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Mathematical Models of Evolution of Cooperation

Bachelor thesis

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I do hereby declare that the entire bachelor thesis is solely my original work and that I have used only the cited sources.

Pilsen, 29.května 2013 Author's signature: ....

Preface

Many population models just approximate or neglect the spatial structure of the population. The aim of this bachelor thesis is to study evolutionary dynamic graphs. Evolutionary dynamic graph considers either quantitative and qualitative point of view of this problem. Quantitative point of view regards the size of specific sections of the population. Qualitative point of view uses a graph to describe the structure of population.

Chapter 1 introduces basic principles of static games from game theory. Furthermore, it describes specific classes studied in this thesis. The last section introduces the evolutionary dynamic graph theory. The evolutionary dynamic graph is a graph whose vertices play static games with each other. This structure if formally defined in Chapter 1. The aim of Chapter 2 is to search for specific evolutionary dynamic graph with a periodic behaviour. The crucial property of the searched evolutionary dynamic graph is the number of its vertices with regard to its periodic behaviour. Some estimates were made in this topic. The aim of this thesis to make these estimates more accurate. Chapter 3 has a similar aim with Chapter 2. However, we made the requirements on the evolutionary dynamic graph more strict.

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Keywords: graph, dynamic graph, evolutionary dynamic graph, games on graphs, static game, game theory

Contents

Chapter 1

Introduction

1.1 Static game theory

1.1.1 General static game theory

A game can be either: an activity made to entertain, improve one's ability at a certain skill, or it can be understood as a situation when players interact and make decisions. The last concept of game can be easily used to describe usual board or sports games i.e. chess, football. Furthermore, the last concept describes every situation of making decisions and interacting. Thus, business meetings, interaction of animals in nature can be described by this concept. A specific class of games is a static game. In a static game, decision of each player (subject interacting and making decision) is made without knowledge of the decision made by other players.

Definition 1. A *static game* is one in which a single decision is made by each player, and each player has no knowledge of the decision made by the other players before making their own decision. [4]

Each game can contain many decisions and it can be very difficult to monitor all of them. For the sake of brevity, decisions of players can be summarized in strategies. Strategies are rules for decision making. This concept simplifies the analysis and description of a game. Such a complex and delicate interaction such as a business meeting can be simplified, for example the interaction of two subjects where either of subjects can act aggressively or passively. Each of the two subjects in the meeting chooses a strategy of being aggressive or being passive.

Definition 2. *Strategy* is a rule for choosing an action at every point that a decision might have to be made. A *pure strategy* is one in which there is no randomisation. The set of all possible pure strategies is denoted by S. [5]

Remark 1. The set of all possible pure strategies S is called a *strategy set*.

The result of interaction of players' strategies can be measured by a utility. A utility is a numerical interpretation of result. It includes all positive and negative factors of the resulting situation for specific player. In this thesis we focus on static games of two players with finite strategy sets. Players are denoted A and B. The appropriate way of description of such game is an utility matrix.

Remark 2. The utility of player A (or B) as a result of combining i^{th} strategy of player A and j^{th} strategy of player B is denoted by $u^A_{i,j}$ (or $u^B_{i,j}$). All possible utilities can easily be described by *utility matrix.*

Example 1. Let us have a static game with two players A, B. Strategy set for player A is $S_A = \{s^A_1, s^A_2, \ldots, s^A_M\}$ and for player B is $S_B = \{s^B_1, s^B_2, \ldots, s^B_N\}$. The utility matrix of such game is $M \times N$ matrix whose entries are ordered pairs of utilities.

$$\begin{array}{c|cccc} & s^B_1 & s^B_2 & \ldots & s^B_N \\ \hline s^A_1 & (u^A_{1,1}, u^B_{1,1}) & (u^A_{1,2}, u^B_{1,2}) & \cdots & (u^A_{1,N}, u^B_{1,N}) \\ s^A_2 & (u^A_{2,1}, u^B_{2,1}) & (u^A_{2,2}, u^B_{2,2}) & \cdots & (u^A_{2,N}, u^B_{2,N}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s^A_M & (u^A_{M,1}, u^B_{M,1}) & (u^A_{M,2}, u^B_{M,2}) & \cdots & (u^A_{M,N}, u^B_{M,N}). \end{array} \tag{1.1}$$

In addition, a randomisation can be added to a decision process. Each player with a finite strategy set can choose each strategy with certain probability. The vector of probabilities of choosing particular strategies is called a *mixed strategy.*

Definition 3. A *mixed strategy* σ specifies the probability $p(s)$ with which each of the pure strategies $s \in S$ is used. [6]

Remark 3. The set of all possible mixed strategies is denoted by Σ.

Each reasonable player wants to choose strategy which maximises its utility. The simplest method is to exclude dominated strategies.

Definition 4. A strategy for player A, σ_A , is *strictly dominated* by σ^*_A if

$$u_A(\sigma^*_A, \sigma_B) > u_A(\sigma_A, \sigma_B),\ \forall \sigma_B \in \Sigma_B. \tag{1.2}$$

That is, whatever player B does, player A is always better off using σ^*_A rather than σ_A . Similarly, a strategy for player B, σ_B , is *strictly dominated* by σ^*_B if

$$u_B(\sigma_A, \sigma^*_B) > u_B(\sigma_A, \sigma_B),\ \forall \sigma_A \in \Sigma_A. \tag{1.3}$$

[7]

However, there exist some static games which cannot be solved by elimination of dominated strategies. Thus, it is convenient for either of players to choose such strategy none of the players can increase his pay-off. Such combination of strategies is called Nash equilibrium. A concept of Nash equilibrium is named after John Forbes Nash who proved a theorem regarding the existence of Nash equilibrium in a game with finite structure [2].

Definition 5. A *Nash equilibrium* (for two player games) is a pair of strategies (σ^*_A, σ^*_B) such that

$$u_A(\sigma^*_A, \sigma^*_B) \geq u_A(\sigma_A, \sigma^*_B) \qquad \forall \sigma_A \in \Sigma_A,$$

and

$$u_B(\sigma^*_A, \sigma_B) \geq u_B(\sigma^*_A, \sigma^*_B) \qquad \forall \sigma_B \in \Sigma_B.$$

In other words, given the strategy adopted by the other player, neither player could do strictly better (i.e., increase their pay-off) by adopting another strategy. [8]

1.1.2 Specific classes

The main purpose of this thesis is to study the problem of cooperation. To do so, we restrict a set of static games to the following:

1. Game has two players (A, B).
2. Players have equivalent strategy set, $S = \{\text{Cooperation, Defection}\}$.
3. Utilities of both players are symmetric, i.e. $u^A_{(s^A_i, s^B_j)} = u^B_{(s^A_j, s^B_i)}, \quad \forall s^A_i, s^A_j \in S^A, s^B_i, s^B_j \in S^B$.

Utility matrix of such a game can be expressed:

$$\begin{array}{c|cc} & C & D \\ \hline C & (a,a) & (b,c) \\ D & (c,b) & (d,d), \end{array} \tag{1.4}$$

or, further, simplified:

$$\begin{array}{c|cc} & C & D \\ \hline C & a & b \\ D & c & d. \end{array} \tag{1.5}$$

To maintain an accurate relation to reality of cooperation problems, we make the following assumptions citeEvol:

1. For the sake of brevity, we assume that no two parameters are equal.
2. It is always better if both players cooperate than if they both defect, i.e. $a > d$.
3. If only one cooperates, it is more advantageous to be the defector, i.e. $c > b$.
4. No matter what strategy a player chooses, it is always better for him if his opponent cooperate, i.e. $a > b$ and $c > d$.
5. Finally, we assume that a, c are positive, b, d can be non-positive.

These assumptions lead to 4 possible scenarios [1]:

Structure	Abbr.	Scenario	Nash Equilibria
$a > c > b > d$	FC	Full Cooperation	(C, C)
$c > a > b > d$	HD	Hawk & Dove	$(C, D), (D, C)$ and mixed equilibrium
$a > c > d > b$	SH	Stag Hunt	$(C, C), (D, D)$ and mixed equilibrium
$c > a > d > b$	PD	Prisoner's Dilemma	(D, D)

Table 1.1: List of all admissible static games and their properties

The topics of this thesis and of the [1] are connected closely. For the sake of brevity, the names and abbreviations of scenarios were adopted from [1].

Inequalities describing each of these scenarios contain 4 independent variables. Therefore, inequalities can be captured as four-dimensional regions which are difficult to plot. Without loss of generality, we fix parameters to be $a = 1, d = 0$. Regions can now be pictured as two-dimensional regions as in Figure 1.1.

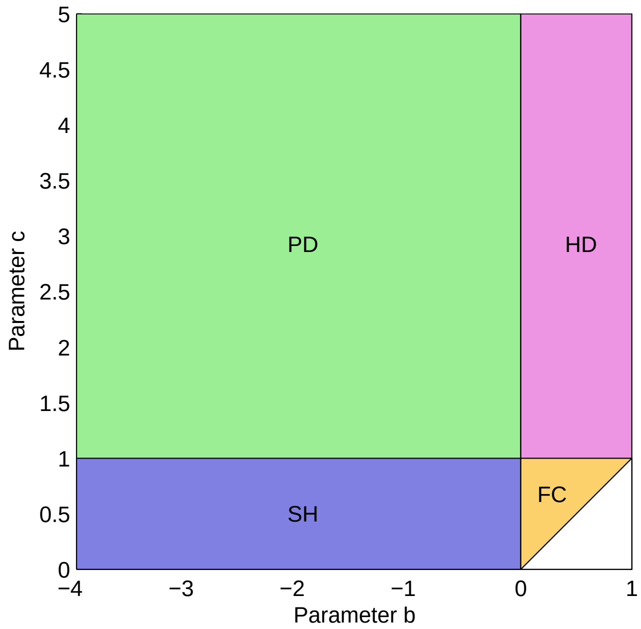

Figure 1.1: Parameter regions given $a = 1$ and $d = 0$.

Full Cooperation

Full Cooperation is a scenario with one pure equilibrium (players play the equilibrium strategy without randomisation). The equilibrium strategy can be simply found by eliminating dominated strategies. Given $a > c > b > d$ (Table 1.1.2), defection is strictly dominated by cooperation. Thus, the best choice for either of players is to cooperate.

Hawk & Dove

Hawk & Dove scenario has exactly 2 pure equilibria and one mixed equilibrium. This scenario cannot be simply solved by elimination of dominated strategies. On the other side, the combination of strategies (C, D) and (D, C) satisfy the conditions of Nash equilibrium.
Let us suppose the player A plays a mixed strategy $\sigma_A = (p, 1-p)$, where $p \in [0, 1]$ and the player B plays mixed strategy $\sigma_B = (q, 1-q)$, where $q \in [0, 1]$. Thus, the utility of player A can be expressed as:

$$u_A = apq + bp(1-q) + c(1-p)q + d(1-p)(1-q). \tag{1.6}$$

Since we want to find an optimal value of p, we express terms containing the variable p only:

$$u_A = p\left(q(a-b-c+d) + b - d\right) + C, \text{ where } C \in \mathbb{R}. \tag{1.7}$$

Now, if player B plays strategy $\sigma_B = (q, 1-q)$ with:

$$q = -\frac{b-d}{a-b-c+d}, \tag{1.8}$$

player A has the same utility u_A no matter, what strategy he plays. The same procedure with utility u_B of player B results in the value of variable p:

$$p \;=\; -\frac{b-d}{a-b-c+d}. \tag{1.9}$$

If either of players A, B cooperates with the probability $p = q = -\frac{b-d}{a-b-c+d}$, the other player cannot increase his utility by the change of his strategy. This combination of strategies is Nash equilibrium. Finally, the properties of parameters of Hawk & Dove scenario: $c > a > b > d$ ensures: $0 < p = q < 1$.

Stag Hunt

The method of finding the pure equilibria and one mixed equilibrium of Stag Hunt scenario is similar to the searching of the equilibria of the Hawk & Dove scenario.

Prisoner's Dilemma

Prisoner's Dilemma scenario has exactly one pure equilibrium. Since $c > a > d > b$, cooperation is strictly dominated by defection. Thus, combination of strategies (D, D) of players A, B is an equilibrium point.

1.2 Evolutionary dynamic graph theory

This section focuses on introducing the evolutionary graph theory. The main idea is to consider vertices of given graph as a players of static games. The games are played each time $t \in \mathbb{N}_0$. An edge linking two vertices means the vertices play a static game of two players each round. Utilities of vertices are summed and their strategies are updated with respect to the sums of utilities each round. In the first part we define basic structures and we begin with a *graph* as a basic structure.

Definition 6. Graph is an pair $G = (V, E)$ where V is a finite set and $E \subset \binom{V}{2} \cup V^2$, where

$$\binom{V}{2} = \{\{x, y\} : x, y \in V \text{ and } x \neq y\} \tag{1.10}$$

is a set of all two-member sets (unordered pairs) of members of set V. Members of set V are called *vertices*. Members of set E are called *edges* of graph G. Vertices x, y are neighbours if $\{x, y\} \in E$. [3]

Remark 4. The number of all vertices $|V(G)|$ is denoted by n. The set of all neighbours of i^{th} vertex is denoted by $N(i)$.

Next we define a mapping assigning each vertex in graph a strategy and graph with strategy mapping-a *strategy graph*. We define a specific notation for the sake of time distinction.

Definition 7. Given a strategy set S, we define a *strategy mapping* $s : V(G) \to S$ to be a function assigning each vertex a strategy from S. A *strategy graph* $\mathfrak{G}$ is a graph with a strategy mapping.

Observation 1. Strategy mapping s defined in Definition 7 assigns only pure strategy to each vertex.

Remark 5. Since we study strategies across time, we denote by $s_i(t)$ the strategy of the i^{th} vertex at time t.

Utility strategy graph is a strategy graph which assigns utility to each of its vertices. Utilities are dependent on the strategies assigned to vertices. This formal definition of utility strategy graph is very wide, since utility of i^{th} vertex can depend on the strategy of all vertices. Furthermore, utility functions of utility strategy graph is defined specifically.

Definition 8. Utility strategy graph $\mathfrak{U}$ is a strategy graph with an utility functions $u_i : S^n \to \mathbb{R}, i = 1, 2, \ldots, n$.

All previous definitions enable us to define a *dynamic graph*. *Dynamic graph* is a utility strategy graph whose vertices' strategies are updated each time $t \in \mathbb{N}_0$ in pre-defined time-dependent sets. Updates are governed by time-independent *update rule*.

Definition 9. Dynamic graph $\mathfrak{D}$ is a utility strategy graph with

- *time-dependent update sets* $\tau : \mathbb{N}_0 \to 2^{V(G)}$, which assign to each time $t \in \mathbb{N}_0$ a set of all vertices $\tau(t)$ whose strategies are going to be updated at t,
- *update rules* $\rho_i : \Sigma^n \to \Sigma$, which assign to the vertex i a new strategy at time $t + 1$, i.e. $s_i(t + 1) = \rho_i (s_1(t), \ldots, s_n(t))$.

Remark 6. We assume that the update rules ρ_i are time-independent.

Definition 10. Dynamic graph $\mathfrak{D}$ with $\tau(t) = V(G),\ t \in \mathbb{N}_0$ is called a dynamic graph with *synchronous updating*.

Strategies of vertices in a dynamic graph are updated throughout time, thus, we can define a *strategy trajectory* and *utility trajectory* of dynamic graph. *Strategy trajectory* is a sequence of vectors containing strategies assigned to each vertex of dynamic graph with given *initial condition*. *Initial condition* is a strategy trajectory at time $t = 0$. Similarly, *utility trajectory* is a sequence of vectors containing utilities of vertices of dynamic graph.

Definition 11. Given the *initial condition* $s(0) = (s_1(0), s_2(0), \ldots, s_n(0))$, we call $s(t) = ((s_1(t), s_2(t), \ldots, s_n(t))$ a *strategy trajectory* of the dynamic graph.
Similarly, $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))$ is called a *utility trajectory* of the dynamic graph.

Remark 7. Given time t, the vector $s = (s_1(t), \ldots, s_n(t))$ is called a *strategy vector at time* t.

Next we define symbol *argmax*. Suppose we have an indexed set $\mathbf{X}$. Symbol *max* gives us the greatest value of all members of the set. It is not always useful to know the greatest value of the set only. Sometimes we need to know the index of the member of the set with the highest value. Symbol *argmax* gives us the index of the member of set $\mathbf{X}$ with the greatest value. Note that *argmax* can result in the set of indices.

Definition 12. Suppose x is an arbitrary member of some set $\mathbf{X}$. Let $f(x)$ be some function that is defined $\forall x \in \mathbf{X}$. Then the symbol *argmax* is defined by the following equivalence.

$$x^* \in \mathrm{argmax}_{x\in\mathbf{X}} f(x) \iff f(x^*) = \max_{x\in\mathbf{X}} f(x). \tag{1.11}$$

[9]

Finally, we can introduce *evolutionary dynamic graph* which is a dynamic graph with two member strategy set $S = \{0, 1\}$. The utility of i^{th} vertex depends only on strategies of the i^{th} vertex itself and its neighbours. Similarly, an update rule updating i^{th} vertex depends on utilities of the i^{th} vertex itself and its neighbours. This definition accentuates the importance of the graph structure of vertices. Although evolutionary dynamic graph is a complex structure, short-term behaviour of each vertex depends only on strategies of vertex itself and its neighbours.

Definition 13. We say that a dynamic graph $\mathfrak{E}$ is evolutionary if

- $S = \{0, 1\}$,
- the utility of each vertex depends only on strategies of the vertex itself and its neighbours,
- the update rule ρ_i has the following form

$$\rho_i = s_k \in s_{\mathrm{argmax}_{j\in N(i)\cup i} u_j(s)}, \tag{1.12}$$

  where k is the only element of the *argmax* set if $|\mathrm{argmax}_{j\in N(i)\cup i}| = 1$, otherwise :

  1. k is arbitrary element from $\mathrm{argmax}_{j\in N(i)\cup i}$ if $\forall l \in \mathrm{argmax}_{j\in N(i)\cup i} : s_l \neq s_i$,
  2. $k = i$ if $\exists l \in \mathrm{argmax}_{j\in N(i)\cup i} : s_l = s_i$.

Remark 8. The update rule either chooses the only element of the *argmax* set or it preserves the strategy of the updated vertex.

Remark 9. Strategy denoted 0 is called *Defection* a the one denoted 1 is called *Cooperation.*

Remark 10. There are properties common to all evolutionary dynamic graphs: strategy set, update rule. On the other side, there are many properties which differentiate evolutionary dynamic graphs. One of the most significant is the structure of utility strategy graph. By the structure we understand the number of vertices and their connection by edges. For the sake of brevity, we say that evolutionary dynamic graph $\mathfrak{E}$ *has* a graph $G_{\mathfrak{E}}$. Graph $G_{\mathfrak{E}}$ has exactly the same number of vertices and the same edges as evolutionary dynamic graph. Or, we can say that graph $G_{\mathfrak{E}}$ *is the graph of* evolutionary dynamic graph $\mathfrak{E}$.

Now, we can make an observation which is quite natural. Still it simplifies further explanation of behaviour of evolutionary dynamic graphs. It is not necessary to examine behaviour of every vertex of evolutionary dynamic graph at every time. Vertices with the same strategy as their neighbouring vertex at time t cannot change its strategy at time t. Update rule (1.12) can only choose vertex with the same strategy as the updated vertex have.

Observation 2. If $s_j(t) = s_l(t)$ for each $l \in N(j)$ for some time $t \in \mathbb{N}_0$ then $\rho_j(t) = s_j(t)$.

Let us introduce *aggregate utility function* $u_{i,\text{agg}}(t)$ which comply with the properties of update function ρ of evolutionary dynamic graph in Definition 13. The main idea of *aggregate utility function* is to sum utilities of i^{th} vertex in all static games played with its neighbours.

Definition 14. Function $u_{i,\text{agg}} : S^n \to \mathbb{R}$ satisfying (dropping the dependence of strategies on t for the sake of brevity):

$$u_{i,\text{agg}}(t) = a \sum_{j \in N(i)} s_i s_j + b \sum_{j \in N(i)} s_i(1-s_j) + c \sum_{j \in N(i)} (1-s_i)s_j + d \sum_{j \in N(i)} (1-s_i)(1-s_j) \quad (1.13)$$

is called *aggregate utility function.*

Remark 11. To study cooperation problems on the graph, the parameters a, b, c, d are identical with parameters from Table 1.1. They also share the same properties and thus, they form 4 scenarios given in the same subsection.

Other possible choice of a utility function is the *mean aggregate utility function.*

Definition 15. Function $u_{i,\text{mean}} : S^n \to \mathbb{R}$ satisfying:

$$u_{i,\text{mean}}(t) = \frac{u_{i,\text{agg}}(t)}{|N(i)|} \quad (1.14)$$

is called *mean aggregate utility function.*

Let us take a deeper look into properties of strategy trajectories and vectors of dynamic graphs. Properties are defined for strategy trajectories and vectors of dynamic graphs only. Since evolutionary dynamic graphs are a special case of dynamic graphs, the properties can also be observed for evolutionary dynamic graphs. It is natural to look for a strategy vector of a dynamic graph whose strategies remain the same after updating. This vector is called a *fixed point.* Furthermore, we can observe "stronger" fixed point called *limit* of a dynamic graph. If strategy trajectories of a dynamic graph are constant from some time $t > 0$, this constant vector is called a *limit* of a dynamic graph.

Definition 16. We say that a strategy vector s_f is a fixed point of a dynamic graph $\mathfrak{D}$, if the strategy trajectories are constant for the initial condition $s(0) = s_f$.

Definition 17. We say that a fixed point s_f is a limit of a dynamic graph $\mathfrak{D}$ with initial condition $s(0)$, if there exists $t_0 \in \mathbb{N}_0$, such that for all $t > t_0$ we have $s(t) = s_f$.

In addition, we can observe a periodic behaviour of dynamic graphs. If strategy vectors of a dynamic graph repeat periodically from some time $t_0 \in \mathbb{N}$, we say that the dynamic graph *converges to a limit cycle*. Similarly to the fixed point we define a "stronger" property. If initial condition of dynamic graph is in the cycle, we can say that the dynamic graph *generates a cycle*.

Definition 18. We say that a dynamic graph $\mathfrak{D}$ with initial condition $s(0)$ converges to a limit cycle of length T if there exists $t_0 \in \mathbb{N}_0$ such that for all $t > t_0$ we have $s(t) = s(t+T)$.

Definition 19. We say that a dynamic graph $\mathfrak{D}$ with an initial condition $s(0)$ generates a cycle of length T if for all $t \geq 0$ we have $s(t) = s(t+T)$.

It is important to pay attention to notation. The number of vertices of a graph G is denoted by type n, $|V(G)|= n$. The length of a cycle of a dynamic graph $\mathfrak{D}$ is denoted by type k. Since both types k and n are usually used as indices of sequences or series they can be easily mistaken.
Here ends the part devoted to properties of general dynamic graphs. Next we define some terms concerning evolutionary dynamic graphs only. The main difference between dynamic graphs and evolutionary dynamic graphs is in the strategy set S of evolutionary dynamic graphs, $S_{\mathfrak{E}} = \{0,1\}$ and thus $|S_{\mathfrak{E}}|= 2$. It is sometimes more simple to study sets of strategy vectors of evolutionary dynamic graph rather than single ones. Thus, we define a set $\mathfrak{m}$ which groups strategy vectors of evolutionary dynamic graph with the same number of cooperators.

Definition 20. Let $m \in \{0,1,\dots,n\}$. By $\mathfrak{m}$ we denote a set of all strategy vectors $s \in S^n$ such that $m = \sum_{i=1}^{n} s_i$. We call $\mathfrak{m}$ a strategy profile of the evolutionary dynamic graph $\mathfrak{E}$.

Remark 12. Given evolutionary dynamic graph $\mathfrak{E}$ with $|V(G)|= n$ the set $\mathfrak{m}$ contains exactly one strategy vector for $m \in \{0,n\}$. The set $\mathfrak{m}$ for $m = 0$ will be denoted by $\mathfrak{o}$ and its only full defective strategy vector can be also denoted $\mathfrak{o}$. The same holds for $m = n$ and its full cooperative strategy vector $\mathfrak{n}$.

Observation 3. Strategy trajectories are constant for the initial conditions $\mathfrak{o}$ or $\mathfrak{n}$, thus strategy vectors $\mathfrak{o}, \mathfrak{n}$ are fixed points.

The main subject of this thesis is to examine a periodic behaviour of evolutionary dynamic graphs. One of the topics examines the number of vertices of evolutionary dynamic graphs generating a cycle of given length k. Thus, we define sequences $C_s(k)$ and $D_s(k)$. Note that subjects of our interest are evolutionary dynamic graphs with synchronous updating.

Definition 21. The minimal number of the vertices of the evolutionary dynamic graph $\mathfrak{E}$ with synchronous updating which generates a cycle of the length at least k is denoted by $C_s(k)$.

Definition 22. The minimal number of the vertices of the evolutionary dynamic graph $\mathfrak{E}$ with synchronous updating which generates a cycle of the length at least k and there exist times t_1, t_2 and strategy profiles $\mathfrak{m}_1, \mathfrak{m}_2$ such that $s(t_1) \in \mathfrak{m}_1, s(t_2) \in \mathfrak{m}_2$ and $\mathfrak{m}_1 \neq \mathfrak{m}_2$ is denoted by $D_s(k)$.

Lemma 1.

$$C_s(k) \geq \log_2(k+2) \tag{1.15}$$

Proof. Suppose we have an evolutionary dynamic graph $\mathfrak{E}$ with at least $C_s(k)$ vertices which converges to the limit cycle of length at least k. The number of all strategy vectors of evolutionary dynamic graph $\mathfrak{E}$ is at least $|S^{C_s(k)}|= 2^{C_s(k)}$. The uniqueness of the update rule ρ implies every strategy vector in the cycle unique: $|S^{C_s(k)}|\geq n$.
Observation 3 excludes vectors $\mathfrak{o}, \mathfrak{n}$ from being in cycle. Thus, $|S^{C_s(k)}|= 2^{C_s(k)} \geq k+2$. Furthermore, $C_s(k) \geq \log_2(k+2)$. □

Lemma 2.

$$C_s(k) \leq D_s(k) \tag{1.16}$$

Proof. Let us denote the set of all evolutionary dynamic graphs generating a cycle of length at least k by $A(\mathfrak{E}_{C_s(k)})$. Similarly, let us denote the set of all evolutionary dynamic graphs generating a cycle of the length at least k and there exist times t_1, t_2 and strategy profiles $\mathfrak{m}_1, \mathfrak{m}_2$ such that $s(t_1) \in \mathfrak{m}_1, s(t_2) \in \mathfrak{m}_2$ and $\mathfrak{m}_1 \neq \mathfrak{m}_2$ by $A(\mathfrak{E}_{D_s(k)})$. There exists a evolutionary dynamic graph $\mathfrak{E}_{\min} \in \mathfrak{E}_{D_s(k)}$ such that $|V(G_{\mathfrak{E}_{\min}})|= D_s(k)$. Since $A(\mathfrak{E}_{C_s(k)}) \supset A(\mathfrak{E}_{D_s(k)})$, then $\mathfrak{E}_{\min} \in A(\mathfrak{E}_{C_s(k)})$ and thus $C_s(k) \leq D_s(k)$ for each $k \in \mathbb{N}$. □

Chapter 2

Cycle of arbitrary length

This chapter looks into a problem of construction of an evolutionary dynamic graph which generates a cycle of arbitrary length k. Furthermore, we look into properties of such evolutionary dynamic graphs. Especially, the number of vertices of such evolutionary dynamic graph dependent on the cycle length k. The first section of this chapter shows the example of evolutionary dynamic graph constructed by Epperlein, Siegmund and Stehlík in [1]. This example also states the sequence which estimates the sequence $D_s(k)$ from above. The second section explains the construction of evolutionary dynamic graph which generates a cycle of arbitrary length k. However, every strategy vector in the cycle belongs to the same strategy profile. Thus, this example provides the upper restrictive sequence to the sequence $C_s(k)$. The main question of this chapter is following:

Question 1. *Given $k \in \mathbb{N}$, can we find an evolutionary dynamic graph which generates a cycle of length k?*

Remark 13. The form of the question is crucial here. Exchange of word "given" with "arbitrary" in Question 1 means we want to create an evolutionary dynamic graph which generates a cycle of arbitrary length $k \in \mathbb{N}$ when k is not known at time of construction. Thus, we keep this formulation of Question 1. For the sake of clarity, if we create an evolutionary dynamic graph which generates a cycle of arbitrary length k, it *always* means k is *given* at the time of construction of evolutionary dynamic graph.

2.1 Exponential dependence

The article [1] provided a manual for construction of the evolutionary dynamic graph $\mathfrak{E}_A$ and its initial condition $s_{\mathfrak{E}_A}(0)$ which generates a cycle of arbitrary length k. The example of such evolutionary dynamic graph also stated the sequence estimating $D_s(k)$ from above.

Remark 14.

$$D_s(k) \leq 2^{k+1} + 2. \tag{2.1}$$

Remark 14 states an exponential restrictive function. Since exponential function grows very quickly, the question arises: Can we estimate sequence $D_s(k)$ by a sequence with a slower growth? We conjecture that there is a polynomial function which restricts $D_s(k)$ from above. Note that the sequence which bounds sequence $D_s(k)$ from above automatically bounds the sequence $C_s(k)$ from above (Lemma 2). Contrarily, the upper restrictive function of $C_s(k)$ does not state anything about the restriction of $D_s(k)$.

Conjecture 1. There exists a function $B_u(k)$ for which $D_s(k) \leq B_u(k)$ and $B_u(k) \in O(k^a)$ for some $a \in \mathbb{R}^+$.

2.2 Walking structure

This example of evolutionary dynamic graph and its initial vector was inspired by "walker" introduced by M. A. Nowak, p. 159 [2]. "Walker" is a cluster of cooperators which can "walk" through the spatial grid of defectors. This section explains construction of such evolutionary dynamic graph, its initial condition and derives conditions on parameters a, b, c, d under which this evolutionary dynamic graph generates a cycle. Finally, we prove this evolutionary dynamic graph generates a cycle under given conditions. An example of this evolutionary dynamic graph generating a cycle of length $k = 8$ is presented in Appendix A.

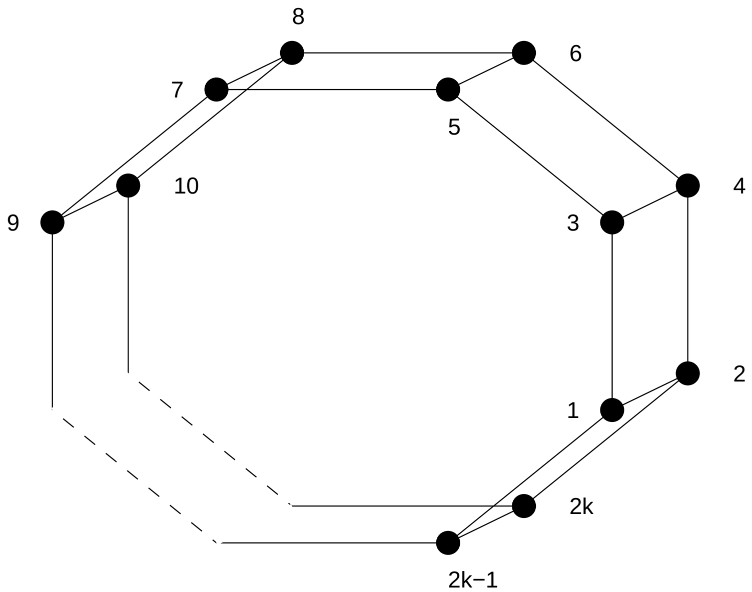

Figure 2.1: Graph $G_{\mathfrak{E}_1}$ of evolutionary dynamic graph $\mathfrak{E}_1$ with $|V(G_{\mathfrak{E}_1})| = 2k$ vertices.

Example 2. Let us introduce an evolutionary dynamic graph $\mathfrak{E}_1$ with

- 3-regular graph $G_{\mathfrak{E}_1}$ with $|V(G_{\mathfrak{E}_1})| = 2k$ described in Construction 1 and captured in Figure 2.1 with $|V(G_{\mathfrak{E}_1})| = 2k$.
- the mean aggregate utility function (1.14).
- synchronous updating.

- initial condition $s_{\mathfrak{E}_1}(0) = (1,1,1,1,1,0,\ldots,0)$ (Figure 2.2).

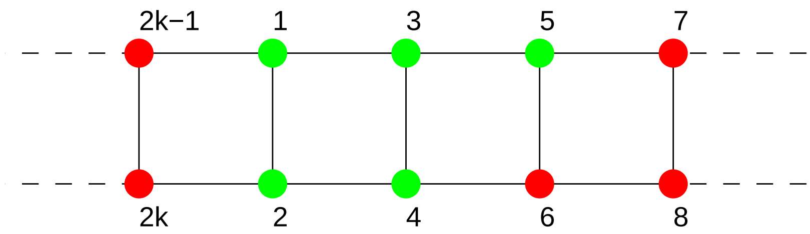

Figure 2.2: Initial condition $s_{\mathfrak{E}_1}(0)$ of evolutionary dynamic graph $\mathfrak{E}_1$. Vertices with the strategy of cooperation (0) are green and vertices with the strategy of defection (1) are red.

First, we clearly specify the construction of the graph $G_{\mathfrak{E}_1}$ with $2k$ vertices:

Construction 1. Let us have the graph with $2k$ vertices. Each of vertices has exactly 3 neighbours. Edges containing i^{th} vertex for each $i \in \{1,\ldots,2k\}$:

1. $(i, i+1)$ if i is odd, $(i, i-1)$ if i is even,
2. $(i, (i+1)(\text{mod } 2k) + 1)$,
3. $(i, (i+2k-1)(\text{mod } 2k) - 1)$.

We have graph $G_{\mathfrak{E}_1}$ constructed as in Construction 1. We state initial condition $s_{\mathfrak{E}_1}(0)$ with 5 vertices being cooperators and other vertices being defectors, i.e. $s_{\mathfrak{E}_1}(0) = (1,1,1,1,1,0,\ldots,0)$. If $k \geq 5$ then we can compute following utilities (at time $t = 0$):

$$
\begin{aligned}
u_1 &= \frac{2a+b}{3} \\
u_2 &= \frac{2a+b}{3} \\
u_3 &= a \\
u_4 &= \frac{2a+b}{3} \\
u_5 &= \frac{a+2b}{3} \\
u_6 &= \frac{2c+d}{3} \\
u_7 &= \frac{c+2d}{3} \\
u_{2k-1} &= \frac{c+2d}{3} \\
u_{2k} &= \frac{c+2d}{3} \\
u_i &= d,\ i = \{8,9,\ldots,2k-2\}
\end{aligned}
$$

We formally define the cyclic behaviour of evolutionary dynamic graph $\mathfrak{E}_1$. Furthermore, we use this description of behaviour to derive conditions on parameters a, b, c, d. We want the cluster of cooperators to "slide" compactly through the graph i.e.

$$s_i(t) = s_{(i+1)(\bmod 2k)+1}(t-1), \qquad t \in \mathbb{N} \tag{2.2}$$

The rule (2.2) specifies the strategy of the neighbour of i^{th} vertex with the highest utility. Table 2.1 clearly shows the strategy of each vertex in time $t = 1$ and the inequality needed. Keeping the properties of the parameters a, b, c, d in mind (Table 1.1) we can

Index	New strategy	Inequality
1	1	$a > \frac{c+2d}{3}$
2	1	$\frac{2a+b}{3} > \frac{c+2d}{3}$
3	1	Every neighbour is cooperator
4	0	$a < \frac{2c+d}{3}$
5	0	$a < \frac{2c+d}{3}$
6	0	$\frac{2a+b}{3} < \frac{2c+d}{3}$
7	0	$\frac{a+2b}{3} < \frac{c+2d}{3}$
8	0	Every neighbour is defector
$2k-1$	1	$\frac{2a+b}{3} > \frac{c+2d}{3}$
$2k$	1	$\frac{2a+b}{3} > \frac{c+2d}{3}$

Table 2.1: The table of indices of vertices of evolutionary dynamic graph $\mathfrak{E}_1$, their new strategies at time $t = 1$ and inequality needed

reduce the list of all inequalities to:

$$\frac{2a+b}{3} > \frac{c+2d}{3} \tag{2.3}$$

$$\frac{2c+d}{3} > a \tag{2.4}$$

$$\frac{c+2d}{3} > \frac{a+2b}{3}. \tag{2.5}$$

Observation 4. Since inequalities (2.3), (2.4), (2.5) are satisfied, the strategy vector $s_{\mathfrak{E}_1}(1)$ of evolutionary dynamic graph $\mathfrak{E}_1$ at time $t = 1$ with initial condition $s_{\mathfrak{E}_1}(0)$ is $s_{\mathfrak{E}_1}(1) = (1, 1, 1, 0, \ldots, 0, 1, 1)$

Observation 4 defines the change of initial condition $s_{\mathfrak{E}_1}(0)$ to strategy vector at time $t = 1$, $s_{\mathfrak{E}_1}(1)$. We can show that evolutionary dynamic graph $\mathfrak{E}_1$ with initial condition $s_{\mathfrak{E}_1}(0)$ generates a cycle. Observation 4 can prove the existence of a cycle in evolutionary dynamic graph $\mathfrak{E}_1$ after some modifications.

Lemma 3. The evolutionary dynamic graph $\mathfrak{E}_1$ with graph $G_{\mathfrak{E}_1}$ given by Construction 1 where $|V(G_{\mathfrak{E}_1})| = 2k$ (Figure 2.1), synchronous updating, initial condition $s_{\mathfrak{E}_1}(0)$ and parameters a, b, c, d satisfying inequalities (2.3), (2.4), (2.5) generates a cycle of the length k for $k \geq 5$.

Proof. The change of strategy vector from $s_{\mathfrak{C}_1}(0)$ to $s_{\mathfrak{C}_1}(1)$ is clarified by Observation 4. We shift the indices of vertices of graph $G_{\mathfrak{C}_1}$ following way:

$$i \longrightarrow (i+1)(\text{mod } 2k)+1, \quad i=\{1,2,\dots,2k-1,2k\}. \tag{2.6}$$

Thus, we obtain evolutionary a new dynamic graph $\mathfrak{C}_1^1$, with graph $G_{\mathfrak{C}_1^1}$. The strategy vector of the evolutionary dynamic graph $\mathfrak{C}_1^1$ is $s^1_{\mathfrak{C}_1}(1) = (1,1,1,1,1,0,\dots,0)$. Using Observation 4 we can update the evolutionary dynamic graph $\mathfrak{C}_1^1$ to the state at time $t=2$. Shift indices again and repeat the procedure.
After k^{th} repetition we obtain the evolutionary dynamic graph $\mathfrak{C}_1^n$ whose graph $G_{\mathfrak{C}_1^k}$ has exactly same labelling as graph $G_{\mathfrak{C}_1}$. Strategy vectors are equal, i.e. $s_{\mathfrak{C}_1}(0) = s^k_{\mathfrak{C}_1}(k) = s_{\mathfrak{C}_1}(k)$.
Thus, the evolutionary dynamic graph $\mathfrak{C}_1$ generates a cycle of length k. $\square$

Observation 5. The evolutionary dynamic graph $\mathfrak{C}_1$ with $|V(G_{\mathfrak{C}_1})|=2k$ vertices generates a cycle for $n \geq 5$.

We proved that evolutionary dynamic graph $\mathfrak{C}_1$ with graph $G_{\mathfrak{C}_1}$ generates a cycle of length k. Furthermore, we want to prove that every graph constructed as described in Construction 1 can be modified such that evolutionary dynamic graph with modified graph generates a cycle of length $k+1$. First, the modification of graph $G_{\mathfrak{C}_1}$ must be defined.

Construction 2. Let us have graph $G_{\mathfrak{C}_1}$ with $|V(G_{\mathfrak{C}_1})|=2k$ constructed as in Construction 1. Now modify vertices a and edges following way:

1. delete edges $(2k,2)$ and $(2k-1,2)$,
2. add vertices $2k+1, 2k+2$,
3. add edges $(2k-1,2k+1),(2k,2k+2),(2k+1,1)$ and $(2k+2,2)$.

Lemma 4. If we modify a graph $G_{\mathfrak{C}_1}$ of evolutionary dynamic graph $\mathfrak{C}_1$ described in Lemma 3 the way described in Construction 2 the evolutionary dynamic graph $\mathfrak{C}_1$ with initial condition $s_{\mathfrak{C}_1}(0) = (1,1,1,1,1,0,\dots,0)$ (Figure 2.3) generates a cycle.

Proof. The proof is similar to the proof of Lemma 3 with two small modifications:

1. shift rule (2.6) is changed to
$$i \longrightarrow (i+1)(\text{mod } 2k+2)+1, \quad i=\{1,2,\dots,2k+1,2k+2\}. \tag{2.7}$$
2. we obtain the original strategy vector after $(k+1)^{\text{th}}$ repetition.

Thus, the evolutionary dynamic graph $\mathfrak{C}_1'$ generates a cycle of length $k+1$. $\square$

Observation 6. Every strategy vector $s_{\mathfrak{C}_1}$ of an evolutionary dynamic graph $\mathfrak{C}_1$ with initial condition $s_{\mathfrak{C}_1}(0)$ contains the same number of cooperators, i.e.

$$\forall t \in \mathbb{N}_0 : s_{\mathfrak{C}_1}(t) \in \mathfrak{m}, \text{ where } m=5. \tag{2.8}$$

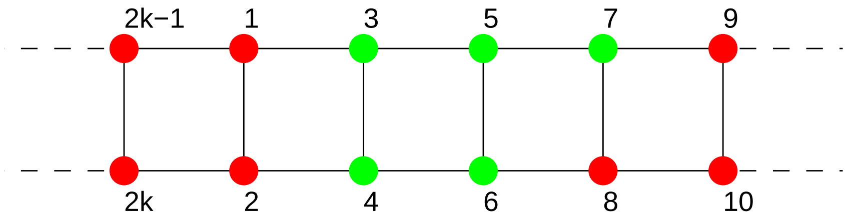

Figure 2.3: Initial condition of modified evolutionary dynamic graph $\mathfrak{E}_1'$. Vertices with the strategy of cooperation (0) are green and vertices with the strategy of defection (1) are red.

Every strategy vector in the cycle of evolutionary dynamic graph $\mathfrak{E}_1$ belongs to the same strategy profile. Thus, this example can make restriction from above of $C_s(k)$ only.

Corollary 1. *The Example 2 provide that:*

$$C_s(k) \leq 2k, \ k \geq 5. \tag{2.9}$$

Chapter 3

Cycle changing strategy profile

An evolutionary dynamic graph stated in example in this chapter shares many properties with the evolutionary dynamic graph described in the article [1] (here denoted by $\mathfrak{E}_A$). The main idea of the functionality of this evolutionary dynamic graph is almost the same. In this section we introduce construction of such evolutionary dynamic graph. For the sake of brevity, we divide vertices of evolutionary dynamic graph into 4 sets. Furthermore, we explain behaviour of every vertex (change of strategy or keeping its strategy) throughout the time of evolutionary dynamic graph and thus, derive conditions on parameters a, b, c, d needed. An example of this evolutionary dynamic graph generating a cycle of length $k = 7$ is presented in Appendix B.

3.1 Introduction of the graph

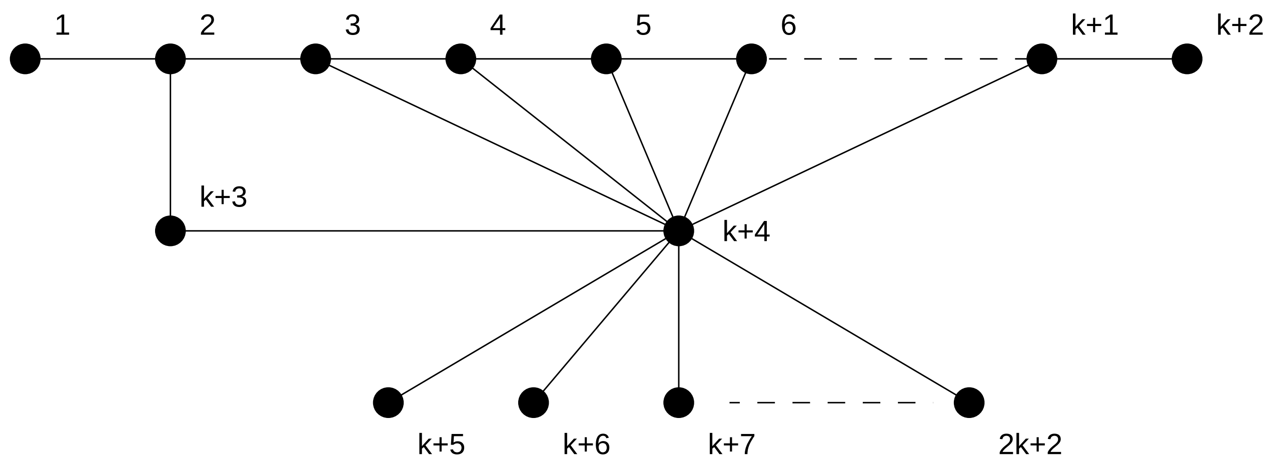

Figure 3.1: Graph $G_{\mathfrak{E}_2}$ of evolutionary dynamic graph $\mathfrak{E}_2$ with $|G_{\mathfrak{E}_2}|= 2k + 2$.

Example 3. Let us have an evolutionary dynamic graph $\mathfrak{E}_2$ with:

1. graph $G_{\mathfrak{E}_2}$ depicted in Figure 3.1 (Construction 3).

2. the mean aggregate utility function (1.14).

3. synchronous updating.

4. initial condition $s_{\mathfrak{E}_2}(0) = (1, 1, 0, \ldots, 0)$ (Figure 3.2).

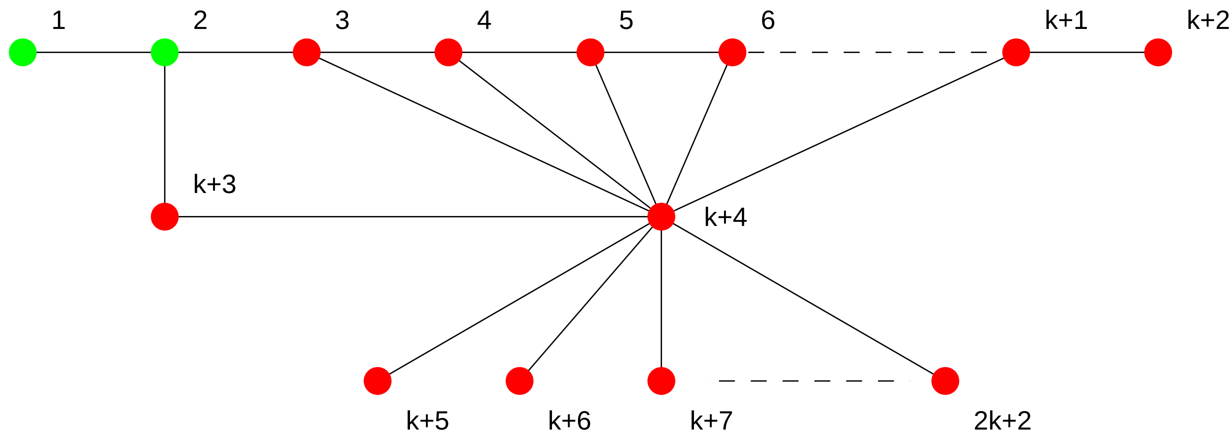

Figure 3.2: Initial condition $s_{\mathfrak{E}_2}(0)$ of evolutionary dynamic graph $\mathfrak{E}_2$. Vertices with the strategy of cooperation (0) are green and vertices with the strategy of defection (1) are red.

Construction 3. Let us have a graph on $2k+2$ vertices with edges:

1. $(i, i+1)$ for $i \in \{1, \ldots, k+1, k+3\}$,

2. $(k+4, i)$ for $i \in \{3, \ldots, k+1\} \cup \{k+5, \ldots, 2k+2\}$,

3. $(2, k+3)$.

Remark 15. For the sake of further computation, the number of neighbours of vertex $k+4$ is $d_{k+4} = 1 + (k-1) + (k-2) = 2k-2$.

3.2 Properties and existence of cycle

We have graph $G_{\mathfrak{E}_2}$ with $|V(G_{\mathfrak{E}_2})| = 2k+2$ constructed as in Construction 3 and initial condition $s_{\mathfrak{E}_2}(0) = \{1, 1, 0, \ldots, 0\}$ of evolutionary dynamic graph $\mathfrak{E}_2$. Next, we want the update function ρ to respect following rules:

$$\begin{aligned} s_i(0) &= s_i(t),\ t \in \mathbb{N},\ i \in \{1, 2, k+2, k+3, \ldots, 2k+1, 2k+2\} && (3.1) \\ s_i(t) &= \begin{cases} 1, & t(\mathrm{mod}\ k) \geq i-3, \quad i \in \{3, 4, \ldots, k, k+1\}, \\ 0, & t(\mathrm{mod}\ k) < i-3, \quad i \in \{3, 4, \ldots, k, k+1\}. \end{cases} && (3.2) \end{aligned}$$

In other words, vertices mentioned in (3.1) never change their strategy and vertices mentioned in (3.2) repeatedly become cooperators one after another and then collectively change to defection.

Remark 16. We study a periodic behaviour of evolutionary dynamic graph $\mathfrak{E}_2$. Thus, it can be easily seen that $s_{\mathfrak{E}_2}(t) = s_{\mathfrak{E}_2}(t+k)$ for $t \in \mathbb{N}_0$ and k given by the number of vertices of graph $G_{\mathfrak{E}_2}$: $|V(G_{\mathfrak{E}_2})|= 2k+2$ (Rules (3.1) and (3.2)). For the sake of brevity we consider only values of time t in $t \in \{0, 1, \dots, k-1\}$.

Now, we divide vertices into 4 main sets (Table 3.1, Figure 3.3). This simplifies further explanation of behaviour of the evolutionary dynamic graph $\mathfrak{E}_2$. The sets:

- vertices that do not change their strategy except of vertex k+4 and every their neighbour has the same strategy (Υ_1),
- vertices that do not change their strategy except of vertex k+4 and some of their neighbours have different strategy (Υ_2),
- vertices that do change their strategy (Υ_3),
- vertex k+4 (Υ_4).

Set	Vertices
Υ_1	$\{1, k+5, \dots, 2k+2\}$
Υ_2	$\{2, k+2, k+3\}$
Υ_3	$\{3, \dots, k+1\}$
Υ_4	$\{k+4\}$

Table 3.1: Elements of sets $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4$

Following paragraphs explain the conditions on parameters a, b, c, d needed to strategy vectors satisfy Rules (3.1), (3.2). Furthermore, we explain behaviour of every vertex of evolutionary dynamic graph $\mathfrak{E}_2$ for all values of time $t \in \{0, \dots, k-1\}$. For the sake of brevity, we omit each vertex $j \in V(G_{\mathfrak{E}_2})$ at time t such that $s_j(t) = s_{l \in N(j)}(t)$. Strategy of these vertices cannot be changed at time t (Observation 2).

Vertices in Υ_1 : Neighbours of every vertex in this set have exactly same strategy as the vertex itself. Thus, strategy of vertices in set Υ_1 cannot be changed.

Vertices in Υ_2 : Each of vertices 2 and $k+3$ does not change strategy. Thus, for each $t \in \{0, 1, \dots, k-1\}$ the utilities are

$$u_1(t) > u_{k+3}(t) > u_2(t). \tag{3.3}$$

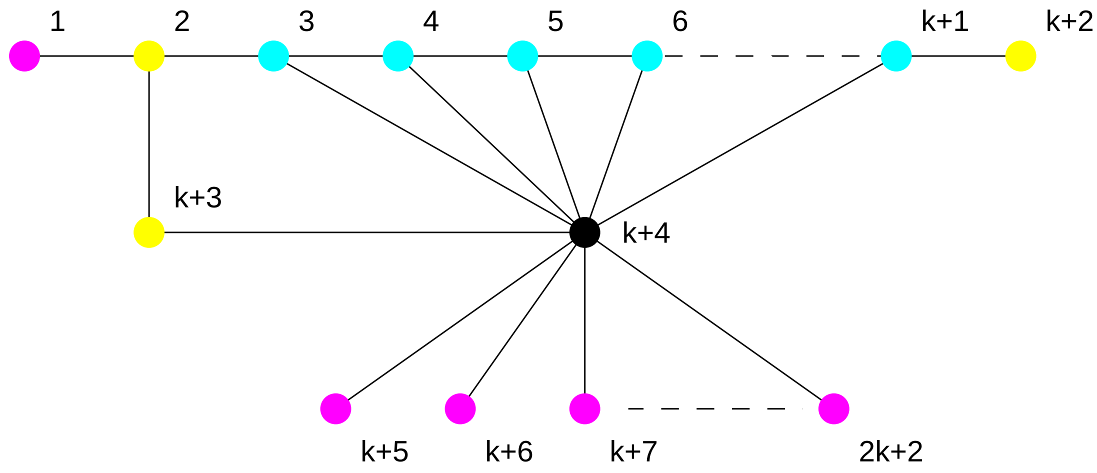

Figure 3.3: Division of the vertices into sets from Table 3.1. Vertices from Υ_1 are magenta (purple), Υ_2 are yellow, Υ_3 are cyan (blue), Υ_4 are black.

Sequences $u_1(t)$ and $u_{k+3}(t)$ are constant.

$$
\begin{aligned}
u_1 &= a \\
u_{k+3} &= \frac{c+d}{2} \\
u_2(t) &= \begin{cases} \frac{a+2b}{3} & t = 0 \\ \frac{2a+b}{3} & t \in \{1,\ldots,k-1\}. \end{cases}
\end{aligned}
$$

Inequalities (3.3) are satisfied for each $t \in \{0,1,\ldots,k-1\}$ if:

$$a > \frac{c+d}{2} \tag{3.4}$$

$$\frac{2a+b}{3} < \frac{c+d}{2}. \tag{3.5}$$

The strategy of vertex $k+2$ can be changed in the time t_c only if $s_{k+2}(t_c-1) \neq s_{k+1}(t_c-1)$. In other words, the strategy of vertex $k+2$ can be changed to cooperation only if strategy of it's neighbour is cooperation. Still the utility of cooperator must be higher than the utility of defector. This occurs if $t_c = k-1$. Thus, $u_{k+1}(t_c) < u_{k+2}(t_c)$. This leads to inequality $\frac{a+2b}{3} < c$ which is directly implied by (3.5) and assumptions of static games in Subsection 1.1.2.

Vertices in Υ_3 : Vertices in set Υ_3 are the ones described by rule (3.2). The cycle consists of two stages: cooperation spreading ($t \in \{0,\ldots,k-2\}$) and resetting ($t = k-1$). Verbal explanation of behaviour of evolutionary graph may lead to confusion. Thus, we stick to formal notation. Any time, please refer to Figure 3.2. It captures the strategies of vertices at time $t = i-2$ for given vertex index $i \in \{3,\ldots,k\}$. Further explanation uses a specific property of an evolutionary dynamic graph with update rule ρ (1.12). First we introduce new notation:

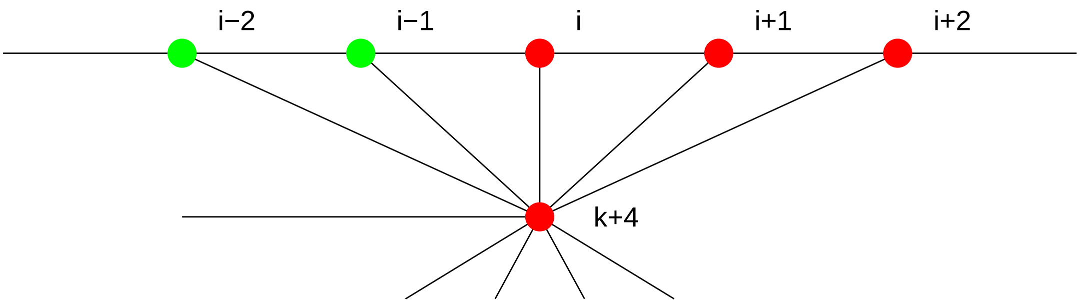

Figure 3.4: Strategies of i^{th} vertex and its closely neighbouring vertices at time $t = i - 2$.

Remark 17. Let us denote $N_0(i,t)$ the set of all neighbours of i^{th} vertex which have strategy 0 at time t and vice versa $N_1(i,t)$ for strategy 1.

Now we make observation directly derived from the form of update rule ρ (1.12). It describes the update of evolutionary dynamic graph with two vertices with different strategy neighbouring under certain conditions on neighbours of either of them.

Observation 7. Let us have two vertices p and q of arbitrary evolutionary dynamic graph with synchronous updating such that p, q are neighbours. Let their strategy be $s_p(t) = 1$ and $s_q(t) = 0$ at arbitrary time t . Let $N_0(q,t) \cup \{p\} = N(q)$. If $u_p(t) > \max_{j \in N_0(q,t) \cup N_0(p,t) \cup q}(u_j(t))$ then $s_p(t+1) = s_q(t+1) = 1$.

Cooperation spreading ($t \in \{0, \dots, k-2\}$): The proper course of cooperation spreading is described by two elementary actions:

1. strategy of i^{th} vertex from set Υ_3 is changed at time t, where $t = i-3$ from defection to cooperation, (3.2).
2. strategy of i^{th} vertex from set Υ_3 is not changed at time t, where $t \in \{i-2, \dots, k-2\}$, (3.2).

Using Observation 7, the strategy of i^{th} vertex from set Υ_3 is changed at time $t = i - 3$ if:

$$u_{i-1}(t) \quad > \max_{j \in N_0(i,t) \cup N_0(i-1,t) \cup \{i\}} u_j(t), \; t = i - 3, \; i \in \Upsilon_3. \tag{3.6}$$

The notation of inequality (3.6) can be confusing. Thus, it can be simplified by evaluating the right side:

$$\max_{j \in N_0(i,t) \cup N_0(i-1,t) \cup \{i\}} u_j(t) = \max\{u_{k+4}(t), u_i(t), u_{i+1}(t)\}, \; t = i - 3, \; i \in \Upsilon_3. \tag{3.7}$$

Next, the utilities of vertices in inequality (3.6) must be specifically stated as the functions of parameters a, b, c, d. The Rules (3.1), (3.2) directly implies vertex i from $i \in \Upsilon_3$ to have one neighbouring cooperator and two neighbouring defectors at time $t = i - 3$. Thus:

$$u_{i-1}(t) = \frac{a + 2b}{3}, \text{ for } t = i - 3. \tag{3.8}$$

Similarly,

$$\begin{aligned} u_i(t) &= \frac{c+2d}{3},\ t=i-3,\ i\in\Upsilon_3, &(3.9)\\ u_{i+1}(t) &= d,\ t=i-3,\ i\in\Upsilon_3. &(3.10)\end{aligned}$$

The utility of vertex $k+4$ is described by sequence $u_{k+4}(t)$ (using Remark 15):

$$u_{k+4}(t)=\frac{tc+(k-2-t)d}{2k-2},\ t\in\{0,\dots,k-1\}. \tag{3.11}$$

Observation 8. Since $c > d$ (Subsection 1.1.2), sequence $u_{k+4}(t)$ is increasing for $t\in\{0,\dots,k-1\}$.

Now, we can evaluate inequality (3.6) using (3.7), (3.8), (3.9), (3.10) and (3.11):

$$\frac{a+2b}{3} > \max\left\{\frac{tc+(k-2-t)d}{2k-2},\frac{c+2d}{3},d\right\},\ t=i-3,\ i\in\Upsilon_3. \tag{3.12}$$

Inequality (3.6) is true for all $i\in\Upsilon_3$. Thus, we have a system of $|\Upsilon_3|=k-2$ inequalities which can be simplified to one (t is substituted by $i-3$):

$$\frac{a+2b}{3} > \max_{i\in\Upsilon_3}\left\{\max\left\{\frac{(i-3)c+(k-2-(i-3))d}{2k-2},\frac{c+2d}{3},d\right\}\right\}. \tag{3.13}$$

Using Observation 8 can be (3.13) evaluated:

$$\frac{a+2b}{3} > \frac{(k-2)c+kd}{2k-2}. \tag{3.14}$$

Now, we clarified the first part of the cooperation spreading stage. Still we must show the inequalities (3.4), (3.5) and (3.14) are sufficient to prevent the vertices in Υ_3 from prematurely changing their strategy to defection. Particularly, the strategy of vertex $i\in\Upsilon_3$ is not changed at time $t\in\{i-2,\dots,k-2\}$. The vertex $i\in\Upsilon_3$ has at time $t\in\{i-2,\dots,k-2\}$ two neighbouring cooperators (vertices $i-1, i+1$) and one neighbouring defector (vertex $k+4$). Thus, $u_{i\in\Upsilon_3}(t)=\frac{2a+b}{3}$, where $t\in\{i-2,\dots,k-2\}$. As mentioned above, the only neighbouring defector of such vertex in Υ_3 is vertex $k+4$. Thus:

$$u_i(t) > u_{k+4}(t),\ t\in\{i-2,\dots,k-2\},\ i\in\Upsilon_3. \tag{3.15}$$

Since $u_i(t)$ is constant for the conditions mentioned in (3.15) we search for highest possible value of $u_{k+4}(t)$ satisfying conditions. Therefore:

$$\frac{2a+b}{3} > \frac{(k-2)c+kd}{2k-2}. \tag{3.16}$$

Since $\frac{2a+b}{3} > \frac{a+2b}{3}$ (properties of a,b,c,d in Subsection 1.1.2), inequality (3.16) is directly implied by (3.14).

Resetting ($t = k-1$): According to Rule (3.2) $s_{i\in\Upsilon_3}(k-1) = 1$ and $s_{i\in\Upsilon_3}(k) = 0$ (the cyclic property of evolutionary dynamic graph $\mathfrak{E}_2$ gives $s(k) = s(0)$). The only defecting neighbour of all vertices in Υ_3 at time $t = k-1$ is vertex $k+4$. Therefore:

$$u_{k+4}(k-1) > \max_{i\in\Upsilon_3} u_i(k-1). \tag{3.17}$$

The cooperating vertices in Υ_3 can have assigned just two values of utility: $u_{i\in\Upsilon_3\setminus\{k+1\}} = \frac{2a+b}{3}$ and $u_{k+1} = \frac{a+2b}{3}$. Thus (3.17) is equal to:

$$\frac{c+d}{2} > \frac{2a+b}{3}, \tag{3.18}$$

which is exactly the same as (3.5).

Vertices in Υ_4 : Vertex $k+4$ does not change its strategy: $u_{k+4}(t) = 0$ for all $t \in \{0,\dots,k-1\}$. The cooperators with the highest utility in its neighbourhood are vertices from Υ_3. Particularly, vertices $i \in \Upsilon_3$ at time $t \in \{i-2,\dots,k-2\}$ (this computation is described in detail in the paragraph considering vertices in Υ_3). The neighbouring defector of vertex $k+4$ with the highest utility is vertex $k+3$ for all $t \in \{0,\dots,k-1\}$ (neither of neighbours of vertex $k+3$ changes its strategy). Thus:

$$u_{k+3}(t) > u_i(t),\ t \in \{i-2,\dots,k-2\},\ i \in \Upsilon_3. \tag{3.19}$$

Furthermore, simplified to:

$$\frac{c+d}{2} > \frac{2a+b}{3}. \tag{3.20}$$

This complicated deduction results in three inequalities (3.4), (3.5), (3.14). These inequalities were computed with the respect to stated Rules (3.1), (3.2). Now these computations need to be summarized by lemma.

Lemma 5. Let us have evolutionary dynamic graph $\mathfrak{E}_2$ with graph $G_{\mathfrak{E}_2}$ on $|V(G_{\mathfrak{E}_2})|= 2k+2$ vertices constructed as in Construction 3, mean aggregate utility function (1.14), synchronous updating, initial condition $s_{\mathfrak{E}_2}(0) = (1,1,0,\dots,0)$ (Figure 3.1) and parameters a,b,c,d satisfying inequalities (3.4), (3.5), (3.14). This evolutionary dynamic graph $\mathfrak{E}_2$ generates a cycle of length k.

Proof. Inequalities (3.4), (3.5), (3.14) were derived a and described in detail in this chapter. Since inequalities are satisfied, strategy trajectory of each vertex of evolutionary dynamic graph $\mathfrak{E}_2$ satisfy rules (3.1) and (3.2). If strategy trajectories of evolutionary dynamic graph satisfy rules (3.1) and (3.2) and these rules describe a strategy trajectory for which $s_{\mathfrak{E}_2}(t) = s_{\mathfrak{E}_2}(t+k)$ for each $t \in \mathbb{N}_0$ then evolutionary dynamic graph $\mathfrak{E}_2$ generates a cycle of length k. $\square$

We can construct graph $G_{\mathfrak{E}_2}$ as in Construction 3 for each $k \in \mathbb{N}$. Subsequently, we can derive conditions (3.4), (3.5), (3.14) dependent on k. Thus, Lemma 5 provides evolutionary dynamic graph $\mathfrak{E}_2$ to generate a cycle of a given length k. We can construct evolutionary dynamic graph which generates a cycle of arbitrary length k. Furthermore, the number of vertices of graph $G_{\mathfrak{E}_2}$ of evolutionary dynamic graph $\mathfrak{E}_2$ is $|V(G_{\mathfrak{E}_2})| = 2k+2$. The graph size is linearly dependent on the cycle length. Thus, we found a sequence from Conjecture 1 such that it belongs to the set $O(k)$ and we can state a new bound of $D_s(k)$.

Remark 18.

$$D_s(k) \leq 2k + 2, \ k \geq 3 \tag{3.21}$$

3.3 Parameter region

Lemma 5 states inequalities (3.4), (3.5), (3.14) which are needed for the existence of the cycle in evolutionary dynamic graph $\mathfrak{E}_2$. These three inequalities describe a region of admissible values of parameters a, b, c, d. This region is plotted in Figure 3.3 for fixed $a = 1$ and $d = 0$ and few chosen values of cycle length k.

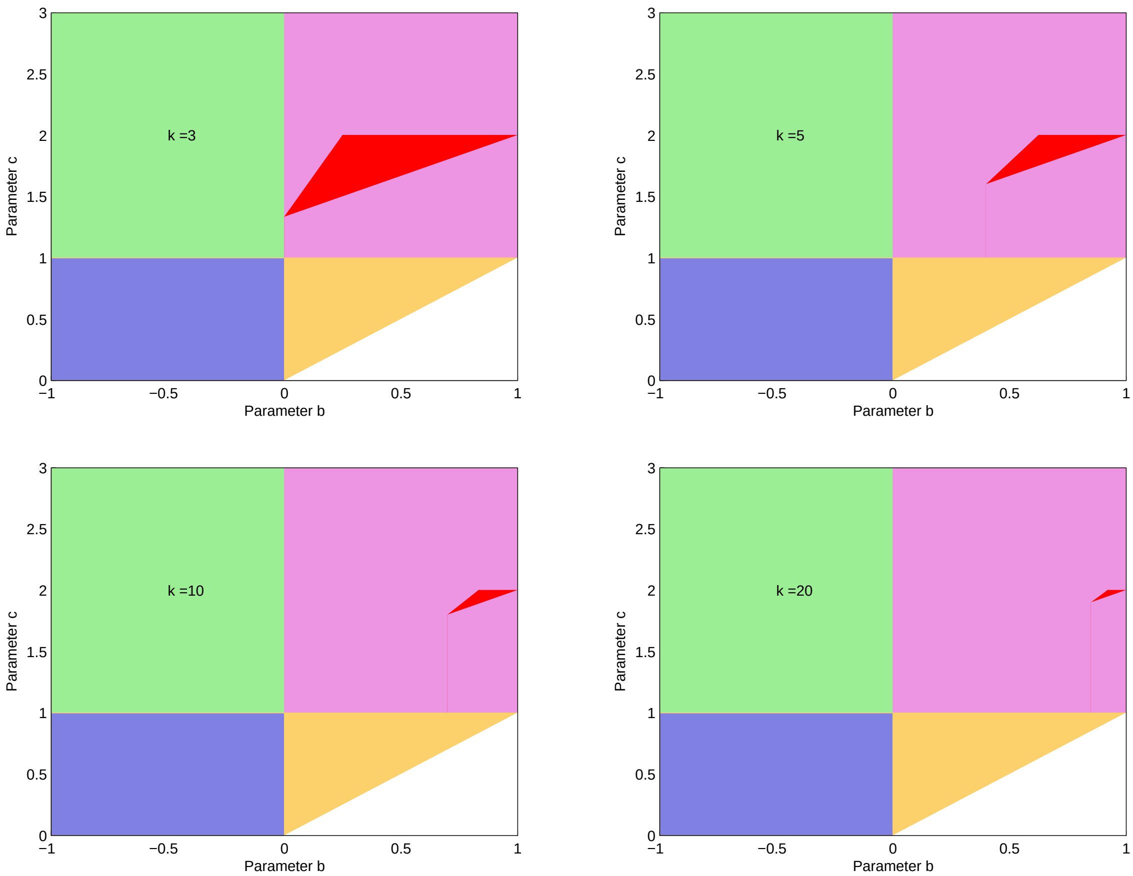

Figure 3.5: Region of admissible values of parameters a, b, c, d (red) of evolutionary dynamic graph $\mathfrak{E}_2$ generating a cycle of length k. Plotted for $k = 2, 5, 10, 20$

It can be easily seen the area of parameter region reducing for increasing k. Furthermore, we can specify the character of reduction of the area of region. For very great k we can observe:

$$\frac{(k-2)c+kd}{2(k-1)} \xrightarrow{k\to\infty} \frac{c+d}{2}. \tag{3.22}$$

Thus, since (3.5), (3.14) gives $\frac{2a+b}{3} < \frac{c+d}{2}$ and $\frac{a+2b}{3} > \frac{(k-2)c+kd}{2k-2}$:

$$\frac{2a+b}{3} \xrightarrow{k\to\infty} \frac{c+d}{2}, \tag{3.23}$$

$$\frac{a+2b}{3} \xrightarrow{k\to\infty} \frac{2a+b}{3}. \tag{3.24}$$

Furthermore, simplified (3.24):

$$a \xrightarrow{k\to\infty} b \tag{3.25}$$

Observation 9. Given $a = 1$ and $d = 0$:

$$b \xrightarrow{k\to\infty} 1 \tag{3.26}$$

$$c \xrightarrow{k\to\infty} 2 \tag{3.27}$$

It can be easily seen that the region of admissible values of parameters a, b, c, d collapses into a single point for fixed $a = 1$ and $d = 0$ and very great k. This is the motivation for further study. Suppose there exists evolutionary dynamic graph $\mathfrak{E}$ which generates a cycle of arbitrary length k and its conditions on parameters a, b, c, d are not dependent on k. Even for very great k, the region of admissible parameters does not collapse into a single point. Assuming the polynomial dependence of the number of vertices on the cycle length is natural. Evolutionary dynamic graph $\mathfrak{E}_1$ satisfy conditions above. Thus, we consider only evolutionary dynamic graph $\mathfrak{E}$ with some initial condition $s_{\mathfrak{E}}(0)$ where exist two strategy vectors $s_{\mathfrak{E}}(t_1), s_{\mathfrak{E}}(t_2)$ such that $s_{\mathfrak{E}}(t_1) \in \mathfrak{m}_1, s_{\mathfrak{E}}(t_2) \in \mathfrak{m}_2$ and $\mathfrak{m}_1 \neq \mathfrak{m}_2$. Contemplations above lead us to next question:

Question 2. *Can we construct an evolutionary dynamic graph $\mathfrak{E}$ which generates a cycle of length k for given value of k with following properties: The inequalities ensuring the existence of a cycle in $\mathfrak{E}$ are not dependent on k. The number of vertices of graph $G_{\mathfrak{E}}$ of evolutionary dynamic graph $G_{\mathfrak{E}}$ is in the set: $|V(G_{\mathfrak{E}})| \in O(k)$. There exist two strategy vectors in the cycle $s_{\mathfrak{E}}(t_1), s_{\mathfrak{E}}(t_2)$ such that $s_{\mathfrak{E}}(t_1) \in \mathfrak{m}_1, s_{\mathfrak{E}}(t_2) \in \mathfrak{m}_2$ and $\mathfrak{m}_1 \neq \mathfrak{m}_2$.*

Chapter 4

Conclusion

This thesis introduced the evolutionary dynamic graphs. The evolutionary dynamic graphs can be described as graphs whose vertices connected with edges play static games with each other. Furthermore, their pure strategy played in the static game is updated in accordance with the success of the strategies of their neighbours. These structures are very complex. Thus, their behaviour is unpredictable. In order to deepen the knowledge of these structures we used simple deductions and examples to describe and investigate their properties.

We study a periodic behaviour of the evolutionary dynamic graphs. Particularly, we study the number of vertices of evolutionary dynamic graph with respect to the length of the cycle they generate. We state two sets of evolutionary dynamic graphs generating a cycle: all evolutionary dynamic graphs generating a cycle and evolutionary dynamic graphs generating a cycle and changing the number of cooperating vertices. We estimate the number of vertices of such evolutionary dynamic graphs generating a cycle of length at least k which is denoted by: $C_s(k)$ for the first case and $D_s(k)$ for the second case (Definitions 21 and 22 on page 10). By examples in Section 2.2 and Chapter 3 and two Lemmas 1 and 2 we stated following inequalities for $k \geq 5$:

$$\begin{aligned} \log_2(k+2) \leq C_s(k) &\leq D_s(k) \leq 2k+2, && (4.1) \\ C_s(k) &\leq 2k. && (4.2) \end{aligned}$$

The estimate of sequences $C_s(k)$ and $D_s(k)$ from above by a sequence with linear growth can considered as the main success of this thesis.

Furthermore, we observed parameters describing static games played. Examples described in this thesis use the same method: we state evolutionary dynamic graph which generates a cycle of length k and subsequently derive conditions providing existence of the cycle. This order of deductions inspires a question:

Question 3. *Can we construct an evolutionary dynamic graph generating a cycle of given length k for given scenario from Table 1.1?*

Furthermore, we can ask question which is even more difficult do answer than the previous one:

Question 4. *Can we construct an evolutionary dynamic graph generating a cycle of given length k for given parameters a, b, c, d?*

Questions stated in this chapter show, that the topic of evolutionary dynamic graphs is very wide. Reader can ask more questions and investigate the evolutionary dynamic graphs from many views. For example, we did consider only evolutionary dynamic graphs with synchronous updating (all vertices are updated at once) or we used mean aggregate utility function only (1.14). Let this paragraph and questions stated be the inspiration for the further research.

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Note to bibliography

The ideas for definitions 7, 8, 9, 10, 13, 11, 14, 15, 16, 17, 18, 19, 20 from Section 1.2 were adopted from article [1]. Since article [1] was a concept, some expressions were changed or improved for the sake of consistence.

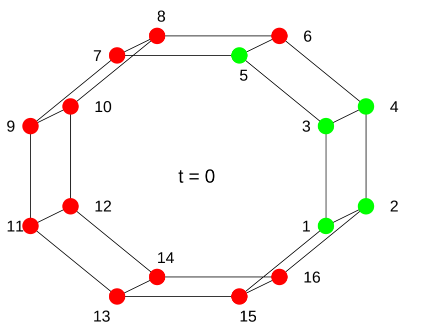
Appendix A

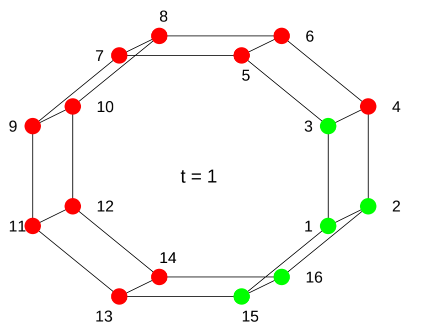
Example from Section 2.2

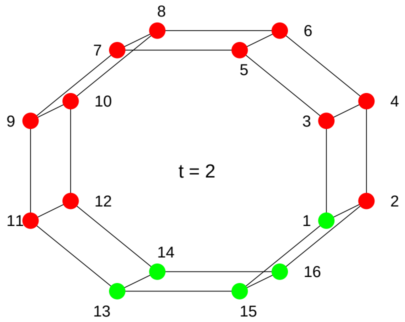
Following figures show example of a cycle generated by evolutionary dynamic graph $\mathfrak{E}_1$ with:

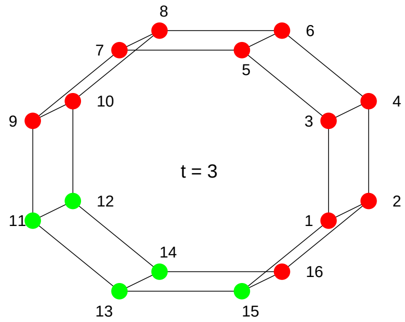
- 3-regular graph $G_{\mathfrak{E}_1}$ with $|V(G_{\mathfrak{E}_1})| = 16$ described in Construction 1.
- initial condition $s_{\mathfrak{E}_1}(0) = (1, 1, 1, 1, 1, 0, \ldots, 0)$ (Figure 2.2).
- paramters a, b, c, d satisfying conditions (2.3), (2.4) and (2.5).
- the mean aggregate utility function (1.14).
- synchronous updating.

  Above described evolutionary dynamic graph generates a cycle of length $k = 8$. It is obvious that strategy vectors at time $t_1 = 0$ and $t_2 = 8$ are equal.






8
6
7
5
10
4
9
3
t = 2
12
2
11
1
14
16
13
15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7
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10
4
9
3
t = 3
12
2
11
1
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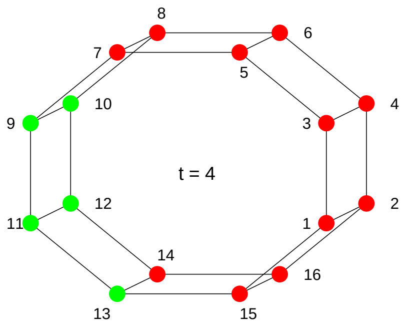
16
13
15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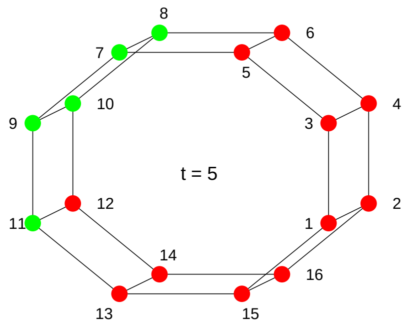
4
9
3
t = 4
12
2
11
1
14
16
13
15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t = 5
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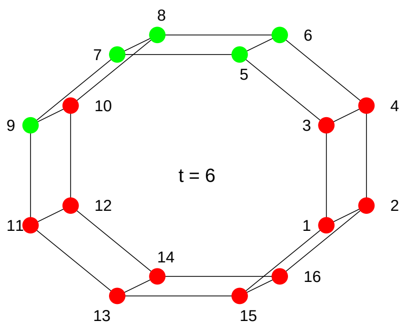
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t = 6
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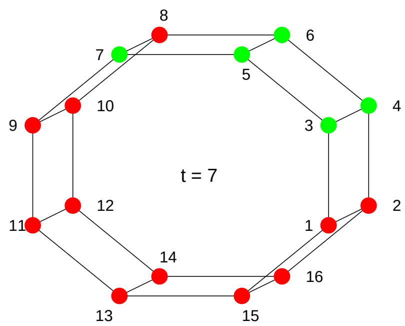
8
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10
4
9
3
t = 7
12
2
11
1
14
16
13
15

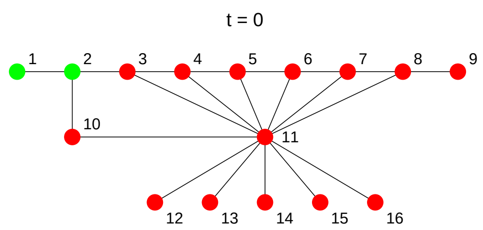
Appendix B

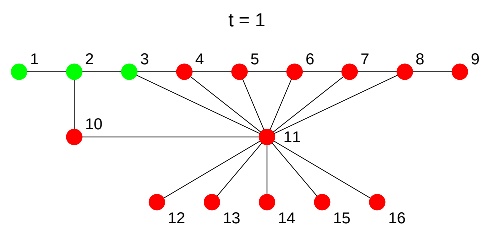
Example from Chapter 3

We captured an example of evolutionary dynamic graph $\mathfrak{E}_2$ from Chapter 3 with:

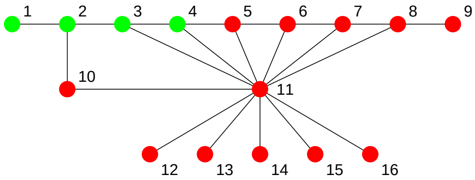
- graph $G_{\mathfrak{E}_2}$, where $|V(G_{\mathfrak{E}_2})|= 16$,
- initial condition $s_{\mathfrak{E}_2}(0) = \{1, 1, 0, \dots, 0\}$,
- parameters a, b, c, d satisfying conditions (3.4), (3.5), (3.14) for $k = 7$,
- the mean aggregate utility function (1.14).
- synchronous updating.

Following figures show the course of the cycle of length $k = 7$ generated by evolutionary dynamic graph $\mathfrak{E}_2$. Note that strategy vectors at times $t_1 = 0$ and $t_2 = 7$ are equivalent, $s_{\mathfrak{E}_2}(0) = s_{\mathfrak{E}_2}(7)$. Also note vertices changing their strategy in accordance with Rules (3.1), (3.2) for $k = 7$.

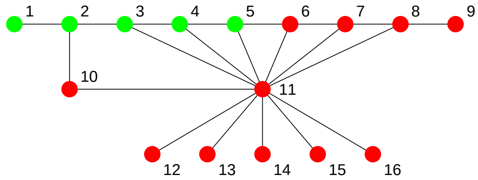




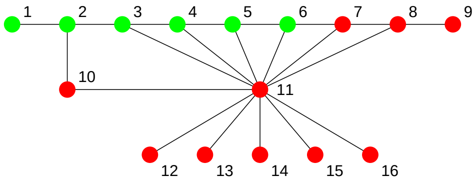
t = 2



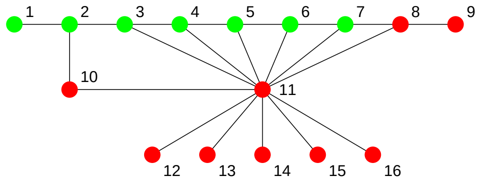
t = 3



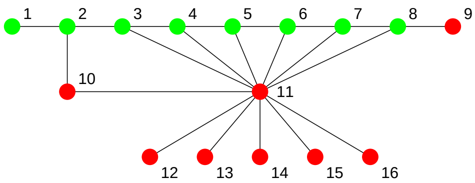
t = 4



t = 5



t = 6



t = 7