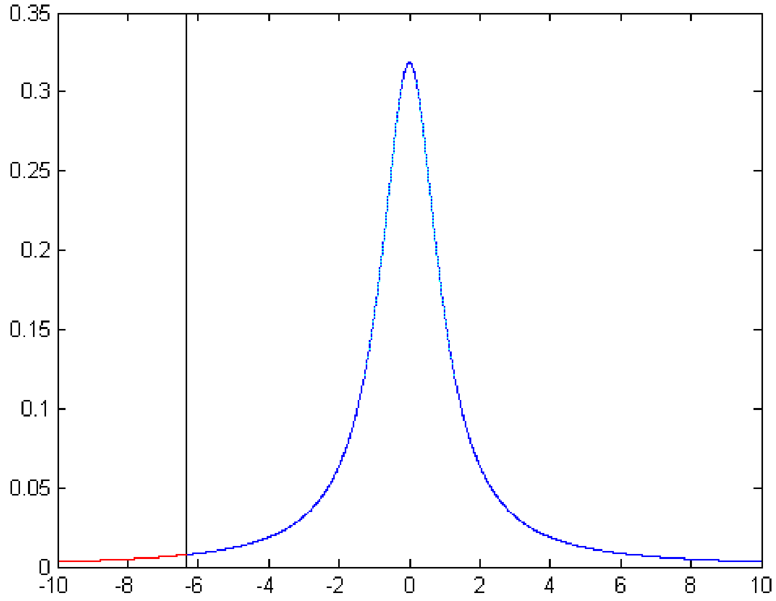

Figure 2.2: 5%VaR diagram for Student's t-distribution

In Figure 2.3 the black line at 0.193 means 5% Value at Risk is 0.193. The left area under the red line represents 5% of the total area under the curve. The right area under the blue line represents 95% of the total area under the curve. The curve represents a hypothetical Profit and Lost probability density function. Log-normal distribution with parameters $\mu = 0, \sigma = 1$. See Appendix A.3.

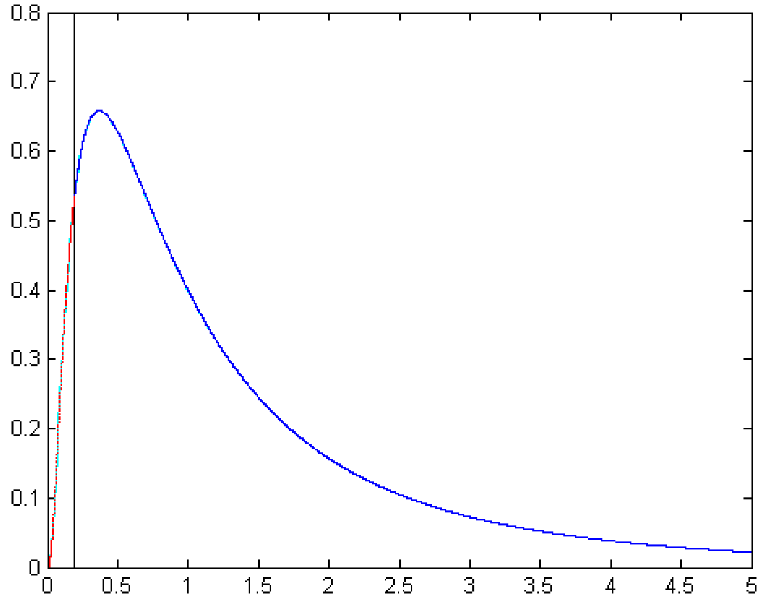

Figure 2.3: 5%VaR diagram for Log-normal distribution

Remark. Daily Volatilities
In option pricing we usually measure time in years and the volatility of an asset also in years ("volatility per year"). In VaR calculations we measure volatility in days ("volatility per day"). Let us denote:
σ_{year} - volatility per year of an asset
σ_{day} - volatility per day of the asset
and let us assume 252 trading days in a year. Then according to equation (2.1) we can write that the relationship between these two volatilities is:

$$\sigma_{year} = \sigma_{day}\sqrt{252} \tag{2.2}$$

or,

$$\sigma_{day} = \frac{\sigma_{year}}{\sqrt{252}} \tag{2.3}$$

We can approximate the equation (2.2) for different time periods.

Example - Single Asset Case

The portfolio consists of a position in a single stock.
The position is worth €5 million in IBM stock. We assume the volatility of IBM 26% per year. We want to be 95% confident that the loss level over 10 days will not be exceeded. We also assume that the expected change in the value of the portfolio is zero (this is possible for short time periods) and that the change in the value of the portfolio is normally distributed.

Parameters:
$P_0 =$ €5 million
$\sigma_{year} = 26\%$
$N = 10$
$\alpha = 95\%$
Firstly following the equation (2.3) we can calculate σ_{day} .

$$\sigma_{day} = \frac{26\%}{\sqrt{252}} = 1.64\%$$

We denote σ_p the standard deviation of the change in the portfolio in 1 day.
Then

$$\sigma_p = P_0 \cdot \sigma_{day} \tag{2.4}$$

In our case it is

$$\sigma_p = 5000000 \cdot 0.0164 = €82000$$

By using formula (2.2) we can calculate the standard deviation of the change in portfolio in 10 days.

$$\sigma_{p,10day} = 82000 \cdot \sqrt{10} = €259306.78$$

As we said we assume that the change is normally distributed. In the Table 2.1 we can see that for $p = 0.95$ we have that $Q_{0.95} = 1.654$.

Table 2.1: Often used quantiles of normal distribution

p	0,5	0,9	0,95	0,975	0,99	0,995
Q_p	0,0	1,2816	1,6449	1,9600	2,3263	2,5758

$$VaR_{95\%,10day} = Q_{0.95} \cdot \sigma_{p,10days}$$

it is

$$VaR_{95\%,10day} = 1.645 \cdot 259306.78 = €426559.63$$

The 10-day 95% VaR for IBM is €426559.63.

Consider next portfolio consisting of €10 million position in Microsoft. Suppose the volatility per year 32% (corresponds to 2% per day). We will use the same calculation like in the IBM example. The standard deviation of the change in the value of the portfolio in 1 day is

$$\sigma_p = 10000000 \cdot 2\% = €200000$$

the change in 10 days is

$$\sigma_{p,10day} = 200000 \cdot \sqrt{10} = €632455.53$$

Assuming the change is normally distributed, the 10-day 95% VaR is

$$VaR_{95\%,10day} = 1.645 \cdot 632455.53 = €1040389.35$$

Example - Two-Asset Case

In this case we will consider the portfolio consisting of both €5 000 000 of IBM shares and €10 000 000 of Microsoft shares. We suppose that the returns on the two shares have a bivariate normal distribution and the correlation is 0.3 .

A standard result in statistics tells that, if we have two variables X and Y, their standard deviations are equal to σ_x and σ_y with correlation coefficient between them equal to ρ, the standard deviation of $X + Y$ is given by

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y\rho} \tag{2.5}$$

We apply this to our previous example and the set X equal to the change in the value of the position in IBM over 1-day period and Y equal to the change in he value of the position in Microsoft over 1-day period. In this case

$$\sigma_X = 82000 \text{ AND } \sigma_Y = 200000$$

Following the equation (2.5) the standard deviation of the change in the portfolio value in 1-day is

$$\sigma_{X+Y} = \sqrt{82000^2 + 200000^2 + 2 \cdot 82000 \cdot 200000 \cdot 0.3} = €237831.87$$

The standard deviation of the change in the portfolio value in 10-days is

$$\sigma_{X+Y,10day} = 237831.87 \cdot \sqrt{10} = €752090.42$$

We assume that the mean of change is zero and the change is normally distributed. Then the 10-day 95% VaR for the portfolio is equal to

$$VaR_{95\%,10day} = 752090.42 \cdot 1.645 = €1237188.74$$

Remark. Benefits of Diversification
In our examples we have just considered:

- The 10-day 95% VaR for the portfolio of IBM shares is €426 559.63
- The 10-day 95% VaR for the portfolio of Microsoft shares is €1 040 389.35
- The 10-day 95% VaR for the portfolio of both is €1 237 188.74

The amount

$$(426559.63 + 1040389.35) - 1237188.74 = €229760.24$$

represents the benefits of diversification. If IBM and Microsoft were perfectly correlated, the amount will be zero. It means the VaR for the portfolio of both would be equal VaR for the IBM portfolio plus the VaR for Microsoft portfolio.

We can use several different way how to calculate VaR. Between most popular method belongs historical method or Monte Carlo simulation. We will talk about these methods in detail later.

2.2 Mathematical definition of VaR

At first we have to say something about quantiles, the following theorem can be found in article [13].

Theorem 2.2.1 (Quantiles). *Given $\alpha \in [0,1]$ the numberq $\in R$ is an α-quantile of the random variable X under the probability distribution P if one of the three equivalent properties below is satisfied:*
1.$P(X \leq q) \geq \alpha \geq P(X < q)$,
2. $P(X \leq q) \geq \alpha$ and $P(X \geq q) \geq 1 - \alpha$,
3. $F_X(q) \geq \alpha$ and $F_X(q-) \leq \alpha$ with $F_X(q-) = \lim_{x \to q, x<q} F(x)$,
where F_X is the cumulative distribution function of X.

Remark. The set of such α-quantiles is a closed interval. Since Ω is finite, there is a finite left- (resp. right-) end point q_α^- (resp. q_α^+) which satisfies $q_\alpha^- = inf\{x \in R : P(X \leq x) \geq \alpha\}$ (resp. $q_\alpha^+ = inf\{x \in R : P(X \leq x) > \alpha\}$). With the exception of at most countably many α the equality $q_\alpha^- = q_\alpha^+$ holds. The quantile q_α^- is the number $F(\alpha) = inf\{x \in R : P(X \leq x) \geq \alpha\}$.

We formally define VaR in the following way:
Given confidence level $\alpha \in [0,1]$ the Value at Risk (VaR_α) at level α of the final net worth X with distribution P is the quantile q_α^+ of X

$$VaR_\alpha(X) = inf\{x \in R : P(X > x) \leq 1 - \alpha\}. \tag{2.6}$$

2.3 Methods of computing VaR

As we already said, there are several different methods of calculating VaR.
We have two basic approach how to calculate VaR:

- **Parametric approach** - Parametric methods, e.g. variance and covariance method (sometimes called also analytical method)
- **Non-parametric approach** - Simulation methods, e.g. historical simulation or Monte Carlo simulation.

Historical simulation is sometimes denoted as empirical approach.
In this chapter we will describe some of these methods for calculating VaR.

2.3.1 Historical simulation

Historical simulation is one of the simplest and most obvious method to estimate VaR for many portfolios. Method works on the principle that, based on historical changes in market factors to determine the possible future profits and losses.

As a first step we have to collect the data which identify market factors that influence the value of portfolio (yields of individual portfolio instruments). These data are from the previous period or from the previous periods (dependent on the institution - banks calculate VaR daily).

Into the resulting time series we include also the present state of the portfolio. Then from these data we estimate VaR as distribution density quantile.

To obtain historical data of market factors represented by the timing series can be time consuming. The historical data should be sampled at the daily frequency and should reach many years into the past.
The risk is measured with price changes:

- **Relative change in price** - if the change is relative to the initial price, than we call it return (rate of return),
- **Absolute change in price**,
- **Logarithmic change in price**.

1-day Period

Denote P_t as a price in time t (represents one trading day). Then relative rate of return (R_t) or just return, between t and $t-1$ is

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}. \tag{2.7}$$

Absolute rate of return (Ra_t) for the same time period is

$$Ra_t = P_t - P_{t-1}. \tag{2.8}$$

Logarithmic rate of return (Rg_t) correspond to

$$Rg_t = ln\left(\frac{P_t}{P_{t-1}}\right) = ln(1 + R_t). \tag{2.9}$$

Furthermore we will use the relative rate of return.
In the text above R_t is described as 1-day return, now we will show how to use it in more then one day period of time.

k-days Period

Return of the k-days period of time is define as

$$R_t(k) = \frac{P_t - P_{t-k}}{P_{t-k}}. \tag{2.10}$$

Linear Portfolio

Let $t = 1, \ldots, T$ be individual time periods. When the portfolio is linear and created by N assets, $R_{j,t}$ is the return of the j-th asset in the time period t and $w_{j,T}$ is the current weight of the j-th asset in this portfolio, then the return of the portfolio, $R_{p,t}$ can be calculated as

$$R_{p,t} = \sum_{j=1}^{\mathbf{N}} w_{j,T} R_{j,t}. \tag{2.11}$$

Non-linear Portfolio

Estimation for the non-linear portfolio is little bit different from the linear. We have to identify the market factors that influence the portfolio and collect the data of change in these market factors during some period of time t. This time is $t = 0, \ldots, T$. Then we set today (the j-th day) T scenarios how will behave the variable tomorrow ($(j+1)$-st day) according to the historical development of the time series.

Let V_i^k be a value of k-th market variable (k=1,...,n) in the i-th day $(i = 1, \ldots, m)$. We suppose that today is the j-th day. Then the i-th scenario of the market variable value tomorrow ($(j+1)$-st day) is equal to

$$V_{j+1,i}^k = V_j^k \frac{V_i^k}{V_{i-1}^k}. \tag{2.12}$$

We calculate the total value of the portfolio for each scenario and we denote it

$$P_{j+1,i} = f(V_{j+1,i}^1, \ldots, V_{j+1,i}^n), \tag{2.13}$$

where f is function of market variables.

Then we can calculate the relative change

$$R_{p,i} = \frac{P_{j+1,i} - P_j}{P_j}, \tag{2.14}$$

where P_j is value of the portfolio in the j-th day.

We showed how to calculate the relative change for linear and non-linear portfolios. The following process to calculate VaR is for both linear and non-linear portfolios the same. The technique is from the book [14].

We rank values calculated using equation (2.11).

$$R_{p,(1)} < \cdots < R_{p,(T)}.$$

And we estimate the empirical α quantile $\tilde{u}_\alpha$ defined for $0 < \alpha < 1$ due to a given confidence level by

$$\tilde{u}_\alpha = \begin{cases} R_{([T\alpha]+1)} & ,\text{if } T\alpha \notin Z \\ \frac{1}{2}[R_{([T\alpha])} + R_{([T\alpha]+1)})] & ,\text{if } T\alpha \in Z \end{cases} \tag{2.15}$$

Then we calculate VaR as

$$VaR_\alpha = \tilde{u}_\alpha P_0,$$

when P_0 is the initial portfolio value.

Advantages and Disadvantages

An advantage of the historical method is that it is non-parametric, which means it does not require assumptions on probability distribution. The disadvantage is that the past may have very different risk characteristics from the future.

2.3.2 Parametric methods

In parametric method, we have to at first decide what we consider a random variable (R,Ra, Rg, or another risk factor). Parametric approach can be used in two levels:

- **Portfolio approach** - work with a Ra (R, Rg, or another risk factor) of the whole portfolio.
- **Position approach** - work with a Ra (R, Rg, or another risk factor) of individual assets in portfolio, which together create multivariate distribution.

Portfolio approach - Univariate distribution

There is a lot of univariate parametric methods. In this work we always consider the conditional distribution to time series R,Ra, Rg, or another risk factor. Factor distribution can be parametrized with the estimation of time series. The time series have always constant length.

In this work we will consider:

- **Normal distribution**,
- **Distributions derived from the normal distribution**
  Student's t-distribution,
  Log-normal distribution.

VaR calculation for Normal distribution

We suppose that the risk factor Ra, absolute return (2.8), has the normal distribution with parameters μ and σ (see Appendix A.1), i.e. $Ra \sim N(\mu, \sigma^2)$. For VaR_α satisfies:

$$P[Ra < -VaR_\alpha] = 1 - \alpha.$$

Parameters μ and σ can be estimated with using the selective mean value and the selective standard deviation.

$$\frac{Ra - m}{s} \sim N(0,1)$$
$$P\left[\frac{Ra - m}{s} < \frac{-VaR_\alpha - m}{s}\right] = 1 - \alpha$$
$$\frac{-VaR_\alpha - m}{s} = \Phi^{-1}(1-\alpha) = -\Phi^{-1}(\alpha) =: -z_\alpha.$$

Now we can easily separate VaR_α

$$VaR_\alpha = -m + s \cdot z_\alpha, \tag{2.16}$$

where
m - selective mean value,
s - selective standard deviation,
$\Phi^{-1}(\alpha) \equiv z_\alpha$ - inverse distribution function of normal distribution (quantile function).

VaR calculation for Student's t-distribution

Now we suppose that the risk factor Ra, absolute return (2.8), has the t-distribution with v degrees of freedom (see Appendix A.2), i.e. $Ra \sim t_v(a,b)$. For VaR_α we have that:

$$P[Ra < -VaR_\alpha] = 1 - \alpha$$

and

$$P\left[\frac{Ra - a}{b} < \frac{-VaR_\alpha - a}{b}\right] = 1 - \alpha$$
$$\frac{-VaR_\alpha - a}{b} = t_v^{-1}(1-\alpha) = -t_v^{-1}(\alpha) =: -tinv_v(\alpha).$$

Now we can separate VaR_α

$$VaR_\alpha = -a + b \cdot tinv_v(\alpha), \tag{2.17}$$

we can write it as

$$VaR_\alpha = -m + s \cdot \sqrt{\frac{v-2}{v}} \cdot tinv_v(\alpha), \tag{2.18}$$

where
$t_v(a,b)$ - generalised t-distribution with v degrees of freedom and parameters(a,b),
m - selective mean value,
s - selective standard deviation,
$tinv_v(\alpha)$ - inverse distribution function of t-distribution.

VaR calculation for Log-normal distribution

Now suppose that the value of the portfolio has the log-normal distribution (see Appendix A.3), i.e. we suppose that the Rg, logarithmic rate of return (2.9), has the normal distribution with parameters μ_{Rg}, σ_{Rg}, i.e. $Rg \sim N(\mu_{Rg}, \sigma^2_{Rg})$. Let Rg_{crit} be the value of Rg, which separate$(1-\alpha)\%$ of the worst rates from the rest of the possible rates. It satisfies:

$$P[Rg < Rg_{krit}] = 1 - \alpha$$

and

$$P\left[\frac{Rg - m_{Rg}}{s_{Rg}} < \frac{Rg_{crit} - m_{Rg}}{s_{Rg}}\right] = 1 - \alpha$$
$$\frac{Rg_{crit} - m_{Rg}}{s_{Rg}} = \phi^{-1}(1-\alpha) = -\phi_v^{-1}(\alpha) =: -z_(\alpha)$$
$$Rg_{crit} = m_{Rg} - s_{Rg} \cdot z_\alpha.$$

Now we can write VaR_α

$$VaR_{\alpha,t} = -1(P_{t-1} \cdot exp(Rg_{crit}) - P_{t-1}) \tag{2.19}$$

it is

$$VaR_{\alpha,t} = P_{t-1}(1 - exp(Rg_{crit})), \tag{2.20}$$

where
$VaR_{\alpha,t}$ - VaR_α calculated in day $t-1$,
P_{t-1} - value of portfolio in the time $t-1$ - current value of the portfolio,
$N(\mu_{Rg}, \sigma_{Rg})$ - normal distribution with parameters μ_{Rg}, σ_{Rg},
m_{Rg} - selective mean value of logarithmic rates,
s_{Rg} - selective standard deviation of logarithmic rates,
$\Phi^{-1}(\alpha) \equiv z_\alpha$ - inverse distribution function of normal distribution.

The assumption of portfolio to have the log-normal distribution and the derivation of VaR is consistent with the theory that value of the portfolio follows the geometric Brownian motion (see Appendix 5).

$$P_t = exp(\mu t + \sigma w_t)$$
$$Rg_t = ln\left(\frac{P_t}{P_{t-1}}\right) = \mu + \sigma(w_t - w_{t-1}) \sim Norm(\mu, \sigma^2),$$

where
P_t - value of the portfolio in time t,
Rg - logarithmic rate of return,
$Norm(\mu, \sigma)$ - normal distribution with parameters μ, σ,
w_t - Wiener process (see Appendix 5), satisfying: $w_t - w_s \sim Norm(0, t-s)$.

Position approach - Multivariate distribution

Calculation position approach is harder then portfolio approach. There are technical demands on collection, saving and calculation. We will not demonstrate the calculation in this work.

More about this methods you can read in [10].

Advantages and Disadvantages

The advantage is that VaR estimation with parametric methods could be more accurate, the accuracy is limited by distribution parameters. The disadvantage is that it may be difficult to estimate the right distribution and parameters of our risk factors.

2.3.3 Monte Carlo simulation method

We will talk about this method in details in the next chapter. Now only a short description of this method. [15] Chapter 3, Page 43.

This method has a lot of in common with the historical simulation method, but the one big difference is that historical simulation use the normal distribution. On the opposite Monte Carlo simulation uses for selected market factors such a statistical distribution that can best affect change in market factors. It is not determined that it will be a normal distribution as in the case of historical simulation. Selection of an adequate statistical distribution is on the financial managers of the company.

After selecting the statistical distribution we generate thousands hypothetical changes in market factors using the pseudo-random numbers generator. Subsequently, we construct hypothetical profits and losses of the portfolio. The maximum expected loss is then determined from the distribution of profits and portfolio losses for the desired confidence level.

Advantages and Disadvantages

The advantage is lots of data in the simulation. You can use various distribution assumptions. The disadvantage is that it may take a lot of computational power (and hence a longer time to estimate results).

2.3.4 Comparison of methods

There is a question which of these approaches or methods is the best for calculating Value at Risk. Unfortunately there is no easy answer. The various methods differ in complex risk measurement capabilities of market factors, in difficulty of implementation, in ways of explanation to the management, flexibility in analysing the effect of changes in the assumptions and reliability of the results.

The choice of method depends on the parameters which the risk manager considers more important. The simplest method to determine the maximum expected loss is a historical simulation. This method is also simple to implement and easily understood by managers. On the other hand, we have to know the time series of relevant market factors. If these series are atypical for a given number of factors, the result is relatively inaccurate.

Parametric approach to calculating VaR is used at the academy rather than in practice. With added value, that lies in a parametric distribution, that we assume to be correct, the estimate of VaR with parametric method is more accurate. Of course, the accuracy is limited by the distribution parameters.

The highest demands on the initial assumptions has analytical method. However the assumption of normal distribution for the individual factors in a portfolio cannot be accepted, especially at longer time intervals.

As for the accuracy of the result, the best method seems to be the Monte Carlo method. The advantage flexibility is particularly large. However, its use may be time consuming and it requires some knowledge and experience of the creators and users.[15]

MC simulation and historical simulation both these methods based on simulations suffer when using a lower number of scenarios by bad convergence to the actual sample quantile. While Monte Carlo method is generating larger number of scenarios,and the limits are given by the computational resources available, the historical simulation method exhibits a more serious problem - a long time series are often not available and

Table 2.2: Comparison of Value at risk methodologies

	Historical simulation	**Parametric methods**	**MC simulation**
Able to measure the risk of portfolios which include options?	Yes, regardless of the options content of the portfolio	Yes, but is better for a short time period and small content options	Yes, regardless of the options content of the portfolio
Easy to implement?	Yes, for portfolios with the available historical data of market factors	Yes, for portfolios that use only the tools of current software.Simply - medium, according to the complexity of software tools and data	Yes, for portfolios that use only the tools of current software.Simply - medium, according to the complexity of software tools and data
Easy to explain to management?	Yes	No	No
Computation performed quickly?	Yes	Relatively yes	Relatively yes
VaR values are misleading if the recent history is atypical?	Yes	Yes, except the case when alternative parameters estimation are used	Yes, except the case when alternative parameters estimation are used

VaR can not be estimated, especially at higher levels of probability.

There are some ways to improve the convergence of these two methods(e.g. using pseudohistorical scenarios), see for example [16].

Comparison of three methods mentioned above is summarized in Table 2.2. Table with more criterion can be found in [17], Page 38.

2.4 Advantages and Disadvantages of VaR

Advantages [2]

- **VaR is easy to understand** - VaR is measured in price units (Dollars, Euros) or as a percentage of portfolio value. This makes VaR very easy to interpret and also to further use in analyses.

- **Comparing VaR of different assets and portfolios** - We can measure and compare VaR of different types of assets and various portfolios (stocks, bonds, currencies, derivatives, or any other assets with price).

- **VaR is often available in financial software** - VaR is a frequent part of various types of financial software.

- **Everybody else uses VaR** - When competitors use it, clients require it, and regulators recommend it.

Disadvantages [3]

- **VaR can be misleading** - Many people think of VaR as "the most I can lose", especially when it is calculated with the confidence parameter set to 99%. In reality 99% is very far from 100% and here is the place where the incomplete understanding of VaR can be fatal.

- **VaR gets difficult to calculate with large portfolios** - When you are calculating VaR of a portfolio, we need to measure or estimate not only the return of individual assets, but also the correlations between them. With growing number and diversity of positions in the portfolio, the difficulty of this task grows exponentially.

- **VaR is not additive** - Correlations between individual risk factors enter the VaR calculation is also the reason why VaR is not simply additive. The portfolio VaR containing assets A and B does not equal to the sum of asset A VaR and asset B VaR.

- **The resulting VaR is only as good as the inputs** - Using unrealistic return distributions as inputs can lead to underestimating the real risk with VaR.

- **Different VaR methods lead to different results** - Different approaches (Historical simulation, Analytical simulation, MC simulation,..) can also lead to very different results with the same portfolio, so the representativeness of VaR can be questioned.

[2] 12th April, 2012. http://www.macroption.com/value-at-risk-var-advantages/
[3] 12th April, 2012. http://www.macroption.com/value-at-risk-var-limitations-disadvantages/

Chapter 3

Monte Carlo method (MC)

In this chapter we follow the book written by Fotr and Hnilica [18], Chapters 2,3.
At first we have to say something about sensitivity analysis, which will be useful later in the MC simulation.

3.1 Sensitivity analysis

The goal of sensitivity analysis in financial management is detection of the companies or projects sensitivity selected financial criterion on possible risk factors value changes. It means that we have to determine, how certain changes (e.g. volume of production, utilization of production capacity, size of the investment costs,...) influence the given criterion.

The basic form of sensitivity analysis is the **single factor analysis**, where we can see the effects of **isolated changes** on the chosen criterion in individual risk factors. In this case all other factors stay on their anticipated (planned, most likely) values.
Changes of individual risk factors values then may have the following character:

- **Pessimistic or Optimistic** values,
- **Deviation** from planned (most likely) values of the certain size (e.g. ± 10).

We can divide risk factors into:

- **Little importance** - risk factors whose changes (of character referred above) produce only slightly change in the selected criterion.
- **Significant** - factors whose changes cause a significant change in the selected criterion. The criterion is very sensitive to changes in these factors.

3.1.1 Example

In this example we want to assess the significance of risk factors which influence profit (profit before tax) of a new project.

Following Example 2.1 from [18], Chapter 2, Pages 32-35, we use for demonstration the sensitivity analysis.
Annual profit before tax is difference between revenues and total costs. Manufactured product is designed to markets in a EU country. Size of revenues is product of annual sales in terms of volume (sales prices in Euro and exchange rate of Crown against Euro).

Costs represent sum of variable and fixed costs for a period.
Variable costs value of material consumption on annual production. Product of annual production, standards of material consumption and the purchase price.
Fixed costs are mostly overhead costs.

Annual profit from the production of a new product is

$$P = S \cdot SP \cdot ER - (S \cdot C \cdot MP + FN), \tag{3.1}$$

where

P - profit before tax (CZK/year),
S - sales, (production - amount of stocks is still on the same level, sales volume is the same as the size of production) (pc/year),
SP - sales price of product (Euro/pc),
ER - exchange rate CZK against Euro (CZK/Euro),
C - standard of material consumption per unit of production (kg/pc),
MP - material price (CZK/kg),
FN- fixed costs (mill. CZK/year).

We assume that the planned values of all six factors are effected by uncertainties. To assess the benefits and risks of the project in addition to the planned values will be processed in two scenarios, the **optimistic** one (positive development of factors) and the **pessimistic**(negative development of factors).

All the values and analysis process summarize the following Table 3.1 from Excel, which is optimal to calculating this analysis. The original file can be found on the attached CD.

Table 3.1: Sensitivity analysis

Sensitivity Analysis

	Risk factor	Unit	Scenario: Pessimistic	Scenario: Most likely	Scenario: Optimistic
S	Sales - (pc/year)	1000	75	100	120
SP	Selling price - (Euro/pc)		135	150	160
ER	Exchange rate - (CZK/Euro)		22	24	25
MP	Material price - (CZK/kg)		46	40	36
FC	Fixed costs - (CZK/year)	1 000 000	85	75	70
C	Standard of material consumption - (kg/pc)		62	60	58
Profit			-76 150 000	45 000 000	159 440 000

Now we will show how sensitive is profit on individual risk factors value changes by isolated transition of each factor from the value of most likely scenario to the value of pessimistic scenario. We have to do six calculations. Always one factor will be changed and the rest stay at the most likely scenario.
e.g. : Decline of sales from 100 000 pc/year to 75 000 pc/year, other factors will be not changed. So we have now the annual profit 15 000 000 CZK, it is absolute decline is 30 000 000 CZK. The relative decline will be 30 000 000/45 000 000 x 100 = 66,7%.

	Risk factor	Unit	Factor value: Most likely	Factor value: Pessimistic	Profit	Decline of profit: A (CZK)	Decline of profit: R (%)
S	Sales - (pc/year)	1000	100	75	15 000 000	30 000 000	66,67
SP	Selling price - (Euro/pc)		150	135	9 000 000	36 000 000	80,00
ER	Exchange rate - (CZK/Euro)		24	22	15 000 000	30 000 000	66,67
MP	Material price - (CZK/kg)		40	46	9 000 000	36 000 000	80,00
FC	Fixed costs - (CZK/year)	1 000 000	75	85	35 000 000	10 000 000	22,22
C	Standard of material consumption - (kg/pc)		60	62	37 000 000	8 000 000	17,78

A	Absolute decline in annual profit
R	Relative decline in annual profit

We can see that the annual profit responds most sensitively on decline of selling price from 150 Euro/pc to 135 Euro/pc and on increase of purchase price of materials from 40 CZK/kg to 46 CZK/kg. These two factors are the most important risk factors of the project. Quite significant are demand and exchange rate (profit decline of 66,7%). Less significant are fixed costs (profit decline of 22,22%). The least significant is standard of material consumption (profit decline of 17,8%).

The second table illustrate profit sensitivity on negative factors development which influence profit from new product production. This project is very risky. That is obvious also from the first table where we can see that the calculation of profit with pessimistic scenario. In this case is project is very loss, the loss will reach up the 76 150 000 CZK.

Advantage of that sensitivity analysis form is that it respects different uncertainty amount of factors that effect the chosen financial criterion for the project.

Disadvantage is that we can use this type only if the pessimistic scenario was created. And the ambiguity of understanding the pessimistic (optimistic) scenario.

In practise financial managers often use a sensitivity analysis based on investigating the impact of certain percentage changes in risk factors (usually $\pm 10\%$).

3.2 Technique of Monte Carlo simulation

We will follow the book [18], Chapter 3, Pages 71,72. We can divided the technique of the simulation into the following steps:

- **Creating mathematical model of the project** - and his processing e.g. in MS Excel. The model has usually the form of profit and loss statements, balance sheets, cash flow and the formulas for calculating individual criterion of project.
- **Determine of key risk factors** - input variables of model, which significantly affect the uncertainty of simulation inputs in form of financial criterion. It means the uncertainty of these factors will be respected in the simulation. Other inputs variables will be constants in the form of their most likely values. The useful tool to determine key factors is sensitivity analysis.
- **Determine of probability distribution of key risk factors** - this is generally complex task. By discrete risk factors has a distribution spreadsheet form. By continuous risk factors is selected a certain type of distribution and it's parameters are entered. It is possible to use historical data or experts experiences.
- **Determine of statistical dependence of key factors** - value of some risk factors may depend on other factors. In the simulation we can not generate these factors independently, but their dependence has to be respected.
- **Process of simulation with using computer program** - consists of a large number of steps which are repeated until the end of the simulation. In each step the program generates values of risk factors on their probability distribution and calculates the model of risk analysis object. User gets results in graphic or numerical form.

We will show the whole simulation on the example.

3.3 Example

The Example 3.6 [18], Chapter 3, Page 73 we will study for the MC simulation.
The task is to process risk analysis of investment project using MC simulation. Project is about production a new product. Profit before tax is affected by six factors. Each factor is loaded by some uncertainty. We consider three scenarios for evaluating the benefits and risks of this project:

- **most likely scenario (basic)** - based on assumptions,
- **optimistic scenario** - very positive development of risk factors,
- **pessimistic scenario** - opposite of optimistic scenario, very negative development of risk factors.

Their values are shown in Table 3.2.

Table 3.2: Risk factor values scenarios

	Risk factor	Unit	Scenario: Pessimistic	Scenario: Most likely	Scenario: Optimistic
S	Sales - (pc/year)	1000	75	100	120
SP	Selling price - (Euro/pc)		135	150	160
ER	Exchange rate - (CZK/Euro)		22	24	25
MP	Material price - (CZK/kg)		46	40	36
FC	Fixed costs - (CZK/year)	1 000 000	85	75	70
C	Standard of material consumption - (kg/pc)		60	60	60
Profit			-69 250 000	45 000 000	150 800 000

Now we will follow the steps from the beginning of this chapter.

3.3.1 Creating mathematical model of the project

We assume that the chosen financial criterion of the project is annual profit before tax, then our model of this project is rather simple. We have four relations:

- **Annual revenues calculation (R)** - from sales of the new product we set

$$R = S \cdot SP \cdot ER, \tag{3.2}$$

where
S is the sale volume ,
SP is the product selling price,
ER is the exchange rate (CZK/Euro).

- **Variable costs calculation (VN)** - we set

$$VC = S \cdot C \cdot MP, \tag{3.3}$$

where
S is the annual sale (we assume not manufactured in inventory, that means the size of sales expresses production volume),
C is the standard of material consumption,
MP is the material price.

- **Total costs calculation (N)** - we set as

$$TC = VC + FC, \tag{3.4}$$

where
VC are variable costs,
FC are fixed costs (we work with them as one item).

- **Profit before tax calculation (Z)** - finally we can set

$$P = R - TC. \tag{3.5}$$

Based on these four relation we can create simple program in MS Excel Table 3.3, which will calculate revenues, variable costs, total costs and profit before tax with dependence on six influencing parameters.

Table 3.3: Production of new product

Production of a new product			
Input variables			
Parameter		**Unit**	**Most likely scenario**
S	Sales - (pc/year)	1000	100
SP	Selling price - (Euro/pc)		150
ER	Exchange rate - (CZK/Euro)		24
MP	Material price - (CZK/kg)		40
FC	Fixed costs - (CZK/year)	1 000 000	75
C	Standard of material consumption - (kg/pc)		60
Output variables			
R	Revenues (CZK)		360 000 000
VC	Variable costs (CZK)		240 000 000
TC	Total costs (CZK)		315 000 000
P	Profit before tax (CZK)		45 000 000

The created Table 3.3 has two parts:

There are input variables of the model - which correspond to the most likely scenario from Table 3.2 .

Second part contains output variables of the model - we can see that annual revenues are with this scenario CZK 360 M., total costs are CZK 315 M. and profit before tax is CZK 45 M.

3.3.2 Key risk factors determination

To determine the key factors (which uncertainty will be accepted in the simulation) we can use results of sensitivity analysis, which we created in the section 3.1 - Sensitivity Analysis.

Example of sensitivity analysis shows that first five factors are significant. Their uncertainty should be respected in the simulation. The least significant factor is standard of material consumption, that's why we can set this factor as a constant on the level his most likely value.

So, we receive a new Table 3.4 with parameters.

Table 3.4: Determine of key risk factors

Determine of key risk factors					
			Scenario		
Risk factor		Unit	Pessimistic	Most likely	Optimistic
S	Sales - (pc/year)	1000	75	100	120
SP	Selling price - (Euro/pc)		135	150	160
ER	Exchange rate - (CZK/Euro)		22	24	25
MP	Material price - (CZK/kg)		46	40	36
FC	Fixed costs - (CZK/year)	1 000 000	85	75	70
C	Standard of material consumption - (kg/pc)		60	60	60

3.3.3 Key factors probability distribution determination

We will use a **triangular distribution** [5] for displaying individual risk factors probability distribution.
For the key uncertainty variable, the Monte Carlo simulation defines the possible values with a probability distribution. The type of distribution is based on the conditions surrounding the variable. In cases, where minimum, most likely, and maximum values are known but the relationship among those points is unknown, a triangular distribution is most appropriate.

Triangular distribution

The user defines the minimum, most likely, and maximum values. Values around the most likely are more likely to occur. Distribution can be either symmetric (the most probable value = mean = median) or asymmetrical. Variables that could be described by a triangular distribution include past sales history per unit of time and inventory levels.
More about the triangular distribution can be found in Appendix 5.
We can see the distribution of our key factors in the Figure 3.1.

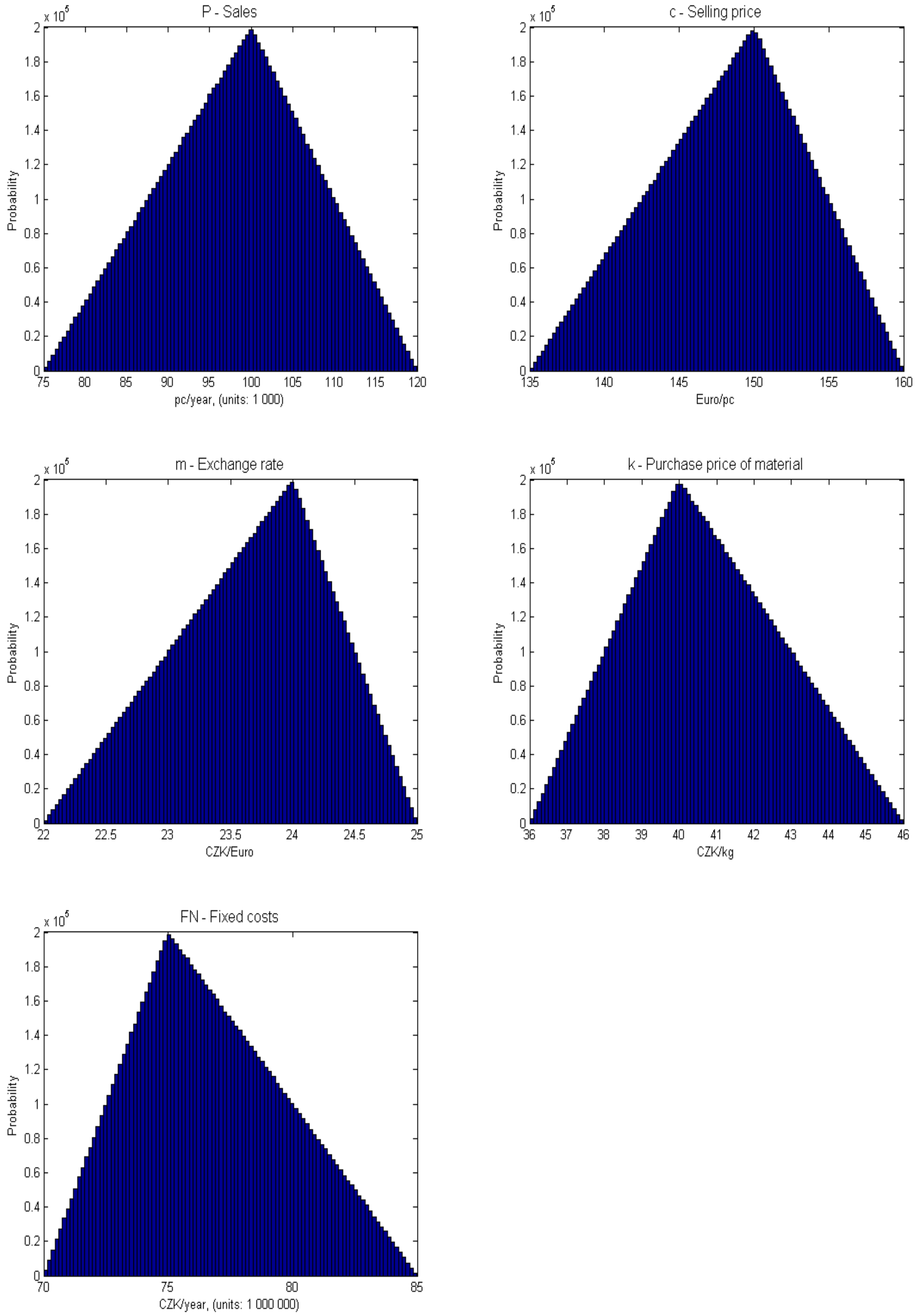

Figure 3.1: Distribution of key factors

Some frequently used probability distributions [1]

- **Normal/Gaussian distribution** - Continuous distribution is applied in situations where the mean and the standard deviation are given and the values in the middle near the mean represent the most probable values of the variable. It is symmetric around the mean and is not bounded. (Examples of variables described by normal distributions include inflation rates and energy prices.)

- **Log-normal Distribution** - Continuous distribution is specified by mean and standard deviation. This is appropriate for a variable ranging from zero to infinity, with positive skewness and with normally distributed natural logarithm. Values are not symmetric like a normal distribution. (Examples of variables described by log-normal distributions include real estate property values, stock prices, and oil reserves.)

- **Uniform Distribution** - Continuous distribution is bounded by known minimum and maximum values. In contrast to the triangular distribution, the probability of occurrence the values between the minimum and maximum is the same. (Examples of variables that could be uniformly distributed include manufacturing costs or future sales revenues for a new product.)

- **Exponential Distribution** - Continuous distribution used to illustrate the time between independent occurrences, provided the rate of occurrences is known.

More can be found in [19].

3.3.4 Determination of statistical dependence of key factors

In this very simple example we will omit dependences. If you want to know something about this problematic you can find it in: *Aplikovaná analýza rizika ve finančním managementu a investičním rozhodování, Jiří Hnilica, Jiří Fotr, Chapter 7*.

[1] 3th November 2012. http://www.investopedia.com/articles/07/monte_carlo_intro.asp#axzz2Bdsl6C27

3.3.5 Simulation process with using computer support

For the simulation we will use program MATLAB 7.1. For easy processing results is very useful to link Matlab with MS Excel with using tool Excel Link. Short instruction how to do it can be found in Appendix 5.

Before we start the simulation we will create a file with input data in Excel. We will use this file for all our simulations. An example file is shown in following Table [3.5].

Table 3.5: MS Excel - Input file

Input file									
			Scenario			Data for Matlab		
Risk factor		Unit	Pessimistic	Most likely	Optimistic			
S	Sales - (pc/year)	1000	75	100	120	75	100	120
SP	Selling price - (Euro/pc)		135	150	160	135	150	160
ER	Exchange rate - (CZK/Euro)		22	24	25	22	24	25
MP	Material price - (CZK/kg)		46	40	36	36	40	46
FC	Fixed costs - (CZK/year)	1 000 000	85	75	70	70	75	85
C	Standard of material consumption - (kg/pc)		60	60	60			
Profit			-69 250 000	45 000 000	150 800 000			

We can see data for Matlab in the small yellow table on the right. These data are sorted from smallest to largest and will be put into the Matlab with using a command *xlsread('fileName','sheet','range')*. The variable range will be in our case *H5:J9*.

Chapter 4

Simulation experiments

In this chapter we will show various simulations. We will watch influence on the simulation accuracy, what is the best simulation technique. We follow the book written by D. Vose [20] and also the book by Fotr, Hnilica [18], Chapter 3.

At the beginning, we have to clarify several terms:

Simulation Experiment - often also called only " simulation " - not to be confused with one particular simulation. Experiments can consist of 100-1 000 simulations.

Simulation - one particular sample contained in the simulation experiment. Simulation can be also called as one iteration in the simulation experiment.

Scenario - generally represents specific images or descriptions of the future, composed of elements and their links within the file. The goal of scenarios is to provide a structured view on the neighbourhood development, in which the company is located.

In practise, we can meet two basic types of scenarios:

- **Qualitative** - are long term oriented (5 to 10, eventually 20). These have usually verbal description. By creation these scenarios are used by external workers and consultants. These scenarios are not used as a support in making decisions under risk or uncertainty. They are rather used to generate new strategic ideas.They have to extended the circle of managers thought (change their corporate stereotypes and isolation).
- **Quantitative** - are generally short-term nature. By creation, these scenarios apply analytical and data-based techniques. They put the emphasis to the internal specialists. These scenarios are often used for determining the impact of risk decisions (e.g. investment projects) for each scenario, evaluation and selection.

More interesting reading about scenarios [18], Chapter 3, Section 3.2.

4.1 Accuracy

We will study how **the number of scenarios** affect the simulation accuracy in the first type of simulation experiment.

We will show three simulations experiments with different parameters alpha (0.95, 0.99, 0.995). For all of them we will use the same number of simulations and we will change the number of scenarios.

As we already mentioned in Section 3.3.5, we will use the program MATLAB 7.1 and the results will be processed in MS Excel by using the Excel Link tool (instruction in Appendix 5).

We start with data loading from prepared Excel file 3.5. Then, we define all variables. Number of simulations will be 1000 in one simulation experiment. Number of scenarios will be changed (10 000, 100 000, 1 000 000), for all values of parameter alpha.
Now we look at the main body of the source code, see Listing 4.1. First, we start time tracing. Second, we perform m simulations. Each simulation has n scenarios. In each scenario we generate 5 numbers from interval [0,1].

We calculate values of each key factors for each of this 5 numbers.

Given a random variate U drawn from the uniform distribution in the interval (0, 1), then the variate

$$\begin{cases} X = a + \sqrt{U(b-a)(c-a)} & \text{for } 0 < U < F(c), \\ X = b - \sqrt{(1-U)(b-a)(b-c)} & \text{for } F(c) < U < 1. \end{cases} \tag{4.1}$$

More information about generating triangular-distributed random variates can be found in Appendix 5.

Table 4.1: Factors

	A	B	C	D	E	F	G	H	I
1									
2									
3		1	2	3			1	2	3
4	1	75	100	120		1	j,1	j,2	j,3
5	2	135	150	160		2	j,1	j,2	j,3
6	3	22	24	25		3	j,1	j,2	j,3
7	4	36	40	46		4	j,1	j,2	j,3
8	5	70	75	85		5	j,1	j,2	j,3
9									

Formula (4.1) for factors from Table 4.1 is implemented in the source code 4.1, on lines 40-42. With using this formula and Table of factors 4.1 we can appoint to formula in the source code, line 40.
After this calculation, we compute the resulting profit in each scenario. Subsequently, we calculate Value at Risk for different value of parameter alpha for each simulation.

With this procedure we get 1000 values of Value at Risk for each parameter alpha. In the rest of the source code, we calculate averages and standard deviations for all values of alpha and in the end we plot graphs of normal distribution.

The whole source code of described procedure follows.

Listing 4.1: Computation of VaR

```

clear

%load input data from excel file
data=xlsread('input_data.xls','MC','H5:J9');

% DEFINITION OF VARIABLES

% number of scenarios (10 000, 100 000, 1 000 000)
n=10000;
% number of simulations
m=1000;
% standard of material consumption (fixed factor)
s=60;

alpha1= 0.95;
alpha2= 0.99;
alpha3= 0.995;

% an auxiliary matrix, (to speed up the calculation)
profit=zeros(n,1);
% an auxiliary matrix, (to speed up the calculation)
x=zeros(5,1);
% an auxiliary matrix, (to speed up the calculation)
VaR=zeros(m,1);

% MAIN BODY

 % starts to trace time
tic
    for k = 1:m

        for i=1:n

         % generetes 5 random numbers from intevral [0,1]
         u=rand(5,1);

            % calculates the corresponding values of each
               variable in the given scenario
            for j=1:5
                if u(j)<=(data(j,2)-data(j,1))/(data(j,3)-data(j
                   ,1))
                x(j)=data(j,1)+sqrt(u(j)*(data(j,2)-data(j,1))*(
```

```
                    data(j,3)-data(j,1)));
                else x(j)=data(j,3)-sqrt((1-u(j))*(data(j,3)-data(j,1))*(data(j,3)-data(j,2)));
                end
            end

        % calculates the resulting profit in the given scenario
        profit(i)=x(1)*1000*x(2)*x(3)-(x(1)*1000*s*x(4)+x(5)*1000000);
        end

    VaR1(k,1)=-quantile(profit,1-alpha1); % VaR value - 0.95
    VaR2(k,1)=-quantile(profit,1-alpha2); % VaR value - 0.99
    VaR3(k,1)=-quantile(profit,1-alpha3); % VaR value - 0.995
    end




average1=mean(VaR1);
standard_deviation1=std(VaR1);

average2=mean(VaR2);
standard_deviation2=std(VaR2);

average3=mean(VaR3);
standard_deviation3=std(VaR3);

%GRAPHS

figure(1)

subplot(3,1,1)
% a histogram with a specified number of classes
hist(VaR1)
ylabel('VaR');
title('Histogram','fontsize',12)
subplot(3,1,2)
% a normal probability plot
normplot(VaR1)
subplot(3,1,3)
% empirical distribution function
cdfplot(VaR1)

figure(2)

subplot(3,1,1)
% a histogram with a specified number of classes
hist(VaR2)
```

```
ylabel('VaR');
title('Histogram','fontsize',12)
subplot(3,1,2)
% a normal probability plot
normplot(VaR2)
subplot(3,1,3)
% empirical distribution function
cdfplot(VaR2)

figure(3)

subplot(3,1,1)
% a histogram with a specified number of classes
hist(VaR3)
ylabel('VaR');
title('Histogram','fontsize',12)
subplot(3,1,2)
% a normal probability plot
normplot(VaR3)
subplot(3,1,3)
% empirical distribution function
cdfplot(VaR3)


% end of time tracing
time = toc;
```

Results

In the first simulation experiment we used **10 000 scenarios**. Results can be seen in Table 4.2. Time of experiment is about 40 seconds, which is negligible. We can also see that the average and standard deviation increase with increasing alpha. In Figure 4.1 we can see how the VaR is close to normal distribution.

In the second experiment we set **100 000 scenarios**.
We can see in Table 4.3 that the experiment time is now about 7 minutes. Average and standard deviation is again increasing with increasing alpha. Contrary to the Simulation 1 the standard deviation is lower. We can see how the VaR is close to normal distribution in Figure 4.2.

In the third simulation experiment we set **1 000 000 scenarios**.
In Table 4.4 the time is almost 1 hour. Average and standard deviation is again increasing with increasing alpha. In this case, the standard deviation is much lower than in the previous two cases. It is even in the order of tens of thousands. Figure 4.3 shows how the VaR is close to normal distribution.

Table 4.2: Simulation accuracy 1 - Results

13	Simulation 1				
14					
15	number of scenarios	10 000			
16	number of simulations	1000			
17					
18	alpha = 0.95	alpha = 0.99	alpha = 0.995		
1000	6588096,989	19620170,87	23156835,24		
1001	6643488,433	19060150,78	23006439,37		
1002	6961470,461	19888808,09	25194597,16		
1003	6349834,2	19667727,82	23567496,07		
1004	6370945,316	19952365,98	23930405,18		
1005	6350256,613	18993136,02	22762825,34		
1006	6519821,664	19361033,76	23180302,9		
1007	6046557,466	18523375,63	23231549,9		
1008	6545812,346	19393004,93	24302784,88		
1009	5813871,054	18743556,35	22429727,63		
1010	6835848,61	19878861,59	24384174,27		
1011	6542977,352	19717418,79	23740002,19		
1012	6457916,808	19675783,1	23772594,66		
1013	7176524,953	19885277,95	25209154,91		
1014	7465249,071	18656843,33	22803809,45		
1015	6198144,759	19635062,78	24652296,95		
1016	6777560,465	19680552,67	24885297,19		
1017	6542754,499	19534146,61	23702276,67		
1018	7153130,83	20254011,66	25083503,29		
1019					
1020					
1021	average				
1022	6 792 417,87	19 659 059,36	24 050 110,96		
1023	standard deviation				
1024	403 139,25	660 864,02	822 572,74		
1025	simulation time [s]			sim. time [min]	
1026	39,67088303			0,661181384	

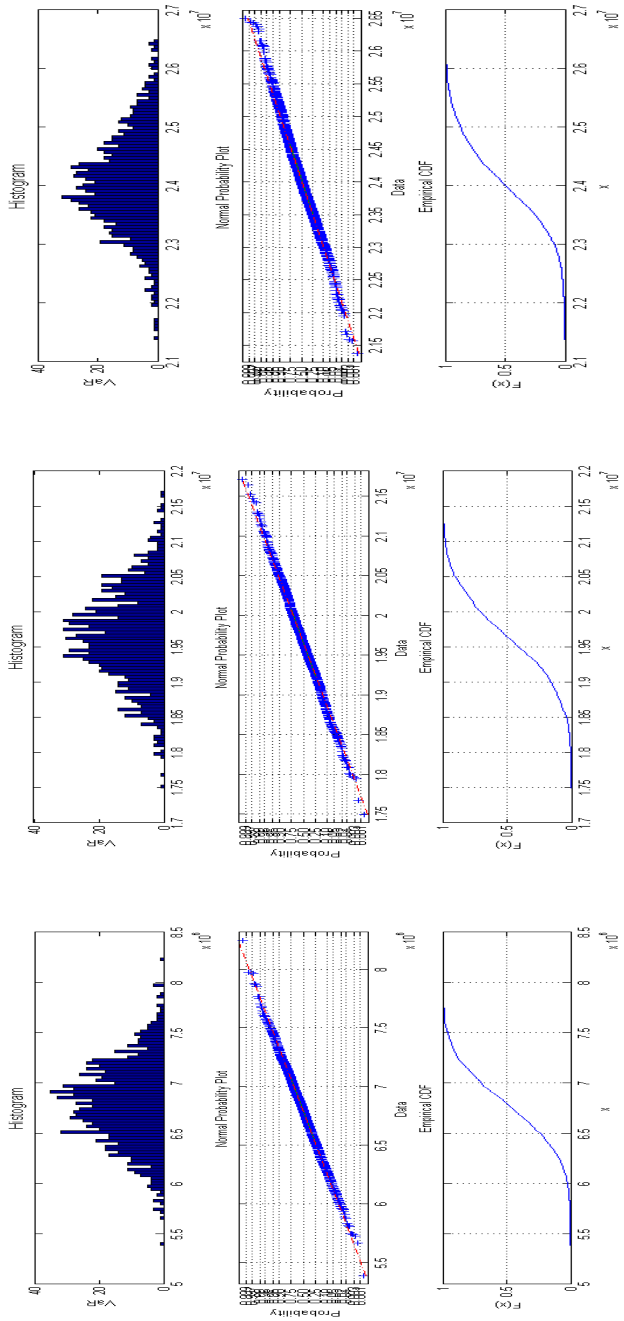

Figure 4.1: Graphs 1 - alpha= 95%, 99%, 995%

Table 4.3: Simulation accuracy 2 - Results

13	Simulation 2			
14				
15	number of scenarios	100 000		
16	number of simulations	1000		
17				
18	alpha = 0.95	alpha = 0.99	alpha = 0.995	
1000	6918055,858	19721431,78	24006673,22	
1001	6732121,337	19979307,34	24452991,3	
1002	6842398,269	19662646,67	24081437,32	
1003	6990015,348	19784715,38	24242399,33	
1004	6880271,716	19682561,49	23781916,47	
1005	6728359,527	19486674,94	23599414,82	
1006	6583661,192	19515414,82	24301655,86	
1007	6829234,921	19697587,04	23966119,46	
1008	6783912,846	19781423,04	23962964,22	
1009	6892291,194	19646155,9	24064197,44	
1010	6686079,337	19575796,94	24014695,35	
1011	6933140,289	19975420,44	24492262,03	
1012	6728225,68	19441313,78	24048890,56	
1013	7008905,962	19664183,68	23947044,72	
1014	6833284,288	19476847,26	23722630,82	
1015	6789164,711	19425242,4	23644158,33	
1016	6648208,026	19278934,44	23921528,95	
1017	6695700,803	19659245,59	23814158,5	
1018	6846860,882	19783254,01	24147940,45	
1019				
1020				
1021	average			
1022	6 801 940,83	19 692 983,78	24 088 193,19	
1023	standard deviation			
1024	133 525,63	210 934,34	266 441,84	
1025	simulation time [s]			sim. time [min]
1026	412,4069461			6,873449102

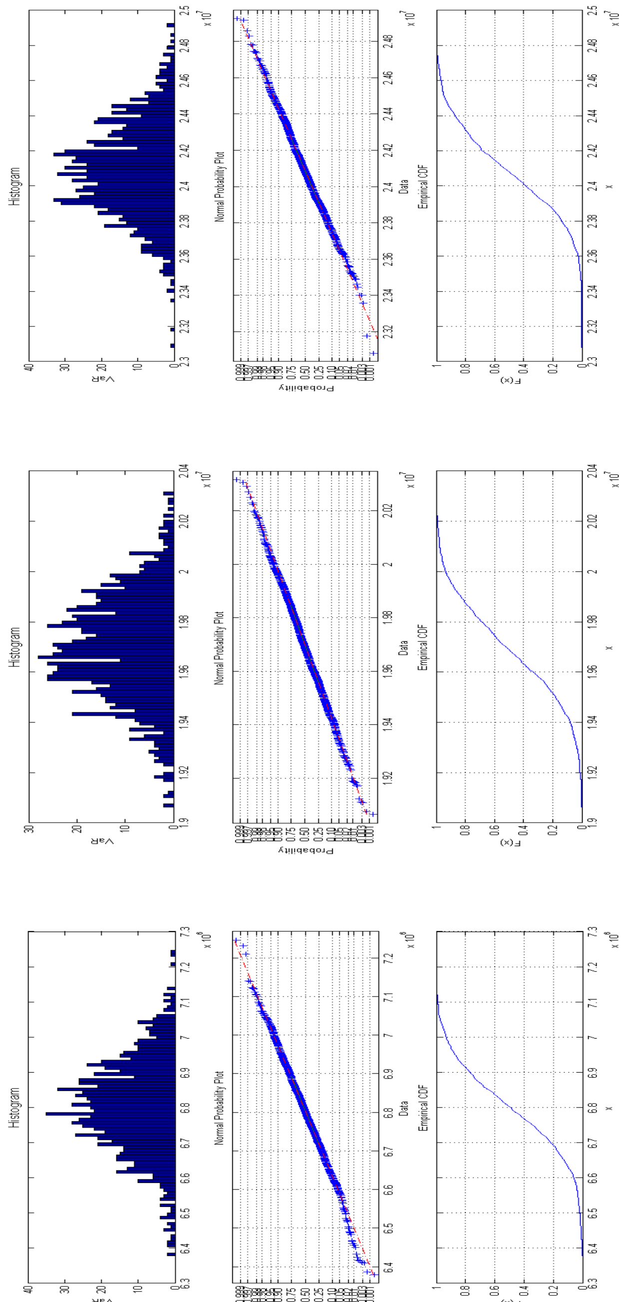

Figure 4.2: Graphs 2 - alpha= 95%, 99%, 995%

Table 4.4: Simulation accuracy 3 - Results

13	Simulation 3				
14					
15	number of scenarios	1 000 000			
16	number of simulations	1000			
17					
18	alpha = 0.95	alpha = 0.99	alpha = 0.995		
1000	6876152,957	19745454,35	24244637,73		
1001	6821056,065	19784732,57	24111229,52		
1002	6769852,044	19606939,97	23925273,45		
1003	6906417,893	19835514,17	24150573,5		
1004	6804066,107	19700154,16	24131330,92		
1005	6780137,258	19745437,42	24142731,83		
1006	6825167,403	19620046,56	24028490,85		
1007	6830767,93	19736077,65	24115904,98		
1008	6803870,647	19666487,36	24079344,38		
1009	6868544,221	19787163,61	24091905,26		
1010	6860865,927	19662169,82	24017471,01		
1011	6788170,026	19808703,84	24201979,51		
1012	6857003,557	19721045,37	24057983,93		
1013	6822772,651	19726147,1	24154742,43		
1014	6815930,291	19587298,39	23973199,68		
1015	6790360,076	19599316,03	24044533,25		
1016	6781964,364	19681938,46	24101449,72		
1017	6757820,991	19729780,37	24028208,54		
1018	6793290,083	19675526,33	24062654,95		
1019					
1020					
1021	average				
1022	6 809 111,79	19 706 003,00	24 105 398,68		
1023	standard deviation				
1024	38 791,31	64 996,52	81 925,01		
1025	simulation time [s]			sim. time [min]	
1026	4161,124634			69,35207723	

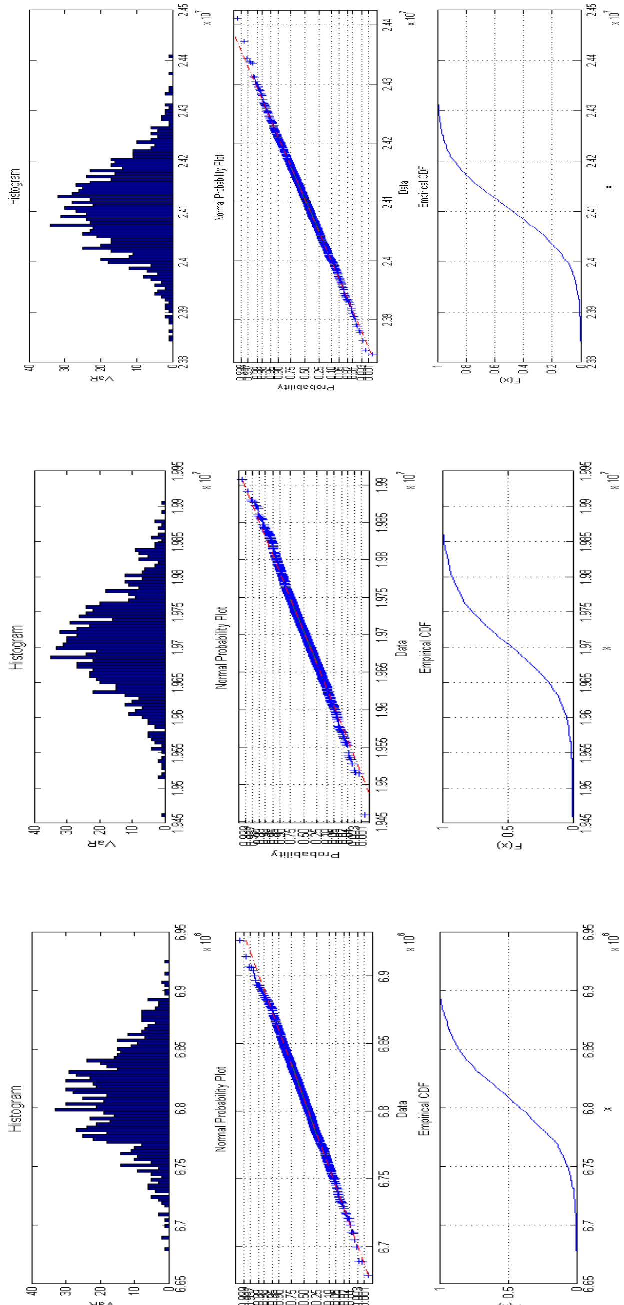

Figure 4.3: Graphs 3 - alpha= 95%, 99%, 995%

4.2 Simulation time

We will look closer to the simulation time in the second type of simulation experiment. We will examine which procedure is better for computing Value at Risk. Again we will show three simulation experiments with different parameter alpha (0.95, 0.99, 0.995).

The source code will be very similar to the source code 4.1.

We start with data loading from the prepared input Excel file - Table 3.5. After that we define variables. Number of simulations will be constant (1 000) for simulation experiment. Number of scenarios will be changed (1 000, 10 000, 100 000), for all values of parameter alpha. This source code will differ in the main body. For n scenarios we will generate 5 numbers from the interval [0,1], similar like in simulation experiment 4.1 we compute the values of individual factors and subsequently we compute the resulting profit in each scenario.

Now the source code starts to differ from the previous code 4.1. We will calculate the value of VaR in the cycle, where variable r is equal to 10. Thus we get 10 values of VaR from which we want to calculate the resulting value of VaR for all m simulations.

On the following pages you can see the whole source code 4.2.

Listing 4.2: Computation of VaR with repetition

```
clear

%load input data from excel file
data=xlsread('input_data.xls','MC','H5:J9');

% DEFINITION OF VARIABLES

% number of scenarios (1 000, 10 000, 100 000)
n=10000;
% number of simulations
m=1000;
% standard of material consumption (fixed factor)
s=60;
% number of repetitions
r=10;

alpha1=0.95;
alpha2=0.99;
alpha3=0.995;

% an auxiliary matrix, (to speed up the calculation)
profit=zeros(n,1);
% an auxiliary matrix, (to speed up the calculation)
x=zeros(5,1);
% an auxiliary matrix, (to speed up the calculation)
VaR=zeros(m,1);
% an auxiliary matrix, (to speed up the calculation)
VaR_result=zeros(r,1);

% MAIN BODY

 % starts to trace time
 tic
     for k = 1:m

          for l=1:r

            for i=1:n

             % generetes 5 random numbers from intevral [0,1]
             u=rand(5,1);

                  % calculates the corresponding values of each
                     variable in the given scenario
                  for j=1:5
                      if u(j)<=(data(j,2)-data(j,1))/(data(j,3)-
                         data(j,1))
                      x(j)=data(j,1)+sqrt(u(j)*(data(j,2)-data(j
```

```
                                ,1))*(data(j,3)-data(j,1)));
                            else x(j)=data(j,3)-sqrt((1-u(j))*(data(j
                                ,3)-data(j,1))*(data(j,3)-data(j,2)));
                            end
                        end
                  % calculates the resulting profit in the given
                      scenario
                  profit(i)=x(1)*1000*x(2)*x(3)-(x(1)*1000*s*x(4)+x
                      (5)*1000000);
                 end

                 VaR1(l,1)=-quantile(profit,1-alpha1);
                 VaR2(l,1)=-quantile(profit,1-alpha2);
                 VaR3(l,1)=-quantile(profit,1-alpha3);

               end

           VaR_result1(k,1)=mean(VaR1);   % VaR value - 0.95
           VaR_result2(k,1)=mean(VaR2);   % VaR value - 0.99
           VaR_result3(k,1)=mean(VaR3);   % VaR value - 0.995
       end

% end of time tracing
time = toc;
```

Results

In the first simulation experiment, we used **1 000 scenarios** with 10 repetitions and we compare it to **10 000 scenarios** without repetition. We can see in Table 4.5 that the simulation time is grater for simulation experiment with repetition. In computing the average, there is a larger part of the demands on computer memory. In this case, the difference between both procedures is very small (5.477 sec).

In the second simulation experiment, we used **10 000 scenarios** with 10 repetitions and we compare it with **100 000 scenarios** without repetition. We can see in Table 4.6 that in this case it is faster to calculate Value at Risk with repetition. The difference between both methods is again negligible (6.362 sec).

In the third simulation experiment, we used **100 000 scenarios** with 10 repetitions and we compare it with **1 000 000 scenarios** without repetition. We can see in Table 4.7 that the method of computing VaR with repetition is again faster like in previous case. Now the difference is grater (1.394 min).

From these experiments, we can conclude that with growing number of scenarios the time difference will be greater. Therefore, for large data sets we recommend to use the method with repetition.

Table 4.5: Simulation time 1

	A	B	C	D	E	F	G	H
1	1000 scenarios	10 000 scenarios		1000 scenarios	10 000 scenarios		1000 scenarios	10 000 scenarios
2	1000 simulations	1000 simulations		1000 simulations	1000 simulations		1000 simulations	1000 simulations
3	10x repetition	no repetition		10x repetition	no repetition		10x repetition	no repetition
4	alpha = 0,95			alpha = 0,99			alpha = 0,995	
970	7130222,278	6279711,425		20113115,07	18968067,13		25339973,24	23552861,04
971	7447753,973	7361513,281		18840174,27	18953789,68		23212895,54	23721127,41
972	6339863,016	7725590,07		19085898,69	21008269,81		24052639,18	25172849,36
973	6753072,968	6316792,129		21183716,65	18518879,64		26373910,59	22608785,86
974	6955732,639	7668783,599		19322111,06	20344796,43		23258430,7	24545849,03
975	7308550,076	6602586,879		19670139,12	18778596,8		24393971,16	24440072,04
976	7191945,858	7083451,823		19436181,36	19773125,74		23691289,53	23708012,34
977	7012384,58	6350761,51		20644980,95	19873279,73		24569250,09	25277528,12
978	6631710,445	7129277,222		19610208,51	19964892,63		24193139,54	23845578,62
979	6998243,41	7335765,897		20831999,46	20581397,77		25431052,81	24711135,85
980	7407978,692	7267160,943		19960139,14	19790125,26		24206544,8	23086364,44
981	6889255,495	6295317,338		19932589,2	19549701,73		25137163,98	24876874,9
982	7401205,944	6619261,277		20279056,07	19938751,02		24496415,06	23606122,77
983	6940764,969	7359040,746		19023907,19	19807275,35		23563049,72	24735916,05
984	6665119,777	6400502,947		20411122,07	19416001,9		24025260,09	23763283,25
985	6512479,654	7529028,394		19150136,08	20132513,99		23787976,02	24296079,85
986	6808492,281	5708976,25		18979610,89	19385003,24		23042026,82	23229033,24
987	7216992,21	6638994,793		19567966,1	19978973,58		24110963,41	25307574,34
988	5501660,822	6975989,298		19207405,35	20933692,3		23546246,41	23902273,84
989	6394149,616	6983762,735		19216495,19	19737589,86		23900570,69	23831027,32
990	6651015,152	7272478,371		19018150,04	20996787,07		23877213,11	25233569,39
991	6779352,893	6475695,452		19903785,29	18839848,66		25294227,38	24700634,77
992	6819012,691	7115780,528		20446677,87	20115270,97		25639765,83	24424790,56
993	6035276,472	7216122,857		19440352,74	19456390,99		23638222,1	24514824,16
994	6495323,881	6820127,814		19090195,83	20615012,48		24134196,04	25060807,83
995	6665727,285	6500464,54		20300293,25	18619740,05		25439082,01	23009333,76
996	6560384,204	6233504,188		19778390,69	18522019,04		23846683,45	23429723,08
997	6590827,967	6005642,663		19121208,24	18908565,36		23089249,89	23188649,42
998	6592162,283	7333997,259		19580904,87	19551621,04		24372842,32	24255979,06
999	6709516,906	6662086,536		19972746,2	19164541,13		24775073,74	23581745,99
1000	7046970,415	6830093,378		19575672,82	19550614,07		23273763,91	23853477,8
1001	7113303,047	6583495,77		19953808,57	20045486,74		24571797,94	24567969,91
1002	6690328,123	6469933,952		19645993,93	19339107,84		23267397,97	23769330,02
1003	7544376,623	7573667,461		20370436,24	21698924,62		24194619,91	24518920,56
1004	6609985,789	7057591,877		19860586,54	20234262,34		24134388,07	24013598,13
1005								
1006	simulation time [s]	simulation time [s]		simulation time [s]	simulation time [s]		simulation time [s]	simulation time [s]
1007	46,39374948	40,91627455		46,39374948	40,91627455		46,39374948	40,91627455
1008	average	average		average	average		average	average
1009	6 788 128,27	6 800 517,83		19 673 097,18	19 706 079,19		24 080 519,38	24 084 864,21
1010	standard deviation	standard deviation		standard deviation	standard deviation		standard deviation	standard deviation
1011	429 954,97	413 122,75		655 469,73	668 916,04		832 873,59	847 071,33
1012								
1013	average difference	12 389,56		average difference	32 982,01		average difference	4 344,83
1014	standard deviations difference	16 832,22		standard deviations difference	13 446,31		standard deviations difference	14 197,75
1015	simulation times difference	5,47747493		simulation times difference	5,47747493		simulation times difference	5,47747493
1016								

Table 4.6: Simulation time 2

	A	B	C	D	E	F	G	H
1	10 000 scenarios	100 000 scenarios		10 000 scenarios	100 000 scenarios		10 000 scenarios	100 000 scenarios
2	1000 simulations	1000 simulations		1000 simulations	1000 simulations		1000 simulations	1000 simulations
3	10x repetition	no repetition		10x repetition	no repetition		10x repetition	no repetition
4	alpha = 0,95			alpha = 0,99			alpha = 0,995	
970	6926422,982	6904333,861		19552635,83	19862985,73		24024128,3	24415081,33
971	6804278,62	6863200,019		19926144,26	19979263,65		24406894,97	24719030,35
972	6640192,285	6829734,091		19375417,93	19381422,77		23707845,27	23791381,16
973	6876712,501	6785262,911		20143523,66	19849939,08		24471583,59	24102036
974	6883182,684	6711299,109		19648085,59	19594727,39		24262005,92	23843281,95
975	6748493,551	6726060,917		19737645,62	19823774,31		23966766,54	24037524,28
976	6613839,096	6713779,396		19812836,39	20022687,92		23980202,53	24441970,69
977	6658297,611	7015782,692		19695458,25	19624338,77		24001655,25	24174942,96
978	6554950,273	7025809,971		19604356,03	20201077,3		24144575,97	24389449,16
979	6981651,305	6657790,751		19302830,86	19589795,24		24019723,37	23751466,39
980	6861270,81	6740506,171		20074564,36	19356625,95		24199915,82	23729467,08
981	6585224,8	6653948,536		19592150,11	19683052,45		24012389,58	24171157,1
982	7015908,239	6434087,863		19673308,82	19557329,04		23948314,98	24209919,34
983	6967951,787	6727280,088		19713289,19	19299958,84		24113494,13	23574203,01
984	6758564,957	6937375,47		19798253,57	19728649,56		24146180,66	24217493,67
985	6738501,055	6660066,841		19596672,5	19471480,32		24152768,99	24045651,4
986	6778560,837	6675966,422		19787133,68	19675483,46		24250222,33	23792898,01
987	6789870,519	6863416,454		19428846,4	19448760,34		23784182,47	23721364,24
988	6687249,588	6700838,434		19839140,05	19820997,38		24285688,23	24171466,93
989	6729949,585	6727176,467		19867171,11	19738816,56		24404208,91	24186997,37
990	6665560,361	6678737,269		19817877,71	19804133,69		24032108,42	24251792,88
991	6667298,799	6522562,883		19649262,23	19322717,91		23953816,96	23662634,61
992	6888406,206	6837928,803		19591559,48	19694608,1		24238778,83	23985311,72
993	6989693,401	6896078,158		20008222,05	19649647,24		24456965,73	24209785,71
994	6826641,346	6728829,771		19781435,07	19735628,92		23862618,86	24130962,99
995	6706080,328	6985750,933		19548485,2	20104717,59		24237021,94	24505362,3
996	6869046,936	6707370,572		19681638,01	19617680,37		24291480,92	24198555
997	6917712,795	6792640,235		19791611,68	19911624,81		24217085,1	24202135,19
998	6542135,18	6775591,53		19568149,59	19725896,34		24054662,65	24338233,18
999	6874757,947	6811078,253		19898792,39	19711488,33		24325412,28	24294643,83
1000	6771812,415	6668635,706		19930840,47	19526835,83		24235801,73	24098652,77
1001	6830046,916	6772994,248		19481270,67	19631366,97		23690045,04	23971345,37
1002	6798067,057	6841451,995		19758155,36	19913335,84		24003495,9	24099935,07
1003	6919503,956	6855946,06		19975145,38	19729817,77		24240845,07	24322969,53
1004	6765004,254	7086643,178		19836698,93	19791407,94		24246567,98	24649457,95
1005								
1006	simulation time [s]	simulation time [s]		simulation time [s]	simulation time [s]		simulation time [s]	simulation time [s]
1007	408,6753207	415,0374176		408,6753207	415,0374176		408,6753207	415,0374176
1008	average	average		average	average		average	average
1009	6 807 016,50	6 810 347,43		19 699 281,79	19 705 381,95		24 099 283,07	24 103 643,67
1010	standard deviation	standard deviation		standard deviation	standard deviation		standard deviation	standard deviation
1011	129 193,03	131 241,24		211 110,31	224 630,27		266 599,94	269 254,26
1012								
1013	average difference	3 330,93		average difference	6 100,16		average difference	4 360,60
1014	standard deviations difference	2 048,22		standard deviations difference	13 519,96		standard deviations difference	2 654,32
1015	simulation times difference	6,362096859		simulation times difference	6,362096859		simulation times difference	6,362096859
1016								

Table 4.7: Simulation time 3

	A	B	C	D	E	F	G	H
1	100 000 scenarios	1 000 000 scenarios		100 000 scenarios	1 000 000 scenarios		100 000 scenarios	1 000 000 scenarios
2	1000 simulations	1000 simulations		1000 simulations	1000 simulations		1000 simulations	1000 simulations
3	10x repetition	no repetition		10x repetition	no repetition		10x repetition	no repetition
4	alpha = 0,95			alpha = 0,99			alpha = 0,995	
970	6793878,341	6796022,787		19696682,82	19680736,68		24045992,94	24016247,52
971	6810126,566	6848114,716		19814758,14	19752012,39		24310259,12	24089859,81
972	6776685,6	6810114,746		19653238,75	19757444,28		24012914,29	24138638,43
973	6859989,045	6804412,093		19717193,43	19687267,06		24083765,14	24013694,38
974	6826830,152	6806154,263		19698127,14	19616854		24118148,03	24091913,04
975	6775067,275	6854245,486		19814534,53	19685009,53		24224297,07	24063350,52
976	6828044,758	6861003,771		19654078,38	19665795,12		24116951,87	24140052,91
977	6770838,992	6730166,664		19634873,06	19589183,67		24082769,7	24005242,21
978	6840673,481	6728822,991		19792493,47	19653639,16		24241446,44	23968799,93
979	6790009,484	6834414,025		19792637,11	19603492,51		24193572,3	24074034,59
980	6805794,011	6749271,247		19767349,9	19778308,11		24270392,75	24227993,96
981	6794904,398	6787418,348		19637957,35	19754253,78		24002115,16	24211767,74
982	6819901,136	6811934,986		19755754,88	19810300,75		24178240,03	24168565,97
983	6820969,138	6763322,812		19653890,75	19667579,03		24043544,86	24049743,22
984	6809914,78	6759267,377		19851250,68	19758480,18		24224649,36	24190504,52
985	6765994,773	6797253,946		19680185,52	19776963,91		24262442,11	24171280,41
986	6748071,982	6782952,915		19635367,76	19793687,51		24100648,69	24136520,39
987	6716576,454	6854235,278		19624586,17	19589022,37		24024551,48	24012078,88
988	6878913,357	6833125,99		19832863,46	19738797,89		24194051,75	24108873,59
989	6829124,706	6845059,019		19814854,74	19667681,29		24237298,53	24067148,45
990	6865849,574	6732709,085		19668853,23	19525164,71		23973150,6	23890421,81
991	6822954,28	6823731,237		19675470,39	19737359,13		24137722,51	24233233,85
992	6914071,29	6848767,666		19899435,62	19655602,98		24313721,44	24067420,97
993	6838013,766	6793386,407		19724153,48	19659946,5		24019988,98	24033219,55
994	6801602,471	6832456,499		19696864,17	19683879,06		24055804,79	24040646,92
995	6830715,836	6790184,168		19630305,5	19673149,11		24063456,38	24100380,18
996	6787796,923	6776371,754		19673522,92	19740212,59		24155232,16	24112472,83
997	6779830,274	6761671,812		19572350,63	19695205,7		23959709,43	24051135,77
998	6771646,613	6787649,368		19631347,29	19638529,03		24102104,26	23945606,1
999	6814162	6787929,197		19698360,67	19723891,29		24100649,02	24090551,09
1000	6865261,072	6766633,472		19842504,89	19669906,12		24262014,96	23999671
1001	6836624,428	6792659,822		19759914,12	19629743,28		24150664,99	24000977
1002	6821149,118	6796784,652		19642303,7	19725123,37		24155059,89	24073932,48
1003	6738562,011	6811696,265		19670083,44	19665058,38		24083959,04	24142195,56
1004	6845632,783	6714611,024		19691035,91	19546382,2		24064833,79	23956721,78
1005								
1006	simulation time [s]	simulation time		simulation time [s]	simulation time [s]		simulation time [s]	simulation time [s]
1007	4163,594474	4247,262343		4163,594474	4247,262343		4163,594474	4247,262343
1008	average	average		average	average		average	average
1009	6 810 083,20	6 811 327,77		19 705 473,23	19 705 244,96		24 104 344,01	24 103 179,13
1010	standard deviation	standard deviation		standard deviation	standard deviation		standard deviation	standard deviation
1011	41 734,55	42 382,37		67 630,75	68 856,56		84 279,46	83 801,53
1012								
1013	average difference	1 244,57		average difference	228,27		average difference	1 164,88
1014	standard deviations difference	647,82		standard deviations difference	1 225,81		standard deviations difference	477,93
1015	simulation times difference	83,66786896		simulation times difference	83,66786896		simulation times difference	83,66786896
1016								

4.3 Number of simulations in experiment

In this experiment, we show how to determine the number of simulations n which we need to run to get some specified level of accuracy.

In the [20], Section 7.3.4 Vose solves the question: How many simulations runs are needed?
The problem is that for one model trying to determine a mean, 500 simulations may be good enough. For another trying to determine a 95th percentile, 100 0000 simulations might be needed. It depends on sensitivity to outputs accuracy of the question decision.

Estimate the number of simulations to run to get sufficient accuracy for the mean

We will follow the Section 7.3.4 in Vose [20].
Monte Carlo simulation estimates the true mean μ of the output distribution such that:

$$\hat{\mu} = \left(\frac{1}{n}\right) \sum_{i=0}^{n} x_i,$$

where
x_i are generating values,
n is number of simulations.

If we used Monte Carlo sampling, each x_i is an iid (independent identically distributed random variable). Then following the central limit theorem we can say that the distribution of the true mean is given by

$$\hat{\mu} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right),$$

where σ is the true standard deviation of the models output.
Using the pivotal method, we can rearrange this equation to make equation for μ :

$$\mu \sim N\left(\hat{\mu}, \frac{\sigma}{\sqrt{n}}\right). \tag{4.2}$$

Specifying the level of confidence, we require for our mean estimation transfer into a relationship between δ, σ and n.
Formally

$$\delta = \frac{\sigma}{\sqrt{n}} \Phi^{-1}\left(\frac{1+\alpha}{2}\right), \tag{4.3}$$

where $\Phi^{-1}(\cdot)$ is an inverse function of normal cumulative distribution. When we rearrange Equation (4.3) and recognise that we want to have at least this accuracy we get a minimum value for n

$$n > \left(\frac{\Phi^{-1}\left(\frac{1+\alpha}{2}\right)\sigma}{\delta}\right)^2. \tag{4.4}$$

But we don't know the standard deviation σ true output.
For our purposes we can estimate this by taking the standard deviation of the first few (e.g. 50) simulations.

We can do this by using function NORMNSINV in Excel, which returns values of Φ^{-1}.

Estimate the number of simulations to run to get sufficient accuracy for the cumulative probability $F(x)$ associated with a particular value x

Percentiles closer to the 50th percentile of an output distribution will reach a stale value relatively quick than percentiles toward the tails. But we often are often interested in what happens in the tails because that is where the risks and opportunities lie.

In the following technique, Vose [20] *,Section 7.3.4* describes how to ensure that the required level of accuracy for the percentile will be associated with a particular value.

Monte Carlo will estimate the cumulative percentile $F(x)$ of the output distribution associated with a value x by the determining fraction of the simulation fell at or below x.

For illustration, we provide following example:
x is actually the 80th percentile of the true output distribution. Then for Monte Carlo experiment the generated value in each simulation independently has an 80% probability of the falling below x: It means it is a binomial process with probability $p = 80\%$. Then, if we have n simulations and s falls at or below x, the distribution $Beta(s+1, n-s+1)$ describes the uncertainty associated with the true cumulative percentile we should associate with x (see [20],Section 8.2.3).

When we are estimating the percentile close to the median of the distribution, or when we are performing a large number of simulations, s and n will both be large. We can use a normal approximation to the beta distribution (4.5):

$$Beta(s+1, n-s+1) \approx N\left(\hat{P}, \sqrt{\frac{\hat{P}(1-\hat{P})}{n}}\right) \tag{4.5}$$

where $\hat{P} = \frac{s}{n}$ is the best-guess estimate for $F(x)$.

We can produce a relationship similar to the Equation (4.3) for determining the number of scenarios to get the required precision for the output mean:

$$\delta = \sqrt{\frac{\hat{P}(1-\hat{P})}{n}}\Phi^{-1}\left(\frac{1+\alpha}{2}\right) \tag{4.6}$$

Rearranging Equation (4.6) and recognising that we want to have at least this accuracy gives a minimum value for n:

$$n > \hat{P}(1-\hat{P})\left(\frac{\Phi^{-1}(\frac{1+\alpha}{2})}{\delta}\right)^2 \tag{4.7}$$

Table 4.8: Number of simulations

α	0,95	s	950
δ	0,005	n	1000
$\hat{P}$	0,95		
n >	7298,772		
α	0,99	s	950
δ	0,005	n	1000
$\hat{P}$	0,95		
n >	12606,304		
α	0,995	s	950
δ	0,005	n	1000
$\hat{P}$	0,95		
n >	14970,933		
α	0,999	s	950
δ	0,005	n	1000
$\hat{P}$	0,95		
n >	20572,376		

In Table 4.8 we summarize the estimated number of simulations n for different confidence values alpha, fixed $\hat{P} = \frac{s}{n} = 0.95$ and fixed $\delta = 0.005$.
To set the VaR to be in the interval $(\hat{P} - \delta, \hat{P} + \delta)$, the smallest number of simulations (at confidence level $\alpha = 0.95$) we need is 7299.
With growing alpha, n also grows.

Chapter 5

Conclusion

This thesis describes the Value at Risk method. It also describes computational methods of Value at Risk. We also demonstrate a computation on several examples. Above mentioned descriptions and demonstrations are described in Chapter 2.

We compared Monte Carlo simulation, analytical method and historical simulation. According to comparison criteria summed up in Table 2.2 we concluded that the selection among these methods depends on given requirements of the analytical expert. At the end of the Chapter 2 we provided advantages and disadvantages of Value at Risk method.

We also concluded that the benefit of this method is its high availability. Thus using this method can also be beneficial for comparing the results with other companies. The main disadvantage of the Value at Risk method is that its various computational methods can produce different results. Which bring us back to computational method selection.

In Chapter 3 we described the Monte Carlo simulation in detail. We provide the whole procedure of this simulation method. Which we applied in Chapter 4, where we showed the simulation experiments.

In the first experiment we showed that standard deviation declines when the number of scenarios grows. When the number of scenarios grows the simulation time grows as well. Thus in the second experiment we focused on the simulation time and we compared two computational methods of Value at Risk.

We conclude that for high number of scenarios it is better to use the method with repetitions, described in the Section 4.2. We consider values higher than approximately 100 000 scenarios as a high number. In the last simulation experiment we showed the minimal number of simulations needed for the experiment.

As a further extension of this work, we can devote a description and the subsequent comparison of other methods to calculate Value at Risk. E.g. the Conditional Autoregressive Value at Risk (CAVaR). It is an alternative semiparametric method to estimate Value at Risk. This model pays attention directly to the quantile. The approach is based on the simple intuition that it is better to model directly the quantile as it evolves through time instead of attempting to model and estimate the entire distribution of portfolio returns. More about this topic can be found in [21] written by F. Engle and S. Manganelli who introduced the the Conditional Autoregressive Value at Risk model in 1999 or in [22] also written by F. Engle and S. Manganelli.

Another possibility is to consider a Conditional Value at Risk (CVaR) eventually on upper Conditional Value at Risk ($CVaR^{+}$) and lower Conditional Value at Risk ($CVaR^{-}$). $CVaR^{+}$ represents expected losses strictly exceeding VaR and $CVaR^{-}$ represents expected losses weakly exceeding VaR. More about this topic can be found in [3].

We can also study VaR tools that are useful for risk management, like marginal VaR, incremental VaR and component VaR. It is possible to analyse some back testing methods to validate the use of VaR model. About these topics we refer to [23] or [2].

Appendix A

Distribution types

A.1 Normal distribution

The normal (or sometimes called Gaussian) distribution is a continuous probability distribution with two parameters $N(\mu, \sigma^2)$.

- $\mu \in R$ is a mean
- $\sigma^2 > 0$ is a variance

The distribution with $\mu = 0$ and $\sigma^2 = 1$ is called the standard normal distribution.

Probability distribution function (PDF) is given by

$$P(x) = f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}, x \in R. \tag{A.1}$$

Figure A.1 represents PDF for various parameters μ and σ^2

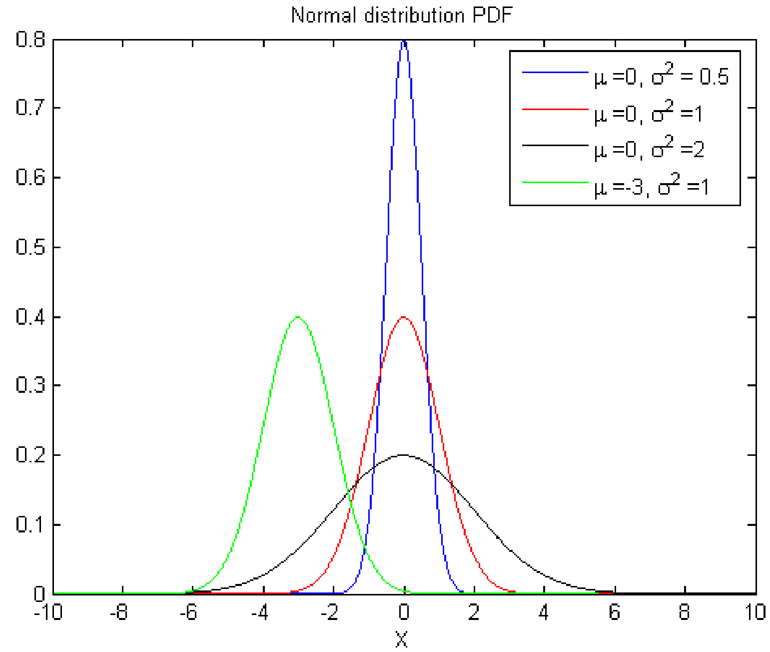

Figure A.1: Probability density function

Cumulative distribution function (CDF) is given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x-\mu}{\sqrt{2\sigma^2}}\right)\right], x \in R, \tag{A.2}$$

where erf is an error function.

For a generic normal random variable with μ and $\sigma^2 > 0$ will be CDF equal to

$$F(x;\mu,\sigma) = \Phi\left(\frac{x-\mu}{\sigma}\right) \tag{A.3}$$

Figure A.2 represents CDF for various parameters μ and σ^2

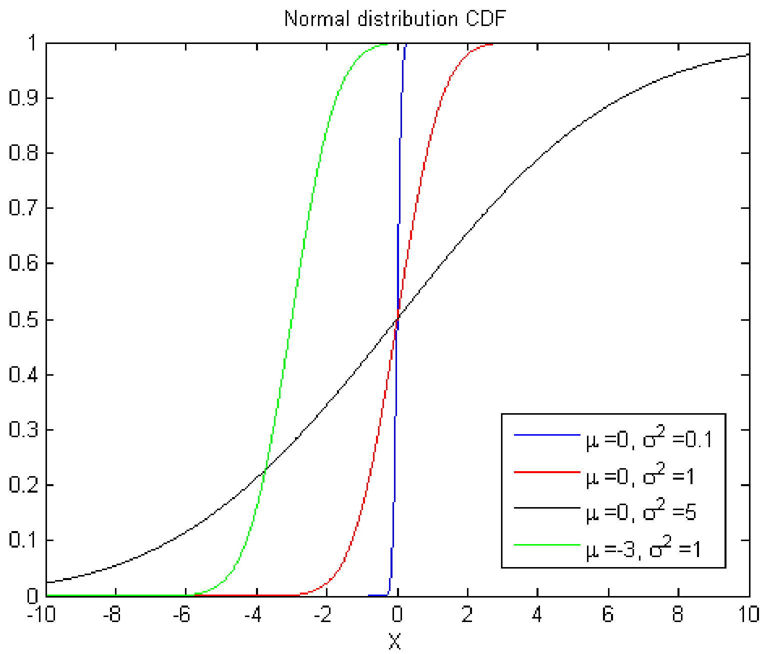

Figure A.2: Cumulative distribution function

A.2 Student's *t*-distribution

Student's t-distribution (or just t-distribution) is a continuous probability distribution with v degrees of freedom and we denote it $t(v)$.
Probability distribution function (PDF) is given by

$$f(x) = \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v}{2})\sqrt{\pi v}} \left(1 + \frac{x^2}{v}\right)^{-\frac{v+1}{2}}, \tag{A.4}$$

where Γ is gamma function and $-\infty < x < \infty$
Figure A.3 represents PDF for various values v.

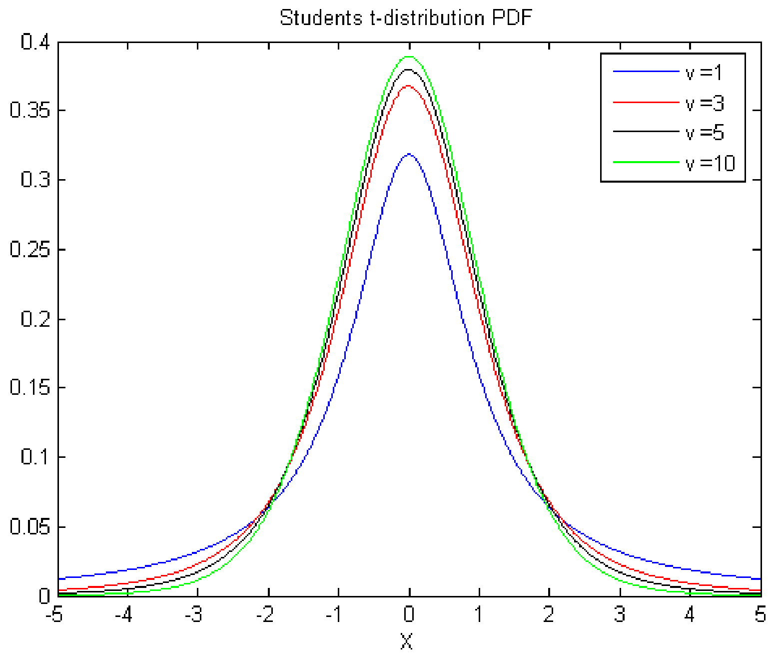

Figure A.3: Probability density function

Cumulative distribution function (CDF) is given by

$$F(x) = \frac{1}{2} + x\Gamma\left(\frac{v+1}{2}\right)\frac{2F_1(\frac{1}{2}, \frac{v+1}{2}; \frac{3}{2}; \frac{x^2}{v})}{\sqrt{\pi v}\Gamma(\frac{v}{2})}, \tag{A.5}$$

where $2F1_1$ is hypergeometric function.
Certain values v give an especially simple form for calculation CDF.
Figure A.4 represents CDF for various values v.

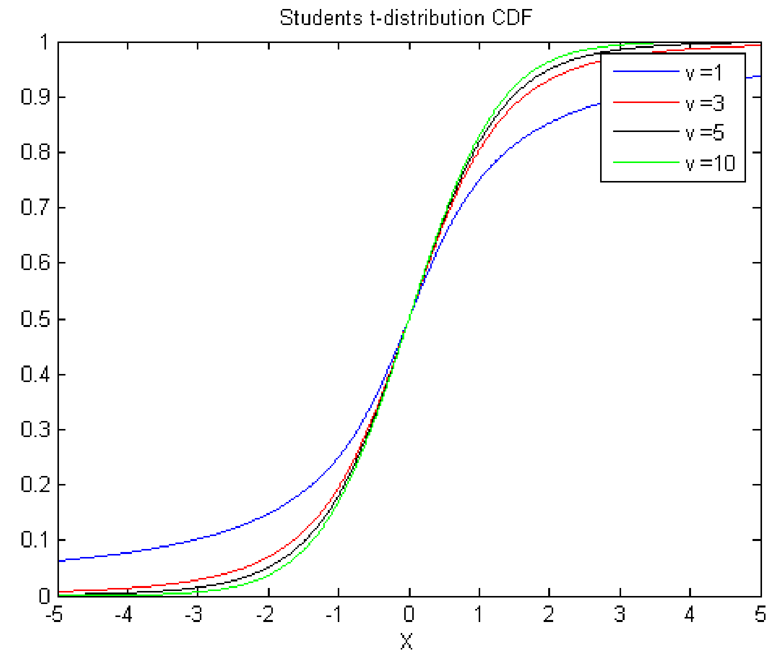

Figure A.4: Cumulative distribution function

There also exist generalised t-distribution, we denote it by $t_v(a,b)$.

$$t_v(a,b) = a + bt_v \tag{A.6}$$

Parameters (a,b) we can estimate such that

$$a = m_x, b = s_x\sqrt{\frac{(v-2)}{2}}, \tag{A.7}$$

where m is selective mean value and s is selective standard deviation.

We will use this generalised t-distribution for this work purposes.

A.3 Log-Normal distribution

Log-normal distribution is a continuous probability distribution of a random variable whose logarithm is normally distributed. Log-normal distribution has two parameters

- $\mu \in R$ is a mean
- $\sigma^2 > 0$ is a variance

Probability distribution function (PDF) is given by

$$f(x;\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{(ln(x)-\mu)^2}{\sigma})^2}, x \in (0,+\infty). \tag{A.8}$$

Figure A.5 represents PDF for various parameters μ and σ^2

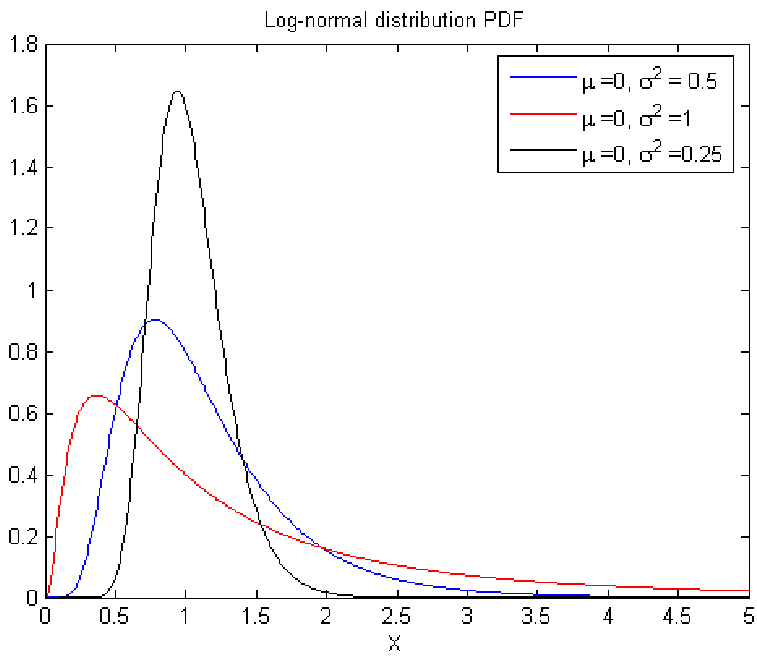

Figure A.5: Probability density function

Cumulative distribution function (CDF) is given by

$$F(x) = \frac{1}{2}\,\text{erf}\left[-\frac{ln(x)-\mu}{\sigma\sqrt{2}}\right], x \in R, \tag{A.9}$$

where erf is an error function.

For a generic normal random variable with μ and $\sigma^2 > 0$ will be CDF equal to

$$F(x;\mu,\sigma) = \Phi\left(\frac{ln(x) - \mu}{\sigma}\right) \tag{A.10}$$

Figure A.6 represents CDF for various parameters μ and σ^2

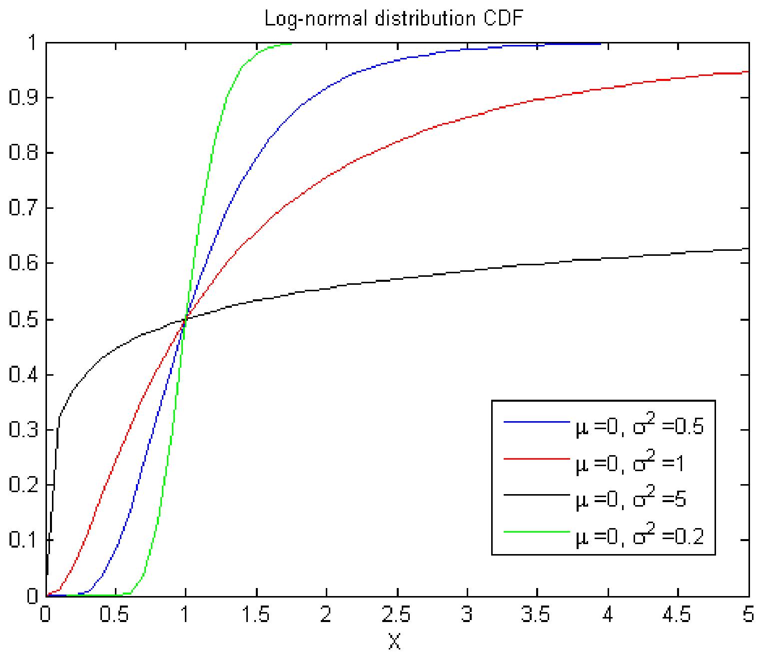

Figure A.6: Cumulative distribution function

Appendix B

Brownian motion

Brownian motion is the random movement of microscopic particles in a fluid (a liquid or a gas). Explanation of Brownian motion is that the molecules in solution is due to thermal motion constantly collide, and the direction and strength of these collisions are random, which makes the instant position of particles random. Brownian motion speed is proportional to the system temperature.

Geometric Brownian motion (GBM)

GBM is also known as exponential Brownian motion. It is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion with drift. It is an important example of stochastic processes satisfying a stochastic differential equation (SDE). It is used in mathematical finance to model stock prices in the Black–Scholes model [24].

Stochastic differential equation (SDE)

A stochastic process S_t follows a GBM if it satisfies the following stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{11}$$

where W_t is a Brownian motion or Wiener process and μ ('the percentage drift') and σ ('the percentage volatility') are constants.

In mathematics, Brownian motion is described by the Wiener process.

Wiener process

Wiener process is a continuous-time stochastic process named in honor of Norbert Wiener. Often called also standard Brownian motion, after Robert Brown. It is one of the best known Lévy processes - stochastic processes with stationary independent increments. Occurs frequently in mathematics, economics, quantitative finance and physics.
Weiner process W_t is characterised by four facts:

- $W_0 = 0$,
- W_t is almost surely continuous,
- W_t has independent increments,
- $W_t - W_s \sim N(0, t-s)$ for $(0 \leq s \leq t)$,

where $N(\mu, \sigma^2)$ is normal distribution [A]

The condition about independent increments means if $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2$ then $W_{t1} - W_{s1}$ and $W_{t2} - W_{s2}$ are independent random variables. Similar condition holds for n increments.

More about the Brownian motion or Wiener process can be found in [14] or in [25], [26].

Appendix C

Triangular distribution

The triangular distribution is a continuous probability distribution with lower limit - a, upper limit - b and mode - c, where $a < b$ and $a \leq c \leq b$.
The probability density function Figure 7 is given by

$$f(x \mid a,b,c) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)} & \text{for } x \in [a,b], \\ \dfrac{2}{b-a} & \text{for } x = c, \\ \dfrac{2(b-x)}{(b-a)(b-c)} & \text{for } x \in (c,b], \\ 0 & \text{for } x \notin [a,b]. \end{cases} \tag{12}$$

whose cases avoid division by zero if c = a or c = b.

Triangular distribution parameters are:
$a = min\{x_1, x_2, \ldots, x_n\}$
$b = max\{x_1, x_2, \ldots, x_n\}$
$c = mode\{x_1, x_2, \ldots, x_n\}$

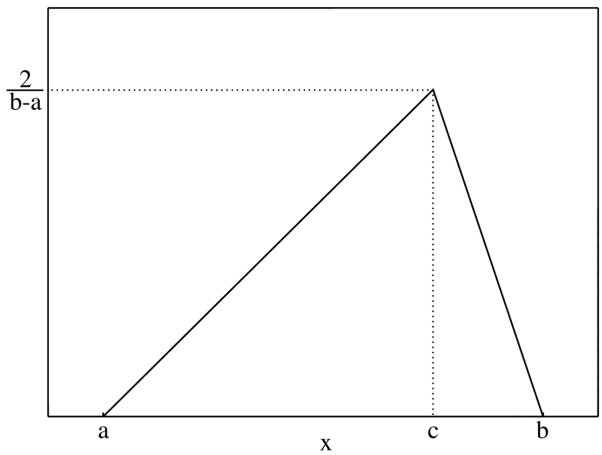

Figure 7: Probability density function

Generating Triangular-distributed random variates

Given a random variate U drawn from the uniform distribution in the interval (0, 1), then the variate

$$\begin{cases} X = a + \sqrt{U(b-a)(c-a)} & \text{for } 0 < U < F(c), \\ X = b - \sqrt{(1-U)(b-a)(b-c)} & \text{for } F(c) < U < 1. \end{cases} \tag{13}$$

Where a, b, c are the parameters of triangular distribution, defined such as:

a (lower limit): $a \in (-\infty, \infty)$,
b (upper limit): $a < b$,
c (mode): $a \leq c \leq b$

and $F(c) = (c-a)/(b-a)$ has a triangular distribution with these three parameters. This can be obtained from the cumulative distribution function (CDF) Figure 8. [1]

CDF is given by:

$$F(x) = \begin{cases} 0 & \text{for } x < a, \\ \dfrac{(x-a)^2}{(b-a)(c-a)} & \text{for } a \leq x \leq c, \\ 1 - \dfrac{(b-x)^2}{(b-a)(b-c)} & \text{for } c < x \leq b, \\ 1 & \text{for } b < x. \end{cases} \tag{14}$$

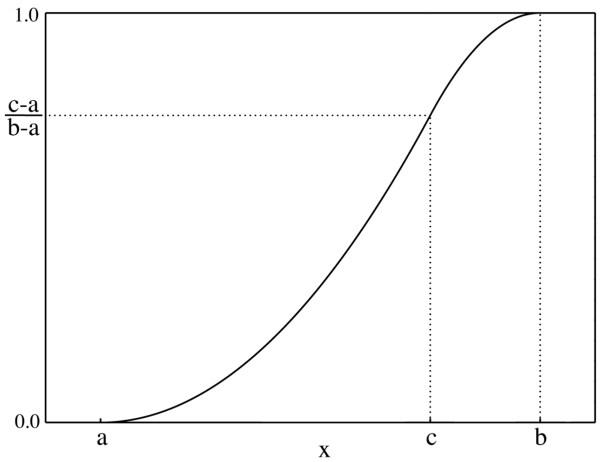

Figure 8: Cumulative distribution function

[1] 19th Nov 2012. http://en.wikipedia.org/wiki/Triangular_distribution

Appendix D

Tool for MATLAB

First open your MS Excel. In the menu top click on the item “Tools” (”Nástroje”) then choose “Accessories” (”Doplňky”).

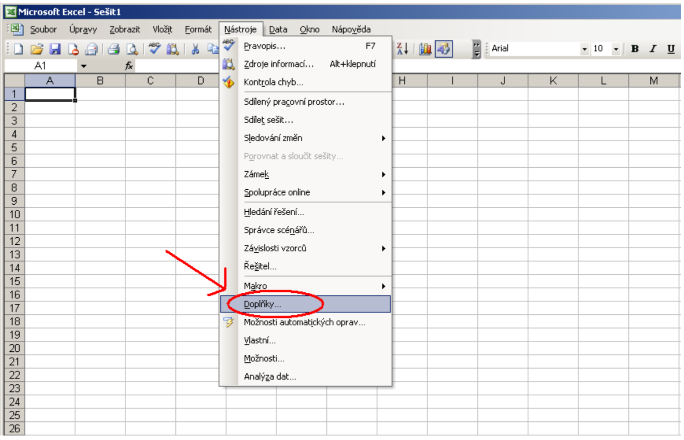

Figure 9: Instruction - part1

Now table appears. Check the check box with name Excel Link 2.3 for use with MATLAB (here could be another name it depends on the version you have).
If you don’t have this option in your offer click on the button “Brows” (”Procházet”) and look in the folder where you have installed your Matlab. There is a folder **toolbox** then **exlink** finally choose the file **excllink**. Then follow the instruction above.

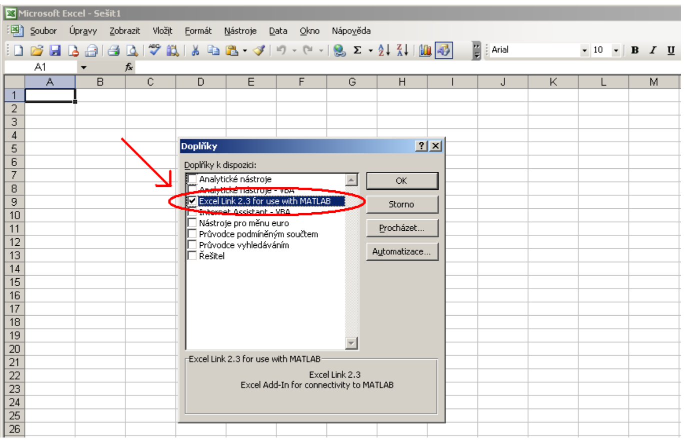

Figure 10: Instruction - part2

After that you can see in the left top new functions.

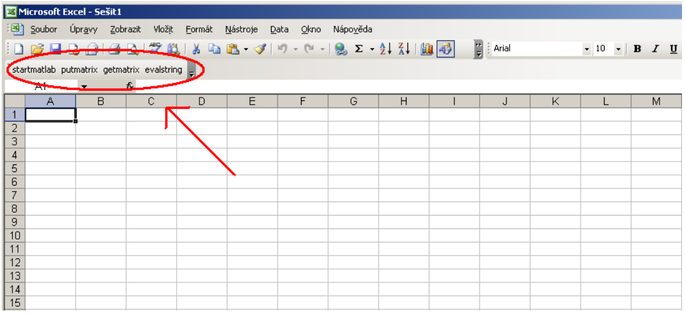

Figure 11: Instruction - part3

Now we can shortly describe new functions in MS Excel:

- **Strartmatlab** - it runs Matlab
- **Putmatrix** - send data to Matlab
- **Getmatrix** - retrieve Matlab matrix
- **Evalstring** - execute the Matlab command

Appendix E

Description of the thesis attachment

The attached CD contains:

- README.txt -
  File which contains information about the CD structure.
- Bachelor_Thesis.pdf -
  The Bachelor thesis in PDF.
- /Matlab/ -
  Folder named Matlab, which contains all scripts for simulation experiments.
- /MS Excel/ -
  Folder named MS Excel, which contains all Excel files used in this thesis.
- /Source_TeX/ -
  Folder named Source_TeX, which contains LATEX files used for generating the thesis, all the settings, graphics and bibliography files included.

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